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Research Article

On the Relationship between Jordan Algebras and Their Universal Enveloping Algebras

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The relationship between JW-algebras (resp. JC-algebras) and their universal enveloping von Neumann algebras (resp. C^*-algebras) can be described as significant and influential. Examples of numerous relationships have been established. In this article, we established a relationship between the set of split faces of the state space (resp. normal states) of a JC-algebra (resp. a JW-algebra) and the set of split faces of the state space (resp. normal states) of its universal enveloping C^*-algebra (resp. von Neumann algebra), and we tied up this relationship with the correspondence between the classes of invariant faces, closed ideals, and central projections of these Jordan algebras and of their universal enveloping algebras.

1. Introduction

Let $\mathscr{A}$ be a C^*-algebra with a unit element denoted by I. As an important consequence of the GNS-construction, $\mathscr{A}$ is contained in the algebra $B(H)$ of all bounded linear operators on some complex Hilbert space H ([1], 4.5.6; [2], 1.9.8). The weak closure of $\mathscr{A}$ in $B(H)$ will be denoted by $\overline{\mathscr{A}}^w$. We denote by $\mathscr{A}_{s.a}$ and $\mathscr{A}^+$ the set of all self-adjoint elements and the set of all positive elements in $\mathscr{A}$, respectively. A bounded linear functional $\rho\colon \mathscr{A} \longrightarrow \mathbb{C}$ on $\mathscr{A}$ is said to be *positive* if $\rho(x^*x) \geq 0$ for every $x \in \mathscr{A}$. It is called *a state* if ρ is positive and $\rho(I) = 1$. The set of all states of $\mathscr{A}$ will be denoted by $\mathscr{S}(\mathscr{A})$ and is called *the state space* of $\mathscr{A}$; it is convex and weak$*$ compact. A *von Neumann algebra* $\mathscr{M}$ is a strongly (=weakly) closed C^*-subalgebra of the algebra $B(H)$ of bounded linear operators on a complex Hilbert space H. Every von Neumann algebra has a unit, and it is the norm closure of the linear span of its projections ([3], 2.2.6). A bounded linear functional φ on a von Neumann algebra $\mathscr{M}$ is said to be *normal* if for each bounded increasing net $\{x_i\}$ in $\mathscr{M}_{s.a}$ with limit x, the net $\{\varphi(x_i)\}$ converges to $\varphi(x)$. It is known that the set $\mathscr{M}_*$ of all normal bounded linear functionals on $\mathscr{M}$ is the predual of $\mathscr{M}$, that is, $\mathscr{M} = (\mathscr{M}_*)^*$ ([3], 3.6.5). If $\mathscr{A}\subseteq B(H)$ is a C^*-algebra, its weak closure $\overline{\mathscr{A}}^w \subseteq B(H)$ and its second dual $\mathscr{A}^{**}$ are von Neumann algebras ([4], 7.1.11). Every bounded linear functional on $\mathscr{A}$ extends to a normal bounded linear functional on $\overline{\mathscr{A}}^w$ and $\mathscr{A}^{**}$. The set of all normal states on a von Neumann algebra $\mathscr{M}$ will be denoted by $\mathscr{S}(\mathscr{M})$.

A *JC-algebra* A (with a unit element 1) is a norm (uniformly) closed Jordan subalgebra of the Jordan algebra $B(H)_{s.a}$ of all bounded self-adjoint operators on a complex Hilbert space H. The Jordan product is given by $a \circ b = ((ab + ba)/2)$. The self-adjoint part of a C^*- algebra is a JC-algebra. A *JW-algebra* $M \subseteq B(H)_{s.a}$ is a weakly closed JC-algebra. If $A \subseteq B(H)_{s.a}$ is a JC-algebra, its weak closure $\overline{A}^w \subseteq B(H)_{s.a}$ and its second dual A^{**} are JW-algebras ([4], 7.1.11). As in the context of C^*-algebras, every bounded linear functional on a JC-algebra A extends to a normal bounded linear functional on $\overline{A}^w$ and A^{**} ([5], 4.7.3). The set of all states (resp. normal states) of a JC-algebra (resp. JW-algebra) A will be denoted by $S(A)$. An element $a \in A$ is called *positive*, written as $a \geq 0$, if a is of a square ([4], 3.3.3). The set of all positive elements of A is denoted by A^+. A linear map $\varphi\colon A \longrightarrow B$ between JC-algebras A and B is called *a (Jordan) homomorphism* if it preserves the Jordan product. A *representation* of a JC-algebra A is a (Jordan) homomorphism $\pi\colon A \longrightarrow B(H)_{s.a}$, for some complex Hilbert space H. It is known that a (Jordan) homomorphism φ between JC-algebras A and B is continuous, and $\varphi(A)$ is a JC-subalgebra B ([4], 3.3.3). A JC-algebra A is said to be *reversible* if $a_1a_2\dots a_n + a_na_{n-1}\dots a_1 \in A$ whenever $a_1, a_2, \dots, a_n \in A$ and is said to be *universally reversible* if $\pi(A)$ is reversible for every representation π of A ([6], p. 5). A *multiplication operator* T_a: $A \longrightarrow A$, $a \in A$ on a Jordan algebra A is a linear operator given by $T_a b = a \circ b$, for all $b \in A$. It is clear that for all $a, b \in A$, $T_a b = T_b a$. Two elements a and b in this algebra are said to *operator commute* if $T_aT_b = T_bT_a$. The *Jordan triple operator* $U_{a,c}$: $A \longrightarrow A$, $a, c \in A$ is defined by $U_{a,c}(b) = \{abc\}$ for all $b \in A$, where $U_{a,c} = T_aT_c + T_cT_a - T_{a\circ c}$. The operator $U_{a,a}$ is denoted by U_a. The *center* $Z(A)$ of A is the set of all elements of A which operator commutes with every other element of A, that is, $Z(A) = \{a \in A\colon T_aT_b = T_bT_a, \text{for all } b \in A\}$. A JC-algebra $A \subseteq B(H)_{s.a}$ is called *irreducible* if A acts irreducibly on H, or equivalently, $\overline{A}^w$ is a JW-factor (i.e., its center consists of scalar multiples of the identity). A Jordan subalgebra J of a JC-algebra A is called *Jordan ideal* if $T_b(A) \subseteq J$ for every $b \in J$, and it is called a *quadratic ideal* if $U_b(A) \subseteq J$ whenever $b \in J$ (see [7]).

