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Research Article

Sequential Properties over Negative Pauli Pascal Table

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For polynomials $(x+y)^i$ and $(x+y)^{-i}$ satisfying the noncommutative multiplication $yx=-xy$, let $C^{(-1)}$ and $N^{(-1)}$ be the arithmetic tables, respectively. We investigate sequential properties of various diagonal sums over the tables $C^{(-1)}$ and $N^{(-1)}$ and prove that they are types of interlocked Fibonacci sequence and Padovan sequence.

1. Introduction

The Pascal table $C^{(1)}=[c_{i,j}^{(1)}]$ is an arithmetic table of a polynomial $(x+y)^i=\sum_{j=0}^{i} c_{i,j}^{(1)} x^{i-j} y^j$, $(i\geq 0)$, in which $yx=xy$ is assumed tactically. The Pauli Pascal table $C^{(-1)}=[c_{i,j}^{(-1)}]$ is an arithmetic table of $(x+y)^i=\sum_{j=0}^{i} c_{i,j}^{(-1)} x^{i-j} y^j$ with noncommuting variables x, y such that $yx=-xy$. The $c_{i,j}^{(1)}$ and $c_{i,j}^{(-1)}$ satisfy the following (see [1]):

$$\begin{aligned} c_{i+1,j+1}^{(1)} &= c_{i,j}^{(1)} + c_{i,j+1}^{(1)},\\ c_{i+1,j+1}^{(-1)} &= c_{i,j}^{(-1)} + (-1)^{j+1} c_{i,j+1}^{(-1)},\\ & i, j\geq 0. \end{aligned} \tag{1}$$

It is known that diagonal sums over $C^{(1)}$ give a Fibonacci sequence $\{f_1, f_2, f_3, \ldots,\}$, while those over $C^{(-1)}$ yield a sequence $\{f_1, f_0, f_2, f_1, f_3, f_2, \ldots,\}$ that is interlocked by two Fibonacci sequences [2].

Consider a polynomial $(x+y)^{-i}$ with negative exponent satisfying either $yx=xy$ or $yx=-xy$ and denote the corresponding arithmetic table by either $N^{(1)}=[n_{i,j}^{(1)}]$ or $N^{(-1)}=[n_{i,j}^{(-1)}]$ with $(x+y)^{-i}=\sum_{j=0}^{\infty} n_{i,j}^{(\pm 1)} x^{-i-j} y^j$, respectively.

By a t/s-slope diagonal over a table $(t, s\geq 1)$, we mean a generalized diagonal moving s steps along x-axis and t steps along y-axis. Over $C^{(\pm 1)}$, let $d_{\langle t/s\rangle,i}^{(\pm 1)}$ denote the t/s-slope diagonal set starting from $c_{i,0}^{(\pm 1)}$ $(i\geq 0)$ toward northeast direction, and $D_{\langle t/s\rangle,i}^{(\pm 1)}$ be the sum of elements in $d_{\langle t/s\rangle,i}^{(\pm 1)}$. We call it the t/s-slope ith diagonal sum. Similarly, over $N^{(\pm 1)}$, let $g_{\langle t/s\rangle,j}^{(\pm 1)}$ be the t/s-slope diagonal set starting from $n_{1,j}^{(\pm 1)}$ $(j\geq 0)$ toward southwest direction, and $G_{\langle t/s\rangle,j}^{(\pm 1)}$ be the t/s-slope jth diagonal sum. When $s=1$, we simply say t-slope diagonal sums $D_{\langle t\rangle,i}^{(\pm 1)}$ and $G_{\langle t\rangle,j}^{(\pm 1)}$.

A purpose of the work is to study arithmetic tables $C^{(\pm 1)}$ and $N^{(\pm 1)}$. We investigate sequential properties of generalized diagonal sums $D_{\langle t/s\rangle,i}^{(\pm 1)}$ and $G_{\langle t/s\rangle,j}^{(\pm 1)}$ and find their interrelationships. We particularly give attention to $G_{\langle t/s\rangle,j}^{(-1)}$ of $N^{(-1)}$ and prove $\left\{G_{\langle t/s\rangle,j}^{(-1)}\right\}$ is a type of interlocked Padovan sequence. The results of the work provide interesting connections of sequences over the arithmetic tables of $(x+y)^{\pm i}$ having either commutative $yx=xy$ or noncommutative $yx=-xy$ rules.

2. Arithmetic Table and Its Diagonal Sum

The arithmetic table $N^{(1)}=[n_{i,j}^{(1)}]$ of $(x+y)^{-i}$ with $yx=xy$ can be obtained by Taylor series expansion. Every element $n_{i,j}^{(1)}$ $(i\geq 1, j\geq 0)$ of $N^{(1)}$ in Table 1 satisfies a recurrence rule $n_{i+1,j+1}^{(1)}=n_{i,j+1}^{(1)}-n_{i+1,j}^{(1)}$.

By flipping $N^{(1)}$ and passing it over $C^{(1)}$, the pile-up table

Table 1: $N^{(1)} = [n_{i,j}^{(1)}]$.

1	1	−1	1	−1	1	−1	1	−1	⋯
2	1	−2	3	−4	5	−6	7	−8	⋯
3	1	−3	6	−10	15	−21	28	−36	⋯
4	1	−4	10	−20	35	−56	84	−120	⋯
5	1	−5	15	−35	70	−126	210	−330	⋯

$[N^{(1)}/C^{(1)}] =$

−5	1	−5	15	−35	70	−126	⋯
−4	1	−4	10	−20	35	−56	⋯
−3	1	−3	6	−10	15	−21	⋯
−2	1	−2	3	−4	5	−6	⋯
−1	1	−1	1	−1	1	−1	⋯
0	1						
1	1	1					

follows the Pascal rule (1). For example, we get $(x+y)^{-4} = x^{-4}\quad(1 - 4x^{-1}y + 10x^{-2}y^2 - 20x^{-3}y^3 + 35x^{-4}y^4+,\ldots,)$. Some recurrence rules of t-slope diagonal sum $D_{\langle t\rangle,i}^{(\pm 1)}$ over $C^{(\pm 1)}$ and $1/t$-slope diagonal sum $G_{\langle 1/t\rangle,j}^{(1)}$ over $N^{(1)}$ were studied as follows.

Lemma 1 (see [3, 4]). *$D_{\langle t\rangle,i}^{(1)} + D_{\langle t\rangle,i+t}^{(1)} = D_{\langle t\rangle,i+t+1}^{(1)}$ with $t+1$ initials $1,\ldots,1$. And $G_{\langle 1/t\rangle,i}^{(1)} - G_{\langle 1/t\rangle,i+t-1}^{(1)} = G_{\langle 1/t\rangle,i+t}^{(1)}$ with t initials $1,-1,1,-1,\ldots,$. Moreover, $D_{\langle t\rangle,i}^{(-1)} + D_{\langle t\rangle,i+2t}^{(-1)} = D_{\langle t\rangle,i+2t+2}^{(-1)}$ $(i \geq 2(t+1))$ with initials $\underbrace{1,\ldots,1}_{t+1}$, $\underbrace{2,1,2,1,2,1,\ldots,}_{t+1}$.*

The proof of Lemma 1 is mainly due to

$$n_{i,j}^{(1)} = (-1)^j c_{i+j-1,j}^{(1)}, n_{i,0}^{(1)} = c_{i,0}^{(1)} = 1 \text{ and } n_{i,1}^{(1)} = -c_{i,1}^{(1)} = -i. \tag{2}$$

Let $\{D_{\langle 1\rangle,j}^{(-1)e}\}$ and $\{D_{\langle 1\rangle,j}^{(-1)o}\}$ be subsequences having only eventh and oddth terms, respectively, in $\{D_{\langle 1\rangle,i}^{(-1)} \mid i \geq 0\}$. From Table 2, we observe

$$\begin{aligned}\{D_{\langle 1\rangle,i}^{(-1)} \mid i \geq 0\} &= \{1,2,3,5,8,13,21,\ldots,\} \cup \{1,1,2,3,5,8,13,\ldots,\}\\ &= \{D_{\langle 1\rangle,i}^{(-1)e} \mid j \geq 0\} \cup \{D_{\langle 1\rangle,j}^{(-1)o} \mid j \geq 0\}.\end{aligned} \tag{3}$$

Since both $\{D_{\langle 1\rangle,j}^{(-1)e}\}$ and $\{D_{\langle 1\rangle,j}^{(-1)o}\}$ are Fibonacci sequences, we say $\{D_{\langle 1\rangle,i}^{(-1)}\}$ is an interlocked Fibonacci sequence. For any $t \geq 1$, a sequence $\{f_n\}$ satisfying $f_n + f_{n+t} = f_{n+t+1}$ with $t+1$ initials is called a Fibo t-sequence [5].

