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Research Article

Inequalities for the Derivative of Rational Functions with Prescribed Poles

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In this paper, we consider a class of rational functions $r(s(z))$ of degree mn where $s(z)$ is a polynomial of degree m and establish some inequalities for rational functions with prescribed poles which generalize and refine the result of I. Qasim and A. Liman.

1. Introduction

Let P_n denote the class of all complex polynomials of degree at most n and let k be a positive real number. We denote $T_k = \{z: |z| = k\}, \{D_{k-} = z: |z| < k\}$, and $D_{k+} = \{z: |z| > k\}$. Consider a polynomial $p(z)$ of degree n. In 1926, Bernstein [1] presented the following well-known inequality:

$$\max_{|z|=1}|p'(z)| \le n\max_{|z|=1}|p(z)|. \tag{1}$$

Equality holds in (1) only for $p(z) = az^n$, where $a \neq 0$. If we restrict to the class of polynomials having no zeros in D_{1-}, inequality (1) can be sharpened. In fact, it was conjectured by P. Erdö s and later proved by Lax [2] that if $p(z)$ has no zeros in D_{1-}, then

$$\max_{|z|=1}|p'(z)| \le \frac{n}{2}\max_{|z|=1}|p(z)|. \tag{2}$$

For the class of polynomials having no zeros in D_{1+}, Turán [3] proved that

$$\max_{|z|=1}|p'(z)| \ge \frac{n}{2}\max_{|z|=1}|p(z)|. \tag{3}$$

For $a_j \in \mathbb{C}$ $(1 \le j \le n)$, we let $w(z) = \prod_{j=1}^{n}(z - a_j)$ and

$$B(z) = \prod_{j=1}^{n}\left(\frac{1 - \overline{a_j}z}{z - a_j}\right),$$
$$R_n = R_n(a_1, a_2, \ldots, a_n) := \left\{\frac{p(z)}{w(z)}: p \in P_n\right\}. \tag{4}$$

The product $B(z)$ is known as a Blaschke product.

Then, R_n is the set of rational functions with at most n poles $a_1, a_2, \ldots, a_n$ and with finite limit at infinity. For f defined on T_1, we denote $\|f\| = \sup_{z\in T_1}|f(z)|$, the Chebyshev norm of f on T_1. Throughout this paper, we assume that all poles $a_1, a_2, \ldots, a_n$ are in D_{1+}.

In 1995, Li et al. [4] proved some inequalities similar to (1), (2), and (3) for rational functions. Among other things, they proved the following result.

Theorem 1 (see [4]). *Let $r \in R_n$ with all its zeros lying in $T_1 \cup D_{1+}$. Then, for $z \in T_1$,*

$$|r'(z)| \le \frac{1}{2}|B'(z)| \cdot \|r\|. \tag{5}$$

Equality holds for $r(z) = aB(z) + b$ with $|a| = |b| = 1$.

In 1997, inequality (5) was improved by Aziz and Shah [5] under the same hypothesis. They obtained the following theorem.

Theorem 2 (see [5]). *Let $r \in R_n$ with all its zeros lying in $T_1 \cup D_{1+}$. Then, for $z \in T_1$,*

$$|r'(z)| \leq \frac{1}{2}|B'(z)|(\|r\| - m), \quad (6)$$

where $m = \min_{|z|=1}|r(z)|$. Equality holds for $r(z) = B(z) + he^{i\alpha}$ where $h \geq 1$ and α is real.

In 1999, Aziz and Zarger [6] considered a class of rational functions R_n not vanishing in $T_k \cup D_{k-}$, where $k \geq 1$, and established the following generalization of Theorem 1.

Theorem 3 (see [6]). *Let $r \in R_n$ with all its zeros lying in $T_k \cup D_{k+}$, where $k \geq 1$. Then, for $z \in T_1$,*

$$|r'(z)| \leq \frac{1}{2}\left[|B'(z)| - \frac{n(k-1)}{k+1} \cdot \frac{|r(z)|^2}{\|r\|^2}\right] \cdot \|r\|. \quad (7)$$

Equality holds for $r(z) = ((z+k)/(z-a))^n$ and $B(z) = ((1-az)/(z-a))^n$ evaluated at $z = 1$, where $a > 1$ and $k \geq 1$.