Let A be a JC-algebra. Then, there exists a complex C^*-algebra $C^*(M)$ and an embedding π_A: $A \longrightarrow C^*(A)$ such that $\pi_A(A)$ generates $C^*(A)$, and for any Jordan homomorphism π: $A \longrightarrow \mathscr{A}_{s.a}$, where $\mathscr{A}$ is a complex C^*-algebra, there is a (unique) complex $*$-homomorphism $\tilde{\pi}$: $C^*(A) \longrightarrow \mathscr{A}$ such that $\tilde{\pi} \circ \pi_A = \pi$. The composition of π_A: $A \longrightarrow C^*(A)$, and the identity map from $C^*(M)$ onto its opposite algebra $C^*(M)^{\mathrm{op}}$ induces via the universal property an involutory $*$-anti-automorphism θ_A on $C^*(A)$. The C^*-algebra $C^*(A)$ is called *the universal enveloping complex C^*-algebra of A* and θ_A is called *the canonical $*$ anti-automorphism of* $C^*(A)$ ([4], 7.1.8). Also, given a JW-algebra M, there exists a von Neumann algebra $W^*(M)$ and an embedding π_M: $M \longrightarrow W^*(M)$ such that $\pi_M(M)$ generates $W^*(M)$, and for any normal Jordan homomorphism π: $M \longrightarrow \mathscr{N}_{s.a}$, where $\mathscr{N}$ is a von Neumann algebra, there is a (unique) normal $*$--homomorphism $\tilde{\pi}$: $W^*(M) \longrightarrow \mathscr{N}$ such that $\tilde{\pi} \circ \pi_M = \pi$. The composition of π_M: $M \longrightarrow W^*(M)$, and the identity map $W^*(M) \longrightarrow W^*(M)^{\mathrm{op}}$ induces via the universal property an involutory $*$-anti-automorphism Φ_M on $W^*(M)$. The von Neumann algebra $W^*(M)$ is called *the universal enveloping von Neumann algebra of M* and Φ_M is called *the canonical $*$ anti-automorphism of* $W^*(M)$ ([4], 7.1.9).

If M is a JC-algebra (resp. JW-algebra), let $C^*(M)$ (resp. $W^*(M)$) be the universal enveloping C^*-algebra (resp. von Neumann algebra) of M, and let θ_M (resp. Φ_M) be the canonical involutive $*$-anti-automorphism of $C^*(M)$ (resp. $W^*(M)$). Usually, we will regard M as a generating Jordan subalgebra of $C^*(M)$ (resp. $W^*(M)$) so that θ_M (resp. Φ_M) fixes each point of M. If M is a JC-algebra, the real C^*-algebra $R^*(M) = \{x \in C^*(M)\colon \theta_M(x) = x^*\}$ satisfies

$$\begin{gathered} R^*(M) \cap iR^*(M) = 0, \\ C^*(M) = R^*(M) \oplus iR^*(M), \end{gathered} \tag{1}$$

and if M is a JW-algebra, the real von Neumann algebra $RW^*(M) = \{x \in W^*(M)\colon \Phi_M(x) = x^*\}$ satisfies

$$\begin{gathered} RW^*(M) \cap iRW^*(M) = 0, \\ W^*(M) = RW^*(M) \oplus iRW^*(M). \end{gathered} \tag{2}$$

It is known that the universal enveloping C^*-algebra $C^*(M)$ of a JW-algebra M can be realized as the C^*-subalgebra of $W^*(M)$ generated by M so that $W^*(M)$ is the weak closure $\overline{C^*(M)}^w$ of $C^*(M)$ ([8], Theorem 2.7), and $M = W^*(M)^{\Phi}_{\mathrm{sa}} = \{x \in W^*(M)\colon \Phi(x) = x = x^*\}$ when M is universally reversible ([4], Lemma 7.3.3).

Given a subspace M of a Banach space E and a subset G of its dual E^*, let $M^{\perp} = \{\rho \in E^*\colon \rho(x) = 0,\ x \in M\}$ and $G^{\circ} = \{x \in E\colon \rho(x) = 0,\ \rho \in G\}$ be the annihilators of M and G, respectively. It is easy to see that $M^{\perp}$ is weak$*$-closed (i.e., $\sigma(E^*, E)$-closed) in E^* and G° is a norm closed subspace of E. The closed unit ball $E_1 = \{x \in E\colon x \le 1\}$ of a normed linear space E is weak*-dense (i.e., $\sigma(E^{**}, E^*)$-dense) in $(E^{**})_1$, and hence E is weak*-dense in E^{**} ([4], 1.1.19).

A face F of a convex subset X of a vector space E over $\mathbb{C}$ is a nonempty convex subset of X such that the conditions $(1-\lambda)x_1 + \lambda x_2 \in F$, $0 \le \lambda \le 1$, $x_1, x_2 \in X$, imply that $x_1, x_2 \in F$ ([1], 1.4). A face F of X is called *a split face* if there exists a face F' such that X is a direct convex sum of F and F'. That is, each $x \in X$ can be written uniquely of the form $x = \lambda y + (1-\lambda)z$, where $0 \le \lambda \le 1$, $y \in F$ and $z \in F'$. The face F' is called the *complement of* F and is uniquely determined by F. If F is a subset of a cone $C \subset X$, then F is a face if and only if $0 \le y \le x \in F \Rightarrow y \in F$ ([9], p. 3).