Theorem 1. *$\{D_{\langle t\rangle,i}^{(-1)}\}$ is an interlocked Fibo t-sequence for $t \geq 1$.*

Proof. Let $\{D_{\langle t\rangle,j}^{(-1)e}\}$ and $\{D_{\langle t\rangle,j}^{(-1)o}\}$ be subsequences consisting of eventh or oddth terms, respectively, in $\{D_{\langle t\rangle,i}^{(-1)} \mid i \geq 0\}$. When $t = 1$, $\{D_{\langle 1\rangle,i}^{(-1)}\}$ is clearly an interlocked Fibo 1-sequence. When $t = 2, 3$, Table 2 shows that

$$\begin{aligned}\{D_{\langle 2\rangle,i}^{(-1)} \mid i \geq 0\} &= \{1,1,1,2,3,4,6,9,13,19,\ldots,\}\\ &\cup \{1,2,2,3,5,7,10,15,\ldots,\}\\ &= \{D_{\langle 2\rangle,j}^{(-1)e} \mid j \geq 0\} \cup \{D_{\langle 2\rangle,j}^{(-1)o} \mid j \geq 0\},\end{aligned} \tag{4}$$

where $\{D_{\langle 2\rangle,j}^{(-1)e}\}$ and $\{D_{\langle 2\rangle,j}^{(-1)o}\}$ are Fibo 2-sequences with initials 1, 1, 1 and 1, 2, 2. Similarly, $\{D_{\langle 3\rangle,i}^{(-1)} \mid i \geq 0\} = \{D_{\langle 3\rangle,j}^{(-1)e} \mid j \geq 0\} \cup \{D_{\langle 3\rangle,j}^{(-1)o} \mid j \geq 0\}$ shows $\{D_{\langle 3\rangle,j}^{(-1)e}\} = \{1,1,2,2,3,4,6,8,11,\ldots,\}$ and $\{D_{\langle 3\rangle,j}^{(-1)o}\} = \{1,1,1,1,2,3,4,5,7,10,\ldots,\}$ are Fibo 3-sequences having initials 1, 1, 2, 2 and 1, 1, 1, 1. So, $\{D_{\langle t\rangle,i}^{(-1)}\}$ with $t = 2, 3$ is an interlocked Fibo t-sequence.

In general, for any $t \geq 1$, Lemma 1 shows that the sequence

$$\{\ldots, D_{\langle t\rangle,i}^{(-1)}, D_{\langle t\rangle,i+1}^{(-1)}, \ldots, D_{\langle t\rangle,i+2t}^{(-1)}, D_{\langle t\rangle,i+2t+1}^{(-1)}, D_{\langle t\rangle,i+2t+2}^{(-1)}, \ldots,\} \tag{5}$$

holds $D_{\langle t\rangle,i}^{(-1)} + D_{\langle t\rangle,i+2t}^{(-1)} = D_{\langle t\rangle,i+2t+2}^{(-1)}$ $(i \geq 2(t+1))$ with initials $\underbrace{1,\ldots,1}_{t+1}, \underbrace{2,1,2,1,\ldots,}_{t+1}$. Thus, $\{D_{\langle t\rangle,j}^{(-1)e} \mid j \geq 0\}$ and $\{D_{\langle t\rangle,j}^{(-1)o} \mid j \geq 0\}$ satisfy recurrences $D_{\langle t\rangle,j}^{(-1)e} + D_{\langle t\rangle,j+t}^{(-1)e} = D_{\langle t\rangle,j+t+1}^{(-1)e}$ and $D_{\langle t\rangle,j}^{(-1)o} + D_{\langle t\rangle,j+t}^{(-1)o} = D_{\langle t\rangle,j+t+1}^{(-1)o}$. So, they are Fibo t-sequences having initials $\begin{cases}\{1,\ldots,1\}_{(t+1)\text{tuples}} & 2 \mid t\\ \{\underbrace{1,\ldots,1}_{((t+1)/2)}, \underbrace{2,\ldots,2}_{((t+1)/2)}\} & 2 \nmid t\end{cases}$ and $\begin{cases}\{\underbrace{1,\ldots,1}_{(t/2)}, \underbrace{2,\ldots,2}_{(t/2)+1}\} & 2 \mid t\\ \{1,\ldots,1\}_{(t+1)\text{tuples}} & 2 \nmid t\end{cases}$, respectively. Hence, $\{D_{\langle t\rangle,i}^{(-1)}\}$ is an interlocked Fibo t-sequence.

Now, in order to have the table $N^{(-1)} = [n_{i,j}^{(-1)}]$ of $(x+y)^{-i}$ with $yx = -xy$, look at the piled-up table

$[N^{(-1)}/C^{(-1)}] =$

−4	1	0	−2	0	3	0	−4	⋯
−3	1	1	−2	−2	3	3	−4	⋯
−2	1	0	−1	0	1	0	−1	⋯
−1	1	1	−1	−1	1	1	−1	⋯
0	1							
1	1	1						
2	1	0	1					

satisfying the Pauli rule (1).

Then, by flipping the upper part upside down, we get $N^{(-1)}$ (Table 3) holding

$$n_{i+1,j+1}^{(-1)} = (-1)^{j+1}\left(n_{i,j+1}^{(-1)} - n_{i+1,j}^{(-1)}\right) \quad \text{for } i \geq 1, j \geq 0. \tag{6}$$

Thus, $C^{(-1)}$ and $N^{(-1)}$ yield expansions of $(x+y)^{\pm i}$ with $yx = -xy$, for instance, $(x+y)^{-5} = x^{-5}\ (1 + x^{-1}y - 3x^{-2}y^2 - 3x^{-3}y^3 + 6x^{-4}y^4 + 6x^{-5}y^5+,\cdots,)$.

Table 2: $G^{(1)}_{\langle 1/t\rangle,i}$ and $D^{(-1)}_{\langle t\rangle,i}$.

t/i	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
1	0	0	0	0	0	0	0	0	0	1	1	2	1	3	2	5	3	8
2	−1	2	−3	5	−8	13	−21	34	−55	1	1	1	2	1	2	2	3	3
3	−1	1	0	−1	2	−2	1	1	−3	1	1	1	1	2	1	2	1	3
4	−1	1	−1	2	−3	4	−5	7	−10	1	1	1	1	1	2	1	2	1
5	−1	1	−1	1	0	−1	2	−3	4									

Table 3: $N^{(-1)} = [n^{(-1)}_{i,j}]$.

	0	1	2	3	4	5	6	7	8	9	10···
1	1	1	−1	−1	1	1	−1	−1	1	1	−1···
2	1	0	−1	0	1	0	−1	0	1	0	−1···
3	1	1	−2	−2	3	3	−4	−4	5	5	−6···
4	1	0	−2	0	3	0	−4	0	5	0	−6···
5	1	1	−3	−3	6	6	−10	−10	15	15	−21···

Theorem 2. $n^{(-1)}_{i,j} = \begin{cases} c^{(-1)}_{i+j-1,j} & j \equiv 0,1 \,(\mathrm{mod}\,4) \\ -c^{(-1)}_{i+j-1,j} & j \equiv 2,3 \,(\mathrm{mod}\,4) \end{cases}$. *In particular, $n^{(-1)}_{2i,2j+1} = 0$ and $n^{(-1)}_{2i-1,2j} = n^{(-1)}_{2i-1,2j+1} = n^{(-1)}_{2i,2j}$ for any $i, j \geq 1$.*

Proof. Since $\begin{cases} n^{(-1)}_{5,4} = 6 \\ c^{(-1)}_{8,4} = 6 \end{cases}$ $\begin{cases} n^{(-1)}_{5,5} = 6 \\ c^{(-1)}_{9,5} = 6 \end{cases}$ $\begin{cases} n^{(-1)}_{5,6} = -10 \\ c^{(-1)}_{10,6} = 10 \end{cases}$ $\begin{cases} n^{(-1)}_{5,7} = -10 \\ c^{(-1)}_{11,7} = 10 \end{cases}$ $\begin{cases} n^{(-1)}_{5,8} = 15 \\ c^{(-1)}_{12,8} = 15 \end{cases}$, we may assume $n^{(-1)}_{i,j} = \begin{cases} c^{(-1)}_{i+j-1,j}, & j \equiv 0,1 \,(\mathrm{mod}\,4) \\ -c^{(-1)}_{i+j-1,j}, & j \equiv 2,3 \,(\mathrm{mod}\,4) \end{cases}$ for some i, j.

If $j \equiv 0 \,(\mathrm{mod}\,4)$, then recurrence (6) implies

$$\begin{aligned} n^{(-1)}_{i+1,j} &= (-1)^j \left(n^{(-1)}_{i,j} - n^{(-1)}_{i+1,j-1} \right) = n^{(-1)}_{i,j} - n^{(-1)}_{i+1,j-1} \\ &= c^{(-1)}_{i+j-1,j} - (-1) c^{(-1)}_{i+1,j-1} = (-1)^j c^{(-1)}_{i+j-1,j} + c^{(-1)}_{i+1,j-1} = c^{(-1)}_{i+j,j}. \end{aligned} \tag{7}$$

Similarly, if $j \equiv 1 \,(\mathrm{mod}\,4)$, then

$$n^{(-1)}_{i+1,j} = -\left(c^{(-1)}_{i+j-1,j} - c^{(-1)}_{i+1,j-1} \right) = (-1)^j c^{(-1)}_{i+j-1,j} + c^{(-1)}_{i+1,j-1} = c^{(-1)}_{i+j,j}. \tag{8}$$

The other cases $j \equiv 2,3 \,(\mathrm{mod}\,4)$ can be proved analogously. In particular, $(-1)(n^{(-1)}_{2i-1,2j+1} - n^{(-1)}_{2i,2j}) = n^{(-1)}_{2i,2j+1} = 0$, so $n^{(-1)}_{2i,2j} = n^{(-1)}_{2i-1,2j+1}$.

Theorem 2 can be compared to $n^{(1)}_{i,j} = \begin{cases} c^{(1)}_{i+j-1,j} & j \equiv 0 \,(\mathrm{mod}\,2) \\ -c^{(1)}_{i+j-1,j} & j \equiv 1 \,(\mathrm{mod}\,2) \end{cases}$ in (2).

3. Diagonal Sum over $N^{(-1)}$

We will discuss t-slope diagonal $g^{(-1)}_{\langle t\rangle,i}$ and its sum $G^{(-1)}_{\langle t\rangle,i}$ on $N^{(-1)}$.