Recently, inequalities (6) and (7) were improved by Arunrat and Nakprasit [7] under the same hypothesis. They obtained the following theorem.

Theorem 4 (see [7]). *Let $r \in R_n$, where r has exactly n poles at $a_1, a_2, \ldots, a_n$ and all its zeros lie in $T_k \cup D_{k+}$, $k \geq 1$. Then, for $z \in T_1$,*

$$|r'(z)| \leq \frac{1}{2}\left[|B'(z)| - \frac{(n(1+k)-2t)(|r(z)|-m)^2}{(1+k)(\|r\|-m)^2}\right](\|r\| - m), \quad (8)$$

where t is the number of zeros of r with counting multiplicity and $m = \min_{|z|=k}|r(z)|$. Equality holds for $r(z) = ((z+k)^t/(z-a)^n)$ and $B(z) = ((1-az)/(z-a))^n$ evaluated at $z = 1$, $a > 1$, and $k \geq 1$.

In 2015, Qasim and Liman [8] considered a class of rational functions $r(s(z)) \in R_{mn}$ with all poles $a_1, a_2, \ldots, a_{mn}$ lying in D_{1+}, defined by

$$(r \circ s)(z) = r(s(z)) := \frac{p(s(z))}{w(s(z))}, \quad (9)$$

where $s(z)$ is a polynomial of degree m with all its zeros lying in $T_1 \cup D_{1-}$ and $r \in R_n$. Let

$$w(s(z)) = \prod_{j=1}^{mn}(z - a_j), \quad (10)$$

and the Blaschke product

$$B(z) = \frac{w^*(s(z))}{w(s(z))} = \frac{z^{mn}\overline{w(s(1/\overline{z}))}}{w(s(z))} = \prod_{j=1}^{mn}\left(\frac{1-\overline{a}_j z}{z-a_j}\right). \quad (11)$$

They proved the following generalization of inequality (5).

Theorem 5. *(see [8]). Let $r(s(z)) \in R_{mn}$, where $r(s(z))$ has no zeros in D_{1-} and all zeros of $s(z)$ lie in $T_1 \cup D_{1-}$. Then, for $z \in T_1$,*

$$|r'(s(z))| \leq \frac{1}{2mm'}|B'(z)| \cdot \|r \circ s\|, \quad (12)$$

where $m' = \min_{z \in T_1}|s(z)|$. The inequality is sharp and equality holds for $r(s(z)) = aB(z) + b$ with $a, b \in T_1$ and $s(z) = z^m$.

Observe that if $s(z)$ has a zero on T_1, then $m' = 0$, and we obtain a trivial inequality:

$$0 = 2mm' \cdot |r'(s(z))| \leq |B'(z)|\|r \circ s\|. \quad (13)$$

In this paper, we consider the class of rational functions R_{mn} having no zeros in D_{k-}, where $k \geq 1$, and prove the generalization of the result of Qasim and Liman [8].

2. Lemmas

For the proof of our main theorems, we need the following lemmas. These two lemmas are due to Li et al. [4].

Lemma 1 (see [4]). *Let $r \in R_n$. If all zeros of r lie in $T_1 \cup D_{1+}$, then, for $z \in T_1$,*

$$\mathrm{Re}\left(\frac{zr'(z)}{r(z)}\right) \leq \frac{|B'(z)|}{2}, \quad (14)$$

where $r(z) \neq 0$.

Lemma 2 (see [4]). *If $r \in R_n$ and $r^*(z) = B(z)\overline{r(1/\overline{z})}$, then, for $z \in T_1$,*

$$|(r^*(z))'| + |r'(z)| \leq |B'(z)| \cdot \|r\|. \quad (15)$$

Equality holds for $r(z) = aB(z)$ with $a \in T_1$.

Lemma 3 is due to Aziz and Dawood [9].

Lemma 3 (see [9]). *If $p \in P_n$ and $p(z)$ has all its zeros in $T_1 \cup D_{1-}$, then*

$$\min_{z \in T_1}|p'(z)| \geq n \cdot \min_{z \in T_1}|p(z)|. \quad (16)$$

The inequality is sharp and equality holds for polynomials having all zeros at the origin.