Theorem 1 (see [9], Corollary 3.41, Corollary 3.63). *Let $\mathscr{M}$ be a von Neumann algebra (resp. a C^*-algebra). Then, there is a 1-1 correspondence between the set of σ-weakly closed (resp. norm closed two sided) ideals in $\mathscr{M}$ and the set of norm closed split faces (resp. weak$*$- closed split faces) of $\mathcal{S}(\mathscr{M})$ given by $\mathcal{J} \longrightarrow \mathcal{F} = \mathcal{J}^{\perp} \cap \mathcal{S}(\mathscr{M}) = \{\rho \in \mathcal{S}(\mathscr{M})\colon \rho(x^*x) = 0, \forall x \in \mathcal{J}\}$ and $\mathcal{F} \longrightarrow \mathcal{J} = \mathcal{F}^{\circ} = [x \in \mathscr{M}\colon \rho(x^*x) = 0, \forall \rho \in \mathcal{F}]$.*

Theorem 2 (see [10], Proposition 5.36, Corollary 5.37; [11], Theorem 2.3). *Let M be a JW-algebra (resp. a JC-algebra). Then, there is a 1-1 correspondence between the set of weakly (resp. norm) closed Jordan ideals in M and the set of split faces (resp. weak$*$- closed split faces) of $S(M)$ given by $J \longrightarrow F = J^{\perp} \cap S(M) = \{\rho \in S(M)\colon \rho(x) = 0, \forall x \in J\}$ and $F \longrightarrow J = F^{\circ} = \{x \in M\colon \rho(x) = 0, \forall \rho \in F\}$.*

Since JC-algebras and JW-algebras are so close to C^*-algebras and von Neumann algebras, respectively (see [3, 8, 12–17]), and in view of the similarity of the correspondences given in Theorems 1 and 2 above, a desire of studying the relationship of the sets in Theorem 2 for JW-algebras (resp. a JC-algebras) and the corresponding sets in Theorem 1 for their enveloping von Neumann algebras (resp. a C^*-algebras) arises naturally. One can expect that more similarities involving other classes of these algebras can be established. The most significant outcome of this paper is the collective correspondences given in Theorem 12 and Theorem 13.

Throughout this paper, we keep the assumption that our C^*-algebras and JC-algebras have units. The reader is referred to [4, 6, 7, 13, 14, 18] for a detailed account of the theory of JC-algebras and JW-algebras. The relevant background on the theory of C^*-algebras and von Neumann algebras can be found in [1, 2, 19, 20].

2. Main Results

In view of the remarkable connection between a JW-algebra (resp. a JC-algebra) M and its universal enveloping von Neumann algebra $W^*(M)$ (resp. C^*-algebra $C^*(M)$), we investigate the relationship between certain classes of these Jordan algebras and the corresponding classes of their universal enveloping algebras. A nice feature of our discussions is that it is conceptually simple and notably involves well-known results in the literature.

For sake of completeness and clarity, we state the following known lemmas which will be used frequently.

Lemma 1 (see [21], Lemma 3.6). *Let M be a universally reversible JW-algebra such that the center $Z(W^*(M))$ of $W^*(M)$ is pointwise fixed under the $*$-anti-automorphism $\Phi_M: W^*(M) \longrightarrow W^*(M)$. Then, each nonzero σ-weakly closed ideal $\mathcal{J}$ of $W^*(M)$ intersects M, i.e., $\mathcal{J} \cap M \neq \{0\}$.*

Lemma 2 (see[22], Proposition 2.10). *Let M be a universally reversible JW-algebra with no abelian part. If M contains no weakly closed Jordan ideal isomorphic to the self-adjoint part of a von Neumann algebra, then $Z(M) = Z(W^*(M))_{sa}$.*

Lemma 3 (see [21], Lemma 3.7). *Let A be an irreducible universally reversible JC-algebra. If $\mathcal{J}$ is a norm closed two-sided ideal in $C^*(A)$ such that $\mathcal{J} \cap A = \{0\}$, then $\mathcal{J} = \{0\}$.*

Theorem 3. *Let M be a universally reversible JW-algebra such that the center $Z(W^*(M))$ of $W^*(M)$ is pointwise fixed under the $*$-anti-automorphism $\Phi_M: W^*(M) \longrightarrow W^*(M)$. Then, there is a 1-1 correspondence $\mathfrak{I}_M^w \leftrightarrows \mathcal{J}_{W^*(M)}^w$ between the set $\mathfrak{I}_M^w$ of weakly closed Jordan ideals in M and the set $\mathcal{J}_{W^*(M)}^w$ of σ-weakly closed two-sided ideals in $W^*(M)$.*

Proof. Let J be a weakly closed Jordan ideal of M; then, the C^*-algebra $[J]$ generated by J in is an ideal in $C^*(A)$, $[J] \cong C^*(J)$, and $J = [J] \cap M$ ([11], Theorem 2). Since $C^*(M)$ can realized as the C^*-subalgebra of $W^*(M)$ generated by M, $W^*(M)$ is the weak closure $\overline{C^*(M)}^w$ of $C^*(M)$, and $W^*(J) \cong \overline{C^*(J)}^w \cong \overline{[J]}^w$ is the weakly closed ideal in $W^*(M)$ corresponding to J.

