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Research Article

Fixed-Point Theorems for $\theta-\phi$-Contraction in Generalized Asymmetric Metric Spaces

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In the last few decades, a lot of generalizations of the Banach contraction principle had been introduced. In this paper, we present the notion of θ-contraction and $\theta-\phi$-contraction in generalized asymmetric metric spaces to study the existence and uniqueness of the fixed point for them. We will also provide some illustrative examples. Our results improve many existing results.

1. Introduction

The problem of the existence of the solution of many mathematical models is equivalent to the existence of a fixed-point problem for a certain map. The study of fixed points, therefore, has a central role in many disciplines of applied sciences. The most essential and key part of the theory of fixed points is the existence of the solution of operator equations satisfying certain conditions, for example, Fredholm integral equations, Volterra integral equations, and two-point boundary-value problems in differential equations, as well as some eigenvalue problems [1–3]. A beautiful blend of analysis, topology, and geometry has laid down the foundation of the theory of fixed points.

The Banach contraction principle [4] has become a powerful tool in modern analysis, and it is an important tool for solving existence problems in mathematics and physics. Many authors have established the theory of fixed points particularly in two directions: one by stating the conditions on the mapping T and second, taking the set X as a more general structure [5–8].

Many generalizations of the concept of metric spaces are defined, and some fixed-point theorems are proved in these spaces. In particular, asymmetric metric spaces were introduced by Wilson [9] as metric spaces, but without the requirement that the asymmetric metric d has to satisfy $d(x,y)=d(y,x)$.

Asymmetric metric spaces have numerous recent applications both in pure and applied mathematics, for example, in rate-independent models for plasticity [10], shape-memory alloys [11], models for material failure [12], and the questions of the existence and uniqueness of Hamilton–Jacobi equations [13].

Many mathematicians worked on this interesting space. For more details, refer [14, 15].

A. Branciari in [16] initiated the notions of a generalized metric space as a generalization of a metric space, where the triangular inequality of metric spaces was replaced by $d(x,y)\le d(x,u)+d(u,v)+d(v,y)$ (quadrilateral inequality). Various fixed-point results were established on such spaces, see [4, 17–21] and references therein.

Combining conditions used for definitions of asymmetric metric and generalized metric spaces, Piri et al. [22] announced the notions of the generalized asymmetric metric space.

In this paper, we introduce the notion of θ-contraction and $\theta-\phi$-contraction and establish some new fixed-point theorems for mappings in the setting of complete generalized asymmetric metric spaces. Our result generalizes, improves, and extends the corresponding results due to Kannan and Reich. Moreover, illustrative examples are presented to support the obtained results.

2. Preliminaries

In the following, we recollect some definitions which will be useful in our main results.

Definition 1 (see [16]). Let X be a nonempty set and $d: X \times X \longrightarrow \mathbb{R}^+$ be a mapping such that, for all $x, y \in X$ and for all distinct points $u, v \in X$, each of them different from x and y, one has

(i) $d(x, y) = 0$ if and only if $x = y$
(ii) $d(x, y) = d(y, x)$ for all distinct points $x, y \in X$
(iii) $d(x, y) \leq d(x, u) + d(u, v) + d(v, y)$ (quadrilateral inequality)

Then, (X, d) is called a generalized metric space.

Definition 2 (see [22]). Let X be a nonempty set and $d: X \times X \longrightarrow \mathbb{R}^+$ be a mapping such that, for all $x, y \in X$ and for all distinct points $u, v \in X$, each of them different from x and y, one has

(i) $d(x, y) = 0$ if and only if $x = y$
(ii) $d(x, y) \leq d(x, u) + d(u, v) + d(v, y)$ (quadrilateral inequality)

Then, (X, d) is called a generalized asymmetric metric space.

Definition 3 (see [22]). Let (X, d) be a generalized asymmetric metric space and $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in X, and $x \in X$. Then,

(i) We say that $\{x_n\}_{n\in\mathbb{N}}$ forward (backward) converges to x if and only if

$$\lim_{n\longrightarrow+\infty} d(x, x_n) = \lim_{n\longrightarrow+\infty} d(x_n, x) = 0. \tag{1}$$

(ii) We say that $\{x_n\}_{n\in\mathbb{N}}$ forward (backward) Cauchy if

$$\lim_{n,m\longrightarrow+\infty} d(x_n, x_m) = \lim_{n,m\longrightarrow+\infty} d(x_m, x_n) = 0. \tag{2}$$

Example 1. Let $X = A \cup B$, where $A = \{0, 2\}$ and $B = \{(1/n), n \in \mathbb{N}^*\}$ and $d: X \times X \longrightarrow [0, +\infty[$ defined by

$$\begin{cases} d(0,2) = d(2,0) = 1, \\ d\left(\frac{1}{n}, 0\right) = \frac{1}{n}, \\ d\left(0, \frac{1}{n}\right) = 1, \\ d\left(\frac{1}{n}, 2\right) = 1, \\ d\left(2, \frac{1}{n}\right) = \frac{1}{n}, \\ d\left(\frac{1}{n}, \frac{1}{m}\right) = d\left(\frac{1}{m}, \frac{1}{n}\right) = 1, \end{cases} \tag{3}$$

for all $n, m \in \mathbb{N}^*, n \neq m$. Then, (X, d) is a generalized asymmetric metric space. However, we have the following:

(1) (X, d) is not a metric space as $d((1/n), 0) \neq d(0, (1/n))$, for all $n > 1$
(2) (X, d) is not a asymmetric metric space as $d(2, 0) = 1 > (1/2) = d(2, (1/4)) + d((1/4), 0)$
(3) (X, d) is not a rectangular metric space as $d((1/n), 2) \neq d(2, (1/n))$, for all $n > 1$

Remark 1. Let (X, d) be as in Example 1 and $\{1/n\}_{n\in\mathbb{N}^*}$ be a sequence in X. However, we have the following:

(i) $\lim_{n\longrightarrow+\infty} d((1/n), 0) = 0$, $\lim_{n\longrightarrow+\infty} d((1/n), 2) = 1$ and $\lim_{n\longrightarrow+\infty} d(0, (1/n)) = 1$, $\lim_{n\longrightarrow+\infty} d(2, (1/n)) = 0$. Then, the sequence $\{1/n\}$ forward converges to 2 and backward converges to 0, so the limit is not unique.
(ii) $\lim_{n\longrightarrow+\infty} d((1/m), (1/n)) = \lim_{n\longrightarrow+\infty} d((1/m), (1/n)) = 1$. So, forward (backward) convergence does not imply forward (backward) Cauchy.

Lemma 1 (see [22]). Let (X, d) be a generalized asymmetric metric space and $\{x_n\}_n$ be a forward (or backward) Cauchy sequence with pairwise disjoint elements in X. If $\{x_n\}_n$ forward converges to $x \in X$ and backward converges to $y \in X$, then $x = y$.

Definition 4 (see [22]). Let (X, d) be a generalized asymmetric metric space. X is said to be forward (backward) complete if every forward (backward) Cauchy sequence $\{x_n\}_n$ in X forward (backward) converges to $x \in X$.

Definition 5 (see [22]). Let (X, d) be a generalized asymmetric metric space. X is said to be complete if X is forward and backward complete.The following definition was given by Jleli and Samet in [23].

Definition 6 (see [23]). Let Θ be the family of all functions θ: $]0,+\infty[\longrightarrow]1,+\infty[$ such that

$(\theta)_1$ θ is increasing,

$(\theta)_2$ for each sequence $x_n\in]0,+\infty[$,

$$\lim_{n\longrightarrow 0} x_n = 0 \quad \text{if and only if} \quad \lim_{n\longrightarrow\infty}\theta(x_n) = 1. \tag{4}$$

$(\theta)_3$ θ is continuous.

Recently, Zheng et al. [24] introduced the new type of contractive mappings as follows:

Definition 7 (see [24]). Let Φ be the family of all functions ϕ: $[1,+\infty[\longrightarrow[1,+\infty[$ such that

$\phi)_1$ ϕ is increasing

$\phi)_2$ for each $t\in]1,+\infty[$, $\lim_{n\longrightarrow+\infty}\phi^n(t)=1$

$\phi)_3$ ϕ is continuous

Lemma 2 (see [24]). If $\phi\in\Phi$, then $\phi(1)=1$, and $\phi(t)<t$ for all $t\in]1,+\infty[$.

Definition 8 (see [24]). Let (X,d) be a metric space and T: $X\longrightarrow X$ be a mapping.

T is said to be a $\theta-\phi$-contraction if there exist $\theta\in\Theta$ and $\phi\in\Phi$ such that, for any $x,y\in X$,

$$d(Tx,Ty)>0\Longrightarrow\theta[d(Tx,Ty)]\le\phi(\theta[N(x,y)]), \tag{5}$$

where

$$N(x,y)=\max\{d(x,y),d(x,Tx),d(y,Ty)\}. \tag{6}$$

Theorem 1 (see [22]). Let (X,d) be a generalized asymmetric metric space and T: $X\longrightarrow X$ be a mapping. Suppose that there exist $\theta\in\Theta$ and $k\in]0,1[$ such that, for any $x,y\in X$,

$$\max\{d(Tx,Ty),d(Ty,Tx)\}>0\Rightarrow\theta\left[\frac{d(Tx,Ty)+d(Ty,Tx)}{2}\right]\le\theta\left[\frac{d(x,y)+d(y,x)}{2}\right]^k. \tag{7}$$

Then, T has a unique fixed point.

3. Main Result

Motivated and inspired by Piri et al. [22] and Zheng et al. [24], we define the notions of θ-contraction and $\theta-\phi$-contraction on the generalized asymmetric metric space, and we give some results on such space.

Definition 9 Let (X,d) be a generalized asymmetric metric space and T: $X\longrightarrow X$ be a mapping.