Theorem 3. *When $t = 1$, $G^{(-1)}_{\langle 1\rangle,0} = 1$, $G^{(-1)}_{\langle 1\rangle,1} = 2$, and $G^{(-1)}_{\langle 1\rangle,i} = 0$ for $i \geq 2$.*

When $t = 1/2$, $(-1)^i (G^{(-1)}_{\langle 1/2\rangle,i} - G^{(-1)}_{\langle 1/2\rangle,i+1}) = G^{(-1)}_{\langle 1/2\rangle,i+2}$ for $i \geq 1$.

Proof. The first few 1-slope diagonal sets and sums in $N^{(-1)}$ are as follows:

$$\begin{array}{l|l} g^{(-1)}_{\langle 1\rangle,0} = \{1\}, \quad G^{(-1)}_{\langle 1\rangle,0} = 1 & g^{(-1)}_{\langle 1\rangle,2} = \{-1, 0, 1\}, \quad G^{(-1)}_{\langle 1\rangle,2} = 0 \\ \hline g^{(-1)}_{\langle 1\rangle,1} = \{1, 1\}, \; G^{(-1)}_{\langle 1\rangle,1} = 2 & g^{(-1)}_{\langle 1\rangle,3} = \{-1, -1, 1, 1\}, \; G^{(-1)}_{\langle 1\rangle,3} = 0 \end{array} \tag{9}$$

In general, by (6) and $n^{(-1)}_{2i,2j+1} = 0$ for all i, j (Theorem 2), we have

$$\begin{aligned} G^{(-1)}_{\langle 1\rangle,2l} &= n^{(-1)}_{1,2l} + n^{(-1)}_{2,2l-1} + n^{(-1)}_{3,2l-2} +, \ldots, + n^{(-1)}_{2l,1} + n^{(-1)}_{2l+1,0} \\ &= n^{(-1)}_{1,2l} + n^{(-1)}_{3,2l-2} + n^{(-1)}_{5,2l-4} + n^{(-1)}_{7,2l-6} +, \ldots, + n^{(-1)}_{2l-1,2} + n^{(-1)}_{2l+1,0} \\ &= n^{(-1)}_{1,2l} + \left(n^{(-1)}_{2,2l-2} - n^{(-1)}_{3,2l-3} \right) +, \ldots, + \left(n^{(-1)}_{2l-2,2} - n^{(-1)}_{2l-1,1} \right) + n^{(-1)}_{2l+1,0} \\ &= \left(n^{(-1)}_{1,2l} + n^{(-1)}_{2,2l-2} \right) -, \ldots, - \left(n^{(-1)}_{2l-3,3} - n^{(-1)}_{2l-2,2} \right) - n^{(-1)}_{2l-1,1} + n^{(-1)}_{2l+1,0} = 0, \end{aligned} \tag{10}$$

because $n^{(-1)}_{2k-1,2l+1} = n^{(-1)}_{2k,2l}$, $n^{(-1)}_{1,2l} = -n^{(-1)}_{1,2l-1} = -n^{(-1)}_{2,2l-2}$ and $n^{(-1)}_{2l-1,1} = n^{(-1)}_{2l,0} = n^{(-1)}_{2l+1,0}$ in Theorem 2.

Similarly, $n^{(-1)}_{2k-1,2l+1} = n^{(-1)}_{2k,2l} = n^{(-1)}_{2k-1,2l}$ and $n^{(-1)}_{2k,2l+1} = 0$ also show

$$\begin{aligned} G^{(-1)}_{\langle 1\rangle,2l+1} &= \left(n^{(-1)}_{1,2l+1} + n^{(-1)}_{2,2l} \right) + \left(n^{(-1)}_{3,2l-1} + n^{(-1)}_{4,2l-2} \right) +, \cdots, + \left(n^{(-1)}_{2l+1,1} + n^{(-1)}_{2l+2,0} \right) \\ &= 2\left(n^{(-1)}_{2,2l} + n^{(-1)}_{4,2l-2} + n^{(-1)}_{6,2l-4} +, \cdots, + n^{(-1)}_{2l,2} + n^{(-1)}_{2l+2,0} \right) \\ &= 2\left(n^{(-1)}_{1,2l} + n^{(-1)}_{3,2l-2} + n^{(-1)}_{5,2l-4} +, \cdots, + n^{(-1)}_{2l-1,2} + n^{(-1)}_{2l+1,0} \right) \\ &= 2\left(n^{(-1)}_{1,2l} + n^{(-1)}_{2,2l-1} + n^{(-1)}_{3,2l-2} +, \cdots, + n^{(-1)}_{2l-1,2} + n^{(-1)}_{2l,1} + n^{(-1)}_{2l+1,0} \right) \\ &= 2G^{(-1)}_{\langle 1\rangle,2l} = 0. \end{aligned} \tag{11}$$

Therefore, we have $\left\{ G^{(-1)}_{\langle 1\rangle,i} \mid i \geq 0 \right\} = \{1, 2, 0, 0, 0, 0, 0, \ldots\}$.

The 1/2-slope diagonals sets and their sums $G^{(-1)}_{\langle 1/2\rangle,i}$ in $N^{(-1)}$ are as follows:

i	$g^{(-1)}_{\langle 1/2\rangle,i}$	$G^{(-1)}_{\langle 1/2\rangle,i}$	i	$g^{(-1)}_{\langle 1/2\rangle,i}$	$G^{(-1)}_{\langle 1/2\rangle,i}$	i	$g^{(-1)}_{\langle 1/2\rangle,i}$	$G^{(-1)}_{\langle 1/2\rangle,i}$
4	{1, −1, 1}	1	6	{−1, 1, −2, 1}	−1	8	{1, −1, 3, −2, 1}	2
5	{1, 0, 1}	2	7	{−1, 0, −2, 0}	−3	9	{1, 0, 3, 0, 1}	5

(12)

Thus, $\left\{ G^{(-1)}_{\langle 1/2\rangle,i} \mid i \geq 1 \right\} = \{1, 0, -1, 1, 2, -1, -3, 2, 5, -3, \ldots\}$ satisfies

$$\begin{aligned} G^{(-1)}_{\langle 1/2\rangle,5} - G^{(-1)}_{\langle 1/2\rangle,6} &= -G^{(-1)}_{\langle 1/2\rangle,7}, \\ G^{(-1)}_{\langle 1/2\rangle,6} - G^{(-1)}_{\langle 1/2\rangle,7} &= G^{(-1)}_{\langle 1/2\rangle,8}. \end{aligned} \tag{13}$$

So, we have $G^{(-1)}_{\langle 1/2\rangle,i} - G^{(-1)}_{\langle 1/2\rangle,i+1} = (-1)^i G^{(-1)}_{\langle 1/2\rangle,i+2}$ for $1 \leq i \leq 7$.

Now, when $i = 2l + 1$, we have

$$\begin{aligned}
G^{(-1)}_{\langle 1/2\rangle,2l+1} &= n^{(-1)}_{1,2l+1} + n^{(-1)}_{2,2l-1} + n^{(-1)}_{3,2l-3} +, \ldots, + n^{(-1)}_{l,3} + n^{(-1)}_{l+1,1} \\
&= n^{(-1)}_{1,2l+1} + (-1)\left(n^{(-1)}_{1,2l-1} - n^{(-1)}_{2,2l-2}\right) \\
&\quad + (-1)\left(n^{(-1)}_{2,2l-3} - n^{(-1)}_{3,2l-4}\right) + (-1)\left(n^{(-1)}_{3,2l-5} - n^{(-1)}_{4,2l-6}\right) \\
&\quad +, \ldots, +(-1)\left(n^{(-1)}_{l-1,3} - n^{(-1)}_{l,2}\right) + n^{(-1)}_{l+1,1} \\
&= -\left(n^{(-1)}_{1,2l-1} + n^{(-1)}_{2,2l-3} + n^{(-1)}_{3,2l-5} +, \ldots, + n^{(-1)}_{l-1,3} + n^{(-1)}_{l+1,1}\right) \\
&\quad + \left(n^{(-1)}_{1,2l} + n^{(-1)}_{2,2l-2} + n^{(-1)}_{3,2l-4} + n^{(-1)}_{4,2l-6} +, \ldots, + n^{(-1)}_{l,2}\right) \\
&= -G^{(-1)}_{\langle 1/2\rangle,2l-1} + G^{(-1)}_{\langle 1/2\rangle,2l},
\end{aligned} \tag{14}$$

for $n^{(-1)}_{1,2l+1} = n^{(-1)}_{1,2l}$. Hence, $G^{(-1)}_{\langle 1/2\rangle,2l-1} - G^{(-1)}_{\langle 1/2\rangle,2l} = -G^{(-1)}_{\langle 1/2\rangle,2l+1}$.

On the other hand, when $i = 2l$, we have

$$\begin{aligned}
G^{(-1)}_{\langle 1/2\rangle,2l} &= n^{(-1)}_{1,2l} + \left(n^{(-1)}_{1,2l-2} - n^{(-1)}_{2,2l-3}\right) +, \cdots, + \left(n^{(-1)}_{l-1,2} - n^{(-1)}_{l,0}\right) + n^{(-1)}_{l+1,0} \\
&= \left(n^{(-1)}_{1,2l-2} + n^{(-1)}_{2,2l-4} + \cdots + n^{(-1)}_{l+1,0}\right) \\
&\quad - \left(n^{(-1)}_{1,2l-1} + n^{(-1)}_{2,2l-3} + \cdots + n^{(-1)}_{l,0}\right) \\
&= G^{(-1)}_{\langle 1/2\rangle,2l-2} - G^{(-1)}_{\langle 1/2\rangle,2l-1},
\end{aligned} \tag{15}$$

for $n^{(-1)}_{1,2l} = -n^{(-1)}_{1,2l-1}$. Thus, $G^{(-1)}_{\langle 1/2\rangle,2l-2} - G^{(-1)}_{\langle 1/2\rangle,2l-1} = G^{(-1)}_{\langle 1/2\rangle,2l}$.