Conversely, let $\mathcal{J} \neq \{0\}$ be a weakly closed two-sided ideal of $W^*(M)$, and note that since M is universally reversible $M = W^*(M)_{sa}^{\Phi} = \{x \in W^*(M): \Phi(x) = x = x^*\}$, by ([4], Proposition 7.3.3), $Z(W^*(M))_{s.a} = Z(M)$. Therefore, $J = M \cap \mathcal{J} \neq \{0\}$ is a weakly closed Jordan ideal of M by ([21], Lemma 3.6), and the weak closure $\overline{[J]}^w$ of the C^*-algebra $[J]$ generated by J in $W^*(M)$ is $\mathcal{J}$ (see [11], Theorem 2). □

Corollary 1. *Let M be a universally reversible JW-algebra with no abelian part such that M contains no weakly closed Jordan ideal isomorphic to the self-adjoint part of a von Neumann algebra. Then, there is a 1-1 correspondence between the set $\mathfrak{I}_M^w$ of weakly closed Jordan ideals in M and the set $\mathcal{J}_{W(M)}^w$ of σ-weakly closed two-sided ideals in $W^*(M)$.*

Proof. Since M contains no weakly closed Jordan ideal isomorphic to the self-adjoint part of a von Neumann algebra, $Z(W^*(M))_{s.a} = Z(M)$ by ([22], Theorem 7). The proof is completed by Theorem 3.

Applying Lemma 3 and a similar argument of Theorem 3, we have the following. □

Theorem 4. *Let A be an irreducible universally reversible JC-algebra. Then, there is a 1-1 correspondence between the set $\mathfrak{I}_A^n$ of norm closed Jordan ideals in A and the set $\mathcal{J}_{C^*(A)}^n$ of norm closed two-sided ideals in $C^*(A)$.*

Recall that if M is a von Neumann algebra (or a JW-algebra) and F is a nonempty set of positive normal functionals on M, then the smallest projection p in M such that $\sigma(p) = \sigma$ for all σ in F is called the support projection or the carrier projection of F, denoted by car (F). Given a projection $p \in M$, the set $F_p = \{\sigma \in S(M): \sigma(p) = 1\}$ associated with p is a normed closed face of $S(M)$, called a projective face. Every norm closed face F of $S(M)$ is a projective face ([9], Corollary 3.31; [10], Theorem 5.32), that is, $F = F_p$ for some projection $p \in M$, and car $(F_p) = p$, and $F \subseteq S(M)$ is a norm closed split face of $S(M)$ if and only if p is a central projection in M ([9], Proposition 3.40; [10], Corollary 5.35).

Theorem 5. *Let M be a universally reversible JW-algebra such that the center $Z(W^*(M))$ of $W^*(M)$ is pointwise fixed under the $*$-anti-automorphism $\Phi_M: W^*(M) \longrightarrow W^*(M)$. Then, there is a 1-1 correspondence $\mathcal{F}_{S(M)}^n \leftrightarrows \mathcal{F}_{\mathcal{S}(W^*(M))}^n$ between the set $\mathcal{F}_{S(M)}^n$ of norm closed split faces of $S(M)$ and the set $\mathcal{F}_{\mathcal{S}(W^*(M))}^n$ of norm closed split faces of $\mathcal{S}(W^*(M))$.*

Proof. Let $\mathcal{F}$ be a norm closed split face of $\mathcal{S}(W^*(M))$. Then, $\mathcal{F} = \mathcal{F}_p = \{\sigma \in \mathcal{S}(W^*(M)): \sigma(p) = 1\}$ where $p = \text{car}(\mathcal{F})$. Since $\mathcal{F}$ is a split face, $p \in Z(W^*(M))_{s.a}$ by ([9], Proposition 3.40) which implies that $p \in Z(M)$ since $Z(W^*(M))_{s.a} = Z(M)$. Let $G = \{\sigma|_M(p), \forall \sigma \in \mathcal{F}\}$; then, it is clear that $G \subseteq F_p = \{\rho \in S(M): \rho(p) = 1\}$. On the other hand, if $\rho \in F_p$, then ρ extends to a normal state $\hat{\rho}$ on $W^*(M)$, and $\hat{\rho}(p) = \rho(p) = 1$ (see [4], Theorem 7.1.9), which implies that $\hat{\rho} \in \mathcal{F}_p = \mathcal{F}$ and $\rho \in G$. Hence, $G = F_p$, so G is a split face of $S(M)$.

Conversely, Let F be a norm closed split face of $S(M)$; then, $F = F_p = \{\rho \in S(M): \rho(p) = 1\}$ where $p = \text{car}(F) \in Z(M) = Z(W^*(M))_{s.a}$. Let $\hat{G}$ be the set of all normal extensions $\hat{\rho}$ of states ρ in F; then, $\hat{\rho}(p) = \rho(p) = 1, \forall \hat{\rho} \in \hat{G}$, which implies that $\hat{G} \subseteq \mathcal{F}_p = \{\sigma \in \mathcal{S}(W^*(M)): \sigma(p) = 1\}$. Since $\sigma|_M(p) = 1, \forall \sigma \in \mathcal{F}_p$, we see that $\mathcal{F}_p \subseteq \hat{G}$. Hence, $\hat{G} = \mathcal{F}_p$

and $\hat{G} = \left\{\hat{\rho} \in \mathcal{S}(W^*(M)) \colon \hat{\rho}|_M \in F\right\}$ is a split face of $\mathcal{S}(W^*(M))$. □

Remark 1

(i) Recall that if M is a JW-algebra, then $U_a(b) = \{aba\} \geq 0$ for all $a \in M$ and $b \in M^+ = \{a^2 \colon a \in M\}$ ([4], Proposition 3.3.6), that is, $U_a(M^+) \subseteq M^+$ for all $a \in M$, and hence M^+ is a proper convex cone of M (see [4], Lemma 3.3.7). Also, recall that a norm closed subspace J of a JC-algebra A is a Jordan ideal of A if and only if $U_a(J) \subseteq J$ for all $a \in A$ ([23], Lemma 2.4), and J is a quadratic ideal of A if and only if it is a hereditary subalgebra of A (a subalgebra $B \subseteq A$ is said to be hereditary if the cone B^+ is a face of the cone A^+) ([23], Theorem 2.3). It is easy to see a Jordan ideal J of A is a quadratic ideal by applying the identity $U_a = 2T_a^2 + -T_{a^2}$. Hence, if J is a norm closed Jordan ideal J of a JC-algebra A, then J^+ is a norm closed invariant face of A^+ and vice versa (see [23], Corollary 2.5).

