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Research Article

Direction Curves Associated with Darboux Vectors Fields and Their Characterizations

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In this paper, we consider the Darboux frame of a curve α lying on an arbitrary regular surface and we use its unit osculator Darboux vector $\overline{D}_o$, unit rectifying Darboux vector $\overline{D}_r$, and unit normal Darboux vector $\overline{D}_n$ to define some direction curves such as $\overline{D}_o$-direction curve, $\overline{D}_r$-direction curve, and $\overline{D}_n$-direction curve, respectively. We prove some relationships between α and these associated curves. Especially, the necessary and sufficient conditions for each direction curve to be a general helix, a spherical curve, and a curve with constant torsion are found. In addition to this, we have seen the cases where the Darboux invariants δ_o, δ_r, and δ_n are, respectively, zero. Finally, we enrich our study by giving some examples.

1. Introduction

In the theory of curves in differential geometry, characterizing the curves and giving general information about their structure in terms of curvature is a very interesting and important problem that is developed by many differential geometers in different ambient spaces. The most popular and important curves that have been studied in many types of research are spherical curve [1], general helix [2], relatively normal slant helix [3], isophote curve [4], Salkowski curve [5], and anti-Salkowski curve [6].

Recently, the problem of deriving a new curve from a given curve and obtaining new characterizations for them has taken its place among popular topics. This category of curves called associated curve has been investigated in various research. Bertrand curve [7], involute-evolute curve [8], Mannheim curve [9], and spherical indicatrix [10] are among the leading examples.

In this sense, in the Euclidean 3-dimensional space, a new version of the associated curve called direction curve was introduced by Choi and Kim in [11]. They defined the principal-direction curve and binormal-direction curve as the integral curve of principal normal N and binormal B of a Frenet curve, respectively, and they use this concept to characterize general helices, slant helices, and PD-rectifying curve in E^3. However, Deshmukh et al. defined in [12] the natural mate curve and the conjugate mate curve which are similar to the principal-direction curve and binormal-direction curve, respectively, from algebraic viewpoint, but are more accurate and comprehensive from the geometric viewpoint since the integral curve is defined only for vector fields on a region which contains a curve, not along a curve. Then, they derived some new characterizations of helices, slant helices, spherical curves, and rectifying curves. Furthermore, in [13], the authors expressed new direction curves such as evolute direction curves, Bertrand direction curves, and Mannheim directon curves by means of a vector field generated by Frenet vectors of normal indicatrix of a given curve. These direction curves were used to give a new approach to construct slant helices and C-slant helices. In [14], Macit and Duldul defined W-direction curves and W-rectifying curves of a Frenet curve in $\mathbb{R}^3$ by utilizing the Darboux vector of the curve. They also introduced V-direction curve of a given curve on a surface by using the Darboux frame. This curve was used to characterize the relatively normal-slant helix in [3].

Motivated by this, in the present study, we consider the Darboux frame of a curve α lying on an arbitrary regular surface, and we define in Euclidean 3-dimensional space the $\overline{D}_o$-direction curve, the $\overline{D}_r$-direction curve and the $\overline{D}_n$-direction curve of α as the integral curve of the unit osculator Darboux vector $\overline{D}_o$, the unit rectifying Darboux vector $\overline{D}_r$, and the unit normal Darboux vector $\overline{D}_n$, respectively. Then, we give some relationships between the curve α and each direction curve. Especially, we obtain necessary and sufficient conditions for that these direction curves be a general helix, a spherical curve, and a curve with constant torsion. Beside, we discuss the cases where the Darboux invariants δ_o, δ_r, and δ_n are, respectively, zero. Finally, two examples are illustrated.

2. Preliminaries

In this section, we recall some basic concepts and properties on classical differential geometry of curves lying on a regular surface, in the Euclidean 3-dimensional space.

(i) We denote by E^3 the Euclidean 3-dimensional space, with the usual metric

$$\langle x, y\rangle = x_1 y_1 + x_2 y_2 + x_3 y_3, \tag{1}$$

where $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ are two vectors of E^3.

(ii) Let $\alpha = \alpha(s)$: $I \subset \mathbb{R} \longrightarrow E^3$ be a regular curve in E^3, with nonnull curvature, we also assume that is parametrized by arc-length s, i.e., $\langle \alpha'(s), \alpha'(s)\rangle = 1$ for all $s \in I$.

The Frenet frame along the curve α is an orthonormal frame $(T(s), N(s), B(s))$

$$\begin{aligned} T(s) &= \frac{d\alpha(s)}{ds},\\ N(s) &= \frac{T'(s)}{\|T'(s)\|},\\ B(s) &= T(s)\wedge N(s), \end{aligned} \tag{2}$$

where T is the unit tangent, N is the unit principal normal, B is the unit binormal, and k and τ are, respectively, the curvature and the torsion of the curve α, given by

$$\begin{aligned} k(s) &= \|\alpha''(s)\|,\\ \tau(s) &= -\langle B'(s), N(s)\rangle. \end{aligned} \tag{3}$$

The curve α is called a Frenet curve if $\tau \neq 0$.

It is known that a Frenet curve α in E^3 is a spherical curve if and only if

$$(p'q)' + \frac{p}{q} = 0, \tag{4}$$

holds, where $p = k^{-1}$, $q = \tau^{-1}$. Moreover, if the Frenet curve α: $I \longrightarrow E^3$ is a spherical curve lying on a sphere of radius a, then we have [15]

$$p^2 + (p'q)^2 = a^2, \tag{5}$$

(iii) Let M be a regular surface, and $\alpha = \alpha(s)$: $I \subset \mathbb{R} \longrightarrow M$ be a unit speed curve on the surface M. The Darboux frame along the curve α is an orthonormal frame $(T(s), V(s), U(s))$, where T is the unit tangent, U is the unit normal on the surface M, and $V = U\wedge T$. Then, the Darboux equations are given by the following relations:

$$\frac{d}{ds}\begin{bmatrix} T(s)\\ V(s)\\ U(s)\end{bmatrix} = \begin{bmatrix} 0 & k_g(s) & k_n(s)\\ -k_g(s) & 0 & \tau_g(s)\\ -k_n(s) & -\tau_g(s) & 0\end{bmatrix}\begin{bmatrix} T(s)\\ V(s)\\ U(s)\end{bmatrix}, \tag{6}$$

where k_g, k_n, and τ_g are, respectively, the geodesic curvature, the normal curvature, and the geodesic torsion of the curve α and are given by

$$\begin{cases} k_g = k \sin\varphi,\\ k_n = k\cos\varphi,\\ \tau_g = \tau + \dfrac{d\varphi}{ds}, \end{cases} \tag{7}$$

with φ denote the angle between the surface normal U and the normal N.