Let us continue to work with 1/3-slope diagonals in $N^{(-1)}$.

Theorem 4. *For* $i \geq 1$, $G^{(-1)}_{\langle 1/3\rangle,i} + (-1)^i G^{(-1)}_{\langle 1/3\rangle,i+2} = G^{(-1)}_{\langle 1/3\rangle,i+3}$.

Proof. Observe $G^{(-1)}_{\langle 1/3\rangle,5} - G^{(-1)}_{\langle 1/3\rangle,7} = G^{(-1)}_{\langle 1/3\rangle,8}$ and $G^{(-1)}_{\langle 1/3\rangle,6} + G^{(-1)}_{\langle 1/3\rangle,8} = G^{(-1)}_{\langle 1/3\rangle,9}$ from

i	$g^{(-1)}_{\langle 1/3\rangle,i}$	$G^{(-1)}_{\langle 1/3\rangle,i}$	i	$g^{(-1)}_{\langle 1/3\rangle,i}$	$G^{(-1)}_{\langle 1/3\rangle,j}$	i	$g^{(-1)}_{\langle 1/3\rangle,i}$	$G^{(-1)}_{\langle 1/3\rangle,j}$
5	{1, −1}	0	7	{−1, 1, 1}	1	9	{1, −1, −2, 1}	−1
6	{−1, 0, 1}	0	8	{1, 0, −2}	−1	10	{−1, 0, 3, 0}	2

(16)

Let $2 \nmid i$. Since $n^{(-1)}_{1,i+3} = -n^{(-1)}_{1,i+2}$ and $n^{(-1)}_{p,q} = 0$ for $2|p$, $2 \nmid q$, we have

$$\begin{aligned}
&G^{(-1)}_{\langle 1/3\rangle,i} + (-1)^i G^{(-1)}_{\langle 1/3\rangle,i+2} \\
&= \left(n^{(-1)}_{1,i} + n^{(-1)}_{2,i-3} + n^{(-1)}_{3,i-6} + \cdots\right) - \left(n^{(-1)}_{1,i+2} + n^{(-1)}_{2,i-1} + n^{(-1)}_{3,i-4} + \cdots\right) \\
&= -n^{(-1)}_{1,i+2} + \left(n^{(-1)}_{1,i} - n^{(-1)}_{2,i-1}\right) + \left(n^{(-1)}_{2,i-3} - n^{(-1)}_{3,i-4}\right) + \left(n^{(-1)}_{3,i-6} - n^{(-1)}_{4,i-7}\right) + \cdots \\
&= -n^{(-1)}_{1,i+2} + (-1)^i n^{(-1)}_{2,i} + n^{(-1)}_{3,i-3} + (-1)^i n^{(-1)}_{4,i-6} + n^{(-1)}_{5,i-9} + \cdots \\
&= n^{(-1)}_{1,i+3} + n^{(-1)}_{2,i} + n^{(-1)}_{3,i-3} + n^{(-1)}_{4,i-6} + n^{(-1)}_{5,i-9} + \cdots \\
&= G^{(-1)}_{\langle 1/3\rangle,i+3}.
\end{aligned} \tag{17}$$

Now, let $2|i$. Again with $n^{(-1)}_{p,q} = 0$ $(2|p, 2 \nmid q)$, we also have

$$\begin{aligned}
&G^{(-1)}_{\langle 1/3\rangle,i} + (-1)^i G^{(-1)}_{\langle 1/3\rangle,i+2} \\
&= \left(n^{(-1)}_{1,i} + n^{(-1)}_{2,i-3} + n^{(-1)}_{3,i-6} + \cdots\right) - \left(n^{(-1)}_{1,i+2} + n^{(-1)}_{2,i-1} + n^{(-1)}_{3,i-4} + \cdots\right) \\
&= -n^{(-1)}_{1,i+2} + \left(n^{(-1)}_{1,i} - n^{(-1)}_{2,i-1}\right) + \left(n^{(-1)}_{2,i-3} - n^{(-1)}_{3,i-4}\right) + \left(n^{(-1)}_{3,i-6} - n^{(-1)}_{4,i-7}\right) + \cdots \\
&= n^{(-1)}_{1,i+3} + n^{(-1)}_{2,i} + n^{(-1)}_{3,i-3} + n^{(-1)}_{4,i-6} + n^{(-1)}_{5,i-9} + \cdots \\
&= G^{(-1)}_{\langle 1/3\rangle,i+3}.
\end{aligned} \tag{18}$$

We now have recurrence rules of $1/t$-slope diagonal sums over $N^{(-1)}$.

Theorem 5. $G^{(-1)}_{\langle 1/t\rangle,i} - G^{(-1)}_{\langle 1/t\rangle,i+t-1} = (-1)^i G^{(-1)}_{\langle 1/t\rangle,i+t}$ *with even* $t > 1$.

And $G^{(-1)}_{\langle 1/t\rangle,i} + (-1)^i G^{(-1)}_{\langle 1/t\rangle,i+t-1} = G^{(-1)}_{\langle 1/t\rangle,i+t}$ *with odd* $t > 1$.

Proof. Let $2|t$. Since $n^{(-1)}_{1,i+t-1} = (-1)^i n^{(-1)}_{1,i+t}$, recurrence (6) yields

$$\begin{aligned}
&G^{(-1)}_{\langle 1/t\rangle,i} - G^{(-1)}_{\langle 1/t\rangle,i+t-1} \\
&= \left(n^{(-1)}_{1,i} + n^{(-1)}_{2,i-t} + n^{(-1)}_{3,i-2t} + n^{(-1)}_{4,i-3t} + \cdots + n^{(-1)}_{\lfloor (i/t)\rfloor,i-(\lfloor (i/t)\rfloor-1)t}\right) \\
&\quad - \left(n^{(-1)}_{1,i+t-1} + n^{(-1)}_{2,i-1} + n^{(-1)}_{3,i-t-1} + \cdots + n^{(-1)}_{\lfloor (i/t)\rfloor+1,i-(\lfloor (i/t)\rfloor-1)t-1}\right) \\
&= -n^{(-1)}_{1,i+t-1} + \left(n^{(-1)}_{1,i} - n^{(-1)}_{2,i-1}\right) + \left(n^{(-1)}_{2,i-t} - n^{(-1)}_{3,i-t-1}\right) \\
&\quad + \left(n^{(-1)}_{3,i-2t} - n^{(-1)}_{4,i-2t-1}\right) + \cdots + \left(n^{(-1)}_{\lfloor (i/t)\rfloor,i-(\lfloor (i/t)\rfloor-t1)t} - n^{(-1)}_{\lfloor (i/t)\rfloor+1,i-(\lfloor (i/t)\rfloor-t1)t-1}\right) \\
&= (-1)^i n^{(-1)}_{1,i+t} + (-1)^i n^{(-1)}_{2,i} + (-1)^i n^{(-1)}_{3,i-t} + \cdots + (-1)^i n^{(-1)}_{\lfloor (i/t)\rfloor+1,i-(\lfloor (i/t)\rfloor-1)t} \\
&= (-1)^i G^{(-1)}_{\langle 1/t\rangle,i+t}.
\end{aligned} \tag{19}$$

Now, assume $2 \nmid t$ and $2 \mid i$. Then, $n^{(-1)}_{2,i-1} = n^{(-1)}_{2,i-t} = n^{(-1)}_{4,i-2t-1} = n^{(-1)}_{4,i-3t} =, \cdots, = 0$ for $2 \nmid i - t$. However, since $n^{(-1)}_{1,i+t-1} = n^{(-1)}_{1,i+t}$, we have

$$\begin{aligned}
&G^{(-1)}_{\langle 1/t\rangle,i} + (-1)^i G^{(-1)}_{\langle 1/t\rangle,i+t-1} \\
&= \left(n^{(-1)}_{1,i} + n^{(-1)}_{2,i-t} + n^{(-1)}_{3,i-2t} + \cdots\right) - \left(n^{(-1)}_{1,i+t-1} + n^{(-1)}_{2,i-1} + n^{(-1)}_{3,i-t-1} + \cdots\right) \\
&= n^{(-1)}_{1,i+t-1} + \left(n^{(-1)}_{1,i} + n^{(-1)}_{2,i-1}\right) + \left(n^{(-1)}_{2,i-t} + n^{(-1)}_{3,i-t-1}\right) + \cdots \\
&= n^{(-1)}_{1,i+t-1} + \left(n^{(-1)}_{1,i} - n^{(-1)}_{2,i-1}\right) - \left(n^{(-1)}_{2,i-t} - n^{(-1)}_{3,i-t-1}\right) + \cdots \\
&= n^{(-1)}_{1,i+t} + (-1)^i n^{(-1)}_{2,i} - (-1)^{i-t} n^{(-1)}_{3,i-t} + (-1)^i n^{(-1)}_{4,i-2t} + \cdots \\
&= n^{(-1)}_{1,i+t} + n^{(-1)}_{2,i} + n^{(-1)}_{3,i-t} + n^{(-1)}_{4,i-2t} + n^{(-1)}_{5,i-3t} + \cdots \\
&= G^{(-1)}_{\langle 1/t\rangle,i+t}.
\end{aligned} \tag{20}$$