(ii) A face $\mathcal{D}$ in the positive cone $\mathcal{M}^+$ of a von Neumann algebra $\mathcal{M}$ is said to be invariant if whenever $x \in \mathcal{D}$, $u \in \mathcal{M}$ is a unitary element, then $uxu^* \in \mathcal{D}$, or equivalently, $x \in \mathcal{M}$, $x^*x \in \mathcal{D} \Longrightarrow xx^* \in \mathcal{D}$ (see [20], p. 83).

It is known that if $\mathcal{M}$ is a von Neumann algebra, then there is a 1-1 correspondence between the set $\mathcal{J}_{\mathcal{M}}^w$ of σ-weakly closed two-sided ideals of $\mathcal{M}$ and the set $\mathcal{D}_{\mathcal{M}}^w$ of weakly closed invariant faces of $\mathcal{M}^+$ ([20], Corollary 3.21.2). The following theorem is the Jordan analogue of this result.

Theorem 6. *Let M be a JW-algebra. Then, the mapping $J \longrightarrow J^+$ is a 1-1 correspondence between the set $\mathfrak{J}_M^w$ of weakly closed Jordan ideals J of M and the set $\mathcal{D}_M^w$ of weakly closed invariant faces of M^+.*

Proof. Let J be a weakly closed Jordan ideal of M and let a and b be elements of M^+ such that $0 \leq a \leq b \in J^+$. By ([23], Lemma 2.1), there is an element $c \in M^+$ such that $a = U_{b^{1/3}}c$, which implies that $a = \{b^{1/3}cb^{(1/3)}\} \geq 0$ ([4], Proposition 3.3.6), that is, $a \in J^+$, and hence J^+ is a an invariant face of M^+. Clearly, J^+ is weakly closed since M^+ is weakly closed and $J^+ = J \cap M^+$. Therefore, $J \mapsto J^+$ is an injection from the set $\mathfrak{J}_M^w$ of weakly closed Jordan ideals J of M into the set $\mathcal{D}_M^w$ of weakly closed invariant faces of M^+. Conversely, given a weakly closed invariant face H of M^+, H is norm closed since the weak topology is weaker than the norm topology and $U_a(H) \subseteq H$ for all $a \in M$. By ([23], Corollary 2.5), the set $J \equiv J_H = \{a \in M \colon a^2 \in H\}$ is a norm closed Jordan ideal in M and $J^+ = H$. Since the weak closure $\overline{J}^w$ of a Jordan ideal J of M is ideal in M ([11], p. 314), there is a unique central projection $e \in M$ such that $\overline{J}^w = eM$ ([4], Proposition 4.3.6; [10], Proposition 2.39). Since the multiplication operator T_a is weakly continuous for all $a \in M$, $(\overline{J}^w)^+ = \overline{J^+}^w = \overline{H}^w = H$ ([4], Corollary 4.1.6), and so we have $H = (\overline{J}^w)^+ = eM^+$. Now, let $c \in \overline{J}^w = eM$; then, $c = ea$ for some $a \in M$, and $c^2 \in (\overline{J}^w)^+ = H$. Note that $c = ea = e \circ a$ by ([10], Proposition 1.47 and Proposition 1.49), which implies that

$$\begin{aligned} c^2 &= (a \circ e)^2 = (a \circ e) \circ (a \circ e) = T_{e \circ a}(e \circ a) \\ &= T_e(T_a(T_e a)) = T_e(T_e a^2) \\ &= T_e T_{a^2} e = T_{a^2} T_e e = a^2 \circ e = ea^2. \end{aligned} \tag{3}$$

Therefore, $c^2 = ea^2 \in eM^+ = H$, which implies that $c \in J$, and hence J is a weakly closed Jordan ideal of M, proving that the mapping $H \mapsto \{a \in M \colon a^2 \in H\}$ is an injection form the set $\mathcal{D}_M^w$ of weakly closed invariant faces of M^+ into the set $\mathfrak{J}_M^w$ of weakly closed Jordan ideals of M which is clearly the inverse of the injection map $J \mapsto J^+ \colon \mathfrak{J}_M^w \longrightarrow \mathcal{D}_M^w$, proving the theorem.

Note that given a von Neumann algebra $\mathcal{M}$, there is a 1-1 correspondence between the set $\mathcal{F}_{\mathcal{S}(\mathcal{M})}^n$ of norm closed split faces of $\mathcal{S}(\mathcal{M})$ and the set $\mathcal{D}_{\mathcal{M}}^w$ of σ-weakly closed invariant faces of $\mathcal{M}^+$, which is an immediate result of the reciprocal bijections $\mathcal{J}_{\mathcal{M}}^w \leftrightarrows \mathcal{D}_{\mathcal{M}}^w$ between the set $\mathcal{J}_{\mathcal{M}}^w$ of σ-weakly closed two-sided ideals of $\mathcal{M}$ and the set $\mathcal{D}_{\mathcal{M}}^w$ of σ-weakly closed invariant faces of $\mathcal{M}^+$ (see [2], Corollary 3.21.2) and the mutual bijections $\mathcal{J}_{\mathcal{M}}^w \leftrightarrows \mathcal{F}_{\mathcal{S}(\mathcal{M})}^n$ between the set $\mathcal{J}_{\mathcal{M}}^w$ of σ-weakly closed ideals in $\mathcal{M}$ and the set $\mathcal{F}_{\mathcal{S}(\mathcal{M})}^n$ of norm closed split faces of $\mathcal{S}(\mathcal{M})$ ([9], Corollary 3.41 and Corollary 3.63).

