

Research Article

Long Time Behavior of the Solution of the Two-Dimensional Dissipative QGE in Lei–Lin Spaces

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In this paper, we study the asymptotic behavior of the two-dimensional quasi-geostrophic equations with subcritical dissipation. More precisely, we establish that $\|\theta(t)\|_{2^{1-2\alpha}}$ vanishes at infinity.

1. Introduction and Statement of Main Results

In this paper, we consider the initial value problem for the 2D quasi-geostrophic equations with subcritical dissipation $(\text{QG})_\alpha$:

$$(\text{QG})_\alpha \begin{cases} \partial_t \theta + u \cdot \nabla \theta + k \Lambda^{2\alpha} \theta = 0, & x \in \mathbb{R}^2, t > 0, \\ u = R^\perp \theta = (-\mathfrak{R}_2 \theta, \mathfrak{R}_1 \theta), \\ \theta(0, x) = \theta^0(x), \end{cases} \quad (1)$$

where $1/2 < \alpha \leq 1$ is a real number and $k > 0$ is a dissipative coefficient. Λ is the operator defined by the fractional power of $-\Delta$:

$$\Lambda = (-\Delta)^{1/2}, \quad (2)$$

$$\widehat{\Lambda g} = (-\Delta)^{1/2} g = |\xi| \widehat{g},$$

and more generally

$$\widehat{\Lambda^{2\alpha} g} = (-\Delta)^\alpha g = |\xi|^{2\alpha} \widehat{g}, \quad (3)$$

where $\widehat{g}(\xi)$ denotes the Fourier transform of g . $\theta(x, t)$ is an unknown scalar function representing potential temperature, and $u = (u_1, u_2)$ is the divergence free velocity which is determined by the Riesz transformation of θ in the following way:

$$\begin{aligned} u_1 &= -\mathfrak{R}_2 \theta = -\partial_2 (-\Delta)^{-1/2} \theta, \\ u_2 &= -\mathfrak{R}_1 \theta = -\partial_1 (-\Delta)^{-1/2} \theta. \end{aligned} \quad (4)$$

Let us fix $k = 1$ for the rest of the paper.

The 2D quasi-geostrophic fluid is an important model in geophysical fluid dynamics, which are special cases of the general quasi-geostrophic approximations for atmospheric and oceanic fluid flow with the small local Rossby number which ensures the validity of the geostrophic balance between the pressure gradient and the Coriolis force (see [1]). Furthermore, this quasi-geostrophic fluid motion equation shares many features with fundamental fluid motion equations. When $k = 0$, this equation is comparable to the vorticity formulation of the Euler equations (see [2]). $(\text{QG})_\alpha$ with $\alpha = 1/2$ shares similar features with the three-dimensional Navier–Stokes equations. Thus, $\alpha = 1/2$ is therefore referred as the critical case, while the cases $\alpha < 1/2$ and $\alpha > 1/2$ are subcritical and supercritical, respectively.

The existence of a global weak solution was established by several researchers. The reader is referred to [3–7] and their references. Furthermore, in the subcritical case, Constantin and Wu [8] proved that every sufficiently smooth initial data give rise to a unique global smooth solution. For the critical case, $\alpha = 1/2$, Constantin et al. [9] proved that there exists of a unique global classical solution for any small initial data in L^∞ . The hypothesis requiring smallness in L^∞ was removed by Caffarelli and Vasseur [10]

and Dong and Du [5]. In [7], the authors proved persistence of a global solution in \mathcal{C}^∞ for to any \mathcal{C}^∞ periodic initial data. Chae and Lee [6] established the global existence and uniqueness of solution for any small initial data in the Besov space $B_{2,1}^{2-2\alpha}$.