Similarly, if $2 \nmid t$ and $2 \nmid i$, then $n^{(-1)}_{2,i} = n^{(-1)}_{4,i-2t} = n^{(-1)}_{6,i-4t} =, \cdots, = 0$, so

$$\begin{aligned}
&G^{(-1)}_{\langle 1/t\rangle,i} + (-1)^i G^{(-1)}_{\langle 1/t\rangle,i+t-1} \\
&= \left(n^{(-1)}_{1,i} + n^{(-1)}_{2,i-t} + n^{(-1)}_{3,i-2t} + \cdots\right) - \left(n^{(-1)}_{1,i+t-1} + n^{(-1)}_{2,i-1} + n^{(-1)}_{3,i-t-1} + \cdots\right) \\
&= -n^{(-1)}_{1,i+t-1} + \left(n^{(-1)}_{1,i} - n^{(-1)}_{2,i-1}\right) + \left(n^{(-1)}_{2,i-t} - n^{(-1)}_{3,i-t-1}\right) + \cdots \\
&= -n^{(-1)}_{1,i+t} + (-1)^i n^{(-1)}_{2,i} - (-1)^{i-t} n^{(-1)}_{3,i-t} + (-1)^i n^{(-1)}_{4,i-2t} + \cdots \\
&= n^{(-1)}_{1,i+t} + n^{(-1)}_{2,i} + n^{(-1)}_{3,i-t} + n^{(-1)}_{4,i-2t} + n^{(-1)}_{5,i-3t} + \cdots \\
&= G^{(-1)}_{\langle 1/t\rangle,i+t}.
\end{aligned} \tag{21}$$

A more explicit relation of the diagonal sets $d^{(-1)}_{\langle t\rangle,i}$ and $g^{(-1)}_{\langle 1/t\rangle,i}$ is as follows.

Theorem 6. *Let* $d_{\langle t\rangle,i,k}$ *and* $g_{\langle 1/t\rangle,i,k}$ *be the kth elements of the sets* $d^{(-1)}_{\langle t\rangle,i}$ *and* $g^{(-1)}_{\langle 1/t\rangle,i}$*, respectively. Then,* $g_{\langle 1/2\rangle,i,k} =$

$\begin{cases} (-1)^k d_{\langle 1\rangle,i,k} & \text{if } i \equiv 0,1\,(\text{mod}4) \\ (-1)^{k+1} d_{\langle 1\rangle,i,k} & \text{if } i \equiv 2,3\,(\text{mod}4) \end{cases}$ *and* $g_{\langle 1/3\rangle,i,k} = \begin{cases} d_{\langle 2\rangle,i,k} & \text{if } i+k \equiv 0,1\,(\text{mod}4) \\ -d_{\langle 2\rangle,i,k} & \text{if } i+k \equiv 2,3\,(\text{mod}4) \end{cases}$. *In general, any kth elements* $g_{\langle 1/(t+1)\rangle,i,k}$ *in* $g^{(-1)}_{\langle 1/(t+1)\rangle,i}$ *and* $d_{\langle t\rangle,i,k}$ *in* $d^{(-1)}_{\langle t\rangle,i}$ *are the same, except for signs.*

Proof. Note $d^{(-1)}_{\langle t\rangle,i} = \{d_{\langle t\rangle,i,k} | k \geq 0\} = \{c^{(-1)}_{i,0}, c^{(-1)}_{i-t,1}, \ldots, c^{(-1)}_{i-kt,k}, \ldots,\}$ of $C^{(-1)}$ and $g^{(-1)}_{\langle 1/t\rangle,i} = \{g_{\langle 1/t\rangle,i,k} | k \geq 0\} = \{n^{(-1)}_{1,i}, n^{(-1)}_{2,i-t}, \ldots, n^{(-1)}_{k+1,i-kt}, \ldots,\}$ of $N^{(-1)}$. Theorem 2 and the symmetricity of $C^{(-1)}$ imply

$$g_{\langle 1/2\rangle,i,0} = n^{(-1)}_{1,i} = \begin{cases} c^{(-1)}_{i,i} = c^{(-1)}_{i,0} = d_{\langle 1\rangle,i,0}, & i \equiv 0,1\,(\text{mod}4), \\ -c^{(-1)}_{i,i} = -c^{(-1)}_{i,0} = -d_{\langle 1\rangle,i,0}, & i \equiv 2,3\,(\text{mod}4), \end{cases}$$
$$g_{\langle 1/2\rangle,i,1} = n^{(-1)}_{2,i-2} = \begin{cases} c^{(-1)}_{i-1,1} = d_{\langle 1\rangle,i,1}, & i \equiv 2,3\,(\text{mod}4), \\ -c^{(-1)}_{i-1,1} = -d_{\langle 1\rangle,i,1}, & i \equiv 0,1\,(\text{mod}4), \end{cases}$$
$$g_{\langle 1/2\rangle,i,2} = n^{(-1)}_{3,i-4} = \begin{cases} c^{(-1)}_{i-2,2} = d_{\langle 1\rangle,i,2}, & i \equiv 0,1\,(\text{mod}4), \\ -c^{(-1)}_{i-2,2} = -d_{\langle 1\rangle,i,2}, & i \equiv 2,3\,(\text{mod}4), \end{cases}$$
$$g_{\langle 1/2\rangle,i,3} = n^{(-1)}_{4,i-6} = \begin{cases} c^{(-1)}_{i-3,3} = d_{\langle 1\rangle,i,3}, & i \equiv 2,3\,(\text{mod}4), \\ -c^{(-1)}_{i-3,3} = -d_{\langle 1\rangle,i,3}, & i \equiv 0,1\,(\text{mod}4), \end{cases} \tag{22}$$

which shows $g_{\langle 1/2\rangle,i,k} = \begin{cases} (-1)^k d_{\langle 1\rangle,i,k}, & i \equiv 0,1\,(\text{mod}4) \\ (-1)^{k+1} d_{\langle 1\rangle,i,k}, & i \equiv 2,3\,(\text{mod}4) \end{cases}$ for $0 \leq k \leq 3$.

Now, for any kth element in the diagonal set $g^{(-1)}_{\langle 1/2\rangle,i}$, we note that

$$g_{\langle 1/2\rangle,i,k} = n^{(-1)}_{k+1,i-2k} = \begin{cases} c^{(-1)}_{i-k,i-2k} = c^{(-1)}_{i-k,k} & i-2k \equiv 0,1\,(\text{mod}4), \\ -c^{(-1)}_{i-k,i-2k} = -c^{(-1)}_{i-k,k} & i-2k \equiv 2,3\,(\text{mod}4). \end{cases} \tag{23}$$

By mod 4, if $i - 2k \equiv 0$, then $i \equiv 0$ or 2 according to $k \equiv 0,2$ or $k \equiv 1,3$. Thus, $i - 2k \equiv 1$ implies $i \equiv 1$ or 3 according to $k \equiv 0,2$ or $k \equiv 1,3$. Similarly, $i - 2k \equiv 2$ means $i \equiv 0$ (if $k \equiv 1,3$) or $i \equiv 2$ (if $k \equiv 0,2$). And $i - 2k \equiv 3$ says $i \equiv 1$ (if $k \equiv 1,3$) or $i \equiv 3$ (if $k \equiv 0,2$).

Thus, we have the following 4 cases (all congruences are by mod 4):

(i) Let $i \equiv 0$. If $k \equiv 0,2$ then $i - 2k \equiv 0$, so $g_{\langle 1/2\rangle,i,k} = c^{(-1)}_{i-k,k} = (-1)^k c^{(-1)}_{i-k,k} = (-1)^k d_{\langle 1\rangle,i,k}$. If $k \equiv 1,3$ then $i - 2k \equiv 2$, so $g_{\langle 1/2\rangle,i,k} = (-1)^k c^{(-1)}_{i-k,k} = (-1)^k d_{\langle 1\rangle,i,k}$. Thus, $g_{\langle 1/2\rangle,i,k} = (-1)^k\, d_{\langle 1\rangle,i,k}$ for any k.

(ii) Let $i \equiv 1$. If $k \equiv 0,2$ then $i - 2k \equiv 1$, so $g_{\langle 1/2\rangle,i,k} = (-1)^k c^{(-1)}_{i-k,k} = (-1)^k d_{\langle 1\rangle,i,k}$. If $k \equiv 1,3$ then $i - 2k \equiv 3$, so $g_{\langle 1/2\rangle,i,k} = -c^{(-1)}_{i-k,k} = (-1)^k d_{\langle 1\rangle,i,k}$. Thus, $g_{\langle 1/2\rangle,i,k} = (-1)^k d_{\langle 1\rangle,i,k}$ for any k.

(iii) Let $i \equiv 2$. If $k \equiv 1,3$ then $i - 2k \equiv 0$, so $g_{\langle 1/2\rangle,i,k} = (-1)^{k+1} c^{(-1)}_{i-k,k} = (-1)^{k+1} d_{\langle 1\rangle,i,k}$. If $k \equiv 0,2$ then $i - 2k \equiv 2$, so $g_{\langle 1/2\rangle,i,k} = -c^{(-1)}_{i-k,k} = (-1)^{k+1}\, d_{\langle 1\rangle,i,k}$.