Lemma 4 is due to Aziz and Shah [5], and Lemma 5 is due to Arunrat and Nakprasit [7].

Lemma 4 (see [5]). *If $B(z)$ is Blaschke product and α is real, $0 \leq \alpha < 2\pi$, then $B(z) + he^{i\alpha}$ has all its zeros in $T_1 \cup D_{1+}$, for every $h \geq 1$.*

Lemma 5 (see [7]). *Assume that $r \in R_n$, where r has exactly n poles at $a_1, a_2, \ldots, a_n$. Let t be the number of zeros of r with counting multiplicity. If all zeros of r lie in $T_k \cup D_{k+}$, where $k \geq 1$, and $z \in T_1$ with $r(z) \neq 0$, then*

$$\mathrm{Re}\left(\frac{zr'(z)}{r(z)}\right) \leq \frac{|B'(z)|}{2} + \frac{2t - n(1+k)}{2(1+k)}. \quad (17)$$

3. Main Theorems

In this section, we state and prove main results. One of them generalizes the result of Qasim and Liman [8].

Theorem 6. *Let $r(s(z)) \in R_{mn}$ with $r(s(z)) \neq 0$ in D_{1-} and all zeros of $s(z)$ lie in $T_1 \cup D_{1-}$. Then, for $z \in T_1$,*

$$\left|r'(s(z))\right| \leq \frac{1}{2mm'}\left|B'(z)\right| \cdot (\|r \circ s\| - m^*), \tag{18}$$

where $m' = \min_{z\in T_1}|s(z)|$ and $m^* = \min_{z\in T_1}|r(s(z))|$.

Equality holds for $r(s(z)) = B(z) + he^{i\alpha}$ where $s(z) = z^m$, $h \geq 1$, and α is real.

Proof. Let $r(s(z)) \in R_{mn}$ without zeros in $|z| < 1$ and $m^* = \min_{z\in T_1}|r(s(z))|$.

Therefore, $m^* \leq |r(s(z))|$ for $z \in T_1$. If $r(s(z))$ has a zero on T_1, then $m^* = 0$, and hence, for every α with $|\alpha| < 1$, we get that $r(s(z)) - \alpha m^* = r(s(z))$. In case $r(s(z))$ has no zeros on T_1, we have for every α with $|\alpha| < 1$ that $| - \alpha m^*| = |\alpha| \cdot m^* < |r(s(z))|$ for $|z| = 1$. It follows from Rouche's theorem that rational functions $R(z) = r(s(z)) - \alpha m^*$ and $r(s(z))$ have the same number of zeros in D_{1-}. That is, for every α with $|\alpha| < 1$, $R(z)$ has no zeros in D_{1-}. We first assume that $R(z) \neq 0$. Lemma 1 yields that for $z \in T_1$,

$$\operatorname{Re}\left(\frac{zR'(z)}{R(z)}\right) \leq \frac{\left|B'(z)\right|}{2}. \tag{19}$$

Let $R^*(z) = B(z)\overline{R(1/\overline{z})} = B(z)\overline{R}(1/z)$. Then, $(R^*(z))' = B'(z)\overline{R}(1/z) - (B(z)/z^2) \cdot \overline{R}'(1/z)$.

Consequently, $z(R^*(z))' = zB'(z)\overline{R}(1/z) - (B(z)/z) \cdot \overline{R}'(1/z)$.

Since $z \in T_1$, we have $\overline{z} = (1/z)$, $|B(z)| = 1$, $((zB'(z))/(B(z))) = |B'(z)|$, and so

$$\left|z(R^*(z))'\right| = \left|zB'(z)\overline{R(z)} - B(z)\overline{zR'(z)}\right| = \left|\frac{zB'(z)}{B(z)} \cdot \overline{R(z)} - \overline{zR'(z)}\right| = \left|\left|B'(z)\right|\overline{R(z)} - \overline{zR'(z)}\right|. \tag{20}$$

Since $|B'(z)|$ is real, we obtain that $|z(R^*(z))'| = ||B'(z)|R(z) - zR'(z)|$.