The global existence for the quasi-geostrophic equation has been studied in the previous work of Benameur and Benhamed [3]. The authors have introduced new spaces $\mathcal{X}^{1-2\alpha}(\mathbb{R}^2)$ defined as follows:

$$\mathcal{X}^{1-2\alpha}(\mathbb{R}^2) = \left\{ f \in S'(\mathbb{R}^2); \left(\xi \mapsto |\xi|^{1-2\alpha} \widehat{f}(\xi) \right) \in L^1(\mathbb{R}^2) \right\},$$

$$\frac{1}{2} < \alpha \leq 1, \tag{5}$$

which is equipped with the norm

$$\|f\|_{\mathcal{X}^{1-2\alpha}} = \int_{\mathbb{R}^2} |\xi|^{1-2\alpha} |\widehat{f}(\xi)| d\xi. \tag{6}$$

More precisely, their result is as follows..

Theorem 1. *Let $\theta^0 \in \mathcal{X}^{1-2\alpha}(\mathbb{R}^2)$. There is a time $T > 0$ and unique solution $\theta \in \mathcal{C}([0, T], \mathcal{X}^{1-2\alpha}(\mathbb{R}^2))$ of $(QG)_\alpha$; moreover, $\theta \in L^1([0, T], \mathcal{X}^1(\mathbb{R}^2))$.*

If $\|\theta^0\|_{\mathcal{X}^{1-2\alpha}} < 1/4$, then the solution is global and

$$\|\theta(t)\|_{\mathcal{X}^{1-2\alpha}} + (1 - 4\|\theta^0\|_{\mathcal{X}^{1-2\alpha}}) \int_0^t \|\theta(s)\|_{\mathcal{X}^1} ds \leq \|\theta^0\|_{\mathcal{X}^{1-2\alpha}},$$

$$\forall t \geq 0. \tag{7}$$

We will recall further that the paper titled ‘‘Behavior of solutions of 2D quasi-geostrophic equations’’ by Constantin and Wu [8] (published in the SIAM Journal of Mathematical Analysis 30, 1999) has several results along the same lines. In particular, it includes the following result.

Theorem 2. *Let $0 < \alpha \leq 1/2$ and $\theta_0 \in L^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$. Then, there exists a weak solution θ of the $(QG)_\alpha$ equation with initial data θ_0 such that*

$$\|\theta(\cdot, t)\|_{L^2(\mathbb{R}^2)} \leq C(1 + t)^{-(1/2\alpha)}, \tag{8}$$

Proof. where C is a constant depending on L^1 and L^2 norms of θ_0 .

The decay of L^2 and Sobolev norms and asymptotic behaviour of solutions to the quasi-geostrophic equations have also been addressed in many articles (see, for example, [5, 11–22])

Global-in-time well-posedness, time-decay, and asymptotic behavior of solutions are core properties in understanding how fluid mechanics models work. In fact, there is a rich literature about those properties for fluid dynamics PDEs via several approaches and different frameworks. In this direction, there are studies in frameworks containing singular data and invariant under the scaling (critical spaces), where the smallness

conditions are taken in the weak norms of the critical spaces (e.g., see the review book [23, 24]). Of particular interest is the analysis of PDEs in critical frameworks whose structure is based on the Fourier transform. Navier–Stokes and quasi-geostrophic equations have been studied in several spaces such as PM^q [13, 25, 26], Fourier–Besov $\mathcal{FB}_{q,r}^{2-(3/q)}$ [7, 27, 28], Lei–Lin spaces \mathcal{X}^σ [3, 29], and Fourier–Besov–Morrey $\mathcal{FB}_{p,\mu,r}^s$ [11, 30]. In the case $q = 1$, Fourier–Besov spaces $\mathcal{FB}_{q,r}^s(\mathbb{R}^n)$ were introduced by Iwabuchi [31] in the context of parabolic-elliptic Keller–Segel system. Later, Iwabuchi and Takada [7] used critical $\mathcal{FB}_{1,2}^{-1}$ -spaces in order to obtain a global well-posedness class (uniformly with respect to the angular velocity) for the Navier–Stokes–Coriolis system. Taking in particular $\Omega = 0$, they also obtained a global well-posedness result for the 3D Navier–Stokes equations with small initial data in $\mathcal{FB}_{1,2}^{-1}$. Konieczny and Yoneda [28] also showed global well-posedness and asymptotic stability of small solutions for 3D Navier–Stokes equations (and Navier–Stokes–Coriolis) in critical Fourier–Besov spaces $\mathcal{FB}_{p,\infty}^{2-(3/q)}$, with $1 < p \leq \infty$. Extensions of those results to critical Fourier–Besov–Morrey spaces $\mathcal{FB}_{p,\mu,r}^s$ (larger than $\mathcal{FB}_{p,r}^s$) were obtained in [30, 32] for Navier–Stokes equations (and Navier–Stokes–Coriolis) and active scalar equations with fractional dissipation $(-\Delta)^\alpha$ (including the 2D quasi-geostrophic equation $(QG)_\alpha$), respectively.