The curve α is called a general helix (resp., a relatively normal slant helix and an isophote curve) if the vector field T (resp., V and U) makes a constant angle with a fixed direction, i.e., there exists a fixed unit vector d and a constant angle θ such that $\langle T, d\rangle = \cos\theta$ (resp., $\langle V, d\rangle = \cos\theta$ and $\langle U, d\rangle = \cos\theta$).

Let us give some theorems characterizing these curves.

Theorem 1 (see [2]). *A curve α is a general helix if and only if the following function*

$$H = \frac{\tau}{k}, \tag{8}$$

is constant.

Theorem 2 (see [3]). *A unit speed curve α on a surface M with $(\tau_g, k_g) \neq (0,0)$ is a relatively normal slant helix if and only if the following function*

$$\sigma_r = \frac{(\tau_g' k_g - k_g'\tau_g) - k_n(\tau_g^2 + k_g^2)}{(\tau_g^2 + k_g^2)^{3/2}}, \tag{9}$$

is constant.

Theorem 3 (see [4]). *A unit speed curve α on a surface M with $(\tau_g, k_n) \neq (0,0)$ is an isophote curve if and only if the following function*

$$\sigma_o = \frac{\tau_g' k_n - k_n' \tau_g + k_g\left(\tau_g^2 + k_n^2\right)}{\left(\tau_g^2 + k_n^2\right)^{3/2}}, \tag{10}$$

is constant.

3. Direction Curves Associated with Darboux Vector Fields

In this section, we introduce the direction curves associated with Darboux vector fields of a curve lying on a regular surface, and we determine some of their properties.

Let $\alpha = \alpha(s)$ be a unit speed regular curve lying on a regular surface M. We denote by (T, V, U) its Darboux frame, k_n, k_g, and τ_g the normal curvature, the geodesic curvature, and the geodesic torsion of the curve α and by $\overline{D}_o = (\tau_g T - k_n V/\sqrt{k_n^2 + \tau_g^2})$ (resp. $\overline{D}_r = (\tau_g T + k_g U/\sqrt{k_g^2 + \tau_g^2})$ and $\overline{D}_n = (-k_n V + k_g U/\sqrt{k_n^2 + k_g^2})$) the unit osculator (resp., rectifying and normal) Darboux vector with $(k_n, \tau_g) \neq (0,0)$ (resp. $(k_g, \tau_g) \neq (0,0)$ and $(k_g, k_n) \neq (0,0)$) and $\delta_o = ((\tau_g' k_n - k_n' \tau_g) + k_g(\tau_g^2 + k_n^2)/(\tau_g^2 + k_n^2))$ (resp. $\delta_r = ((\tau_g' k_g - k_g' \tau_g) - k_n(\tau_g^2 + k_g^2)/(\tau_g^2 + k_g^2)))$ and $\delta_n = ((k_n' k_g - k_g' k_n) + \tau_g(k_n^2 + k_g^2)/(k_n^2 + k_g^2))$ the Darboux invariant associated with the Darboux frame.

Definition 1. The integral curve β_o (resp., β_r and β_n) of the osculator (resp., rectifying and normal) Darboux vectors field $\overline{D}_o$ (resp., $\overline{D}_r$ and $\overline{D}_n$) is called $\overline{D}_o$-direction (resp., $\overline{D}_r$-direction and $\overline{D}_n$–direction) curve of α. In other words,

$$\begin{aligned} \beta_o'(s) &= \overline{D}_o(s), \\ \beta_r'(s) &= \overline{D}_r(s), \\ \beta_n'(s) &= \overline{D}_n(s). \end{aligned} \tag{11}$$

3.1. Frenet Frame of Direction Curves Associated with Darboux Vector Fields. In this paragraph, we determine the Frenet frame of β_o, β_r, and β_n, respectively, assuming for each case that the curve is well defined.

Theorem 4. *Let β_o be the $\overline{D}_o$-direction curve of α. If $\delta_o \neq 0$, the Frenet vector fields $(T_{\beta_o}, N_{\beta_o}, B_{\beta_o})$, curvature k_{β_o}, and torsion τ_{β_o} of β_o are given by*

$$\begin{cases} T_{\beta_o} = \overline{D}_o \\ N_{\beta_o} = -\varepsilon \dfrac{U'}{\|U'\|}, \\ B_{\beta_o} = \varepsilon U \end{cases} \tag{12}$$

$$\begin{cases} k_{\beta_o} = \varepsilon \delta_o \\ \tau_{\beta_o} = \sqrt{k_n^2 + \tau_g^2} \end{cases}, \tag{13}$$

where $\varepsilon = \pm 1\,(\varepsilon\delta_o > 0)$.

Proof. From the definition of the curve β_o, we have $T_{\beta_o} = \overline{D}_o = (\tau_g T - k_n V/\sqrt{k_n^2 + \tau_g^2})$.

Using (6) and after calculation, we get

$$\begin{aligned} \overline{D}_o' &= \frac{k_n\left[\left(\tau_g' k_n - k_n' \tau_g\right) + k_g\left(\tau_g^2 + k_n^2\right)\right]}{\left(\tau_g^2 + k_n^2\right)^{3/2}} T \\ &\quad + \frac{\tau_g\left[\left(\tau_g' k_n - k_n' \tau_g\right) + k_g\left(\tau_g^2 + k_n^2\right)\right]}{\left(\tau_g^2 + k_n^2\right)^{3/2}} V \\ &= k_n \sigma_o T + \tau_g \sigma_o V \\ &= -\sigma_o U'. \end{aligned} \tag{14}$$

Then,

$$\overline{D}_o' = -\delta_o \frac{U'}{\|U'\|}. \tag{15}$$

So, by taking the norm of (15), we obtain $\|\overline{D}_o'\| = \varepsilon\delta_o$, where $\varepsilon = \pm 1\,(\varepsilon\delta_o > 0)$. Thus,

$$N_{\beta_o} = \frac{\overline{D}_o'}{\|\overline{D}_o'\|} = -\varepsilon \frac{U'}{\|U'\|}. \tag{16}$$

The cross production of T_{β_o} and N_{β_o} leads us the binormal vector as follows:

$$B_{\beta_o} = \varepsilon U. \tag{17}$$

On the other hand, we have

$$k_{\beta_o} = \|\overline{D}_o'\| = \varepsilon\delta_o, \tag{18}$$

$$\begin{aligned} \tau_{\beta_o}(s) &= -\langle B_{\beta_o}'(s), N_{\beta_o}(s)\rangle = -\langle \varepsilon U', -\varepsilon \frac{U'}{\|U'\|}\rangle \\ &= \|U'\| = \sqrt{\tau_g^2 + k_n^2}. \end{aligned} \tag{19}$$