Then,

$$\left|\frac{z(R^*(z))'}{R(z)}\right|^2 = \left|\left|B'(z)\right| - \frac{zR'(z)}{R(z)}\right|^2 = \left|B'(z)\right|^2 - 2\left|B'(z)\right| \cdot \operatorname{Re}\left(\frac{zR'(z)}{R(z)}\right) + \left|\frac{zR'(z)}{R(z)}\right|^2 \geq \left|B'(z)\right|^2 - 2\left|B'(z)\right|\left(\frac{\left|B'(z)\right|}{2}\right) + \left|\frac{zR'(z)}{R(z)}\right|^2 = \left|\frac{zR'(z)}{R(z)}\right|^2, \tag{21}$$

where the inequality comes from (19).

This implies that for $z \in T_1$ which are not the zeros of $R(z)$,

$$\left|R'(z)\right| \leq \left|(R^*(z))'\right|, \tag{22}$$

where $R^*(z) = B(z)\overline{R(1/\overline{z})} = r^*(s(z)) - \overline{\alpha}m^*B(z)$ with $r^*(s(z)) = B(z)\overline{r(s(1/\overline{z}))}$.

Moreover, $(R^*(z))' = (r^*(s(z)))' - \overline{\alpha}m^*B'(z)$ and $R'(z) = (r(s(z)))'$.

Applying these relations into (22), we obtain that

$$\left|(r(s(z)))'\right| \leq \left|(r^*(s(z)))' - \overline{\alpha}m^*B'(z)\right|, \tag{23}$$

for $z \in T_1$ with $R(z) \neq 0$ and every α with $|\alpha| < 1$.

Choose the argument of α so that

$$\left|(r^*(s(z)))' - \overline{\alpha}m^*B'(z)\right| = \left|(r^*(s(z)))'\right| - m^*|\alpha|\left|B'(z)\right|, \tag{24}$$

for $z \in T_1$ with $R(z) \neq 0$.

Substituting relation (24) into (23), we obtain that

$$\left|(r(s(z)))'\right| \leq \left|(r^*(s(z)))'\right| - m^*|\alpha|\left|B'(z)\right|. \tag{25}$$

Letting $|\alpha| \longrightarrow 1$, we obtain

$$\left|(r(s(z)))'\right| \leq \left|(r^*(s(z)))'\right| - m^*\left|B'(z)\right|. \tag{26}$$

Lemma 2 implies that

$$\left|(r(s(z)))'\right| \leq \left|B'(z)\right| \cdot \|r \circ s\| - \left|(r(s(z)))'\right| - m^*\left|B'(z)\right|. \tag{27}$$

Thus,

$$\left|(r(s(z)))'\right| \leq \frac{1}{2}\left|B'(z)\right| \cdot (\|r \circ s\| - m^*). \tag{28}$$

For $z \in T_1$ with $R(z) \neq 0$, we have $|(r(s(z)))'| \geq |r'(s(z))| \cdot \min_{z\in T_1}|s'(z)|$.

From Lemma 3, we obtain that

$$\left|(r(s(z)))'\right| \geq \left|r'(s(z))\right| \cdot \left(m \cdot \min_{z\in T_1}|s(z)|\right) = mm'\left|r'(s(z))\right|. \tag{29}$$

It follows from (28) that

$$\left|r'(s(z))\right| \leq \frac{1}{2mm'}\left|B'(z)\right| \cdot (\|r \circ s\| - m^*). \tag{30}$$

This proves inequality for $R(z) \neq 0$. In case $R(z) = 0$, we obtain that $(r(s(z)))' = 0$.

This implies that the above inequality is trivially true. Therefore, inequality (18) holds for all $z \in T_1$.

Next, we show that equality holds for $r(s(z)) = B(z) + he^{i\alpha}$ where $h \geq 1$ and $s(z) = z^m$. Lemma 4 implies that $B(z) + he^{i\alpha}$ has all its zeros in $T_1 \cup D_{1+}$. Moreover, we obtain that

$$\begin{aligned} \|r \circ s\| &= \max_{z\in T_1}\left|B(z) + he^{i\alpha}\right| = h + 1, \\ m^* &= \min_{z\in T_1}\left|B(z) + he^{i\alpha}\right| = h - 1, \\ m' &= \min_{z\in T_1}|s(z)| = 1. \end{aligned} \tag{31}$$

Consider $B'(z) = (r(s(z)))' = r'(s(z)) \cdot s'(z) = mz^{m-1} \cdot r'(s(z))$.