(1) T is said to be a θ-contraction if there exist $\theta\in\Theta$ and $r\in]0,1[$ such that, for any $x,y\in X$, we have

$$\max\{d(x,y),d(y,x)\}>0\Rightarrow\theta\left[\frac{d(Tx,Ty)+d(Ty,Tx)}{2}\right]\le(\theta[M(x,y)])^r, \tag{8}$$

where

$$M(x,y)=\max\left\{\frac{d(x,y)+d(y,x)}{2},\frac{d(x,Tx)+d(Tx,x)}{2},\frac{d(y,Ty)+d(Ty,y)}{2}\right\}. \tag{9}$$

(2) T is said to be a $\theta-\phi$-contraction if there exist $\theta\in\Theta$ and $\phi\in\Phi$ such that, for any $x,y\in X$, we have

$$\max\{d(x,y),d(y,x)\}>0\Longrightarrow\theta\left[\frac{d(Tx,Ty)+d(Ty,Tx)}{2}\right]\le\phi(\theta[M(x,y)]), \tag{10}$$

where

$$M(x,y)=\max\left\{\frac{d(x,y)+d(y,x)}{2},\frac{d(x,Tx)+d(Tx,x)}{2},\frac{d(y,Ty)+d(Ty,y)}{2}\right\}. \tag{11}$$

(3) T is said to be a $\theta-\phi$-Kannan-type contraction if there exist $\theta \in \Theta$ and $\phi \in \Phi$ such that

$$\max\{d(Tx,Ty),d(Ty,Tx)\}>0, \tag{12}$$

for any $x, y \in X$, we have

$$\theta\left[\frac{d(Tx,Ty)+d(Ty,Tx)}{2}\right] \le \phi\left[\theta\left(\frac{d(Tx,x)+d(x,Tx)+d(Ty,y)+d(y,Ty)}{4}\right)\right]. \tag{13}$$

(4) T is said to be a $\theta-\phi$-Reich-type contraction if there exist $\theta \in \Theta$ and $\phi \in \Phi$ such that

$$\max\{d(Tx,Ty),d(Ty,Tx)\}>0, \tag{14}$$

for any $x, y \in X$, we have

$$\theta\left[\frac{d(Tx,Ty)+d(Ty,Tx)}{2}\right] \le \phi\left[\theta\left(\frac{d(x,y)+d(y,x)+d(Tx,x)+d(x,Tx)+d(Ty,y)+d(y,Ty)}{6}\right)\right]. \tag{15}$$

Theorem 2. *Let (X,d) be a complete generalized asymmetric metric space, and let $T\colon X \longrightarrow X$ be a θ-contraction, i.e., there exist $\theta \in \Theta$ and $r \in]0,1[$ such that, for any*

$$x, y \in X,\ \max\{d(Tx,Ty),d(Ty,Tx)\}>0 \Longrightarrow \theta\left[\frac{d(Tx,Ty)+d(Ty,Tx)}{2}\right] \le (\theta[M(x,y)])^r. \tag{16}$$

Then, T has a unique fixed point.

Proof. Let $x_0 \in X$ be a fixed point, and define a sequence $\{x_n\}$ by $x_{n+1} = Tx_n = T^{n+1}x_0$, for all $n \in \mathbb{N}$. If there exists $n_0 \in \mathbb{N}$ such that $d(x_{n_0}, x_{n_0+1}) = 0$ or $d(x_{n_0+1}, x_{n_0}) = 0$, then proof is finished.

We can suppose that $d(x_n, x_{n+1}) > 0$ and $d(x_{n+1}, x_n) > 0$ for all $n \in \mathbb{N}$; then, we have

$$\max\{d(x_n,x_{n+1}),d(x_{n+1},x_n)\}>0. \tag{17}$$

Substituting $x = x_{n-1}$ and $y = x_n$, from (16), for all $n \in \mathbb{N}$, we have

$$\theta\left[\frac{d(x_n,x_{n+1})+d(x_{n+1},x_n)}{2}\right] \le [\theta(M(x_{n-1},x_n))]^r, \quad \forall n \in \mathbb{N}, \tag{18}$$

where

$$\begin{aligned} M(x_{n-1},x_n) &= \max\left\{\frac{d(x_{n-1},x_n)+d(x_n,x_{n-1})}{2}, \frac{d(x_{n-1},x_n)+d(x_n,x_{n-1})}{2}, \frac{d(x_n,x_{n+1})+d(x_{n+1},x_n)}{2}\right\} \\ &= \max\left\{\frac{d(x_{n-1},x_n)+d(x_n,x_{n-1})}{2}, \frac{d(x_n,x_{n+1})+d(x_{n+1},x_n)}{2}\right\}. \end{aligned} \tag{19}$$

Now, we set $D(x_n, x_m) = d(x_n, x_m) + d(x_m, x_n)$. Therefore,

$$M(x_{n-1}, x_n) = \max\left\{\frac{D(x_{n-1}, x_n)}{2}, \frac{D(x_n, x_{n+1})}{2}\right\}. \tag{20}$$

If $M(x_{n-1}, x_n) = \{D(x_n, x_{n+1})/2\}$, then from the assumption of the theorem, we have

$$\theta\left(\frac{D(x_n, x_{n+1})}{2}\right) \le \left(\theta\left(\frac{D(x_n, x_{n+1})}{2}\right)\right)^r < \theta\left(\frac{D(x_n, x_{n+1})}{2}\right), \tag{21}$$

which is a contradiction. Hence, $M(x_{n-1}, x_n) = \{D(x_{n-1}, x_n)/2\}$.

Thus,

$$\theta\left(\frac{D(x_n, x_{n+1})}{2}\right) \le \left(\theta\left(\frac{D(x_{n-1}, x_n)}{2}\right)\right)^r < \theta\left(\frac{D(x_{n-1}, x_n)}{2}\right). \tag{22}$$

Repeating this step, we conclude that

$$\theta\left(\frac{D(x_n, x_{n+1})}{2}\right) \le \left(\theta\left(\frac{D(x_{n-1}, x_n)}{2}\right)\right)^r < \theta\left(\frac{D(x_{n-1}, x_n)}{2}\right) < \cdots < \theta\left(\frac{D(x_0, x_1)}{2}\right)^{r^n}. \tag{23}$$

From (θ_1), we get

$$D(x_n, x_{n+1}) < D(x_{n-1}, x_n). \tag{24}$$

Therefore, $\{D(x_n, x_{n+1})\}_{n\in\mathbb{N}}$ is a monotone strictly decreasing sequence of nonnegative real numbers. Consequently, there exists $\alpha \ge 0$ such that

$$\lim_{n\longrightarrow\infty} D(x_n, x_{n+1}) = \alpha. \tag{25}$$

Now, we claim that $\alpha = 0$. Arguing by the contraction, we assume that $\alpha > 0$. Since $\{D(x_n, x_{n+1})\}_{n\in\mathbb{N}}$ is a nonnegative decreasing sequence, we have

$$D(x_n, x_{n+1}) \ge \alpha, \quad \forall n \in \mathbb{N}. \tag{26}$$

From the property of θ, we get

$$1 < \theta\left(\frac{\lambda}{2}\right) \le \theta\left(\frac{D(x_0, x_1)}{2}\right)^{r^n}. \tag{27}$$

By letting $n \longrightarrow \infty$ in inequality (27), we obtain

$$1 < \theta\left(\frac{\lambda}{2}\right) \le 1. \tag{28}$$

It is a contradiction. Therefore,

$$\lim_{n\longrightarrow\infty} D(x_n, x_{n+1}) = 0. \tag{29}$$

Next, we shall prove that

$$\lim_{n\longrightarrow\infty} d(x_n, x_{n+2}) = 0,$$
$$\lim_{n\longrightarrow\infty} d(x_{n+2}, x_n) = 0. \tag{30}$$

We assume that $x_n \ne x_m$ for every $n, m \in \mathbb{N}, n \ne m$. Indeed, suppose that $x_n = x_m$ for some $n = m + k$ with $k > 0$, so we have $x_{n+1} = Tx_n = Tx_m = x_{m+1}$.

So, from the assumption of the theorem, we get

$$\theta\left(\frac{D(x_m, x_{m+1})}{2}\right) = \theta\left(\frac{D(x_n, x_{n+1})}{2}\right) \le \theta\left(\frac{D(x_{n-1}, x_n)}{2}\right)^r < \theta\left(\frac{D(x_{n-1}, x_n)}{2}\right). \tag{31}$$

Since θ is increasing, we have

$$D(x_n, x_{n+1}) < D(x_{n-1}, x_n). \tag{32}$$

Continuing this process, we can say that

$$D(x_m, x_{m+1}) < D(x_m, x_{m+1}). \tag{33}$$

It is a contradiction. Therefore,

$$\max\{d(x_m, x_n), d(x_n, x_m)\} > 0, \tag{34}$$

for every $n, m \in \mathbb{N}$, $n \ne m$.