□

Theorem 5. *Let β_r be the $\overline{D}_r$-direction curve of α. If $\delta_r \neq 0$, the Frenet vector fields $(T_{\beta_r}, N_{\beta_r}, B_{\beta_r})$, curvature k_{β_r}, and torsion τ_{β_r} of β_r are given by*

$$\begin{cases} T_{\beta_r} = \overline{D}_r \\ N_{\beta_r} = -\varepsilon \dfrac{V'}{\|V'\|}, \\ B_{\beta_r} = \varepsilon V \end{cases} \tag{20}$$

$$\begin{cases} k_{\beta_r} = \varepsilon \delta_r \\ \tau_{\beta_r} = \sqrt{k_g^2 + \tau_g^2} \end{cases}, \tag{21}$$

where $\varepsilon = \pm 1\,(\varepsilon\delta_r > 0)$.

Proof. We have $T_{\beta_r} = \overline{D}_r = (\tau_g T + k_g U/\sqrt{k_g^2 + \tau_g^2})$.

Differentiating $\overline{D}_r$, we get

$$\overline{D}_r' = -\sigma_r\left(-k_g T + \tau_g U\right) = -\sigma_r V' = -\delta_r \frac{V'}{\|V'\|}. \quad (22)$$

Therefore, we have $\|\overline{D}_r'\| = \varepsilon\delta_r$, where $\varepsilon = \pm 1$ $(\varepsilon\delta_r > 0)$. Thus,

$$N_{\beta_r} = \frac{\overline{D}_r'}{\|\overline{D}_r'\|} = -\varepsilon \frac{V'}{\|V'\|}, \quad (23)$$

$$B_{\beta_r} = T_{\beta_r} \wedge N_{\beta_r} = \varepsilon V. \quad (24)$$

On the other hand, we have

$$k_{\beta_r} = \|\overline{D}_r'\| = \varepsilon\delta_r, \quad (25)$$

$$\begin{aligned} \tau_{\beta_r}(s) &= -\langle B_{\beta_r}'(s), N_{\beta_r}(s)\rangle = -\langle \varepsilon V', -\varepsilon \frac{V'}{\|V'\|}\rangle \\ &= \sqrt{\tau_g^2 + k_g^2}. \end{aligned} \quad (26)$$

□

Theorem 6. *Let β_n be the $\overline{D}_n$-direction curve of α. We denote by k and τ the curvature and the torsion of α. If $\tau \neq 0$, the Frenet vector fields $(T_{\beta_n}, N_{\beta_n}, B_{\beta_n})$, curvature k_{β_n}, and torsion τ_{β_n} of β_n are given by*

$$\begin{cases} T_{\beta_n} = \overline{D}_n \\ N_{\beta_n} = -\varepsilon \dfrac{T'}{\|T'\|}, \\ B_{\beta_n} = \varepsilon T \end{cases} \quad (27)$$

$$\begin{cases} k_{\beta_n} = \varepsilon\tau, \\ \tau_{\beta_n} = k, \end{cases} \quad (28)$$

where $\varepsilon = \pm 1$ $(\varepsilon\delta_n > 0)$.

Proof. In this case, we have $T_{\beta_n} = \overline{D}_n = (-k_n V + k_g U/\sqrt{k_n^2 + k_g^2})$.

Differentiating $\overline{D}_n$ by using (6), we get $\overline{D}_n' = -\delta_n (T'/\|T'\|)$.

By using (7), we prove that

$$\delta_n = \frac{k_n' k_g - k_n k_g' + \tau_g\left(k_n^2 + k_g^2\right)}{k_n^2 + k_g^2} = \tau. \quad (29)$$

Then,

$$\overline{D}_n' = -\tau \frac{T'}{\|T'\|}. \quad (30)$$

So, by taking the norm of (30), we have $\|\overline{D}_n'\| = \varepsilon\tau$, where $\varepsilon = \pm 1$ $(\varepsilon\delta_n > 0)$. Consequently,

$$N_{\beta_n} = -\varepsilon \frac{T'}{\|T'\|}, \quad (31)$$

$$B_{\beta_n} = T_{\beta_n} \wedge N_{\beta_n} = \varepsilon T. \quad (32)$$

On the other hand, we have

$$k_{\beta_n} = \|\overline{D}_n'\| = \varepsilon\tau, \quad (33)$$

$$\tau_{\beta_n}(s) = -\langle B_{\beta_n}'(s), N_{\beta_n}(s)\rangle = \sqrt{k_g^2 + k_n^2} = k. \quad (34)$$

Using Theorems 1–3, we conclude the following results. □

Corollary 1

(1) β_o is a general helix if and only if α is an isophote curve

(2) β_r is a general helix if and only if α is a relatively normal-slant helix

(3) β_n is a general helix if and only if α is a general helix

Proof. The proof can be done by using (13), (21), and (28), respectively. □

3.2. Spherical Direction Curves of Darboux Vector Fields. In this paragraph, we propose to determine a necessary and sufficient condition for the direction curve β_o (resp., β_r and β_n) to be a spherical curve.

Theorem 7. *Let $\alpha = \alpha(s)$ be a unit speed regular curve lying on a regular surface M, such that $\delta_o \neq 0$ and $\beta_o = \beta_o(s)$ its $\overline{D}_o$-direction curve. The curve β_o is a spherical curve if and only if*

$$\delta_o = \frac{1}{a}\sec\left(\int \sqrt{k_n^2 + \tau_g^2}\, ds\right), \quad (35)$$

where a is the radius of the sphere.