This implies that $r'(s(z)) = (B'(z)/mz^{m-1})$. Then, for $z \in T_1$, $|r'(s(z))| = (|B'(z)|/m)$.

The right side of inequality (18) is

$$\frac{1}{2mm'}\left|B'(z)\right| \cdot (\|r \circ s\| - m^*) = \frac{1}{2m(1)}\left|B'(z)\right| \cdot ((h+1) - (h-1)) = \left|r'(s(z))\right|. \tag{32}$$

Thus, this bound is best possible. □

Theorem 7. *Let $r(s(z)) \in R_{mn}$ with $r(s(z)) \neq 0$ in $D_{k-}, k \geq 1$, and all zeros of $s(z)$ lie in $T_1 \cup D_{1-}$. Then, for $z \in T_1$,*

$$\left|r'(s(z))\right| \leq \frac{1}{2mm'}\left[\left|B'(z)\right| - \frac{(mn(1+k) - 2mt)(|r(s(z))| - m^*)^2}{(1+k)(\|r \circ s\| - m^*)^2}\right](\|r \circ s\| - m^*), \tag{33}$$

where mt is the number of zeros of $r \circ s$ with counting multiplicity, $m' = \min_{z\in T_1}|s(z)|$, and $m^* = \min_{z\in T_k}|r(s(z))|$. Equality holds for $r(s(z)) = ((z+k)^{mt}/(z-a)^{mn})$, where $s(z) = z^m$ and $B(z) = ((1-az)/(z-a))^{mn}$, $a > 1$ and $k \geq 1$ at $z = 1$.

Proof. Let $r(s(z)) \in R_{mn}$ without zeros in $|z| < k$, where $k \geq 1$.

Let $m^* = \min_{z\in T_k}|r(s(z))|$ and mt be the number of zeros of $r \circ s$ with counting multiplicity. Therefore, $m^* \leq |r(s(z))|$ for $z \in T_k$. If $r(s(z))$ has a zero on T_k, then $m^* = 0$, and hence, for every α with $|\alpha| < 1$, we obtain that $r(s(z)) - \alpha m^* = r(s(z))$. In case $r(s(z))$ has no zeros on T_k, we have for every α with $|\alpha| < 1$ that $|-\alpha m^*| = |\alpha| \cdot m^* < |r(s(z))|$ for $|z| = k$. Therefore, it follows from Rouche's theorem that rational functions $R(z) = r(s(z)) - \alpha m^*$ and $r(s(z))$ have the same number of zeros in D_{k-}. That is, for every α with $|\alpha| < 1$, $R(z)$ has no zeros in D_{k-}. We first assume that $R(z) \neq 0$. Lemma 5 yields that for $z \in T_1$,

$$\operatorname{Re}\left(\frac{zR'(z)}{R(z)}\right) \leq \frac{\left|B'(z)\right|}{2} + \frac{2mt - mn(1+k)}{2(1+k)}. \tag{34}$$

Let $R^*(z) = B(z)\overline{R}(1/\overline{z}) = B(z)\overline{R}(1/z)$. Then, $(R^*(z))' = B'(z)\overline{R}(1/z) - (B(z)/z^2) \cdot \overline{R}'(1/z)$.

Consequently, $z(R^*(z))' = zB'(z)\overline{R}(1/z) - (B(z)/z) \cdot \overline{R}'(1/z)$.

Since $z \in T_1$, we have $\overline{z} = (1/z)$, $|B(z)| = 1$, $(zB'(z)/B(z)) = |B'(z)|$, and so

$$\left|z(R^*(z))'\right| = \left|zB'(z)\overline{R(z)} - B(z)\overline{zR'(z)}\right| = \left|\frac{zB'(z)}{B(z)} \cdot \overline{R(z)} - \overline{zR'(z)}\right| = \left|\left|B'(z)\right|\overline{R(z)} - \overline{zR'(z)}\right|. \tag{35}$$

Since $|B'(z)|$ is real, we obtain that $|z(R^*(z))'| = ||B'(z)|R(z) - zR'(z)|$.