Substituting $x = x_n$ and $y = x_{n+2}$,

$$\max\{d(x_n, x_{n+2}), d(x_{n+2}, x_n)\} > 0. \tag{35}$$

Applying (16) with $x = x_{n-1}$ and $y = x_{n+1}$, we have

$$\theta\left[\frac{d(x_n, x_{n+2}) + d(x_{n+2}, x_n)}{2}\right] \le \left[\theta(M(x_{n-1}, x_{n+1}))\right]^r, \tag{36}$$

where

$$M(x_{n-1}, x_{n+1}) = \max\left\{\frac{D(x_{n-1}, x_n)}{2}, \frac{D(x_{n-1}, x_{n+1})}{2}, \frac{D(x_{n+1}, x_{n+2})}{2}\right\} = \max\left\{\frac{D(x_{n-1}, x_n)}{2}, \frac{D(x_{n-1}, x_{n+1})}{2}\right\}. \tag{37}$$

So, we get

$$\theta\left(\frac{D(x_n, x_{n+2})}{2}\right) \le \left[\theta\left(\max\left\{\frac{D(x_{n-1}, x_n)}{2}, \frac{D(x_{n-1}, x_{n+1})}{2}\right\}\right)\right]^r. \tag{38}$$

Take $a_n = D(x_n, x_{n+2})$ and $b_n = D(x_n, x_{n+1})$. Thus, by (38), one can write

$$\theta\left(\frac{a_n}{2}\right) \le \left[\theta\left(\max\left\{\frac{a_{n-1}}{2}, \frac{b_{n-1}}{2}\right\}\right)\right]^r. \tag{39}$$

By (θ_1), we get

$$a_n < \max\{a_{n-1}, b_{n-1}\}. \tag{40}$$

By (24), we have

$$b_n \le b_{n-1} \le \max\{a_{n-1}, b_{n-1}\}, \tag{41}$$

which implies that

$$\max\{a_n, b_n\} \le \max\{a_{n-1}, b_{n-1}\}, \quad \forall n \in \mathbb{N}. \tag{42}$$

Therefore, the sequence $\{\max\{a_{n-1}, b_{n-1}\}\}_{n\in\mathbb{N}}$ is monotone nonincreasing. Thus, there exists $\lambda \ge 0$ such that

$$\lim_{n\longrightarrow\infty} \max\{a_n, b_n\} = \lambda. \tag{43}$$

By (29), we assume that $\lambda > 0$; then, we get

$$\lim_{n\longrightarrow\infty} \sup a_n = \lim_{n\longrightarrow\infty} \sup \max\{a_n, b_n\} = \lim_{n\longrightarrow\infty} \max\{a_n, b_n\}. \tag{44}$$

Taking $\lim\sup_n \longrightarrow \infty$ in (38) and using the properties of θ_3, we obtain

$$\theta\left(\lim_{n\longrightarrow\infty} \sup \frac{a_n}{2}\right) < \theta\left(\lim_{n\longrightarrow\infty} \max\left\{\frac{a_{n-1}}{2}, \frac{b_{n-1}}{2}\right\}\right). \tag{45}$$

Therefore,

$$\theta\left(\frac{\lambda}{2}\right) < \theta\left(\frac{\lambda}{2}\right). \tag{46}$$

By (θ_1), we get

$$\lambda < \lambda. \tag{47}$$

It is a contradiction. Therefore,

$$\lim_{n\longrightarrow\infty} D\left(x_{n+2,} x_n\right) = 0. \tag{48}$$

Next, we shall prove that $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence, i.e., $\lim_{n\longrightarrow\infty} D(x_{n,} x_m) = 0$, for all $n, m \in \mathbb{N}$. Suppose to the contrary. Then, there is $\varepsilon > 0$ such that, for an integer k, there exist two sequences $\{n_{(k)}\}$ and $\{m_{(k)}\} m_{(k)} > n_{(k)} > k$ such that

$$\begin{aligned} D\left(x_{m_{(k)}}, x_{n_{(k)}}\right) &\ge \varepsilon, \\ D\left(x_{m_{(k)-1}}, x_{n_{(k)}}\right) &< \varepsilon. \end{aligned} \tag{49}$$

Now, using (29), (48), (49), and the quadrilateral inequality, we find

$$\begin{aligned} \varepsilon \le D\left(x_{m_{(k)}}, x_{n_{(k)}}\right) &\le D\left(x_{m_{(k)}}, x_{m_{(k)+1}}\right) \\ &+ D\left(x_{m_{(k)+1}}, x_{m_{(k)-1}}\right) + D\left(x_{m_{(k)-1}}, x_{n_{(k)}}\right) \\ &\le D\left(x_{m_{(k)}}, x_{m_{(k)+1}}\right) + D\left(x_{m_{(k)+1}}, x_{m_{(k)-1}}\right) + \varepsilon. \end{aligned} \tag{50}$$

Then,

$$\lim_{k\longrightarrow\infty} D\left(m_{(k)}, n_{(k)}\right) = \varepsilon. \tag{51}$$

Now, by the quadrilateral inequality, we have

$$\begin{aligned} D\left(x_{m_{(k)+1}}, x_{n_{(k)+1}}\right) &\le D\left(x_{m_{(k)+1}}, x_{m_{(k)}}\right) + D\left(x_{m_{(k)}}, x_{n_{(k)}}\right) + D\left(x_{n_{(k)}}, x_{n_{(k)+1}}\right), \\ D\left(x_{m_{(k)}}, x_{n_{(k)}}\right) &\le D\left(x_{m_{(k)}}, x_{m_{(k)+1}}\right) + D\left(x_{m_{(k)+1}}, x_{n_{(k)+1}}\right) + D\left(x_{n_{(k)+1}}, x_{n_{(k)}}\right). \end{aligned} \tag{52}$$

Letting $k \longrightarrow \infty$ in the above inequalities and using (29), (48), and (51), we obtain

$$\lim_{k\longrightarrow\infty} D\left(x_{m_{(k)+1}}, x_{n_{(k)+1}}\right) = \varepsilon. \tag{53}$$

Therefore, by (29) and (51), we get that

$$\lim_{k\longrightarrow\infty} M\left(x_{m_{(k)}}, x_{n_{(k)}}\right) = \frac{\varepsilon}{2}. \tag{54}$$

By (53), there exists $n_0 \in \mathbb{N}$ such that

$$\begin{aligned} D\left(x_{m_{(k)+1}}, x_{n_{(k)+1}}\right) &= d\left(x_{m_{(k)+1}}, x_{n_{(k)+1}}\right) \\ &+ d\left(x_{n_{(k)+1}}, x_{m_{(k)+1}}\right) \ge \frac{\varepsilon}{2}, \quad \forall n \ge n_0. \end{aligned} \tag{55}$$

Therefore,

$$\max\left\{d\left(x_{m_{(k)+1}}, x_{n_{(k)+1}}\right), d\left(x_{n_{(k)+1}}, x_{m_{(k)+1}}\right)\right\} \ge \frac{\varepsilon}{4}, \quad \forall n \ge n_0. \tag{56}$$

So,

$$\max\left\{d\left(Tx_{m_{(k)}}, Tx_{n_{(k)}}\right), d\left(x_{n_{(k)}}, Tx_{m_{(k)}}\right)\right\} \ge \frac{\varepsilon}{4}, \quad \forall n \ge n_0. \tag{57}$$

Applying with $x = x_{m_{(k)}}$ and $y = x_{n_{(k)}}$, we have

$$\theta\left[\frac{D\left(x_{m_{(k)+1}}, x_{n_{(k)+1}}\right)}{2}\right] \le \left[\theta\left(M\left(x_{m_{(k)}}, x_{n_{(k)}}\right)\right)\right]^r. \tag{58}$$

Letting $k \longrightarrow \infty$ in the above inequality and using (θ_3), (53), and (54), we obtain

$$\theta\left(\lim_{k\longrightarrow\infty} \frac{D\left(x_{m_{(k)+1}}, x_{n_{(k)+1}}\right)}{2}\right) \le \theta\left(\lim_{k\longrightarrow\infty} M\left(x_{m_{(k)}}, x_{n_{(k)}}\right)\right)^r. \tag{59}$$

Therefore,

$$\theta\left(\frac{\varepsilon}{2}\right) < \theta\left(\frac{\varepsilon}{2}\right). \tag{60}$$

Since θ is increasing, we get

$$\Rightarrow \varepsilon < \varepsilon, \tag{61}$$

which is a contradiction. Then,

$$\lim_{n,m\longrightarrow\infty} D(x_m, x_n) = 0. \tag{62}$$

Equivalently,

$$\lim_{n,m\longrightarrow\infty} d(x_m, x_n) = \lim_{n,m\longrightarrow\infty} d(x_n, x_m) = 0. \tag{63}$$

Hence, $\{x_n\}$ is a forward and backward Cauchy sequence in X. By completeness of (X, d), there exists $z, u \in X$ such that

$$\lim_{x\longrightarrow\infty} d(x_n, z) = \lim_{x\longrightarrow\infty} d(u, x_n) = 0. \tag{64}$$

So, from Lemma 1, we get $z = u$.

Now, we show that $d(Tz, z) = 0 = d(z, Tz)$. Arguing by contradiction, we assume that

$$\begin{aligned} &d(Tz, z) > 0,\\ &d(z, Tz) > 0. \end{aligned} \tag{65}$$

Therefore,

$$\max\{d(Tz, z), d(z, Tz)\} > 0. \tag{66}$$

Now, by the quadrilateral inequality, we get

$$d(Tx_n, Tz) \leq d(Tx_n, x_n) + d(x_n, z) + d(z, Tz), \tag{67}$$

$$d(z, Tz) \leq d(z, x_n) + d(x_n, Tx_n) + d(Tx_n, Tz). \tag{68}$$

By letting $n \longrightarrow \infty$ in inequalities (67) and (68), we obtain

$$d(z, Tz) \leq \lim_{n\longrightarrow\infty} d(Tx_n, Tz) \leq d(z, Tz). \tag{69}$$

Therefore,

$$\lim_{n\longrightarrow\infty} d(Tx_n, Tz) = d(z, Tz). \tag{70}$$

On the contrary,

$$d(Tz, Tx_n) \leq d(Tz, z) + d(z, x_n) + d(x_n, Tx_n), \tag{71}$$

$$d(Tx_n, Tz) \leq d(Tx_n, x_n) + d(x_n, z) + d(z, Tz). \tag{72}$$

By letting $n \longrightarrow \infty$ in inequalities (71) and (72), we obtain

$$d(Tz, z) \leq \lim_{n\longrightarrow\infty} d(Tz, Tx_n) \leq d(Tz, z). \tag{73}$$

Therefore,

$$\lim_{n\longrightarrow\infty} d(Tz, Tx_n) = d(Tz, z). \tag{74}$$

By (70) and from the definition of the limit, there exists $n_1 \in \mathbb{N}$ such that

$$d(Tx_n, Tz) > d(z, Tz) > 0, \quad \forall n \geq n_1. \tag{75}$$

Similarly, by (74), there exists $n_2 \in \mathbb{N}$ such that

$$d(Tz, Tx_n) > d(Tz, z) > 0, \quad \forall n \geq n_2. \tag{76}$$

Let $N = \max\{n_1, n_2\}$; we conclude

$$\max\{d(Tz, Tx_n), d(Tx_n, Tz)\} > 0, \quad \forall n \geq N. \tag{77}$$

Applying (16) with $x = z$ and $y = x_n$, we have

$$\theta\left(\frac{D(Tz, Tx_n)}{2}\right) \leq [\theta(M(z, x_n))]^r, \quad \forall n \geq N, \tag{78}$$

where

$$M(z, x_n) = \max\left\{\frac{D(z, x_n)}{2}, \frac{D(z, Tz)}{2}, \frac{D(x_n, Tx_n)}{2}\right\}. \tag{79}$$