Proof. From (5), since the curve β_o is lying on a sphere of radius a, we have

$$\left(\frac{1}{k_{\beta_o}}\right)^2 + \frac{k_{\beta_o}'^2}{k_{\beta_o}^4 \tau_{\beta_o}^2} = a^2. \quad (36)$$

It follows that

$$k_{\beta_o}' = \pm\tau_{\beta_o} k_{\beta_o} \sqrt{a^2 k_{\beta_o}^2 - 1}, \quad (37)$$

and by using the expression of k_{β_o} and τ_{β_o}, we obtain

$$\varepsilon\delta_o' = \pm\varepsilon\delta_o \sqrt{k_n^2 + \tau_g^2}\sqrt{a^2\delta_o^2 - 1}. \quad (38)$$

We have $a^2\delta_o^2 - 1 \neq 0$; otherwise, $k_{\beta_o} = (1/a) = cste$, and as β_o is assumed spherical, necessarily $\tau_{\beta_o} = \sqrt{k_n^2 + \tau_g^2} = 0$ [1], which is absurd because we assumed $(k_n, \tau_g) \neq (0, 0)$. We get

$$\frac{\varepsilon\delta_o'}{\delta_o\sqrt{a^2\delta_o^2-1}} = \pm\varepsilon\sqrt{k_n^2+\tau_g^2}. \quad (39)$$

Therefore, we get

$$\frac{1}{a\delta_o} = \cos\left(\int\sqrt{k_n^2+\tau_g^2}\,ds\right), \quad (40)$$

and it means that

$$\delta_o = \frac{1}{a}\sec\left(\int\sqrt{k_n^2+\tau_g^2}\,ds\right). \quad (41)$$

Conversely, suppose that $\delta_o = (1/a)\sec(\int\sqrt{k_n^2+\tau_g^2}\,ds)$, we have

$$p = \frac{1}{k_{\beta_o}} = \frac{1}{\varepsilon\delta_o} = \varepsilon a\cos\left(\int\sqrt{k_n^2+\tau_g^2}\,ds\right), \quad (42)$$

$$q = \frac{1}{\tau_{\beta_o}} = \frac{1}{\sqrt{k_n^2+\tau_g^2}}. \quad (43)$$

Then,

$$p'q = -\varepsilon a\sin\left(\int\sqrt{k_n^2+\tau_g^2}\,ds\right). \quad (44)$$

So, $(p'q)' + (p/q) = 0$. Hence the result from (4).

By reasoning similarity, we obtain the following theorem. □

Theorem 8. *Let $\alpha = \alpha(s)$ be a unit speed regular curve lying on a regular surface M, such that $\delta_r \neq 0$ and $\beta_r = \beta_r(s)$ its $\overline{D}_r$-direction curve. The curve β_r is a spherical curve if and only if*

$$\delta_r = \frac{1}{a}\sec\left(\int\sqrt{k_g^2+\tau_g^2}\,ds\right), \quad (45)$$

where a is the radius of the sphere.

Theorem 9. *Let $\alpha = \alpha(s)$ be a unit speed regular curve lying on a regular surface M, such that $\tau \neq 0$ and $\beta_n = \beta_n(s)$ its $\overline{D}_n$-direction curve. The curve β_n is a spherical curve if and only if*

$$\tau = \frac{1}{a}\sec\left(\int k\,ds\right), \quad (46)$$

where a is the radius of the sphere, and k and τ are the curvature and the torsion of α, respectively.

Proof. From (5), since β_n is lying on a sphere of radius a, we have

$$\left(\frac{1}{k_{\beta_n}}\right)^2 + \frac{k_{\beta_n}'^2}{k_{\beta_n}^4\tau_{\beta_n}^2} = a^2. \quad (47)$$

It follows that

$$k_{\beta_n}' = \pm\tau_{\beta_n}k_{\beta_n}\sqrt{a^2k_{\beta_n}^2-1}, \quad (48)$$

by using (28), we obtain

$$\varepsilon\tau' = \pm\varepsilon\tau k\sqrt{a^2\tau^2-1}. \quad (49)$$

Since $a^2\tau^2-1\neq 0$, we write

$$\frac{\varepsilon\tau'}{\tau\sqrt{a^2\tau^2-1}} = \pm\varepsilon k. \quad (50)$$

Therefore, we get

$$\tau = \frac{1}{a}\sec\left(\int k\,ds\right). \quad (51)$$

Conversely, suppose that $\tau = (1/a)\sec(\int k\,ds)$, we have

$$p = \frac{1}{k_{\beta_n}} = \frac{1}{\varepsilon\tau} = \varepsilon a\cos\left(\int k\,ds\right)\text{ and } q = \frac{1}{\tau_{\beta n}} = \frac{1}{k}. \quad (52)$$

Then, it is easy to see that $(p'q)' + (p/q) = 0$. Hence the result from (4). □

3.3. Direction Curves of Darboux Vector Fields with Constant Torsion. In this paragraph, we propose to study the case where the direction curve β_o (resp., β_r and β_n) has a constant torsion.

From formula (13), we can conclude that if there exists a function $\varphi(s)$ such that the normal curvature k_n and the geodesic torsion τ_g of the curve α satisfy

$$k_n = c\cos(\varphi(s))\text{ and }\tau_g = c\sin(\varphi(s)), \quad (53)$$

where c is a constant, then the $\overline{D}_o$-direction curve of the curve α has a constant torsion.

Conversely, we give the following theorem.

Theorem 10. *Let $\alpha = \alpha(s)$ be a unit speed regular curve lying on a regular surface M, such that $\delta_o \neq 0$ and $\beta_o = \beta_o(s)$ its $\overline{D}_o$-direction curve. If the curve β_o has a constant torsion, then the normal curvature k_n and the geodesic torsion τ_g of α satisfy the following equalities:*

$$k_n = c\cos(\varphi(s))\text{ and }\tau_g = c\sin(\varphi(s)), \quad (54)$$

where $c = \tau_{\beta_o} \rangle 0$ and

$$\begin{aligned}\varphi(s) &= \pm\int\left(\varepsilon k_{\beta_o} - k_g\right)ds\text{ if }k_n\neq 0\text{ or }\varphi(s)\\ &= \pm\int\left(\varepsilon k_{\beta_o} - k_g\right)ds + \frac{\pi}{2}\text{ if }\tau_g\neq 0.\end{aligned} \quad (55)$$

Proof. We take

$$c = \tau_{\beta_o} = \sqrt{k_n^2+\tau_g^2}\rangle 0. \quad (56)$$

By differentiating (56), we obtain

$$k_nk_n' + \tau_g\tau_g' = 0. \quad (57)$$

On the other hand, we have

(i) If $k_n\neq 0$, by multiplying (58) by (k_n/k_n), and using (57), we get

$$k_{\beta_o} = \varepsilon\left(\frac{\tau_g'}{k_n} + k_g\right). \tag{59}$$

Since $(\tau_g'/k_n) = \pm((\tau_g'/c)/\sqrt{1-(\tau_g/c)^2})$, we can reformulate the given expression of k_{β_o} as

$$k_{\beta_o} = \pm\varepsilon\frac{(\tau_g'/c)}{\sqrt{1-(\tau_g/c)^2}} + \varepsilon k_g \tag{60}$$

or equivalently

$$\varepsilon k_{\beta_o} - k_g = \pm\frac{(\tau_g'/c)}{\sqrt{1-(\tau_g/c)^2}}. \tag{61}$$