The Jordan analogue of the above result is given in the following. □

Theorem 7. *Let M be a JW-algebra. Then, there is a 1-1 correspondence between the set $\mathcal{F}_{\mathcal{S}(\mathcal{M})}^n$ of norm closed split faces of $S(M)$ and the set $\mathcal{D}_{\mathcal{M}}^w$ of weakly closed invariant faces of M^+.*

Proof. The result follows by Theorem 6, Theorem 2, and the commutative diagram $\mathcal{F}_{S(M)}^n \leftrightarrows \mathfrak{J}_M^w \leftrightarrows \mathcal{D}_{\mathcal{M}}^w$. □

Remark 2. Let A be a JC-algebra with positive cone A^+; using the reciprocal bijections $\mathfrak{J}_A^n \leftrightarrows \mathcal{D}_A^n$ between the set $\mathfrak{J}_A^n$ of norm closed Jordan ideals I in A and the set $\mathcal{D}_A^n$ of norm closed invariant faces D of A^+ ([23], Corollary 2.5) and the reciprocal bijections $\mathfrak{J}_A^n \leftrightarrows \mathcal{F}_{S(A)}^{w^*}$ between the set $\mathfrak{J}_A^n$ of norm closed Jordan ideals I in A and the set $\mathcal{F}_{S(A)}^{w^*}$ of w^*-closed split faces of $S(A)$ (cf. Theorem 2), we have the JC-algebra analogue of Theorem 7.

Theorem 8. *Let A be a JC-algebra. Then, there is a 1-1 correspondence between the set $\mathcal{D}_A^n$ of norm closed invariant faces of A^+ and the set $\mathcal{F}_{S(A)}^{w^*}$ of w^*-closed split faces of $S(A)$.*

An application of Theorems 6, 3, and ([9], Corollary 3.41 and Corollary 3.63) gives the following.

Theorem 9. *Let M be a universally reversible JW-algebra such that the center $Z(W^*(M))$ of $W^*(M)$ is pointwise fixed under the $*$-anti-automorphism $\Phi_M \colon W^*(M) \longrightarrow W^*(M)$. Then, there is a 1-1 correspondence between the set $\mathcal{D}_M^w$ of weakly closed invariant faces of M^+ and the set $\mathcal{D}_{W^*(M)}^w$ of σ-weakly closed invariant faces of $W^*(M)^+$.*

Proof. The result follows from the following commutative diagram:

$$\begin{array}{ccc} \mathfrak{J}^w_M & \leftrightarrows & \mathfrak{J}^w_{W^*(M)} \\ \uparrow\downarrow & & \uparrow\downarrow \\ \mathfrak{D}^w_M & \leftrightarrows & \mathfrak{D}^w_{W^*(M)} \end{array} \quad (4)$$

□

Definition 1. Let $\mathscr{A}$ be a unital C^*-algebra; a face $\mathscr{D} \subset \mathscr{A}^+$ is said to be invariant if $uxu^* \in \mathscr{D}$ whenever $x \in \mathscr{D}$ and u is a unitary element in $\mathscr{A}$.

Recall that if $\mathscr{M}$ is a von Neumann algebra, then there is a reciprocal bijection between the set of all two-sided ideals $\mathscr{J} \subset \mathscr{M}$ and the set of all invariant faces $\mathscr{D} \subset \mathscr{M}^+$ ([20], Corollary 3.21.2). The C^*-algebra analogue of this result is proved in the following theorem.

Theorem 10. *Let $\mathscr{A}$ be a C^*-algebra. Then, there is a 1-1 correspondence between the set $\mathscr{D}^n_A$ of norm closed invariant faces of $\mathscr{A}^+$ and the set $\mathscr{J}^n_A$ of norm closed two-sided ideals of $\mathscr{A}$.*

Proof. Let $\mathscr{J}$ be a norm closed two-sided ideal in $\mathscr{A}$. Then, $\mathscr{J}^+$ is a norm closed face of $\mathscr{A}^+$ by ([9], Theorem 3.46). To see that $\mathscr{J}^+$ is invariant, let $x \in \mathscr{J}^+$ and let u be a unitary element in $\mathscr{A}$. Since $\mathscr{J}$ is two-sided ideal, $uxu^* \in \mathscr{J}$. By ([1], Corollary 4.2.7), $uxu^* \in \mathscr{A}^+$, and hence $uxu^* \in \mathscr{J}^+$. That is, $\mathscr{J}^+$ is a norm closed invariant face of $\mathscr{A}^+$. Conversely, let $\mathscr{D}$ be a norm closed invariant face of $\mathscr{A}^+$. By ([9], Theorem 3.46), $\mathscr{J} = \{d \in \mathscr{A}\colon d^*d \in \mathscr{D}\}$ is a norm closed left ideal in $\mathscr{A}$, and $\mathscr{D} = \mathscr{J}^+$. To see that $\mathscr{J}$ is also a right ideal, first we show that $uxu^* \in \mathscr{J}^+$ whenever $x \in \mathscr{J}^+$ and let u be a unitary element of $\mathscr{A}$. So, let $x \in \mathscr{J}^+$ and u be a unitary element of $\mathscr{A}$. Then, $x = vy$ for some $v \in \mathscr{A}$, and $y \in \mathscr{J}^+$ (see [9], Proposition 4.2.9). Since $\mathscr{J}^+ = \mathscr{D}$ and $\mathscr{D}$ is invariant, we have $uyu^* \in \mathscr{J}^+ \subset \mathscr{J}$. It follows that $xu = vyu = vuu^*yu = vu(u^*yu) \in \mathscr{J}$, since $\mathscr{J}$ is a left ideal. By ([9], Theorem 4.1.7), each element in $\mathscr{A}$ is a finite linear combination of unitary elements in $\mathscr{A}$, which implies that $xz \in \mathscr{J}$ for any $x \in \mathscr{J}$ and any $z \in \mathscr{A}$. Hence, $\mathscr{J}$ is a right ideal.