Therefore,

$$\theta\left(\frac{D(Tz, Tx_n)}{2}\right) \leq \left[\theta\left(\max\left\{\frac{D(z, x_n)}{2}, \frac{D(z, Tz)}{2}, \frac{D(x_n, Tx_n)}{2}\right\}\right)\right]^r. \tag{80}$$

By letting $n \longrightarrow \infty$ in inequality (80) and using (θ_3), we obtain

$$\begin{aligned} \theta\left(\lim_{n\longrightarrow\infty} \frac{D(Tz, Tx_n)}{2}\right) &= \theta\left(\frac{D(z, Tz)}{2}\right)\\ &\leq \left[\theta\left(\lim_{n\longrightarrow\infty} \max\left\{\frac{D(z, x_n)}{2}, \frac{D(z, Tz)}{2}, \frac{D(x_n, Tx_n)}{2}\right\}\right)\right]^r\\ &< \theta \lim_{n\longrightarrow\infty} \max\left\{\frac{D(z, x_n)}{2}, \frac{D(z, Tz)}{2}, \frac{D(x_n, Tx_n)}{2}\right\}. \end{aligned} \tag{81}$$

By (θ_1), we get

$$\begin{aligned} \frac{D(z, Tz)}{2} &< \lim_{n\longrightarrow\infty} \max\left\{\frac{D(z, x_n)}{2}, \frac{D(z, Tz)}{2}, \frac{D(x_n, Tx_n)}{2}\right\}\\ &= \frac{D(z, Tz)}{2}, \end{aligned} \tag{82}$$

which is a contradiction. Hence, $Tz = z$.

Uniqueness: now, suppose that $z, u \in X$ are two fixed points of T such that $u \neq z$. Therefore, we have

$$\begin{aligned} d(Tz, Tu) &= d(z, u) > 0, \\ d(Tu, Tz) &= d(u, z) > 0. \end{aligned} \tag{83}$$

Therefore,

$$\max\{d(Tu, Tz), d(Tz, Tu)\} > 0. \tag{84}$$

Applying (16) with $x = z$ and $y = u$, we have

$$\theta\left(\frac{D(Tz, Tu)}{2}\right) = \theta\left(\frac{D(z, u)}{2}\right) \leq [\theta(M(z, u))]^r, \tag{85}$$

where

$$M(z, u) = \max\left\{\frac{D(z, u)}{2}, \frac{D(z, Tz)}{2}, \frac{D(u, Tu)}{2}\right\} = \frac{D(z, u)}{2}. \tag{86}$$

Therefore, we have

$$\theta\left(\frac{D(z, u)}{2}\right) \leq \left[\theta\left(\frac{D(z, u)}{2}\right)\right]^r < \theta\left(\frac{D(z, u)}{2}\right). \tag{87}$$

Therefore,

$$D(z, u) < D(z, u), \tag{88}$$

is a contradiction. Therefore, $u = z$. □

Corollary 1 (Theorem 2; see [22]). Let (X, d) be a complete generalized asymmetric metric space and $T: X \longrightarrow X$ be a given mapping. Suppose that there exist $\theta \in \Theta$ and $k \in]0, 1[$ such that, for any $x, y \in X$, we have $\max\{d(Tx, Ty), d(Ty, Tx)\} > 0 \Longrightarrow \theta[(d(Tx, Ty) + d(Ty, Tx))/2] \leq [\theta((d(x, y) + d(y, x))/2)]^k$.

Then, T has a unique fixed point.

Example 2. Consider $X = \{0, 1, 2, 3\}$. Let $d: X \times X \longrightarrow [0, +\infty[$ be a mapping defined by the following:

(i) $d(x, y) = 0$ if and only $x = y$
(ii) $d(0, 1) = d(1, 0) = d(2, 1) = d(2, 0) = d(3, 0) = d(2, 3) = d(3, 1) = 1$
(iii) $d(1, 2) = d(0, 2) = 2$
(iv) $d(0, 3) = 3$, and $d(3, 2) = 4$
(v) $d(1, 3) = 2$

Clearly, (X, d) is not an asymmetric metric space. Indeed, $d(3, 2) = 4 > d(3, 0) + d(0, 2) = 3$.

However, it is a complete generalized asymmetric metric space.

Let $T: X \longrightarrow X$ be given by

$$\begin{cases} T(0) = T(1) = 0, \\ T(2) = 1, \\ T(3) = 2. \end{cases} \tag{89}$$

Suppose $\theta(t) = \sqrt{t} + 1$ and $k = 12/13$. Therefore, $\theta \in \Theta$ and $k \in]0, 1[$. First, observe that $\max\{d(Tx, Ty), d(Ty, Tx)\} > 0 \Longleftrightarrow \{x = 0, y = 2\}$, $\{x = 1, y = 2\}$, $\{x = 0, y = 3\}$, $\{x = 1, y = 3\}$, or $\{x = 2, y = 3\}$.

For $x = 0, y = 2$, we have

$$\begin{aligned} &\frac{d(T(0), T(2)) + d(T(2), T(0))}{2} = \frac{d(0, 1) + d(1, 0)}{2} = 1, \\ &M(0, 2) = \max\left\{\frac{d(0, 2) + d(2, 0)}{2}, \frac{d(0, T(0)) + d(T(0), 0)}{2}, \frac{d(2, T(2)) + d(T(2), 2)}{2}\right\} \\ &= \max\left\{\frac{d(0, 2) + d(2, 0)}{2}, \frac{d(0, 0) + d(0, 0)}{2}, \frac{d(2, 1) + d(1, 2)}{2}\right\} = \frac{3}{2}. \end{aligned} \tag{90}$$

Therefore,

$$\begin{aligned} &\theta\left(\frac{d(T(0), T(2)) + d(T(2), T(0))}{2}\right) = 2, \\ &\left[\theta\left(\max\left\{\frac{3}{2}, 0, \frac{3}{2}\right\}\right)\right]^k = 2.08. \end{aligned} \tag{91}$$

So,

$$\theta\left(\frac{d(T(0), T(2)) + d(T(2), T(0))}{2}\right) \leq [\theta(M(0, 2))]^k. \tag{92}$$

For $x = 1, y = 2$, we have

$$\theta\left(\frac{d(T(1),T(2))+d(T(2),T(1))}{2}\right)=\theta\left(\frac{d(0,1)+d(1,0)}{2}\right)=2,$$
$$M(1,2)=\max\left\{\frac{d(1,2)+d(2,1)}{2},\frac{d(1,T(1))+d(T(1),1)}{2},\frac{d(2,T(2))+d(T(2),2)}{2}\right\}=\frac{3}{2}, \tag{93}$$

which implies that

$$\theta\left(\max\left\{\frac{3}{2},1,\frac{3}{2}\right\}\right)^{k}=\left(\sqrt{\frac{3}{2}}+1\right)^{12/13}=2.08. \tag{94}$$

Therefore,

$$\theta\left(\frac{d(T(1),T(2))+d(T(2),T(1))}{2}\right)\leq[\theta(M(1,2))]^{k}. \tag{95}$$

For $x=0$, $y=3$, we have

$$\theta\left(\frac{d(T(0),T(3))+d(T(3),T(0))}{2}\right)=\theta\left(\frac{d(0,2)+d(2,0)}{2}\right)=\sqrt{\frac{3}{2}}+1=2.22,$$
$$M(0,3)=\max\left\{\frac{d(0,3)+d(3,0)}{2},\frac{d(0,T(0))+d(T(0),0)}{2},\frac{d(3,T(3))+d(T(3),3)}{2}\right\} \tag{96}$$
$$=\max\left\{2,0,\frac{5}{2}\right\}=\frac{5}{2}.$$

On the contrary,

$$\theta\left(\max\left\{2,0,\frac{5}{2}\right\}\right)^{k}=\left(\sqrt{\frac{5}{2}}+1\right)^{k}=2.39. \tag{97}$$

Then,

$$\theta\left(\frac{d(T0,T3)+d(T3,T0)}{2}\right)\leq[\theta(M(0,3))]^{k}. \tag{98}$$

For $x=1$, $y=3$, we have

$$\theta\left(\frac{d(T(1),T(3))+d(T(3),T(1))}{2}\right)=\theta\left(\frac{d(0,2)+d(2,0)}{2}\right)=\sqrt{\frac{3}{2}}+1=2.22,$$
$$M(1,3)=\max\left\{\frac{d(1,3)+d(3,1)}{2},\frac{d(1,T(1))+d(T(1),1)}{2},\frac{d(3,T(3))+d(T(3),3)}{2}\right\} \tag{99}$$
$$=\max\left\{\frac{3}{2},1,\frac{5}{2}\right\},\theta\left(\max\left\{\frac{3}{2},1,\frac{5}{2}\right\}\right)^{k}$$
$$=2.39.$$

Therefore,

$$\theta\left(\frac{d(T(1),T(3))+d(T(3),T(1))}{2}\right)\leq[\theta(M(1,3))]^{k}. \tag{100}$$

For $x=2$, $y=3$, we have

$$\theta\left(\frac{d(T(2),T(3))+d(T(3),T(2))}{2}\right)=2.22,$$
$$M(2,3)=\max\left\{\frac{d(2,3)+d(3,2)}{2},\frac{d(2,T(2))+d(T(2),2)}{2},\frac{d(3,T(3))+d(T(3),3)}{2}\right\}=\frac{5}{2},$$
$$\left[\theta\left(\frac{5}{2}\right)\right]^{k}=2.39. \tag{101}$$

Then,

$$\theta\left(\frac{d(T(2),T(3))+d(T(3),T(2))}{2}\right)\leq[\theta(M(2,3))]^{k}. \tag{102}$$

Hence, T satisfies the assumption of the theorem, and $z=0$ is the unique fixed point of T.