By integrating (61), we obtain

$$\tau_g = c\sin\left(\pm\int\left(\varepsilon k_{\beta_o} - k_g\right)ds\right), \tag{62}$$

so immediately, we find

$$k_n = c\cos\left(\pm\int\left(\varepsilon k_{\beta_o} - k_g\right)ds\right). \tag{63}$$

(ii) In a similar way, we make sure of the result for $\tau_g \neq 0$. Then, by multiplying (58) by (τ_g/τ_g) and using (57), we get

$$k_{\beta_o} = \varepsilon\left(\frac{-k_n'}{\tau_g} + k_g\right). \tag{64}$$

Since $(k_n'/\tau_g) = \pm((k_n'/c)/\sqrt{1-(k_n/c)^2})$, we can reformulate the given expression of k_{β_o} as

$$k_{\beta_o} = \varepsilon\left(\mp\frac{(k_n'/c)}{\sqrt{1-(k_n/c)^2}} + k_g\right) \tag{65}$$

or equivalently

$$\varepsilon k_{\beta_o} - k_g = \mp\frac{(k_n'/c)}{\sqrt{1-(k_n/c)^2}}. \tag{66}$$

By integrating (66), we obtain

$$k_n = c\sin\left(\mp\int\left(\varepsilon k_{\beta_o} - k_g\right)ds\right), \tag{67}$$

so immediately, we find

$$\tau_g = c\cos\left(\mp\int\left(\varepsilon k_{\beta_o} - k_g\right)ds\right). \tag{68}$$

By taking $\varphi(s) = \pm\int(\varepsilon k_{\beta_o} - k_g)ds$, we get the result as desired.

Similarly, from formula (21), we can conclude that if there exists a function $\varphi(s)$ such that the geodesic curvature k_g and the geodesic torsion τ_g of the curve α satisfy

$$k_g = c\cos(\varphi(s)) \text{ and } \tau_g = c\sin(\varphi(s)), \tag{69}$$

where c is a constant, then the $\overline{D}_r$-direction curve of the curve α has a constant torsion.

Conversely, we can prove the following theorem, using the same reasoning as Theorem 10. ☐

Theorem 11. *Let $\alpha = \alpha(s)$ be a unit speed regular curve lying on a regular surface M, such that $\delta_r \neq 0$ and $\beta_r = \beta_r(s)$ its $\overline{D}_r$-direction curve. If the curve β_r has a constant torsion, then the geodesic curvature k_g and the geodesic torsion τ_g of α satisfy the following equalities:*

$$k_g = c\cos(\varphi) \text{ and } \tau_g = c\sin(\varphi), \tag{70}$$

where $c = \tau_{\beta_r} > 0$ and

$$\varphi = \pm\int\left(\varepsilon k_{\beta_r} + k_n\right)ds \text{ if } k_g \neq 0, \text{ or } \varphi = \pm\int\left(\varepsilon k_{\beta_r} + k_n\right)ds + \frac{\pi}{2} \text{ if } \tau_g \neq 0. \tag{71}$$

Using formula (28), we can state the following theorem without proof.

Theorem 12. *Let $\alpha = \alpha(s)$ be a unit speed regular curve lying on a regular surface M, with torsion $\tau \neq 0$ and $\beta_n = \beta_n(s)$ its $\overline{D}_n$-direction curve. β_n has a constant torsion if and only if the curvature k of α is constant.*

4. Necessary and Sufficient Condition for the Darboux Invariant δ_o (Resp., δ_r and δ_n) to Be Zero

In Section 3, we defined the direction curve β_o (resp., β_r and β_n), when it exists, and we have given its Frenet frame as well as some properties. All this work was done under the condition $\delta_o \neq 0$ (resp., $\delta_r \neq 0$ and $\delta_n \neq 0$), that is, it was assumed that the curvature of β_o (resp., β_r and β_n) is nonzero. In this section, we propose to study the special case where δ_o (resp., δ_r and δ_n) is zero.

Theorem 13. *Let $\alpha = \alpha(s)$ be a unit speed regular curve lying on a regular surface M, and $\beta_o = \beta_o(s)$ its $\overline{D}_o$-direction curve. Then, $\delta_o = 0$ if and only if the position vector of β_o always lies in the osculating plane of α.*

Proof. Suppose that $\delta_o = 0$, then from (15), we get

$$\overline{D}_o = cte. \tag{72}$$

Consequently, $\langle\beta_o, U\rangle = \langle\int\overline{D}_o ds, U\rangle = \langle\overline{D}_o\int ds, U\rangle = \int ds\langle\overline{D}_o, U\rangle = 0.$

Conversely, suppose that

$$\beta_o = \lambda T + \mu V, \tag{73}$$

where λ and μ are differentiable functions. Since β_o is the $\overline{D}_o$-direction curve of α, by differentiating (73) with respect to s, we obtain

$$\beta_o' = \frac{\tau_g T - k_n V}{\sqrt{k_n^2 + \tau_g^2}} = (\lambda' - \mu k_g) T + (\mu' + \lambda k_g) V + (\mu \tau_g + \lambda k_n) U, \tag{74}$$

which gives us the following system:

$$\begin{cases} \lambda' - \mu k_g = \dfrac{\tau_g}{\sqrt{k_n^2 + \tau_g^2}} \\ \mu' + \lambda k_g = \dfrac{-k_n}{\sqrt{k_n^2 + \tau_g^2}} \\ \mu \tau_g + \lambda k_n = 0. \end{cases} \tag{75}$$

In this case, we assume that $(\tau_g, k_n) \neq (0,0)$. If $\tau_g \neq 0$ (we notice that the result is the same if $k_n \neq 0$), from the third equation of (75), we have

$$\mu = -\frac{k_n}{\tau_g} \lambda. \tag{76}$$

By replacing (76) in the first equation of system (75), we obtain the following differential equation:

$$\lambda' + \frac{k_n k_g}{\tau_g} \lambda = \frac{\tau_g}{\sqrt{k_n^2 + \tau_g^2}}. \tag{77}$$