The main goal of this paper is to study the 2D quasi-geostrophic equation in the framework of critical Lei–Lin spaces \mathcal{X}^σ with $\sigma = 1 - 2\alpha$ and $2/3 < \alpha \leq 1$. We show that solution θ presents the asymptotic property $\|\theta(t)\|_{\mathcal{X}^{1-2\alpha}} \rightarrow 0$ as $t \rightarrow \infty$ provided that $\|\theta(t)\|_{\mathcal{X}^{1-2\alpha}} < 1/4$. For that, we use standard interpolation in the Fourier space, energy estimates in L^2 , and Young’s inequality of convolutions, among others. Our main result is the following. \square

Theorem 3. *Let $2/3 < \alpha < 1$, $\|\theta^0\|_{\mathcal{X}^{1-2\alpha}} < 1/4$ and $\theta \in \mathcal{C}(\mathbb{R}^+, \mathcal{X}^{1-2\alpha}(\mathbb{R}^2))$ be a global solution of $(QG)_\alpha$ given by Theorem 1. Then,*

$$\lim_{t \rightarrow \infty} \|\theta(t)\|_{\mathcal{X}^{1-2\alpha}} = 0. \tag{9}$$

Remark 1. Remark that our main result is not implied by Theorem 2 of Constantin–Wu, because our works concern the asymptotic behavior of solution in the Lei–Lin space who belongs to a class whose definition of the norm is based on Fourier transform, but it is not contained in L^2 while that the result of Theorem 2 concerns the study in the space $L^1 \cap L^2$. The proof techniques (for Theorem 3 and Lemma 1) appeal to fairly standard interpolation in the Fourier space (which creates the need for $\alpha > 2/3$ instead of the more natural $\alpha > 1/2$), energy-type L^2 estimates for θ_n that exploit the natural appearance of the \mathcal{X}^{-1} -norm from Young’s inequality of convolutions, and two uses of (Theorem 3) (proved in the earlier work).

The remaining part of the paper is organized as follows. The main results are given in Section 1. We explain the framework in Section 2. The long time behaviour (Theorem 3) is established in Section 3.

2. Preliminary

Let us recall that in [29], Lei and Lin introduced a new space, named Lei–Lin space \mathcal{X}^{-1} , which belongs to a class whose definition of norm is based on the Fourier transform but is not contained in L^2 . In [3], Benameur and Benhamed defined the spaces that are useful for the study of well-posedness of PDEs of parabolic, elliptic, and dispersive types. More precisely, for $\sigma \in \mathbb{R}$, we define

$$\mathcal{X}^\sigma(\mathbb{R}^2) = \left\{ f \in S'(\mathbb{R}^2); \widehat{f} \in L^1_{loc}(\mathbb{R}^2) \text{ and } (\xi \mapsto |\xi|^\sigma \widehat{f}(\xi)) \in L^1(\mathbb{R}^2) \right\}, \tag{10}$$

equipped with the norm

$$\|f\|_{\mathcal{X}^\sigma} = \int_{\mathbb{R}^2} |\xi|^\sigma |\widehat{f}(\xi)| d\xi. \tag{11}$$

To prove Theorem 3, we need the following lemma.