Then,

$$\begin{aligned} \left|\frac{z(R^*(z))'}{R(z)}\right|^2 &= \left|\left|B'(z)\right| - \frac{zR'(z)}{R(z)}\right|^2 = \left|B'(z)\right|^2 - 2\left|B'(z)\right| \cdot \operatorname{Re}\left(\frac{zR'(z)}{R(z)}\right) + \left|\frac{zR'(z)}{R(z)}\right|^2 \geq \left|B'(z)\right|^2 \\ &\quad - 2\left|B'(z)\right|\left[\frac{1}{2}\left(\left|B'(z)\right| + \frac{2mt - mn(1+k)}{(1+k)}\right)\right] + \left|\frac{zR'(z)}{R(z)}\right|^2 = \left|\frac{zR'(z)}{R(z)}\right|^2 + \frac{mn(1+k) - 2mt}{(1+k)} \cdot \left|B'(z)\right|, \end{aligned} \tag{36}$$

where the inequality comes from (34).

This implies that for $z \in T_1$ which are not zeros of $R(z)$, where $R^*(z) = B(z)\overline{R(1/\overline{z})} = r^*(s(z)) - \overline{\alpha} m^* B(z)$ with $r^*(s(z)) = B(z)\overline{r(s(1/\overline{z}))}$,

$$\left[\left|R'(z)\right|^2 + \frac{mn(1+k) - 2mt}{(1+k)} \cdot |R(z)|^2 \left|B'(z)\right| \right]^{(1/2)} \leq \left|(R^*(z))'\right|, \tag{37}$$

Moreover, $(R^*(z))' = (r^*(s(z)))' - \overline{\alpha} m^* B'(z)$ and $R'(z) = (r(s(z)) - \alpha m^*)' = (r(s(z)))'$.

Applying these relations into (37), we obtain that

$$\left[\left|(r(s(z)))'\right|^2 + \frac{mn(1+k) - 2mt}{(1+k)} \cdot \left|r(s(z)) - \alpha m^*\right|^2 \left|B'(z)\right| \right]^{(1/2)} \leq \left|(r^*(s(z)))' - \overline{\alpha} m^* B'(z)\right|, \tag{38}$$

for $z \in T_1$ with $R(z) \neq 0$ and every α with $|\alpha| < 1$.

Choose the argument of α so that

$$\left|(r^*(s(z)))' - \overline{\alpha} m^* B'(z)\right| = \left|(r^*(s(z)))'\right| - m^* |\alpha| \left|B'(z)\right|, \tag{39}$$

for $z \in T_1$ with $R(z) \neq 0$.

Triangle inequality yields that $|r(s(z)) - \alpha m^*| \geq ||r(s(z))| - m^*|\alpha||$.

Note that $||r(s(z))| - m^*|\alpha||^2 = (|r(s(z))| - m^*|\alpha|)^2$ which implies that

$$\left|r(s(z)) - m^* \alpha\right|^2 \geq \left(|r(s(z))| - m^*|\alpha|\right)^2. \tag{40}$$

Substituting relations (40) and (39) into (38), we obtain that

$$\left[\left|(r(s(z)))'\right|^2 + \frac{mn(1+k) - 2mt}{(1+k)} \cdot \left(|r(s(z))| - m^*|\alpha|\right)^2 \left|B'(z)\right| \right]^{(1/2)} \leq \left|(r^*(s(z)))'\right| - m^*|\alpha| \left|B'(z)\right|. \tag{41}$$

Letting $|\alpha| \longrightarrow 1$, we obtain

$$\left[\left|(r(s(z)))'\right|^2 + \frac{mn(1+k) - 2mt}{(1+k)} \cdot \left(|r(s(z))| - m^*\right)^2 \left|B'(z)\right| \right]^{(1/2)} \leq \left|(r^*(s(z)))'\right| - m^* \left|B'(z)\right|. \tag{42}$$

Lemma 2 implies that

$$\left[\left|(r(s(z)))'\right|^2 + \frac{mn(1+k) - 2mt}{(1+k)} \cdot \left(r(s(z)) - m^*\right)^2 \left|B'(z)\right| \right]^{(1/2)} \leq \left|B'(z)\right| \cdot \|r \circ s\| - \left|(r(s(z)))'\right| - m^* \left|B'(z)\right|. \tag{43}$$