We solve the differential equation, and we find

$$\lambda(x) = \left[\int \frac{\tau_g}{\sqrt{k_n^2 + \tau_g^2}} \exp\left(\int \frac{k_n k_g}{\tau_g} dx \right) dx + K \right] \exp\left(-\int \frac{k_n k_g}{\tau_g} dx \right), \tag{78}$$

where K is an integration constant. So, from (76), we write

$$\mu(s) = -\frac{k_n}{\tau_g} \left[\int \frac{\tau_g}{\sqrt{k_n^2 + \tau_g^2}} \exp\left(\int \frac{k_n k_g}{\tau_g} ds \right) ds + K \right] \exp\left(-\int \frac{k_n k_g}{\tau_g} ds \right), \tag{79}$$

and by differentiating (79), we obtain

$$\mu'(s) = \left[-\left(\frac{k_n}{\tau_g}\right)' + \left(\frac{k_n}{\tau_g}\right)^2 k_g \right] \left[\int \frac{\tau_g}{\sqrt{k_n^2 + \tau_g^2}} \exp\left(\int \frac{k_n k_g}{\tau_g} ds \right) ds + K \right] \exp\left(-\int \frac{k_n k_g}{\tau_g} ds \right) - \frac{k_n}{\sqrt{k_n^2 + \tau_g^2}}. \tag{80}$$

Therefore, by replacing (78) and (80) in the second equation of system (75), we find

$$\left[\left(\frac{k_n}{\tau_g}\right)' - k_g \left(1 + \left(\frac{k_n}{\tau_g}\right)^2 \right) \right] \left[\int \frac{\tau_g}{\sqrt{k_n^2 + \tau_g^2}} \exp\left(\int \frac{k_n k_g}{\tau_g} ds \right) ds + K \right] \exp\left(-\int \frac{k_n k_g}{\tau_g} ds \right) = 0. \tag{81}$$

Hence $\delta_o = 0$, which completes the proof.

By doing the same, we get the following two theorems. □

Theorem 14. *Let $\alpha = \alpha(s)$ be a unit speed regular curve lying on a regular surface M, and $\beta_r = \beta_r(s)$ its $\overline{D}_r$-direction curve. Then, $\delta_r = 0$ if and only if the position vector of β_r always lies in the rectifying plane of α.*

Theorem 15. *Let $\alpha = \alpha(s)$ be a unit speed regular curve lying on a regular surface M and $\beta_n = \beta_n(s)$ its $\overline{D}_n$-direction curve. Then, $\tau = 0$ if and only if the position vector of β_n always lies in the normal plane of α.*

5. Examples

Example 1. Let ${}^1\alpha(s) = (-(3/2)\cos(s/2) - (1/6)\cos(3s/2), -(3/2)\sin(s/2) - (1/6)\sin(3s/2), \sqrt{3}\cos(s/2))$ be a curve lying on the surface ${}^1M$ given by the following parametrization (see Figure 1):

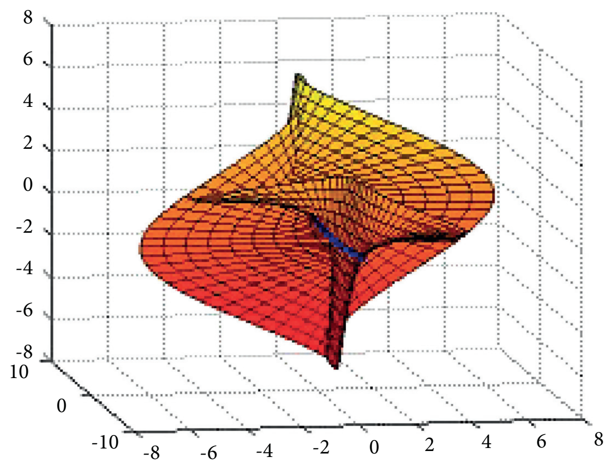

Figure 1: The curve ${}^1\alpha$ (blue) lying on the surface ${}^1M$.

$$
{}^1\varphi(u,v) = \begin{pmatrix} -\frac{3}{2}\cos\left(\frac{u}{2}\right) - \frac{1}{6}\cos\left(\frac{3u}{2}\right) + v\left(\frac{3}{4}\sin\left(\frac{u}{2}\right) + \frac{1}{4}\sin\left(\frac{3u}{2}\right) - \frac{1}{2}\cos\left(\frac{u}{2}\right)\left(2\cos^2\left(\frac{u}{2}\right) - 3\right)\right) \\ -\frac{3}{2}\sin\left(\frac{u}{2}\right) - \frac{1}{6}\sin\left(\frac{3u}{2}\right) + v\left(-\frac{3}{4}\cos\left(\frac{u}{2}\right) - \frac{1}{4}\cos\left(\frac{3u}{2}\right) + \sin^3\left(\frac{u}{2}\right)\right) \\ \sqrt{3}\cos\left(\frac{u}{2}\right) + v\left(-\frac{\sqrt{3}}{2}\sin\left(\frac{u}{2}\right) + \frac{\sqrt{3}}{2}\cos\left(\frac{u}{2}\right)\right) \end{pmatrix}. \tag{82}
$$

The Darboux frame of ${}^1\alpha$ is

$$
\begin{cases} {}^1T(s) = \left(\frac{3}{4}\sin\left(\frac{s}{2}\right) + \frac{1}{4}\sin\left(\frac{3s}{2}\right), -\frac{3}{4}\cos\left(\frac{s}{2}\right) - \frac{1}{4}\cos\left(\frac{3s}{2}\right), -\frac{\sqrt{3}}{2}\sin\left(\frac{s}{2}\right)\right) \\ {}^1V(s) = \left(\frac{1}{2}\cos\left(\frac{s}{2}\right)\left(2\cos^2\left(\frac{s}{2}\right) - 3\right), -\sin^3\left(\frac{s}{2}\right), -\frac{\sqrt{3}}{2}\cos\left(\frac{s}{2}\right)\right) \\ {}^1U(s) = \left(\frac{-\sqrt{3}}{\cos(s/2)}\left(\frac{1}{4}\cos\left(\frac{s}{2}\right) + \frac{1}{4}\cos\left(\frac{3s}{2}\right)\right), \frac{-\sqrt{3}}{\cos(s/2)}\left(\frac{1}{4}\sin\left(\frac{s}{2}\right) + \frac{1}{4}\sin\left(\frac{3s}{2}\right)\right), \frac{1}{2}\right). \end{cases} \tag{83}
$$

We can notice that α^1 is an isophote curve, and according to Corollary 1, β_o^1 is a general helix.