Lemma 1. For $2/3 < \alpha < 1$, $\dot{H}^\alpha(\mathbb{R}^2) \longrightarrow \mathcal{X}^{1-2\alpha}(\mathbb{R}^2)$. More precisely, if $f \in \dot{H}^\alpha(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$. Then, we have

$$\|f\|_{\mathcal{X}^{1-2\alpha}} \leq C \|f\|_{L^2}^{((3\alpha-2)/\alpha)} \|f\|_{\dot{H}^\alpha}^{((2-2\alpha)/\alpha)}. \tag{12}$$

Proof. For $\lambda > 0$, put $\|f\|_{\mathcal{X}^{1-2\alpha}} = \mathbf{A}_\lambda + \mathbf{B}_\lambda$, where

$$\begin{aligned} \mathbf{A}_\lambda &= \int_{|\xi| < \lambda} |\xi|^{1-2\alpha} |\widehat{f}(\xi)| d\xi, \\ \mathbf{B}_\lambda &= \int_{|\xi| > \lambda} |\xi|^{1-2\alpha} |\widehat{f}(\xi)| d\xi. \end{aligned} \tag{13}$$

Let us start by controlling the first term; using Cauchy–Schwarz inequality, we get

$$\begin{aligned} \mathbf{A}_\lambda &= \int_{|\xi| < \lambda} |\xi|^{1-2\alpha} |\widehat{f}(\xi)| d\xi \\ &\leq \left(\int_{|\xi| < \lambda} |\widehat{f}(\xi)|^2 d\xi \right)^{1/2} \left(\int_{|\xi| < \lambda} |\xi|^{2-4\alpha} d\xi \right)^{1/2} \\ &\leq \sqrt{4\pi} \|f\|_{L^2} \left(\int_0^\lambda r^{3-4\alpha} dr \right)^{1/2} \\ &\leq \sqrt{\frac{4\pi}{4-4\alpha}} \|f\|_{L^2} (\lambda^{4-4\alpha})^{1/2}. \end{aligned} \tag{14}$$

Therefore,

$$\mathbf{A}_\lambda \leq \sqrt{\frac{4\pi}{4-4\alpha}} \lambda^{2-2\alpha} \|f\|_{L^2}. \tag{15}$$

A similar calculation to the foregoing yields

$$\begin{aligned} \mathbf{B}_\lambda &= \int_{|\xi| > \lambda} |\xi|^\alpha |\widehat{f}(\xi)| \times |\xi|^{1-3\alpha} d\xi \\ &\leq \left(\int_{|\xi| > \lambda} |\xi|^{2\alpha} |\widehat{f}(\xi)|^2 d\xi \right)^{1/2} \left(\int_{|\xi| > \lambda} |\xi|^{2-6\alpha} d\xi \right)^{1/2} \\ &\leq \sqrt{4\pi} \|f\|_{\dot{H}^\alpha} \left(\int_\lambda^\infty r^{3-6\alpha} dr \right)^{1/2} \\ &\leq \sqrt{\frac{4\pi}{6\alpha-4}} \lambda^{2-3\alpha} \|f\|_{\dot{H}^\alpha}. \end{aligned} \tag{16}$$

Then,

$$\mathbf{B}_\lambda \leq \sqrt{\frac{4\pi}{6\alpha-4}} \lambda^{2-3\alpha} \|f\|_{\dot{H}^\alpha}. \tag{17}$$

Combining (15) and (17), we get

$$\|f\|_{\mathcal{X}^{1-2\alpha}} \leq \sqrt{\frac{4\pi}{4-4\alpha}} \lambda^{2-2\alpha} \|f\|_{L^2} + \sqrt{\frac{4\pi}{6\alpha-4}} \lambda^{2-3\alpha} \|f\|_{\dot{H}^\alpha}. \tag{18}$$

Optimizing $\|f\|_{\mathcal{X}^{1-2\alpha}}$, it suffices to choose $\lambda = (\|f\|_{\dot{H}^\alpha} / \|f\|_{L^2})^{1/\alpha}$, to obtain (12). \square

3. Proof of the Main Theorem

The main aim of this section is to study the asymptotic behavior of the global solutions given by Theorem 1. The proof is inspired from [33].

First, we Take $\theta^0 \in \mathcal{X}^{1-2\alpha}$.