Our next result is the JC-algebra analogue of Theorem 9. □

Theorem 11. *Let A be an irreducible universally reversible JC-algebra. Then, there is a 1-1 correspondence between the set $\mathscr{D}^n_A$ of norm closed invariant faces of A^+ and the set $\mathscr{D}^n_{C^*(M)}$ of norm closed invariant faces of $C^*(M)^+$.*

Proof. By Theorem 10, we have $\mathscr{D}^n_{C^*(A)} \leftrightarrows \mathscr{J}^n_{C^*(M)}$, and from Theorem 4, we have $\mathscr{J}^n_{C^*(M)} \leftrightarrows \mathfrak{J}^n_A$. The result follows since $\mathfrak{J}^n_A \leftrightarrows \mathscr{D}^n_A$ by ([23], Corollary 2.5). □

Remark 3

(i) It is well known that every σ-weakly closed two-sided ideal $\mathscr{J}$ in a von Neumann algebra $\mathscr{M}$ contains a unique central projection e in $\mathscr{M}$ such that $\mathscr{J} = e\mathscr{M}$ (see ([2], Proposition 2.3.12); [19], Theorem 6.8.8; [9], Theorem 3.35 and Theorem 3.40). Obviously, given a central projection e in $\mathscr{M}$, $e\mathscr{M}$ is a σ-weakly closed two-sided ideal in $\mathscr{M}$. The uniqueness of the central projection presenting the weakly closed ideal implies that there is a mutual bijection $\mathscr{J} \leftrightarrow e\mathscr{M}\colon \mathscr{J}^w_{\mathscr{M}} \leftrightarrows \mathscr{P}_{Z(\mathscr{M})}$ between the set $\mathscr{J}^w_{\mathscr{M}}$ of σ-weakly closed two-sided ideals in $\mathscr{M}$ and the set $\mathscr{P}_{Z(\mathscr{M})}$ of central projections in $\mathscr{M}$. On the other hand, since $\mathscr{J}^+ = (e\mathscr{M})^+ = e\mathscr{M}^+$, by ([20], Corollary 3.21.2), there is a 1-1 correspondence between the set $\mathscr{D}^w_{\mathscr{M}}$ of σ-weakly closed invariant faces of $\mathscr{M}^+$ and the set $\mathscr{P}_{Z(\mathscr{M})}$ of central projections in $\mathscr{M}$.

(ii) The similar known result in the context of JW-algebras asserts that a weakly closed subalgebra J of a JW-algebra M is a Jordan ideal if and only if it has the form cM for some central projection $c \in M$ (necessarily unique) ([10], Proposition 2.39) (see also ([4], Proposition 4.3.6)). Hence, there is a mutual bijection $J \longleftrightarrow eM\colon \mathscr{J}^w_M \leftrightarrows \mathscr{P}_{Z(M)}$ between the set $\mathscr{J}^w_M$ of weakly closed Jordan ideals of M and the set $\mathscr{P}_{Z(M)}$ of central projections in M. By Theorem 6, there is a 1-1 correspondence between the set $\mathscr{D}^w_M$ of weakly closed invariant faces of M^+ and the set $\mathscr{P}_{Z(M)}$ of central projections in M.

Combining the results in Theorems 1–3 and 9, Corollary 1, and Remarks 1 and 3, we have the following collective overview correspondences between JW-algebras and their universal enveloping von Neumann algebras.

Theorem 12. *Let M be a universally reversible JW-algebra such that the center $Z(W^*(M))$ of $W^*(M)$ is pointwise fixed under the $*$-anti-automorphism $\Phi_M\colon W^*(M) \longrightarrow W^*(M)$. Then, there is a 1-1 correspondence between the following sets:*

(i) *The set $\mathscr{F}^n_{S(M)}$ of norm closed split faces of $\mathscr{S}(M)$*

(ii) *The set $\mathscr{F}^n_{\mathscr{S}(W^*(M))}$ of norm closed split faces of $\mathscr{S}(W^*(M))$*

(iii) *The set $\mathfrak{J}^w_M$ of weakly closed Jordan ideals of M*

(iv) *The set $\mathscr{J}^w_{W^*(M)}$ of σ-weakly closed two-sided ideals of $W^*(M)$*

(v) *The set $\mathscr{D}^w_M$ of weakly closed invariant faces of M^+*

(vi) *The set $\mathscr{D}^w_{W^*(M)}$ of σ-weakly closed invariant faces of $W^*(M)^+$*

(vii) *The set $\mathscr{P}_{Z(M)}$ of central projections in M*

(viii) *The set $\mathscr{P}_{Z(W^*(M))}$ of central projections in $W^*(M)$*

Proof. It is immediate from following commutative diagram:

$$\begin{array}{ccccc} & & \mathcal{P}_{Z(M)} & & \\ & & \uparrow\downarrow & & \\ \mathfrak{D}^w_M & \leftrightarrows & \mathfrak{J}^w_M & \leftrightarrows & \mathfrak{F}^n_{S(M)} \\ \uparrow\downarrow & & \uparrow\downarrow & & \uparrow\downarrow \\ \mathfrak{D}^w_{W^*(M)} & \leftrightarrows & \mathfrak{J}^w_{W^*(M)} & \leftrightarrows & \mathfrak{F}^n_{S(W^*(M))} \\ & & \uparrow\downarrow & & \\ & & \mathcal{P}_{Z(W^*(M))} & & \end{array} \quad (5)$$