(iv) Let $i \equiv 3$. If $k \equiv 1,3$, then $i - 2k \equiv 1$, so $g_{\langle 1/2\rangle,i,k} = c^{(-1)}_{i-k,k} = (-1)^{k+1} d_{\langle 1\rangle,i,k}$. If $k \equiv 0,2$, then $i - 2k \equiv 3$, so $g_{\langle 1/2\rangle,i,k} = -c^{(-1)}_{i-k,k} = (-1)^{k+1} d_{\langle 1\rangle,i,k}$.

Therefore, we have $g_{\langle 1/2\rangle,i,k} = \begin{cases} (-1)^k d_{\langle 1\rangle,i,k} & \text{if } i \equiv 0,1 \\ (-1)^{k+1} d_{\langle 1\rangle,i,k} & \text{if } i \equiv 2,3 \end{cases}$.

Similarly, in the set $g^{(-1)}_{\langle 1/3\rangle,i}$, the kth element $g_{\langle 1/3\rangle,i,k}$ $(0 \leq k \leq 3)$ are

$$g_{\langle 1/3\rangle,i,0} = n^{(-1)}_{1,i} = \begin{cases} c^{(-1)}_{i,0} = d_{\langle 2\rangle,i,0}, & i \equiv 0,1, \\ -c^{(-1)}_{i,0} = -d_{\langle 2\rangle,i,0}, & i \equiv 2,3, \end{cases}$$
$$g_{\langle 1/3\rangle,i,1} = n^{(-1)}_{2,i-3} = \begin{cases} c^{(-1)}_{i-2,1} = d_{\langle 2\rangle,i,1}, & i \equiv 0,3, \\ -c^{(-1)}_{i-2,1} = -d_{\langle 2\rangle,i,1}, & i \equiv 1,2, \end{cases}$$
$$g_{\langle 1/3\rangle,i,2} = \begin{cases} d_{\langle 2\rangle,i,2}, & i \equiv 2,3, \\ -d_{\langle 2\rangle,i,2}, & i \equiv 0,1, \end{cases}$$
$$g_{\langle 1/3\rangle,i,3} = \begin{cases} d_{\langle 2\rangle,i,3}, & i \equiv 1,2, \\ -d_{\langle 2\rangle,i,3}, & i \equiv 0,3, \end{cases} \tag{24}$$

where all congruences are by mod 4. Thus, we generally have

$$g_{\langle 1/3\rangle,i,k} = n^{(-1)}_{k+1,i-3k} = \begin{cases} c^{(-1)}_{i-2k,k}, & i-3k \equiv 0,1 \\ -c^{(-1)}_{i-2k,k}, & i-3k \equiv 2,3 \end{cases} = \begin{cases} d_{\langle 2\rangle,i,k}, & i+k \equiv 0,1, \\ -d_{\langle 2\rangle,i,k}, & i+k \equiv 2,3. \end{cases} \tag{25}$$

Now, the next table shows $d^{(-1)}_{\langle t\rangle,i}$ and $g^{(-1)}_{\langle 1/(t+1)\rangle,i}$ for the first few t.

i	$d^{(-1)}_{\langle 1\rangle,i}$	$g^{(-1)}_{\langle 1/2\rangle,i}$	$d^{(-1)}_{\langle 2\rangle,i}$	$g^{(-1)}_{\langle 1/3\rangle,i}$	$d^{(-1)}_{\langle 3\rangle,i}$	$g^{(-1)}_{\langle 1/4\rangle,i}$	$d^{(-1)}_{\langle 4\rangle,i}$	$g^{(-1)}_{\langle 1/5\rangle,i}$
4	{1, 1, 1}	{1, −1, 1}	{1, 0}	{1, 0}	{1, 1}	{1, 1}	{1}	{1}
5	{1, 0, 1}	{1, 0, 1}	{1, 1}	{1, −1}	{1, 0}	{1, 0}	{1, 1}	{1, 1}
6	{1, 1, 2, 1}	{−1, 1, −2, 1}	{1, 0, 1}	{−1, 0, 1}	{1, 1}	{−1, 1}	{1, 0}	{−1, 0}
7	{1, 0, 2, 0}	{−1, 0, −2, 0}	{1, 1, 1}	{−1, 1, 1}	{1, 0}	{−1, 0}	{1, 1}	{−1, −1}

(26)

And in general by Theorem 2, we have (congruences are by mod 4)

$$g_{\langle 1/(t+1)\rangle,i,k} = n^{(-1)}_{k+1,i-k(t+1)} = \begin{cases} c^{(-1)}_{i-tk,k} = d_{\langle t\rangle,i,k}, & i-k(t+1) \equiv 0,1, \\ -c^{(-1)}_{i-tk,k} = -d_{\langle t\rangle,i,k}, & i-k(t+1) \equiv 2,3. \end{cases} \tag{27}$$

We are now ready to obtain a recurrence on $G^{(-1)}_{\langle 1/t\rangle,i}$.

Theorem 7. $G^{(-1)}_{\langle 1/t\rangle,i} - G^{(-1)}_{\langle 1/t\rangle,i+2(t-1)} = G^{(-1)}_{\langle 1/t\rangle,i+2t}$ *for any* $t, i > 0$.

Proof. Theorem 3 implies $G^{(-1)}_{\langle 1/2\rangle,i} - G^{(-1)}_{\langle 1/2\rangle,i+1} = (-1)^i G^{(-1)}_{\langle 1/2\rangle,i+2}$ and $G^{(-1)}_{\langle 1/2\rangle,i+1} - G^{(-1)}_{\langle 1/2\rangle,i+2} = (-1)^{i+1} G^{(-1)}_{\langle 1/2\rangle,i+2}$. So, we have

$$\begin{aligned} G^{(-1)}_{\langle 1/2\rangle,i} - G^{(-1)}_{\langle 1/2\rangle,i+2} &= (-1)^i \left(G^{(-1)}_{\langle 1/2\rangle,i+2} - G^{(-1)}_{\langle 1/2\rangle,i+3}\right) \\ &= (-1)^i (-1)^{i+2} G^{(-1)}_{\langle 1/2\rangle,i+4} = G^{(-1)}_{\langle 1/2\rangle,i+4}. \end{aligned} \tag{28}$$

Similarly, Theorem 4 shows $G^{(-1)}_{\langle 1/3\rangle,i} + (-1)^i G^{(-1)}_{\langle 1/3\rangle,i+2} = G^{(-1)}_{\langle 1/3\rangle,i+3}$ and $G^{(-1)}_{\langle 1/3\rangle,i+2} + (-1)^{i+2} G^{(-1)}_{\langle 1/3\rangle,i+4} = G^{(-1)}_{\langle 1/3\rangle,i+5}$. Thus, if $2|i$, then

$$\begin{aligned} G^{(-1)}_{\langle 1/3\rangle,i} - G^{(-1)}_{\langle 1/3\rangle,i+4} &= G^{(-1)}_{\langle 1/3\rangle,i+3} - G^{(-1)}_{\langle 1/3\rangle,i+5} \\ &= G^{(-1)}_{\langle 1/3\rangle,i+3} + (-1)^{i+3} G^{(-1)}_{\langle 1/3\rangle,i+5} = G^{(-1)}_{\langle 1/3\rangle,i+6}, \end{aligned} \tag{29}$$

while if $2\nmid i$, then

$$G^{(-1)}_{\langle 1/3\rangle,i} - G^{(-1)}_{\langle 1/3\rangle,i+4} = G^{(-1)}_{\langle 1/3\rangle,i+3} - G^{(-1)}_{\langle 1/3\rangle,i+5} = G^{(-1)}_{\langle 1/3\rangle,i+6}. \tag{30}$$

Now, consider any $t \geq 1$. If $2|t$, then $G^{(-1)}_{\langle 1/t\rangle,i} - G^{(-1)}_{\langle 1/t\rangle,i+t-1} = (-1)^i G^{(-1)}_{\langle 1/t\rangle,i+t}$ and $G^{(-1)}_{\langle 1/t\rangle,i+t-1} - G^{(-1)}_{\langle 1/t\rangle,i+2(t-1)} = (-1)^{i+t-1} G^{(-1)}_{\langle 1/t\rangle,i+2t-1}$ by Theorem 5. So,

$$\begin{aligned} G^{(-1)}_{\langle 1/t\rangle,i} - G^{(-1)}_{\langle 1/t\rangle,i+2(t-1)} &= (-1)^i \left(G^{(-1)}_{\langle 1/t\rangle,i+t} - G^{(-1)}_{\langle 1/t\rangle,i+2t-1}\right) \\ &= (-1)^i \left((-1)^{i+t} G^{(-1)}_{\langle 1/t\rangle,i+2t}\right) = G^{(-1)}_{\langle 1/t\rangle,i+2t}. \end{aligned} \tag{31}$$

On the other hand, if $2\nmid t$, then $G^{(-1)}_{\langle 1/t\rangle,i} + (-1)^i G^{(-1)}_{\langle 1/t\rangle,i+t-1} = G^{(-1)}_{\langle 1/t\rangle,i+t}$ and $G^{(-1)}_{\langle 1/t\rangle,i+t-1} + (-1)^{i+t-1} G^{(-1)}_{\langle 1/t\rangle,i+2(t-1)} = G^{(-1)}_{\langle 1/t\rangle,i+2t-1}$. However, since the latter identity equals $-(-1)^i G^{(-1)}_{\langle 1/t\rangle,i+t-1} - G^{(-1)}_{\langle 1/t\rangle,i+2(t-1)} = (-1)^{i+t} G^{(-1)}_{\langle 1/t\rangle,i+2t-1}$, and the sum of the two identities yields $G^{(-1)}_{\langle 1/t\rangle,i} - G^{(-1)}_{\langle 1/t\rangle,i+2(t-1)} = G^{(-1)}_{\langle 1/t\rangle,i+2t}$.