We have ${}^1\tau_g = (\sqrt{3}/2)\sin(s/2)$, ${}^1k_n = -(\sqrt{3}/2)\cos(s/2)$, and ${}^1k_g = 0$. It follows

$$
\begin{aligned} {}^1\overline{D}_o &= \left(-\frac{1}{2}\cos(s), -\frac{1}{2}\sin(s), -\frac{\sqrt{3}}{2}\right), \\ {}^1\overline{D}_r &= \left(\frac{3}{4}\sin\left(\frac{s}{2}\right) + \frac{1}{4}\sin\left(\frac{3s}{2}\right), -\frac{3}{4}\cos\left(\frac{s}{2}\right) - \frac{1}{4}\cos\left(\frac{3s}{2}\right), -\frac{\sqrt{3}}{2}\sin\left(\frac{s}{2}\right)\right), \\ {}^1\overline{D}_n &= \left(-\frac{1}{2}\cos\left(\frac{s}{2}\right)\left(2\cos^2\left(\frac{s}{2}\right) - 3\right), \sin^3\left(\frac{s}{2}\right), \frac{\sqrt{3}}{2}\cos\left(\frac{s}{2}\right)\right). \end{aligned} \tag{84}
$$

Consequently, the ${}^1\overline{D}_o$-direction curve, ${}^1\overline{D}_r$-direction curve, and ${}^1\overline{D}_n$-direction curve are given, respectively, by ${}^1\beta_o$, ${}^1\beta_r$, and ${}^1\beta_n$ as follows (see Figures 1–3):

$$\begin{aligned} {}^1\beta_o(s) &= \left(-\frac{1}{2}\sin(s)+c_1, \frac{1}{2}\cos(s)+c_2, -\frac{\sqrt{3}}{2}s+c_3\right), \\ {}^1\beta_r(s) &= \left(-\frac{3}{2}\cos\left(\frac{s}{2}\right)-\frac{1}{6}\cos\left(\frac{3s}{2}\right)+c_4, -\frac{3}{2}\sin\left(\frac{s}{2}\right)-\frac{1}{6}\sin\left(\frac{3s}{2}\right)+c_5, \sqrt{3}\cos\left(\frac{s}{2}\right)+c_6\right), \\ {}^1\beta_n(s) &= \left(\sin\left(\frac{s}{2}\right)\left(1+\frac{2}{3}\sin^2\left(\frac{s}{2}\right)\right)+c_7, -2\cos\left(\frac{s}{2}\right)\left(1-\frac{1}{3}\cos^2\left(\frac{s}{2}\right)\right)+c_8, \sqrt{3}\sin\left(\frac{s}{2}\right)+c_9\right), \end{aligned} \tag{85}$$

where c_i, $i = 1, 2, \ldots 9$ are real constants.

Example 2. Let ${}^2\alpha(s) = ((s/2)\cos(\sqrt{2}\ln(s/2)), (s/2)\sin(\sqrt{2}\ln(s/2)), (s/2))$ be a curve lying on the surface ${}^2M$ given by the following parametrization (see Figure 4):

$${}^2\varphi(u,v) = (u\cos v, u\sin v, u). \tag{86}$$

The Darboux frame of ${}^2\alpha$ is

$$ {}^2T = \begin{pmatrix} \frac{1}{2}\cos\left(\sqrt{2}\ln\frac{s}{2}\right)-\frac{1}{\sqrt{2}}\sin\left(\sqrt{2}\ln\frac{s}{2}\right) \\ \frac{1}{2}\sin\left(\sqrt{2}\ln\frac{s}{2}\right)+\frac{1}{\sqrt{2}}\cos\left(\sqrt{2}\ln\frac{s}{2}\right) \\ \frac{1}{2} \end{pmatrix}, $$

$$ {}^2V = \begin{pmatrix} -\frac{1}{\sqrt{2}}\sin\left(\sqrt{2}\ln\frac{s}{2}\right)-\frac{1}{2}\cos\left(\sqrt{2}\ln\frac{s}{2}\right) \\ \frac{1}{\sqrt{2}}\cos\left(\sqrt{2}\ln\frac{s}{2}\right)-\frac{1}{2}\sin\left(\sqrt{2}\ln\frac{s}{2}\right) \\ -\frac{1}{2} \end{pmatrix}, \tag{87}$$

$$ {}^2U = \begin{pmatrix} -\frac{1}{\sqrt{2}}\cos\left(\sqrt{2}\ln\frac{s}{2}\right) \\ -\frac{1}{\sqrt{2}}\sin\left(\sqrt{2}\ln\frac{s}{2}\right) \\ \frac{1}{\sqrt{2}} \end{pmatrix}. $$

In this example, the curve α is a general helix, a relatively normal slant helix, and an isophote curve. Therefore, according to Corollary 1, the curves ${}^2\beta_o$, ${}^2\beta_r$, and ${}^2\beta_n$ are general helices.

We have ${}^2k_n = (1/\sqrt{2}s)$, ${}^2k_g = (1/s)$, and ${}^2\tau_g = (1/\sqrt{2}s)$. It follows

$$ {}^2\overline{D}_o = \begin{pmatrix} \frac{1}{\sqrt{2}}\cos\left(\sqrt{2}\ln\frac{s}{2}\right) \\ \frac{1}{\sqrt{2}}\sin\left(\sqrt{2}\ln\frac{s}{2}\right) \\ \frac{1}{\sqrt{2}} \end{pmatrix}, $$

$$ {}^2\overline{D}_r = \begin{pmatrix} -\frac{1}{\sqrt{6}}\sin\left(\sqrt{2}\ln\frac{s}{2}\right)-\frac{1}{2\sqrt{3}}\cos\left(\sqrt{2}\ln\frac{s}{2}\right) \\ \frac{1}{\sqrt{6}}\cos\left(\sqrt{2}\ln\frac{s}{2}\right)-\frac{1}{2\sqrt{3}}\sin\left(\sqrt{2}\ln\frac{s}{2}\right) \\ \frac{\sqrt{3}}{2} \end{pmatrix}, \tag{88}$$

$$ {}^2\overline{D}_n = \begin{pmatrix} \frac{1}{\sqrt{6}}\sin\left(\sqrt{2}\ln\frac{s}{2}\right)-\frac{1}{2\sqrt{3}}\cos\left(\sqrt{2}\ln\frac{s}{2}\right) \\ -\frac{1}{\sqrt{6}}\cos\left(\sqrt{2}\ln\frac{s}{2}\right)-\frac{1}{2\sqrt{3}}\sin\left(\sqrt{2}\ln\frac{s}{2}\right) \\ \frac{\sqrt{3}}{2} \end{pmatrix}. $$

Consequently, the ${}^2\overline{D}_o$-direction curve, ${}^2\overline{D}_r$-direction curve, and ${}^2\overline{D}_n$-direction curve are given, respectively, by ${}^2\beta_o$, ${}^2\beta_r$, and ${}^2\beta_n$ as follows (see Figures 5–7):

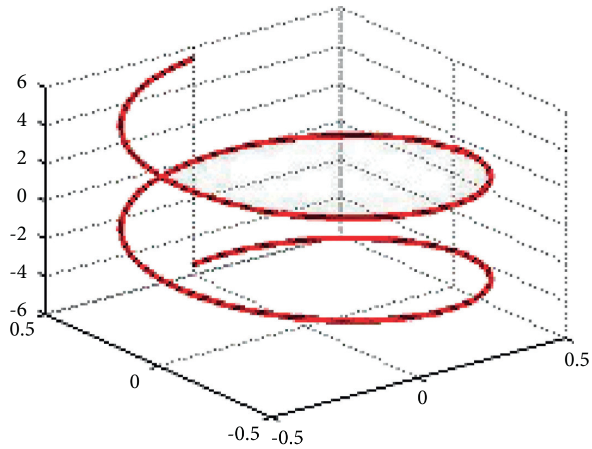

Figure 2: ${}^{1}\overline{D}_{o}$-direction curve of ${}^{1}\alpha$.