Let $\varepsilon > 0$, such that $\varepsilon \leq 1/8$. For $n \in \mathbb{N}$, put

$$\mathbf{A}_n = \left\{ \xi \in \mathbb{R}^2; |\xi| \leq n \text{ and } |\widehat{\theta}_0(\xi)| \leq n \right\}. \tag{19}$$

We have $\mathcal{F}^{-1}(11_{\mathbf{A}_n} \widehat{\theta}^0)$ converges in $\mathcal{X}^{1-2\alpha}$ to θ^0 . Then, there is $n_0 \in \mathbb{N}$ such that

$$\|\theta^0 - \mathcal{F}^{-1}(11_{\mathbf{A}_n} \widehat{\theta}^0)\|_{\mathcal{X}^{1-2\alpha}} \leq \varepsilon/4, \quad \forall n \geq n_0. \tag{20}$$

Let $n \geq n_0$ be a fixed integer. Put

$$\begin{aligned} \theta_n^0 &= \mathcal{F}^{-1}(11_{\mathbf{A}_n} \widehat{\theta}^0), \\ w_n^0 &= \theta^0 - \theta_n^0. \end{aligned} \tag{21}$$

Then,

$$\begin{aligned} \|w_n^0\|_{\mathcal{X}^{1-2\alpha}} &\leq (\varepsilon/4), \\ \theta_n^0 &\in \mathcal{X}^{1-2\alpha}(\mathbb{R}^2) \cap L^2(\mathbb{R}^2). \end{aligned} \tag{22}$$

Consider the system

$$(QG_\alpha)_n \begin{cases} \partial_t w_n + (-\Delta)^\alpha w_n + u_{w_n} \cdot \nabla w_n = 0, & x \in \mathbb{R}^2, t > 0, \\ w_n(0, x) = w_n^0(x). \end{cases} \tag{23}$$

For all $n \geq n_0$, we have

$$\|w_n^0\|_{\mathcal{X}^{1-2\alpha}} \leq \frac{\varepsilon}{4}. \tag{24}$$

Use Theorem 1; then, there exists a unique global solution $w_n \in \mathcal{C}(\mathbb{R}^+, \mathcal{X}^{1-2\alpha}(\mathbb{R}^2)) \cap L^1(\mathbb{R}^+, \mathcal{X}^1(\mathbb{R}^2))$.

Furthermore,

$$\|w_n(t)\|_{\mathcal{X}^{1-2\alpha}} + (1 - 4\|w_n^0\|_{\mathcal{X}^{1-2\alpha}}) \int_0^t \|w_n(s)\|_{\mathcal{X}^1} ds \leq \|w_n^0\|_{\mathcal{X}^{1-2\alpha}}, \quad \forall t \geq 0. \tag{25}$$

Moreover,

$$\begin{cases} \partial_t \theta + (-\Delta)^\alpha \theta + u \cdot \nabla \theta = 0, \\ \theta(0) = \theta^0 \in \mathcal{X}^{1-2\alpha}. \end{cases} \tag{26}$$

Put $\theta = \frac{\theta - w_n}{\theta_n} + w_n$. Then, θ_n is a solution of the following system:

$$\begin{cases} \partial_t \theta_n + (-\Delta)^\alpha \theta_n + u_{\theta_n} \cdot \nabla \theta_n + u_{\theta_n} \cdot \nabla w_n + u_{w_n} \cdot \nabla \theta_n = 0, \\ \theta_n(0) = \theta_n^0 \in \mathcal{X}^{1-2\alpha} \cap L^2. \end{cases} \tag{27}$$

By taking the inner product in $L^2(\mathbb{R}^2)$ with θ_n , we get

$$\frac{1}{2} \frac{d}{dt} \|\theta_n\|_{L^2}^2 + \|\theta_n\|_{\dot{H}^\alpha}^2 \leq |\langle u_{\theta_n} \cdot \nabla w_n, \theta_n \rangle_{L^2}|. \tag{28}$$