4. Extended Sequences of $\{D^{(\pm 1)}_{\langle t\rangle,i}\}$ and $\{G^{(\pm 1)}_{\langle 1/t\rangle,i}\}$

Lemma 1 shows that $\{D^{(1)}_{\langle t\rangle,i} | i > 0\}$ is a Fibo t-sequence. By extending subscripts i backward up to all integers, we have a sequence $\{D'^{(1)}_{\langle t\rangle,i} | i \in \mathbb{Z}\}$ satisfying $D'^{(1)}_{\langle t\rangle,i} + D'^{(1)}_{\langle t\rangle,i+t} = D'^{(1)}_{\langle t\rangle,i+t}$, which is also a Fibo t-sequence. On the contrary, the sequence $\{G^{(1)}_{\langle 1/t\rangle,i} | i > 0\}$ satisfies $G^{(1)}_{\langle 1/t\rangle,i} - G^{(1)}_{\langle 1/t\rangle,i+t-1} = G^{(1)}_{\langle 1/t\rangle,i+t+1}$. By extending i to all integers, we get a sequence $\{G'^{(1)}_{\langle 1/t\rangle,i} | i \in \mathbb{Z}\}$ in [3] satisfying

$$G'^{(1)}_{\langle 1/t\rangle,i} + G'^{(1)}_{\langle 1/t\rangle,i+1} = D'^{(1)}_{\langle 1/t\rangle,i+t}. \tag{32}$$

In fact, from $G^{(1)}_{\langle 1/3\rangle,i} = \{-1, 1, 0, -1, 2, -2, 1, 1, -3, \ldots,\}$, we have

$$\left\{G'^{(1)}_{\langle 1/3\rangle,i}\right\} = \left\{\underbrace{\ldots, 1, 1, -2, 2, -1, 0, 1, -1}_{i<0}, \text{①}, \underbrace{0, 0, 1, 0, 1, 1, 1, 2, 2, 3, 4, 5, \ldots}_{i>0}\right\}, \tag{33}$$

such that $G'^{(1)}_{\langle 1/3\rangle,i} + G'^{(1)}_{\langle 1/3\rangle,i+1} = G'^{(1)}_{\langle 1/3\rangle,i+3}$ (see Table 4).

A sequence $\{p_n\}$ satisfying $p_n + p_{n+1} = p_{n+t}$ with $t > 1$ initials is called a Padovan t-sequence [6, 7]. In particular, it is a Fibonacci sequence if $t = 2$. Identity (32) yields the next lemma immediately.

Lemma 2 (see [3]). $\{G'^{(1)}_{\langle 1/t\rangle,i} | i \in \mathbb{Z}\}$ *is a Padovan t-sequence with initials* $G'^{(1)}_{\langle 1/t\rangle,0} = 1$ *and* $G'^{(1)}_{\langle 1/t\rangle,i} = 0$ $(1 \leq i < t)$.

Now, over $C^{(-1)}$ and $N^{(-1)}$, we consider extended sequences of $\{D^{(-1)}_{\langle t\rangle,i} | i \geq 0\}$ and $\{G^{(-1)}_{\langle 1/t\rangle,i} | i \geq 0\}$, in which subscripts i are extended to all integers. From $D^{(-1)}_{\langle t\rangle,i} + D^{(-1)}_{\langle t\rangle,i+2t} = D^{(-1)}_{\langle t\rangle,i+2t+2}$ in Lemma 1, we easily have an extended sequence $\{D'^{(-1)}_{\langle t\rangle,i} | i \in \mathbb{Z}\}$ of $\{D^{(-1)}_{\langle t\rangle,i} | i \geq 0\}$ satisfying

$$D'^{(-1)}_{\langle t\rangle,i} + D'^{(-1)}_{\langle t\rangle,i+2t} = D'^{(-1)}_{\langle t\rangle,i+2t+2}, \quad i \in \mathbb{Z}. \tag{34}$$

So, $\{D'^{(-1)}_{\langle t\rangle,i} | i \in \mathbb{Z}\}$ is an interlocked Fibo t-sequence as in Theorem 1.

On the contrary, let $G'^{(-1)}_{\langle 1/t\rangle,-i} = G^{(-1)}_{\langle 1/t\rangle,i}$ for $i \geq 0$. Then, $G^{(-1)}_{\langle 1/t\rangle,i} - G^{(-1)}_{\langle 1/t\rangle,i+2(t-1)} = G^{(-1)}_{\langle 1/t\rangle,i+2t}$ in Theorem 7 implies $G'^{(-1)}_{\langle 1/t\rangle,-i} - G'^{(-1)}_{\langle 1/t\rangle,-(i+2(t-1))} = G'^{(-1)}_{\langle 1/t\rangle,-(i+2t)}$. So, by setting $j = -i$, we have

$$G'^{(-1)}_{\langle 1/t\rangle,j} - G'^{(-1)}_{\langle 1/t\rangle,j-2(t-1)} = G'^{(-1)}_{\langle 1/t\rangle,j-2t}. \tag{35}$$

That is, $G'^{(-1)}_{\langle 1/t\rangle,j+2t} = G'^{(-1)}_{\langle 1/t\rangle,j+2} + G'^{(-1)}_{\langle 1/t\rangle,j}$. It shows that $\{G'^{(-1)}_{\langle 1/t\rangle,i} | i \in \mathbb{Z}\}$ is an extended sequence of $\{G^{(-1)}_{\langle 1/t\rangle,i} | i \geq 0\}$ satisfying

Table 4: $\left\{D'^{(1)}_{\langle t\rangle,i} | i \in \mathbb{Z}\right\}$ and $\left\{G'^{(1)}_{\langle 1/t\rangle,i} | i \in \mathbb{Z}\right\}$.

t	$D'^{(1)}_{\langle t\rangle,i}$: $i<0$	0	$i>0$
1	$\ldots, -21, 13, -8, 5, -3, 2, -1, 1, 0,$	1	$1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots$
2	$\ldots, -2, 3, 0, -2, 1, 1, -1, 0, 1, 0, 0,$	1	$1, 1, 2, 3, 4, 6, 9, 13, 19, 28, 41, \cdots$
3	$\ldots, 1, -2, 1, 0, 1, -1, 0, 0, 1, 0, 0, 0,$	1	$1, 1, 2, 3, 4, 5, 7, 10, 14, 19, 26, \ldots$
4	$\ldots, 1, 0, 0, 1, -1, 0, 0, 0, 1, 0, 0, 0, 0,$	1	$1, 1, 1, 1, 2, 3, 4, 5, 6, 8, 11, 15, \ldots$
t	$G'^{(1)}_{\langle 1/t\rangle,i}$: $i<0$	0	$i>0$
2	$\ldots, 34, -21, 13, -8, 5, -3, 2, -1,$	1	$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots$
3	$\ldots, 4, -3, 1, 1, -2, 2, -1, 0, 1, -1,$	1	$0, 0, 1, 0, 1, 1, 1, 2, 2, 3, 4, 5, 7, 9, 12, \ldots$
4	$\ldots, 10, 7, -5, 4, -3, 2, -1, 1, -1,$	1	$0, 0, 0, 1, 0, 0, 1, 1, 0, 1, 2, 1, 1, 2, 4, 6, \ldots$
5	$\ldots, -4, 4, -3, 2, -1, 0, 1, -1, 1, -1,$	1	$,0, 0, 0, 0, 1, 0, 0, 0, 1, 1, 0, 0, 1, 0, 1, 3, \ldots$

Table 5: $\left\{D'^{(-1)}_{\langle t\rangle,i} | i \in \mathbb{Z}\right\}$ and $\left\{G'^{(-1)}_{\langle 1/t\rangle,i} | i \in \mathbb{Z}\right\}$, $(1 \le t \le 5)$.

t	$D'^{(-1)}_{\langle t\rangle,i}$: $i<0$	$0, i>0$
1	$\ldots, 8, 13, 5, -8, -3, 5, 2, -3, -1, 2, 1, -1, 0, 1, 1$	⓪, $1, 1, 2, 1, 3, 2, 5, \ldots$
2	$\ldots, 3, -2, -2, 1, -1, 2, 2, -1, 0, 0, -1, 1, 1, 0, 1, 0,$	⓪, $1, 1, 1, 2, 1, 2, 2, \ldots,$
3	$\ldots, -2, 0, 2, 1, -1, -1, 1, 0, -1, 0, 1, 1, 0, 0, 1, 0, 0,$	⓪, $1, 1, 1, 1, 2, 1, 2, \ldots$
4	$\ldots, -1, -1, 0, 1, 0, -1, 0, 0, 1, 1, 0, 0, 0, 1, 0, 0, 0,$	⓪, $1, 1, 1, 1, 1, 2, 1, \ldots$
t	$G'^{(-1)}_{\langle 1/t\rangle,i}$: $i<0$	$0, i>0$
2	$\ldots, 2, 1, -1, 0, 1,$	①, $0, 1, 1, 2, 1, 3, 2, 5, 3, 8, 5, 13, 8, 21, 13, 34, 21, \ldots$
3	$\ldots, 0, 1, 0, -1, 1,$	①, $0, 0, 1, 0, 1, 1, 1, 0, 2, 1, 2, 1, 3, 1, 4, 2, 5, 2, 7, 3, \ldots$
4	$\ldots, 1, 2, -1, -1, 1,$	①, $0, 0, 0, 1, 0, 0, 1, 1, 0, 1, 0, 1, 1, 1, 1, 2, 0, 2, 1, 2, \ldots$
5	$\ldots, 2, 1, -1, -1, 1,$	①, $0, 0, 0, 0, 1, 0, 0, 0, 1, 1, 0, 0, 1, 0, 1, 0, 1, 1, 1, 1, \ldots$

$$G'^{(-1)}_{\langle 1/t\rangle,i} + G'^{(-1)}_{\langle 1/t\rangle,i+2} = G'^{(-1)}_{\langle 1/t\rangle,i+2t}, \quad i \in \mathbb{Z}. \tag{36}$$