Example 3. Let $X=A\cup B$, where $A=\{(1/n)\colon n\in\{3,4,5,6\}\}$ and $B=[(1/2),(3/2)]$.

Define $d\colon X\times X\longrightarrow[0,+\infty[$ as follows:

$$\begin{cases} d(x,y)=d(y,x)\ \text{for all}\ x,y\in B,\\ d(x,y)=0\Longleftrightarrow y=x\ \text{for all}\ x,y\in X.\end{cases}$$

$$\begin{cases} d\left(\frac{1}{3},\frac{1}{4}\right)=d\left(\frac{1}{5},\frac{1}{6}\right)=0.3,\\ d\left(\frac{1}{3},\frac{1}{5}\right)=d\left(\frac{1}{4},\frac{1}{6}\right)=0.2,\\ d\left(\frac{1}{5},\frac{1}{3}\right)=d\left(\frac{1}{6},\frac{1}{4}\right)=0.35,\\ d\left(\frac{1}{3},\frac{1}{6}\right)=d\left(\frac{1}{3},\frac{1}{6}\right)=0.6,\\ d(x,y)=|x-y|,\quad \text{otherwise.}\end{cases} \tag{103}$$

Then, (X,d) is a generalized asymmetric metric space. However, we have the following:

(1) (X,d) is not a metric space as $d((1/3),(1/6))=0.6>0.5=d((1/3),(1/4))+d((1/4),t(1/6))$

(2) (X,d) is not a generalized metric space as $d((1/6),(1/4))=0.35\neq d((1/4),(1/6))=0.2$

Define mapping $T\colon X\longrightarrow X$ by

$$T(x)=\begin{cases}\sqrt{x}, & \text{if}\ x\in\left[\frac{1}{2},\frac{3}{2}\right],\\ 1, & \text{if}\ x\in A.\end{cases} \tag{104}$$

Then, $T(x)\in[(1/2),(3/2)]$. Let $\theta(t)=e^{t}$ for all $t\in]0,+\infty[$, and $r=4/5$. It is obvious that $\theta\in\Theta$ and $r\in]0,1[$. Consider the following possibilities:

Case 1: $x,y\in[(1/2),(3/2)]$ with $x\neq y$, and assume that $x>y$.

$$\begin{aligned} D(Tx,Ty)&=d(Tx,Ty)+d(Ty,Tx)\\ &=|\sqrt{x}-\sqrt{y}|+|\sqrt{y}-\sqrt{x}|\\ &=2(\sqrt{x}-\sqrt{y}),\\ D(x,y)&=d(x,y)+d(y,x)\\ &=|x-y|+|y-x|\\ &=2(x-y).\end{aligned} \tag{105}$$

Therefore,

$$\theta\left(\frac{D(Tx,Ty)}{2}\right)=e^{\sqrt{x}-\sqrt{y}},$$
$$\left[\theta\left(\frac{D(x,y)}{2}\right)\right]^{4/5}=\left[e^{x-y}\right]^{4/5}. \tag{106}$$

On the contrary,

$$\theta\left(\frac{D(Tx,Ty)}{2}\right)-\left[\theta\left(\frac{D(x,y)}{2}\right)\right]^{4/5}=e^{\sqrt{x}-\sqrt{y}}-[e^{x-y}]^{4/5}\le 0, \quad (107)$$

which implies that

$$\theta\left(\frac{D(Tx,Ty)}{2}\right)\le\left[\theta\left(\frac{D(x,y)}{2}\right)\right]^{4/5}\le\left[\theta\left(\max\left\{\frac{D(x,y)}{2},\frac{D(x,Tx)}{2},\frac{D(y,Ty)}{2}\right\}\right)\right]^{4/5}. \quad (108)$$

Case 2: $x\in[(1/2),(3/2)]$, $y\in A$ or $y\in[(1/2),(3/2)]$, $x\in A$. Therefore, $T(x)=\sqrt{x}$, $T(y)=1$; then, $d(Tx,Ty)=(|\sqrt{x}-1|)$.

In this case, consider two possibilities:

(i) $x\ge 1$: then, $\sqrt{x}\ge 1$. Therefore,

$$D(Tx,Ty)=2(\sqrt{x}-1). \quad (109)$$

So, we have

$$\theta\left(\frac{D(Tx,Ty)}{2}\right)=e^{\sqrt{x}-1},$$
$$M(x,y)=\max\left\{\frac{D(x,y)}{2},\frac{D(x,Tx)}{2},\frac{D(y,Ty)}{2}\right\}\ge\frac{D(y,Ty)}{2}=(1-y)\ge\left(1-\frac{1}{3}\right)=\left(\frac{2}{3}\right),$$
$$\left[\theta\left(\frac{2}{3}\right)\right]^{4/5}=\left(e^{2/3}\right)^{4/5}=e^{8/15}. \quad (110)$$

On the contrary,

$$\theta\left[\frac{D(Tx,Ty)}{2}\right]-\left[\theta\left(\frac{2}{3}\right)\right]^{4/5}=e^{\sqrt{x}-1}-e^{8/15}. \quad (111)$$

Since $x\in[1,(3/2)]$,

$$e^{\sqrt{x}-1}-e^{8/15}\le 0, \quad (112)$$

which implies that

$$\theta\left[\frac{D(Tx,Ty)}{2}\right]\le\left[\theta\left(\frac{D(y,Ty)}{2}\right)\right]^{4/5}\le\left[\theta\left(\max\left\{\frac{D(x,y)}{2},\frac{D(x,Tx)}{2},\frac{D(y,Ty)}{2}\right\}\right)\right]^{4/5}. \quad (113)$$

(ii) $x<1$: then, $\sqrt{x}<1$. Therefore,

$$D(Tx,Ty)=2(1-\sqrt{x}). \quad (114)$$

So, we have

$$\theta\left(\frac{D(Tx,Ty)}{2}\right)=e^{1-\sqrt{x}},$$
$$M(x,y)=\max\left\{\frac{D(x,y)}{2},\frac{D(x,Tx)}{2},\frac{D(y,Ty)}{2}\right\}\ge\frac{D(y,Ty)}{2}=(1-y)\ge\left(1-\frac{1}{3}\right)=\left(\frac{2}{3}\right),$$
$$\left[\theta\left(\frac{2}{3}\right)\right]^{4/5}=\left(e^{2/3}\right)^{4/5}=e^{8/15}. \quad (115)$$

On the contrary,

$$\theta\left(\frac{D(Tx,Ty)}{2}\right)-\left[\theta\left(\frac{2}{3}\right)\right]^{4/5}=e^{1-\sqrt{x}}-e^{8/15}. \quad (116)$$

Since $x\in[(1/2),1]$,

$$e^{1-\sqrt{x}/2}-e^{8/15}\le 0, \quad (117)$$

which implies that

$$\theta\left[\frac{D(Tx,Ty)}{2}\right]\le\left[\theta\left(\frac{D(y,Ty)}{2}\right)\right]^{4/5}\le\left[\theta\left(\max\left\{\frac{D(x,y)}{2},\frac{D(x,Tx)}{2},\frac{D(y,Ty)}{2}\right\}\right)\right]^{4/5}. \quad (118)$$

Hence, T satisfies the assumption of the theorem, and $z=1$ is the unique fixed point of T.

Theorem 3. *Let (X,d) be a complete generalized asymmetric metric space and $T\colon X\longrightarrow X$ be a mapping. Suppose that there exist $\theta\in\Theta$ and $\phi\in\Phi$ such that, for any $x,y\in X$,*

$$\max\{d(Tx,Ty),d(Ty,Tx)\}>0\Longrightarrow\theta\left[\frac{d(Tx,Ty)+d(Ty,Tx)}{2}\right]\leq\phi(\theta[M(x,y)]),\tag{119}$$

where

$$M(x,y)=\max\left\{\frac{d(x,y)+d(y,x)}{2},\frac{d(x,Tx)+d(Tx,x)}{2},\frac{d(y,Ty)+d(Ty,y)}{2}\right\}.\tag{120}$$

Then, T has a unique fixed point.

Proof. Let $x_0\in X$ be a fixed point, and define a sequence $\{x_n\}$ by

$$x_{n+1}=Tx_n=T^{n+1}x_0,\quad \text{for all } n\in\mathbb{N}.\tag{121}$$

If there exists $n_0\in\mathbb{N}$ such that $d(x_{n_0},x_{n_0+1})=0$ or $d(x_{n_0+1,}x_{n_0})=0$, then proof is finished.

We can suppose that $d(x_n,x_{n+1})>0$ and $d(x_{n+1,}x_n)>0$ for all $n\in\mathbb{N}$; then, we have

$$\max\{d(x_n,x_{n+1}),d(x_{n+1,}x_n)\}>0.\tag{122}$$

Substituting $x=x_{n-1}$ and $y=x_n$, from (119), for all $n\in\mathbb{N}$, we have

$$\theta\left[\frac{d(x_n,x_{n+1})+d(x_{n+1,}x_n)}{2}\right]\leq\phi[\theta(M(x_{n-1},x_n))],\quad\forall n\in\mathbb{N},\tag{123}$$

where

$$\begin{aligned}M(x_{n-1},x_n)&=\left\{\frac{d(x_{n-1},x_n)+d(x_n,x_{n-1})}{2},\frac{d(x_{n-1},x_n)+d(x_n,x_{n-1})}{2},\frac{d(x_n,x_{n+1})+d(x_{n+1},x_n)}{2}\right\}\\&=\left\{\frac{d(x_{n-1},x_n)+d(x_n,x_{n-1})}{2},\frac{d(x_n,x_{n+1})+d(x_{n+1},x_n)}{2}\right\}.\end{aligned}\tag{124}$$

Now, we set $D(x_n,x_m)=d(x_n,x_m)+d(x_m,x_n)$.

Therefore,

$$M(x_{n-1},x_n)=\left\{\frac{D(x_{n-1},x_n)}{2},\frac{D(x_n,x_{n+1})}{2}\right\},\tag{125}$$

if for some n, $M(x_{n-1},x_n)=\{(D(x_n,x_{n+1}))/2\}$.