Then,

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\theta_n\|_{L^2}^2 + \|\theta_n\|_{\dot{H}^\alpha}^2 &\leq |\langle u_{\theta_n} \cdot \nabla w_n, \theta_n \rangle_{L^2}| \\ &\leq \|u_{\theta_n} \cdot \nabla w_n\|_{L^2} \|\theta_n\|_{L^2} \\ &\leq C \|\mathcal{F}(u_{\theta_n} \cdot \nabla w_n)\|_{L^2} \|\theta_n\|_{L^2} \\ &\leq C \|\mathcal{F}(u_{\theta_n}) * \mathcal{F}(\nabla w_n)\|_{L^2} \|\theta_n\|_{L^2}. \end{aligned} \tag{29}$$

By Young's inequality, we obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\theta_n\|_{L^2}^2 + \|\theta_n\|_{\dot{H}^\alpha}^2 &\leq C \|\mathcal{F}(u_{\theta_n})\|_{L^2} \|\mathcal{F}(\nabla w_n)\|_{L^1} \|\theta_n\|_{L^2} \\ &\leq C_n \|\mathcal{F}(\nabla w_n)\|_{L^1} \|\theta_n\|_{L^2}^2. \end{aligned} \tag{30}$$

Therefore,

$$\frac{1}{2} \frac{d}{dt} \|\theta_n\|_{L^2}^2 + \|\theta_n\|_{\dot{H}^\alpha}^2 \leq C_n \|w_n\|_{\mathcal{X}^1} \|\theta_n\|_{L^2}^2. \tag{31}$$

Integrating with respect to time, we obtain

$$\|\theta_n\|_{L^2}^2 + 2 \int_0^t \|\theta_n\|_{\dot{H}^\alpha}^2 \leq \|\theta_n^0\|_{L^2}^2 + C_n \int_0^t \|w_n\|_{\mathcal{X}^1} \|\theta_n\|_{L^2}^2. \tag{32}$$

By Gronwall's lemma, we get

$$\begin{aligned} \|\theta_n\|_{L^2}^2 &\leq \|\theta_n^0\|_{L^2}^2 \exp\left(C_n \int_0^t \|w_n\|_{\mathcal{X}^1}\right) \\ &\leq C_n \|\theta_n^0\|_{L^2}^2. \end{aligned} \tag{33}$$

Combining (32) and (33), we obtain

$$\begin{aligned} \|\theta_n\|_{L^2}^2 + 2 \int_0^t \|\theta_n\|_{\dot{H}^\alpha}^2 &\leq \|\theta_n^0\|_{L^2}^2 + C_0 \|\theta_n^0\|_{L^2}^2 \int_0^\infty \|w_n\|_{\mathcal{X}^1} \\ &\leq M_n. \end{aligned} \tag{34}$$

Applying Lemma 1 to θ_n , we infer

$$\|\theta_n\|_{\mathcal{X}^{1-2\alpha}} \leq C \|\theta_n\|_{L^2}^{((3\alpha-2)/\alpha)} \|\theta_n\|_{\dot{H}^\alpha}^{((2-2\alpha)/\alpha)}. \tag{35}$$

Then,

$$\|\theta_n\|_{\mathcal{X}^{1-2\alpha}}^{(\alpha/(1-\alpha))} \leq C \|\theta_n\|_{L^2}^{((3\alpha-2)/(1-\alpha))} \|\theta_n\|_{\dot{H}^\alpha}^2. \tag{36}$$

From (33) it follows that

$$\|\theta_n\|_{\mathcal{X}^{1-2\alpha}}^{(\alpha/(1-\alpha))} \leq C \|\theta_n\|_{\dot{H}^\alpha}^2. \tag{37}$$

Therefore, by integrating in time between 0 and ∞ , we get

$$\int_0^\infty \|\theta_n\|_{\mathcal{X}^{1-2\alpha}}^{(\alpha/(1-\alpha))} \leq C_0 \int_0^\infty \|\theta_n\|_{\dot{H}^\alpha}^2. \tag{38}$$

Indeed consider the following subset of $[0, \infty[$:

$$P_\varepsilon = \left\{ t \geq 0; \|\theta_n(t)\|_{\mathcal{X}^{1-2\alpha}}(\mathbb{R}^2) \geq \frac{\varepsilon}{4} \right\}. \tag{39}$$