□

Remark 4

(i) Given a C^*-algebra $\mathscr{A}$, we shall regard $\mathscr{A}$ as a C^*-subalgebra of its second dual $\mathscr{A}^{**}$ and naturally identify the states of $\mathscr{A}$ with the normal states of $\mathscr{A}^{**}$. Then, we identify the enveloping von Neumann algebra (i.e., the σ-weak closure $\overline{\mathscr{A}}^w \cong \overline{\pi(\mathscr{A})}^w \subseteq B(H)$ of $\mathscr{A}$, where π: $\mathscr{A} \longrightarrow B(H)$ is the universal representation of $\mathscr{A}$ on the Hilbert space H) with $\mathscr{A}^{**}$ (see [2], Theorem 3.2.4, p.122; [9], Corollary 2.1.27). If $\mathscr{J}$ is a norm closed two-sided ideal in $\mathscr{A}$, then the weak closure $\overline{\mathscr{J}}^w$ of $\mathscr{J}$ in $\mathscr{A}^{**}$ is a two-sided ideal, and hence $\overline{\mathscr{J}}^w = e\mathscr{A}^{**}$ for a unique central projection $e \in \mathscr{A}^{**}$ and $\mathscr{J} = \mathscr{A} \cap \overline{\mathscr{J}}^w$. Conversely, if $\mathscr{J}$ is σ-weakly closed two-sided ideal in $\mathscr{A}^{**}$, and since $\mathscr{A}$ is weakly dense in $\mathscr{A}^{**}$, $\mathscr{J} = \mathscr{A} \cap \mathscr{J}$ is a norm closed two-sided ideal in $\mathscr{A}$ such that $\overline{\mathscr{J}}^w = \mathscr{J}$. Note that the map $\mathscr{J} \mapsto \overline{\mathscr{J}}^w$ is a bijection from the set $\mathscr{J}^n_{\mathscr{A}}$ of norm closed two ideals of $\mathscr{A}$ and the set $\mathscr{J}^w_{\mathscr{A}^{**}}$ of weakly closed ideals in $\mathscr{A}^{**}$. Since the map $\mathscr{J} \leftrightarrow e\mathscr{A}^{**}$: $\mathscr{J}^w_{\mathscr{A}^{**}} \leftrightarrows \mathscr{P}_{Z(\mathscr{A}^{**})}$ is a mutual bijection between the set $\mathscr{J}^w_{\mathscr{A}^{**}}$ of weakly closed two-sided ideals of $\mathscr{A}^{**}$ and the set $\mathscr{P}_{Z(\mathscr{A}^{**})}$ of central projections in $\mathscr{A}^{**}$, we have a 1-1 correspondence $\mathscr{J}^n_{\mathscr{A}} \leftrightarrows \mathscr{P}_{Z(\mathscr{A}^{**})}$ between the set $\mathscr{J}_{\mathscr{A}}$ of norm closed two-sided ideals $\mathscr{J}$ in $\mathscr{A}$ and the set $\mathscr{P}_{Z(\mathscr{A}^{**})}$ of central projections in $\mathscr{A}^{**}$.

(ii) A similar argument in the context of JC-algebras leads to a mutual correspondence $\mathfrak{J}^n_A \leftrightarrows \mathscr{P}_{Z(A^{**})}$ between the set $\mathfrak{J}^n_A$ of norm closed Jordan ideals in a JC-algebra A and the set $\mathscr{P}_{Z(\mathscr{A}^{**})}$ of central projections in its second dual A^{**} by using the fact that the weak closure $\overline{J}^w$ in A^{**} of a norm closed Jordan ideal J in A is an ideal, $\overline{J}^w = eA^{**}$, for some central projection $e \in A^{**}$ and $J = A \cap \overline{J}^w$.

(iii) Since $\overline{J^+}^w = (\overline{J}^w)^+ = (eA^{**})^+ = eA^{**+}$, then by Theorem 11 and (ii) above, there is a 1-1 correspondence $\mathscr{D}^n_A \leftrightarrows \mathscr{P}_{Z(A^{**})}$ between the set $\mathscr{D}^n_A$ of norm closed invariant faces of A^+ and the set $\mathscr{P}_{Z(\mathscr{A}^{**})}$ of central projections in A^{**}.

Collecting the results in Theorems 1, 2, 8, 10, and 11 and Remarks 2 and 4, we have the following combined summary of correspondences between JC-algebras and their universal enveloping C^*-algebras.

Theorem 13. *Let A be an irreducible universally reversible JC-algebra. Then, there is a 1-1 correspondence between the following sets:*

(i) The set $\mathscr{F}^{w^*}_{S(A)}$ of weak-closed split faces of $\mathcal{S}(A)$*

(ii) The set $\mathscr{F}^{w^*}_{C^*(A)}$ of weak- closed split faces of $\mathcal{S}(C^*(A))$*

(iii) The set $\mathfrak{J}^n_A$ of norm closed Jordan ideals in A

(iv) The set $\mathscr{J}^n_{C^*(A)}$ of norm closed two-sided ideals in $C^*(A)$

(v) The set $\mathscr{D}^n_A$ of norm closed invariant faces of A^+

(vi) The set $\mathscr{D}^n_{C^*(M)}$ of norm closed invariant faces of $C^*(M)^+$

(vii) The set $\mathscr{P}_{Z(A^{**})}$ of central projections in A^{**}

(viii) The set $\mathscr{P}_{Z(C^*(A)^{**})}$ of central projections in $C^*(A)^{**}$

Proof. It is immediate from the following commutative diagram:

$$\begin{array}{ccccc} & & \mathcal{P}_{Z(A^{**})} & & \\ & & \uparrow\downarrow & & \\ \mathfrak{D}^n_A & \leftrightarrows & \mathfrak{J}^n_A & \leftrightarrows & \mathfrak{F}^{w^*}_{S(A)} \\ \uparrow\downarrow & & \uparrow\downarrow & & \uparrow\downarrow \\ \mathfrak{D}^n_{C^*(M)} & \leftrightarrows & \mathfrak{J}^n_{C^*(A)} & \leftrightarrows & \mathfrak{F}^{w^*}_{C^*(A)} \\ & & \uparrow\downarrow & & \\ & & \mathcal{P}_{Z(C^*(A)^{**})} & & \end{array} \quad (6)$$

□

Data Availability

The data used to support the findings of this study are included within the article.

Disclosure

Most of the results in this article are partly contained in the second author's MS.C. thesis written at King Abdulaziz University under the supervision of Professor F. B. H. Jamjoom.

Conflicts of Interest

The authors declare that they have no conflicts of interest regarding the publication of this paper.

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