From (123), (θ_1), and using Lemma 2, we get

$$\theta\left(\frac{D(x_n,x_{n+1})}{2}\right)\leq\phi\left(\theta\left(\frac{D(x_n,x_{n+1})}{2}\right)\right),\tag{126}$$

which implies that

$$D(x_n,x_{n+1})<D(x_n,x_{n+1}),\tag{127}$$

which is contradiction. Hence, $M(x_{n-1},x_n)=\{(D(x_{n-1},x_n))/2\}$.

Therefore,

$$\theta\left(\frac{D(x_n,x_{n+1})}{2}\right)\leq\phi\left(\theta\left(\frac{D(x_{n-1},x_n)}{2}\right)\right)<\theta\left(\frac{D(x_{n-1},x_n)}{2}\right).\tag{128}$$

As θ is increasing,

$$D(x_n,x_{n+1})<D(x_{n-1},x_n).\tag{129}$$

Therefore, $\{D(x_n,x_{n+1})\}_{n\in\mathbb{N}}$ is a monotone strictly decreasing sequence of nonnegative real numbers. Consequently, there exists $\alpha\geq0$ such that

$$\lim_{n\longrightarrow\infty}D(x_{n+1,}x_n)=\alpha.\tag{130}$$

Now, we claim that $\alpha=0$. Arguing by contradiction, we assume that $\alpha>0$. Since $\{D(x_n,x_{n+1})\}_{n\in\mathbb{N}}$ is a nonnegative decreasing sequence, we have

$$D(x_{n+1,}x_n)\geq\alpha,\quad\forall n\in\mathbb{N}.\tag{131}$$

Thus, we have

$$1<\theta\left(\frac{\lambda}{2}\right)\leq\phi\left[\theta\left(\frac{D(x_n,x_{n+1})}{2}\right)\right]\leq\cdots\leq\phi^n\left[\theta\left(\frac{D(x_0,x_1)}{2}\right)\right].\tag{132}$$

By letting $n\longrightarrow\infty$ in inequality (132) and using (θ_3) and (ϕ_3), we obtain

$$1<\theta\left(\frac{\lambda}{2}\right)\leq1.\tag{133}$$

It is a contradiction. Therefore,

$$\lim_{n\longrightarrow\infty} D\left(x_{n+1}, x_n\right) = 0. \tag{134}$$

Next, we shall prove that

$$\lim_{n\longrightarrow\infty} d\left(x_n, x_{n+2}\right) = 0,$$
$$\lim_{n\longrightarrow\infty} d\left(x_{n+2}, x_n\right) = 0. \tag{135}$$

We assume that $x_n \neq x_m$ for every $n, m \in \mathbb{N}, n \neq m$. Indeed, suppose that $x_n = x_m$ for some $n = m + k$ with $k > 0$, so we have $x_{n+1} = Tx_n = Tx_m = x_{m+1}$.

So, from the assumption of the theorem, we get

$$\theta\left(\frac{D\left(x_m, x_{m+1}\right)}{2}\right) = \theta\left(\frac{D\left(x_n, x_{n+1}\right)}{2}\right) \leq \phi\left(\theta\left(\frac{D\left(x_{n-1}, x_n\right)}{2}\right)\right) < \theta\left(\frac{D\left(x_{n-1}, x_n\right)}{2}\right). \tag{136}$$

Since θ is increasing,

$$D\left(x_n, x_{m+1}\right) < D\left(x_{n-1}, x_n\right). \tag{137}$$

Continuing this process, we can get that

$$D\left(x_m, x_{n+1}\right) < D\left(x_m, x_{m+1}\right), \tag{138}$$

which is a contradiction. Therefore,

$$\max\left\{d\left(x_m, x_n\right), d\left(x_n, x_m\right)\right\} > 0, \quad \text{for every } n,\ m \in \mathbb{N}, n \neq m. \tag{139}$$

Substituting $x = x_n$ and $y = x_{n+2}$,

$$\max\left\{d\left(x_n, x_{n+2}\right), d\left(x_{n+2}, x_n\right)\right\} > 0. \tag{140}$$

Applying (119) with $x = x_{n-1}$ and $y = x_{n+1}$, we have

$$\theta\left[\frac{d\left(x_n, x_{n+2}\right) + d\left(x_{n+2}, x_n\right)}{2}\right] \leq \phi\left[\theta\left(M\left(x_{n-1}, x_{n+1}\right)\right)\right], \tag{141}$$

where

$$M\left(x_{n-1}, x_{n+1}\right) = \max\left\{\frac{D\left(x_{n-1}, x_n\right)}{2}, \frac{D\left(x_{n-1}, x_{n+1}\right)}{2}, \frac{D\left(x_{n+1}, x_{n+2}\right)}{2}\right\} = \max\left\{\frac{D\left(x_{n-1}, x_n\right)}{2}, \frac{D\left(x_{n-1}, x_{n+1}\right)}{2}\right\}. \tag{142}$$

Therefore,

$$\theta\left(\frac{D\left(x_n, x_{n+2}\right)}{2}\right) \leq \phi\left[\theta\left(\max\left\{\frac{D\left(x_{n-1}, x_n\right)}{2}, \frac{D\left(x_{n-1}, x_{n+1}\right)}{2}\right\}\right)\right]. \tag{143}$$

So, from Lemma 2, we have

$$\theta\left(\frac{D\left(x_n, x_{n+2}\right)}{2}\right) < \theta\left(\max\left\{\frac{D\left(x_{n-1}, x_n\right)}{2}, \frac{D\left(x_{n-1}, x_{n+1}\right)}{2}\right\}\right). \tag{144}$$

Take $a_n = D\left(x_n, x_{n+2}\right)$ and $b_n = D\left(x_n, x_{n+1}\right)$. Thus, from (144) and (θ_1), we have

$$a_n < \max\left\{a_{n-1}, b_{n-1}\right\}. \tag{145}$$

Again by (137),

$$b_n \leq b_{n-1} \leq \max\left\{a_{n-1}, b_{n-1}\right\}. \tag{146}$$

Therefore,

$$\max\left\{a_n, b_n\right\} \leq \max\left\{a_{n-1}, b_{n-1}\right\}, \quad \forall n \in \mathbb{N}. \tag{147}$$

Then, the sequence $\left(\max\left\{a_n, b_n\right\}\right)_n$ is monotone nonincreasing, so it converges to some $\beta \geq 0$ such that

$$\lim_{n\longrightarrow\infty} \max\left\{a_n, b_n\right\} = \beta. \tag{148}$$

By (134) and assuming that $\beta > 0$, we have

$$\lim_{n\longrightarrow\infty} \sup a_n = \lim_{n\longrightarrow\infty} \sup \max\left\{a_n, b_n\right\} = \lim_{n\longrightarrow\infty} \max\left\{a_n, b_n\right\}. \tag{149}$$

Taking $\lim\sup_n \longrightarrow \infty$ in (144) and using θ_3, we obtain

$$\theta\left(\lim_{n\longrightarrow\infty} \sup a_n\right) < \theta\left(\lim_{n\longrightarrow\infty} \max\left\{a_{n-1}, b_{n-1}\right\}\right). \tag{150}$$

Therefore,

$$\theta\left(\frac{\beta}{2}\right) < \theta\left(\frac{\beta}{2}\right). \tag{151}$$

From (θ_1), we get

$$\beta < \beta, \tag{152}$$

which is a contradiction. Therefore,

$$\lim_{n\longrightarrow\infty} D\left(x_{n+2}, x_n\right) = 0. \tag{153}$$

Next, We shall prove that $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence, i.e., $\lim_{n\longrightarrow\infty} D\left(x_n, x_m\right) = 0$, for all $n, m \in \mathbb{N}$. Suppose to the contrary. Then, there is $\varepsilon > 0$ such that, for an integer k, there exist two sequences $\left\{n_{(k)}\right\}$ and $\left\{m_{(k)}\right\}$, $m_{(k)} > n_{(k)} > k$, such that

$$D\left(x_{m_{(k)}}, x_{n_{(k)}}\right) \geq \varepsilon,$$
$$D\left(x_{m_{(k)-1}}, x_{n_{(k)}}\right) < \varepsilon. \tag{154}$$

As in the proof of Theorem 2, we conclude that

$$\lim_{k\longrightarrow\infty} D\left(x_{m_{(k)}}, x_{n_{(k)}}\right) = \varepsilon,$$
$$\lim_{k\longrightarrow\infty} D\left(x_{m_{(k)+1}}, x_{n_{(k)+1}}\right) = \varepsilon. \tag{155}$$

Hence, there exists $n_0 \in \mathbb{N}$ such that

$$\max\left\{d\left(Tx_{m_{(k)}}, Tx_{n_{(k)}}\right), d\left(Tx_{n_{(k)}}, Tx_{m_{(k)}}\right)\right\} \geq \frac{\varepsilon}{4}, \quad \forall n \geq n_0. \tag{156}$$

Since T is a $\theta - \phi$-contraction, we derive

$$\theta\left(\frac{D\left(x_{m_{(k)+1}}, x_{n_{(k)+1}}\right)}{2}\right) \leq \phi\left[\theta\left(M\left(x_{m_{(k)}}, x_{n_{(k)}}\right)\right)\right], \tag{157}$$

where

$$M\left(x_{m_{(k)}}, x_{n_{(k)}}\right) = \max\left\{\frac{D\left(x_{m_{(k)}}, x_{n_{(k)}}\right)}{2}, \frac{D\left(x_{m_{(k)}}, x_{m_{(k)+1}}\right)}{2}, \frac{D\left(x_{n_{(k)}}, x_{n_{(k)+1}}\right)}{2}\right\}. \tag{158}$$

As in the proof of Theorem 2, we have

$$\lim_{k\longrightarrow\infty} M\left(x_{m_{(k)}}, x_{n_{(k)}}\right) = \frac{\varepsilon}{2}. \tag{159}$$