We have

$$\|\theta_n(t)\|_{\mathcal{X}^{1-2\alpha}}(\mathbb{R}^2) \geq \mathbf{1}_{P_\varepsilon}(t) \frac{\varepsilon}{4}. \tag{40}$$

Then, for all $1/2 < \alpha \leq 1$,

$$\|\theta_n(t)\|_{\mathcal{X}^{1-2\alpha}}^{(\alpha/(1-\alpha))}(\mathbb{R}^2) \geq \mathbf{1}_{P_\varepsilon}(t) \left(\frac{\varepsilon}{4}\right)^{(\alpha/(1-\alpha))}. \tag{41}$$

Thus,

$$\int_0^\infty \|\theta_n(t)\|_{\mathcal{X}^{1-2\alpha}}^{(\alpha/(1-\alpha))}(\mathbb{R}^2) dt \geq \lambda_1(P_\varepsilon) \left(\frac{\varepsilon}{4}\right)^{(\alpha/(1-\alpha))}. \tag{42}$$

Using the inequality (38), we get $\lambda_1(P_\varepsilon) < \infty$ and $0, \infty[\setminus P_\varepsilon \neq \emptyset$.

Therefore, there exists $t_0 \in [0, \infty[\setminus P_\varepsilon$. Particularly,

$$\|\theta_n(t_0)\|_{\mathcal{X}^{1-2\alpha}}(\mathbb{R}^2) < \frac{\varepsilon}{4}. \tag{43}$$

Now, put

$$\begin{aligned} \theta(t_0) &= \frac{\theta(t_0) - w_n(t_0)}{\theta_n(t_0)} + w_n(t_0) \\ &= \theta_n(t_0) + w_n(t_0). \end{aligned} \tag{44}$$

Thus, by (24), we obtain

$$\begin{aligned} \|\theta(t_0)\|_{\mathcal{X}^{1-2\alpha}} &\leq \|\theta(t_0)\|_{\mathcal{X}^{1-2\alpha}} + \|w_n(t_0)\|_{\mathcal{X}^{1-2\alpha}} \\ &\leq \frac{\varepsilon}{4} + \|w_n^0\|_{\mathcal{X}^{1-2\alpha}} \\ &\leq \frac{\varepsilon}{4} + \frac{\varepsilon}{4}. \end{aligned} \tag{45}$$

Therefore,

$$\|\theta(t_0)\|_{\mathcal{X}^{1-2\alpha}} \leq \frac{\varepsilon}{2} \leq \varepsilon. \tag{46}$$

Let us consider the following equation:

$$\begin{cases} \partial_t \gamma + (-\Delta)^\alpha \gamma + u_\gamma \cdot \nabla \gamma = 0, & x \in \mathbb{R}^2, t > 0, \\ \gamma(0, x) = \gamma^0 = \theta(t_0). \end{cases} \quad (47)$$

Using inequality (46) and Theorem 1, we infer that there exists a unique solution $\gamma \in \mathcal{C}(\mathbb{R}^+, \mathcal{X}^{1-2\alpha}(\mathbb{R}^2)) \cap L^1(\mathbb{R}^+, \mathcal{X}^1(\mathbb{R}^2))$ of $(QG)_\alpha$ such that

$$\|\gamma\|_{\mathcal{X}^{1-2\gamma}} + (1 - 4\|\gamma^0\|_{\mathcal{X}^{1-2\alpha}}) \int_0^t \|\gamma\|_{\mathcal{X}^1} \leq \|\gamma^0\|_{\mathcal{X}^{1-2\alpha}}, \quad \forall t \geq 0. \quad (48)$$

The existence and uniqueness of a solution to the quasi-geostrophic equation gives $\forall t \geq 0 \gamma(t) = \theta(t_0 + t)$. Then,

$$\begin{aligned} \|\theta(t_0 + t)\|_{\mathcal{X}^{1-2\alpha}} &= \|\gamma(t)\|_{\mathcal{X}^{1-2\alpha}} \\ &\leq \|\gamma^0\|_{\mathcal{X}^{1-2\alpha}} \leq \varepsilon. \end{aligned} \quad (49)$$

Thus, Theorem 3 is proved.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Authors' Contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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