Theorem 8. *$\left\{G'^{(-1)}_{\langle 1/t\rangle,i} | i \in \mathbb{Z}\right\}$ is an interlocked Padovan t-sequence with initials $G'^{(-1)}_{\langle 1/t\rangle,0} = 1$ and $G'^{(-1)}_{\langle 1/t\rangle,i} = 0$ $(1 \le i < 2t)$.*

Proof. The subsequence having only eventh terms of $\left\{G'^{(-1)}_{\langle 1/t\rangle,i} | i \in \mathbb{Z}\right\}$ is

$$\left\{G'^{(-1)e}_{\langle 1/t\rangle,i}\right\} = \left\{\ldots, G'^{(-1)}_{\langle 1/t\rangle,0}, G'^{(-1)}_{\langle 1/t\rangle,2}, \ldots, G'^{(-1)}_{\langle 1/t\rangle,2k}, G'^{(-1)}_{\langle 1/t\rangle,2(k+1)}, \ldots\right\}. \tag{37}$$

Then, $G'^{(-1)}_{\langle 1/t\rangle,2k} + G'^{(-1)}_{\langle 1/t\rangle,2(k+1)} = G'^{(-1)}_{\langle 1/t\rangle,2(k+1)}$ in (36) implies that the sum of consecutive two eventh terms equals t distanced eventh term. So, $\left\{G'^{(-1)e}_{\langle 1/t\rangle,i}\right\}$ is a Padovan t-sequence. Similarly, the subsequence $\left\{G'^{(-1)o}_{\langle 1/t\rangle,i}\right\}$ of oddth terms of $\left\{G'^{(-1)}_{\langle 1/t\rangle,i}\right\}$ also satisfies that the sum of consecutive two oddth terms equals t distanced oddth term. Thus, $\left\{G'^{(-1)}_{\langle 1/t\rangle,i} | i \in \mathbb{Z}\right\}$ is an interlocked Padovan t-sequence.

Corollary 1. *The sequences $\left\{G'^{(1)}_{\langle 1/2\rangle,i}\right\}$ and $\left\{G'^{(-1)}_{\langle 1/2\rangle,i}\right\}$ are equal to $\left\{D'^{(1)}_{\langle 1\rangle,i}\right\}$ and $\left\{D'^{(-1)}_{\langle 1\rangle,i}\right\}$, respectively. And $\left\{G'^{(-1)}_{\langle 1/3\rangle,i}\right\}$ is an interlocked Fibo 5-sequence.*

Proof. The proof is due to Tables 4 and 5. And the interlocked Padovan 3-sequence $\left\{G'^{(-1)}_{\langle 1/3\rangle,i}\right\}$ in Theorem 8 satisfies $G'^{(-1)}_{\langle 1/3\rangle,i} + G'^{(-1)}_{\langle 1/3\rangle,i+2} = G'^{(-1)}_{\langle 1/3\rangle,i+6}$. Thus,

$$\begin{aligned} G'^{(-1)}_{\langle 1/3\rangle,i} + G'^{(-1)}_{\langle 1/3\rangle,i+8} &= G'^{(-1)}_{\langle 1/3\rangle,i} + \left(G'^{(-1)}_{\langle 1/3\rangle,i+2} + G'^{(-1)}_{\langle 1/3\rangle,i+4}\right) \\ &= G'^{(-1)}_{\langle 1/3\rangle,i+6} + G'^{(-1)}_{\langle 1/3\rangle,i+4} = G'^{(-1)}_{\langle 1/3\rangle,i+10}, \end{aligned} \tag{38}$$

which is a recurrence of interlocked Fibo 5-sequence.

We note that $\left\{G'^{(-1)}_{\langle 1/3\rangle,i}\right\}$ is interlocked by two Padovan subsequences

$$\left\{G'^{(-1)e}_{\langle 1/3\rangle,i}\right\} = \{\ldots, 0, 1, 1, 1, 2, 2, 3, 4, 5, 7, 9, 12, 16, \ldots\} = \left\{G'^{(-1)o}_{\langle 1/3\rangle,i}\right\}, \tag{39}$$

with initials 1, 1, 2. However, $\left\{G'^{(-1)}_{\langle 1/4\rangle,i}\right\}$ is interlocked by distinct Padovan 4-sequences

$$\left\{G'^{(-1)e}_{\langle 1/4\rangle,i}\right\} = \{\ldots, 0, 1, 1, 0, 1, 2, 1, 1, 3, 3, 2, 4, 6, 5, 6, 10, 11, 11, 16, \ldots\},$$
$$\left\{G'^{(-1)o}_{\langle 1/4\rangle,i}\right\} = \{\ldots, 1, 1, 1, 2, 2, 2, 3, 4, 4, 5, 7, 8, 9, 12, 15, 17, 21, 27, \ldots\}, \quad (40)$$

where these two sequences satisfy $G'^{(-1)e}_{\langle 1/4\rangle,i} + G'^{(-1)e}_{\langle 1/4\rangle,i+2} = G'^{(-1)o}_{\langle 1/4\rangle,i+1}$. Moreover, the sequences are obtained explicitly from Pascal and Pauli tables as follows. Write $C^{(1)} = \langle r_0^{(1)}; r_1^{(1)}; r_2^{(1)}; \ldots\rangle$ and $C^{(-1)} = \langle r_0^{(-1)}; r_1^{(-1)}; r_2^{(-1)}; \ldots\rangle$ by means of ith rows $r_i^{(1)}$ and $r_i^{(-1)}$. Let

$$\mathfrak{C}^{(1)} = \langle \underbrace{r_0^{(1)}; r_0^{(1)}; r_0^{(1)}}; \underbrace{r_1^{(1)}; r_1^{(1)}; r_1^{(1)}}; \ldots \rangle,$$
$$\mathfrak{C}^{(-1)} = \langle \underbrace{r_0^{(-1)}; r_0^{(-1)}}; \underbrace{r_1^{(-1)}; r_1^{(-1)}}; \underbrace{r_2^{(-1)}; \ldots}; \ldots \rangle, \quad (41)$$

be tables having duplicated rows of $C^{(1)}$ and $C^{(-1)}$.

Theorem 9. *In $\mathfrak{C}^{(1)}$, the sequence of 1-slope diagonal sums equals $\left\{G'^{(-1)o}_{\langle 1/4\rangle,i}\right\}$. And in $\mathfrak{C}^{(-1)}$, the sequence of 1-slope diagonal sums is interlocked by two Padovan 3-sequences $P = \{p_i\}$ and $Q = \{q_i\}$, where P is the ordinary Padovan sequence such that $q_{i+4} = p_i + p_{i+3} = 2p_i + p_{i+1}$ for all i.*

Proof. From $\mathfrak{C}^{(1)} = \langle \underbrace{1;1;1}; \underbrace{1,1;1,1;1,1}; \underbrace{1,2,1;1,2,1;1,2,1}; \ldots, \rangle$, the sequence of 1-slope diagonal sums is $\{1,1,1,1,2,2,2,3,4,4,5,7,8,9,12,15,17,21,27,\ldots,\}$ that corresponds to $\left\{G'^{(-1)o}_{\langle 1/4\rangle,i}\right\}$. Similarly, the sequence of 1-slope diagonal sums of $\mathfrak{C}^{(-1)} = \langle \underbrace{1;1}; \underbrace{1,1;1,1}; \underbrace{1,0,1;1,0,1}; \underbrace{1,1,1,1;1,1,1,1}; 1,0,2,0,1; \ldots, \rangle$ is

$$\{\xi_i\} = \{1, 1, 1, 2, 2, 1, 2, 3, 3, 3, 4, 4, 5, 6, 7, 7, 9, 10, 12, 13, 16, 17, \ldots\}. \quad (42)$$

That satisfies $\xi_i + \xi_{i+2} = \xi_{i+6}$ for some i. Note

$$\xi_i = c^{(-1)}_{k,0} + c^{(-1)}_{k-1,1} + c^{(-1)}_{k-1,2} + c^{(-1)}_{k-2,3} + c^{(-1)}_{k-2,4} + \cdots, \quad \text{if } i = 2k,$$
$$\xi_i = c^{(-1)}_{k,0} + c^{(-1)}_{k,1} + c^{(-1)}_{k-1,2} + c^{(-1)}_{k-1,3} + c^{(-1)}_{k-2,4} + c^{(-1)}_{k-2,5} + \cdots, \quad \text{if } i = 2k+1. \quad (43)$$

Then, $\xi_i + \xi_{i+2} = \xi_{i+6}$ for all i by the Pauli recurrence (1). And $\{\xi_i\}$ is an interlocked Padovan 3-sequence by $P = \{p_i\} = \{1, 1, 2, 2, 3, 4, 5, 7, 9, 12, 16, \ldots,\}$ of eventh terms and $Q = \{q_i\} = \{1, 2, 1, 3, 3, 4, 6, 7, 10, 13, 17, \ldots,\}$ of oddth terms. Clearly, P is the ordinary Padovan sequence satisfying $q_{i+4} = p_i + p_{i+3} = 2p_i + p_{i+1}$ for all i.

Data Availability

The data used to support the findings of the study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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