Equivalently,

$$\left|(r(s(z)))'\right|^2 + \frac{mn(1+k) - 2mt}{(1+k)} \cdot \left(|r(s(z))| - m^*\right)^2 \left|B'(z)\right| \leq \left[\left(\|r \circ s\| - m^*\right) \left|B'(z)\right| - \left|(r(s(z)))'\right| \right]^2. \tag{44}$$

Hence,

$$\begin{aligned} \left|(r(s(z)))'\right|^2 + \frac{mn(1+k) - 2mt}{(1+k)} \cdot \left(|r(s(z))| - m^*\right)^2 \left|B'(z)\right| \leq {} & \left(\|r \circ s\| - m^*\right)^2 \left|B'(z)\right|^2 - 2\left(\|r \circ s\| - m^*\right) \left|B'(z)\right| \left|(r(s(z)))'\right| \\ & + \left|(r(s(z)))'\right|^2. \end{aligned} \tag{45}$$

Then,

$$2\left(\|r\circ s\|-m^*\right)\left|B'(z)\right|\left|(r(s(z)))\right|'\leq\left(\|r\circ s\|-m^*\right)^2\left|B'(z)\right|^2-\frac{mn(1+k)-2mt}{(1+k)}\cdot\left(|r(s(z))|-m^*\right)^2\left|B'(z)\right|. \tag{46}$$

That is,

$$2\left(\|r\circ s\|-m^*\right)\left|(r(s(z)))\right|'\leq\left(\|r\circ s\|-m^*\right)^2\left|B'(z)\right|-\frac{mn(1+k)-2mt}{(1+k)}\cdot\left(r(s(z))-m^*\right)^2. \tag{47}$$

Thus,

$$\begin{aligned}\left|(r(s(z)))'\right|&\leq\frac{1}{2}\left[\frac{\left(\|r\circ s\|-m^*\right)^2\left|B'(z)\right|}{\left(\|r\circ s\|-m^*\right)}-\frac{(mn(1+k)-2mt)\left(|r(s(z))|-m^*\right)^2}{(1+k)\left(\|r\circ s\|-m^*\right)}\right]\\&=\frac{1}{2}\left[\left|B'(z)\right|-\frac{(mn(1+k)-2mt)\left(|r(s(z))|-m^*\right)^2}{(1+k)\left(\|r\circ s\|-m^*\right)^2}\right]\left(\|r\circ s\|-m^*\right).\end{aligned} \tag{48}$$

For $z\in T_1$ with $R(z)\neq 0$, we have

$$\begin{aligned}\left|(r(s(z)))'\right|&=\left|r'(s(z))\cdot s'(z)\right|=\left|r'(s(z))\right|\cdot\left|s'(z)\right|\\&\geq\left|r'(s(z))\right|\cdot\min_{z\in T_1}\left|s'(z)\right|.\end{aligned} \tag{49}$$

From Lemma 3, we obtain that

$$\begin{aligned}\left|(r(s(z)))'\right|&\geq\left|r'(s(z))\right|\cdot\left(m\cdot\min_{z\in T_1}|s(z)|\right)\\&=mm'\left|r'(s(z))\right|.\end{aligned} \tag{50}$$

Therefore, it follows from (48) that

$$\left|r'(s(z))\right|\leq\frac{1}{2mm'}\left[\left|B'(z)\right|-\frac{(mn(1+k)-2mt)\left(|r(s(z))|-m^*\right)^2}{(1+k)\left(\|r\circ s\|-m^*\right)^2}\right]\left(\|r\circ s\|-m^*\right), \tag{51}$$

where mt is the number of zeros of $r\circ s$ with counting multiplicity, $m'=\min_{z\in T_1}|s(z)|$, and $m^*=\min_{z\in T_k}|r(s(z))|$. This proves inequality for $R(z)\neq 0$.

In case $R(z)=0$, we obtain that $(r(s(z)))'=0$.

This implies that the above inequality is trivially true.

Therefore, inequality (33) holds for all $z\in T_1$.