Applying (119) with $x = x_{m_{(k)}}$ and $y = x_{n_{(k)}}$, we have

$$\theta\left(\frac{D\left(x_{m_{(k)+1}}, x_{n_{(k)+1}}\right)}{2}\right) \leq \phi\left[\theta\left(\max\left\{\frac{D\left(x_{m_{(k)}}, x_{n_{(k)}}\right)}{2}, \frac{D\left(x_{m_{(k)}}, x_{m_{(k)+1}}\right)}{2}, \frac{D\left(x_{n_{(k)}}, x_{n_{(k)+1}}\right)}{2}\right\}\right)\right]. \tag{160}$$

By letting $k \longrightarrow \infty$ in inequality (160) and using (θ_1), (θ_3), (ϕ_3), and Lemma 2, we obtain

$$\theta\left[\lim_{k\longrightarrow\infty} \frac{D\left(x_{m_{(k)+1}}, x_{n_{(k)+1}}\right)}{2}\right] \leq \phi\left[\theta\left(\lim_{k\longrightarrow\infty} \max\left\{\frac{D\left(x_{m_{(k)}}, x_{n_{(k)}}\right)}{2}, \frac{D\left(x_{m_{(k)}}, x_{m_{(k)+1}}\right)}{2}, \frac{D\left(x_{n_{(k)}}, x_{n_{(k)+1}}\right)}{2}\right\}\right)\right], \tag{161}$$

which implies that

$$\varepsilon < \varepsilon, \tag{162}$$

which is a contradiction. Therefore,

$$\lim_{n,m\longrightarrow\infty} D\left(x_m, x_n\right) = 0. \tag{163}$$

Thus,

$$\lim_{n,m\longrightarrow\infty} d\left(x_m, x_n\right) = \lim_{n,m\longrightarrow\infty} d\left(x_n, x_m\right) = 0. \tag{164}$$

Hence, $\{x_n\}$ is a forward and backward Cauchy sequence in X. By completeness of (X, d), there exists $z, u \in X$ such that

$$\lim_{n\longrightarrow\infty} d\left(x_n, z\right) = \lim_{n\longrightarrow\infty} d\left(u, x_n\right) = 0. \tag{165}$$

So, from Lemma 1, we get $z = u$.

Now, we show that $d(Tz, z) = 0 = d(z, Tz)$. Arguing by contradiction, we assume that

$$\begin{aligned} d(Tz, z) &> 0, \\ d(z, Tz) &> 0. \end{aligned} \tag{166}$$

Therefore,

$$\max\{d(Tz, z), d(z, Tz)\} > 0. \tag{167}$$

As in the proof of Theorem 2, we conclude that

$$\lim_{n\longrightarrow\infty} d\left(Tz, Tx_n\right) = d(Tz, z), \tag{168}$$

$$\lim_{n\longrightarrow\infty} d\left(Tx_n, Tz\right) = d(z, Tz). \tag{169}$$

By (168) and (169), there exists $q \in \mathbb{N}$ such that

$$\max\left\{d\left(Tz, Tx_n\right), d\left(Tx_n, Tz\right)\right\} > 0, \quad \forall n \geq q. \tag{170}$$

Since T is a $\theta - \phi$-contraction, we derive

$$\theta\left(\frac{D\left(Tz, Tx_n\right)}{2}\right) \leq \phi\left(\left[\theta\left(M\left(z, x_n\right)\right)\right]\right), \quad \forall n \geq q, \tag{171}$$

where

$$M\left(z, x_n\right) = \max\left\{\frac{D\left(z, x_n\right)}{2} \frac{D(z, Tz)}{2}, \frac{D\left(x_n, Tx_n\right)}{2}\right\}, \tag{172}$$

which implies that

$$\theta\left(\frac{D\left(Tz, Tx_n\right)}{2}\right) \leq \phi\left[\left(\theta\left(\max\left\{\frac{D\left(z, x_n\right)}{2} \frac{D(z, Tz)}{2}, \frac{D\left(x_n, Tx_n\right)}{2}\right\}\right)\right)\right]. \tag{173}$$

By letting $n \longrightarrow \infty$ in inequality (172) and using (θ_1), (θ_3), (ϕ_3), and Lemma 2, we obtain

$$\begin{aligned}\theta\left(\lim_{n\longrightarrow\infty}\frac{D(Tz,Tx_n)}{2}\right) &\leq \phi\left[\theta\left(\lim_{n\longrightarrow\infty}\max\left\{\frac{D(z,x_n)}{2}\frac{D(z,Tz)}{2},\frac{D(x_n,Tx_n)}{2}\right\}\right)\right]\\ &< \theta\left[\lim_{n\longrightarrow\infty}\left(\max\left\{\frac{D(z,x_n)}{2},\frac{D(z,Tz)}{2},\frac{D(x_n,Tx_n)}{2}\right\}\right)\right].\end{aligned} \tag{174}$$

Therefore,

$$D(z,Tz) < D(z,Tz), \tag{175}$$

which is a contradiction. Thus, $z = Tz$. Thus, T has a fixed point.

Uniqueness: let $z,u \in \mathrm{Fix}(T)$, where $z \neq u$. Then,

$$\begin{aligned}d(Tz,Tu) &= d(z,u) > 0,\\ d(Tu,Tz) &= d(u,z) > 0.\end{aligned} \tag{176}$$

Therefore,

$$\max\{d(Tz,Tu), d(Tu,Tz)\} > 0. \tag{177}$$

From the assumption of the theorem, we get

$$\theta\left(\frac{D(Tz,Tu)}{2}\right) = \theta\left(\frac{D(z,u)}{2}\right) \leq \phi[\theta(M(z,u))], \tag{178}$$

where

$$M(z,u) = \max\left\{\frac{D(z,u)}{2},\frac{D(z,Tz)}{2},\frac{D(u,Tu)}{2}\right\} = \frac{D(z,u)}{2}. \tag{179}$$

Therefore, we have

$$\begin{aligned}\theta\left(\frac{D(z,u)}{2}\right) &\leq \phi\left[\theta\left(\frac{D(z,u)}{2}\right)\right]\\ &< \theta\left(\frac{D(z,u)}{2}\right),\end{aligned} \tag{180}$$

which implies that $D(z,u) < D(z,u)$. It is a contradiction. Therefore, $u = z$.

From Theorem 3, we obtain the following fixed-point theorems for $\theta-\phi$-Reich-type contraction and $\theta-\phi$-Kannan-type contraction. □

Theorem 4. *Let (X,d) be a complete generalized asymmetric space and $T\colon X \longrightarrow X$ be a $\theta-\phi$-Kannan-type contraction; then, T has a unique fixed point.*

Proof. Since T is a $\theta-\phi$-Kannan-type contraction, there exist $\theta \in \Theta$ and $\phi \in \Phi$ such that

$$\begin{aligned}\theta\left[\frac{d(Tx,Ty)+d(Ty,Tx)}{2}\right] &\leq \phi\left[\theta\left(\frac{d(Tx,x)+d(x,Tx)+d(Ty,y)+d(y,Ty)}{4}\right)\right]\\ &\leq \phi\left[\theta\left(\max\left\{\frac{d(x,Tx)+d(Tx,x)}{2},\frac{d(y,Ty)+d(Ty,y)}{2}\right\}\right)\right]\\ &\leq \phi\left[\theta\left(\max\left\{\frac{d(x,y)+d(y,x)}{2},\frac{d(Tx,x)+d(x,Tx)}{2},\frac{d(y,Ty)+d(Ty,y)}{2}\right\}\right)\right].\end{aligned} \tag{181}$$

Therefore, T is a $\theta-\phi$-contraction. As in the proof of Theorem 3, we conclude that T has a unique fixed point. □

Theorem 5. *Let (X,d) be a complete generalized asymmetric space and $T\colon X \longrightarrow X$ be a $\theta-\phi$-Reich-type contraction.*

Then, T has a unique fixed point.

Proof. Since T is a $\theta-\phi$-Reich-type contraction, there exist $\theta\in\Theta$ and $\phi\in\Phi$ such that

$$\begin{aligned}\theta\left[\frac{d(Tx,Ty)+d(Ty,Tx)}{2}\right]&\leq\phi\left[\theta\left(\frac{d(x,y)+d(y,x)+d(Tx,x)+d(x,Tx)+d(Ty,y)+d(y,Ty)}{6}\right)\right]\\&\leq\phi\left[\theta\left(\max\left\{\frac{d(x,y)+d(y,x)}{2},\frac{d(Tx,x)+d(x,Tx)}{2},\frac{d(y,Ty)+d(Ty,y)}{2}\right\}\right)\right].\end{aligned}\tag{182}$$

Therefore, T is a $\theta-\phi$-contraction. As in the proof of Theorem 3, we conclude that T has a unique fixed point. □

Corollary 2. *Let (X,d) be a complete metric space and $T\colon X\longrightarrow X$ be a Kannan-type mapping. i.e., there exists $\alpha\in\,]0,(1/2)[$ such that, for all $x,y\in X$,*

$$\max\{d(Tx,Ty),d(Ty,Tx)\}>0\Longrightarrow\frac{d(Tx,Ty)+d(Ty,Tx)}{2}\leq\alpha\left[\frac{d(Tx,x)+d(x,Tx)+d(Ty,y)+d(y,Ty)}{2}\right].\tag{183}$$

Then, T has a unique fixed point.

Proof. Let $\theta(t)=e^{t}$ for all $t\in\,]0,+\infty[$ and $\phi(t)=t^{2\alpha}$ for all $t\in[1,+\infty[$. We prove that T is a $\theta-\phi$-Kannan-type contraction. Indeed,

$$\begin{aligned}\theta\left(\frac{d(Tx,Ty)+d(Ty,Tx)}{2}\right)&=e^{(d(Tx,Ty)+d(Ty,Tx))/2}\leq e^{\alpha((d(Tx,x)+d(x,Tx)+d(Ty,y)+d(y,Ty))/2)}=e^{2\alpha((d(Tx,x)+d(x,Tx)+d(Ty,y)+d(y,Ty))/4)}\\&=\left[e^{((d(Tx,x)+d(x,Tx)+d(Ty,y)+d(y,Ty))/4)}\right]^{2\alpha}\\&=\phi\left[\theta\left(\frac{d(Tx,x)+d(x,Tx)+d(Ty,y)+d(y,Ty)}{4}\right)\right].\end{aligned}\tag{184}$$

Therefore, as in the proof of Theorem 4, T has a unique fixed point $x\in X$. □

Corollary 3. *Let (X,d) be a complete metric space and $T\colon X\longrightarrow X$ be a Reich-type mapping, i.e., there exists $\lambda\in\,]0,(1/3)[$ such that, for all $x,y\in X$,*

$$\max\{d(Tx,Ty),d(Ty,Tx)\}>0,\tag{185}$$

and we have

$$\frac{d(Tx,Ty)+d(Ty,Tx)}{2}\leq\lambda\left[\frac{d(x,y)+(d(y,x)+d(Tx,x)+d(x,Tx)+d(Ty,y)+d(y,Ty)}{2}\right].\tag{186}$$

Then, T has a unique fixed point.

Proof. Let $\theta(t) = e^t$ for all $t \in]0,+\infty[$ and $\phi(t) = t^{3\lambda}$ for all $t \in [1,+\infty[$.

We prove that T is a $\theta - \phi$-Reich-type contraction.

$$\begin{aligned}\theta\left(\frac{d(Tx,Ty)+d(Ty,Tx)}{2}\right) &= e^{(d(Tx,Ty)+d(Ty,Tx))/2} \le e^{\lambda((d(x,y)+(d(y,x)+d(Tx,x)+d(x,Tx)+d(Ty,y)+d(y,Ty))/2)} \\ &= e^{3\lambda((d(x,y)+(d(y,x)+d(Tx,x)+d(x,Tx)+d(Ty,y)+d(y,Ty))/6)} \\ &= \phi\left[\theta\left(\frac{d(x,y)+(d(y,x)+d(Tx,x)+d(x,Tx)+d(Ty,y)+d(y,Ty)}{6}\right)\right].\end{aligned} \tag{187}$$

Therefore, as in the proof of Theorem 5, T has a unique fixed point $x \in X$. □

Corollary 4 (Theorem 2). *Let (X,d) be a complete generalized asymmetric space and $T: X \longrightarrow X$ be a given mapping. Suppose that there exist $\theta \in \Theta$ and $r \in]0,1[$ such that, for any $x, y \in X$,*

$$\max\{d(Tx,Ty), d(Ty,Tx)\} > 0 \Longrightarrow \theta\left[\frac{d(Tx,Ty)+d(Ty,Tx)}{2}\right] \le (\theta[M(x,y)])^r, \tag{188}$$

where

$$M(x,y) = \max\left\{\frac{d(x,y)+d(y,x)}{2}, \frac{d(x,Tx)+d(Tx,x)}{2}, \frac{d(y,Ty)+d(Ty,y)}{2}\right\}. \tag{189}$$

Then, T has a unique fixed point.

Proof. If $\phi(t) = t^r$, with $r \in]0,1[$, we prove that T is a $\theta - \phi$-contraction. Therefore, as in the proof of Theorem 3, T has a unique fixed point. □

Example 4. Consider $X = \{1,2,3,4\}$. Let $d: X \times X \longrightarrow [0,+\infty[$ be a mapping defined by the following:

(i) $d(x,y) = 0 \Longleftrightarrow x = y$.

(ii) $d(1,2) = 3$, $d(2,1) = 1$.

(iii) $d(2,3) = d(3,2) = d(1,3) = d(3,1) = 1$.

(iv) $d(1,4) = d(4,1) = d(4,2) = d(2,4) = d(3,4) = 4$.

(v) $d(4,3) = 3$.

Clearly, (X,d) is not an asymmetric metric space, from $d(1,2) = 3 > d(1,3) + d(3,2) = 2$.

However, it is a complete generalized asymmetric metric space. Let $T: X \longrightarrow X$ be given by

$$\begin{cases} T(1) = T(2) = T(3) = 1, \\ T(4) = 3. \end{cases} \tag{190}$$

Suppose $\theta(t) = \sqrt{t} + 1$ and $\varphi(t) = 2t + 1/3$. Therefore, $\theta \in \Theta$ and $\phi \in \Phi$.

First, observe that $\max\{d(Tx,Ty), d(Ty,Tx)\} > 0 \Longleftrightarrow \{x = 4, y = 1\}$, $\{x = 4, y = 2\}$, or $\{x = 4, y = 3\}$; for $x = 4$, $y = 1$, we have

$$\frac{d(T4,T1)+d(T1,T4)}{2} = \frac{d(3,1)+d(1,3)}{2} = 1. \tag{191}$$

On the contrary,

$$\max\left\{\frac{d(4,T(4))+d(T(4),4)}{2}, \frac{d(1,T(1))+d(1,T(1))}{2}, \frac{d(4,1)+d(1,4)}{2}\right\} = 4. \tag{192}$$

Therefore,

$$\theta\left(\frac{d(T(4),T(1))+d(T(1),T4)}{2}\right)=2,$$
$$\phi[\theta(4)]=2.33. \tag{193}$$

So,

$$\theta\left(\frac{D(T(4),T(1))}{2}\right)\leq\phi[\theta(M(4,1))]. \tag{194}$$

For $x=4,\ y=2$, we have

$$\theta\left(\frac{d(T(4),T(2))+d(T(2),T(4))}{2}\right)=2,$$
$$\max\left\{\frac{d(4,T(4))+d(T(4),4)}{2},\frac{d(2,T(2))+d(2,T(2))}{2},\frac{d(4,2)+d(2,4)}{2}\right\}=4. \tag{195}$$

Therefore,

$$\varphi[\theta(4)]=2.33. \tag{196}$$

So,

$$\theta\left(\frac{D(T(4),T(2))}{2}\right)\leq\phi[\theta(M(4,2))]. \tag{197}$$

For $x=4,\ y=3$, we have

$$\theta\left(\frac{d(T(4),T(3))+d(T(3),T(4))}{2}\right)=\theta\left(\frac{d(3,1)+d(1,3)}{2}\right)=2,$$
$$\max\left\{\frac{d(4,T(4))+d(T(4),4)}{2},\frac{d(3,T(3))+d(3,T(3))}{2},\frac{d(4,3)+d(3,4)}{2}\right\}=\frac{7}{2}. \tag{198}$$

Therefore,

$$\phi\left[\theta\left(\frac{7}{2}\right)\right]=2.24. \tag{199}$$

So,

$$\theta\left(\frac{D(T(4),T(3))}{2}\right)\leq\phi[\theta(M(4,3))]. \tag{200}$$

Hence, T satisfies the assumption of the theorem, and $z=1$ is the unique fixed point of T.

Example 5. Let $X=[1,+\infty[$ and $d\colon X\times X\longrightarrow[0,+\infty[$ defined by

$$\begin{cases} d(x,y)=y-x & \text{if } y\geq x,\\ d(x,y)=\dfrac{1}{2}(x-y) & \text{if } y<x.\end{cases} \tag{201}$$

Clearly, (X,d) is not metric, asymmetric metric, but it is a complete generalized asymmetric metric space.

Let $T\colon X\longrightarrow X$ be given by

$$T(x)=\sqrt{x}. \tag{202}$$

Let $\theta(t)=\sqrt{t}+1$ and $\phi(t)=((t+1)/2)$. It is obvious that $\theta\in\Theta$ and $\phi\in\Phi$.

Consider the following possibilities:

Case 1: $y\geq x$. We have

$$d(x,y)=y-x,$$
$$d(y,x)=\frac{1}{2}(y-x). \tag{203}$$

Then,

$$\frac{D(x,y)}{2}=\frac{3}{4}(y-x),$$
$$\theta\left(\frac{D(x,y)}{2}\right)=\frac{\sqrt{3}}{2}\left(\sqrt{y-x}\right)+1. \tag{204}$$

So,

$$\phi\left[\theta\left(\frac{D(x,y)}{2}\right)\right]=\frac{\sqrt{3}}{4}\left(\sqrt{y-x}\right)+1,$$
$$\theta\left(\frac{D(Tx,Ty)}{2}\right)=\frac{\sqrt{3}}{2}\left(\sqrt{\sqrt{y}-\sqrt{x}}\right)+1. \tag{205}$$

On the contrary,

$$\begin{aligned}\theta\left(\frac{D(x,y)}{2}\right)-\phi\left[\theta\left(\frac{D(Tx,Ty)}{2}\right)\right]&=\frac{\sqrt{3}}{2}\left(\sqrt{\sqrt{y}-\sqrt{x}}\right)-\frac{\sqrt{3}}{4}\left(\sqrt{y-x}\right)\\&=\frac{\sqrt{3}}{4}\left[\left(2\sqrt{\sqrt{y}-\sqrt{x}}\right)-\left(\sqrt{y-x}\right)\right]\\&=\frac{\sqrt{3}}{4}\left(\sqrt{\sqrt{y}-\sqrt{x}}\right)\left(2-\sqrt{\sqrt{y}+\sqrt{x}}\right).\end{aligned}\tag{206}$$

Since $x, y \in [1, +\infty[$,

$$2-\sqrt{\sqrt{y}+\sqrt{x}}\leq 0,\tag{207}$$

which implies that

$$\begin{aligned}\theta\left(\frac{D(Tx,Ty)}{2}\right)&\leq\phi\left[\theta\left(\frac{D(x,y)}{2}\right)\right]\\&\leq\phi\left[\theta\left(\max\left\{\frac{D(x,y)}{2},\frac{D(x,Tx)}{2},\frac{D(y,Ty)}{2}\right\}\right)\right].\end{aligned}\tag{208}$$

Case 2: $y < x$. Similar to case 1, we conclude that

$$\theta\left(\frac{D(Tx,Ty)}{2}\right)\leq\phi\left[\theta\left(\max\left\{\frac{D(x,y)}{2},\frac{D(x,Tx)}{2},\frac{D(y,Ty)}{2}\right\}\right)\right].\tag{209}$$

Hence, T satisfies the assumption of the theorem, and $z = 1$ is the unique fixed point of T.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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