Next, we show that equality holds for $r(s(z))=((z+k)^{mt}/(z-a)^{mn})$ where $s(z)=z^m$ and $B(z)=((1-az)/(z-a))^{mn}$, $a>1$, and $k\geq 1$ at $z=1$. First, we observe that $\|r\circ s\|=((1+k)^{mt}/(a-1)^{mn})=|r(s(1))|$, $m'=1, m^*=0, |B'(1)|=(mn(a+1)/(a-1))$, and $r'(s(z))=((z+k)^{mt}/mz^{m-1}\ (z-a)^{mn})[(mt/z+k)+(mn/a-z)]$.

Then,

$$\begin{aligned}\left|r'(s(1))\right|&=\frac{(1+k)^{mt}}{m(1)^{m-1}(a-1)^{mn}}\left[\frac{mt}{1+k}+\frac{mn}{a-1}\right]\\&=\left[\frac{t}{1+k}+\frac{n}{a-1}\right]\cdot\|r\circ s\|.\end{aligned} \tag{52}$$

The right side of inequality (33) is

$$\begin{aligned}\frac{1}{2mm'}\left[\left|B'(1)\right|-\frac{(mn(1+k)-2mt)\left(|r(s(1))|-m^*\right)^2}{(1+k)\left(\|r\circ s\|-m^*\right)^2}\right]\left(\|r\circ s\|-m^*\right)&=\frac{1}{2m(1)}\left[\frac{mn(a+1)}{a-1}-\frac{(mn(1+k)-2mt)}{1+k}\right]\cdot\|r\circ s\|\\&=\frac{1}{2}\left[\frac{2n}{a-1}+\frac{2t}{1+k}\right]\cdot\|r\circ s\|=\left|r'(s(1))\right|.\end{aligned} \tag{53}$$

Thus, this bound is best possible.

Theorem 7 simplifies to Theorem 4 when $s(z) = z$.

Observe that $mn(1+k) - 2mt \geq mn(k-1)$ for all $t \leq n$. We obtain an immediately consequence of Theorem 7 as follows. □

Corollary 1. *Let $r(s(z)) \in R_{mn}$ with $r(s(z)) \neq 0$ in $D_{k-}, k \geq 1$, and all zeros of $s(z)$ lie in $T_1 \cup D_{1-}$. Then, for $z \in T_1$,*

$$\left|r'(s(z))\right| \leq \frac{1}{2mm'}\left[\left|B'(z)\right| - \frac{mn(k-1)\left(\left|r(s(z))\right| - m^*\right)^2}{(1+k)\left(\|r\circ s\| - m^*\right)^2}\right]\left(\|r\circ s\| - m^*\right), \tag{54}$$

where $m' = \min_{z\in T_1}|s(z)|$ and $m^* = \min_{z\in T_k}|(r(s(z)))|$. Equality holds for $r(s(z)) = ((z+k)/(z-a))^{mn}$ where $s(z) = z^m$ and $B(z) = ((1-az)/(z-a))^{mn}$, $a > 1$, and $k \geq 1$ at $z = 1$.

From Corollary 1, if $r(s(z))$ has all its zeros in $T_k \cup D_{k+}$ with at least one zero on T_k, we obtain the following corollary.

Corollary 2. *Let $r(s(z)) \in R_{mn}$ and $r(s(z)) \neq 0$ in D_{k-} with at least one zero on T_k, where $k \geq 1$, and all zeros of $s(z)$ lie in $T_1 \cup D_{1-}$. Then, for $z \in T_1$,*

$$\left|r'(s(z))\right| \leq \frac{1}{2mm'}\left[\left|B'(z)\right| - \frac{mn(k-1)\cdot\left|r(s(z))\right|^2}{(1+k)\cdot\|r\circ s\|^2}\right]\cdot\|r\circ s\|, \tag{55}$$

where $m' = \min_{z\in T_1}|s(z)|$. Equality holds for $r(s(z)) = ((z+k)/(z-a))^{mn}$ where $s(z) = z^m$ and $B(z) = ((1-az)/(z-a))^{mn}$, $a > 1$, and $k \geq 1$ at $z = 1$.

When $k = 1$ and $r(s(z))$ has a zero on T_k, Corollary 2 generalizes Theorem 5.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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