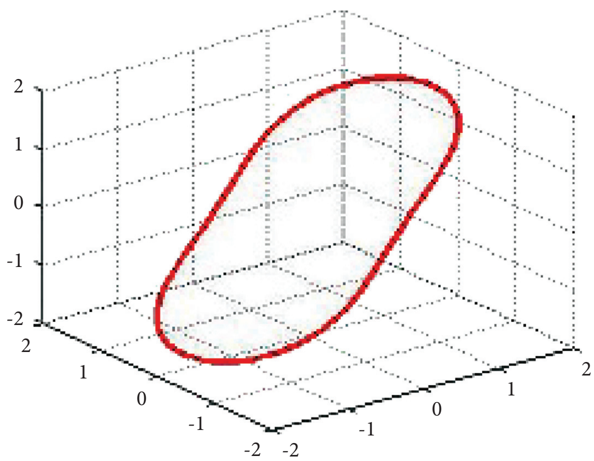

Figure 3: ${}^{1}\overline{D}_{n}$-direction curve of ${}^{1}\alpha$.

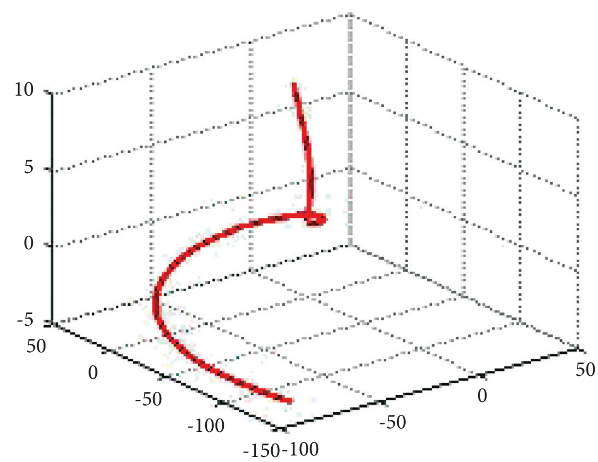

Figure 5: ${}^{2}\overline{D}_{o}$-direction curve of ${}^{2}\alpha$.

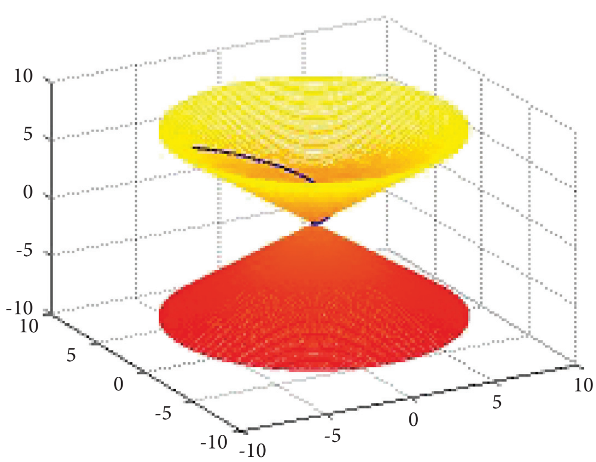

Figure 4: The curve ${}^{2}\alpha$ (blue) lying on the surface ${}^{2}M$.

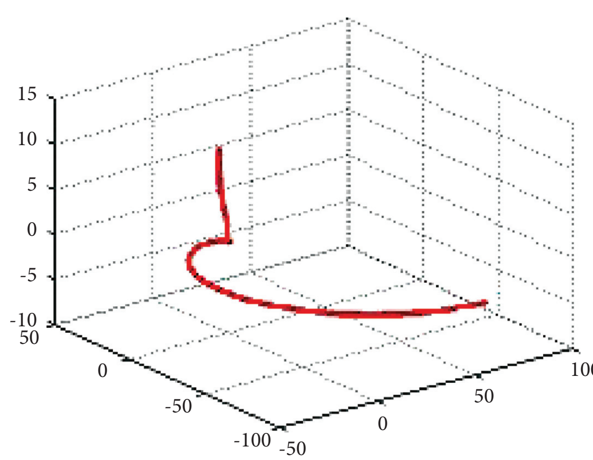

Figure 6: ${}^{2}\overline{D}_{r}$-direction curve of ${}^{2}\alpha$.

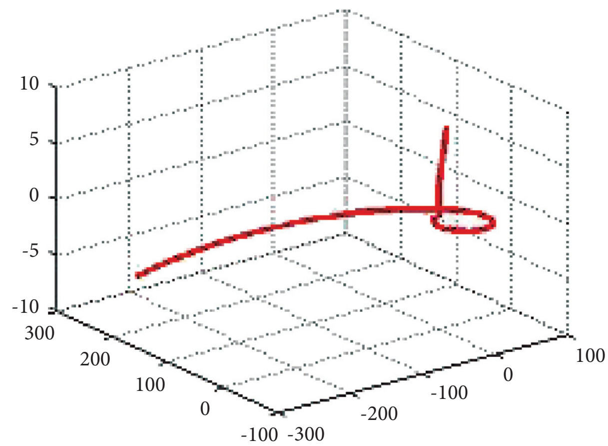

Figure 7: ${}^2\overline{D}_n$-direction curve of ${}^2\alpha$.

$$\begin{aligned}
{}^2\beta_o &= \left(\frac{s}{3}\sin\left(\sqrt{2}\ln\frac{s}{2}\right) + \frac{s}{3\sqrt{2}}\cos\left(\sqrt{2}\ln\frac{s}{2}\right) + c_{10}, \frac{s}{3\sqrt{2}}\sin\left(\sqrt{2}\ln\frac{s}{2}\right) - \frac{s}{3}\cos\left(\sqrt{2}\ln\frac{s}{2}\right) + c_{11}, \frac{s}{\sqrt{2}} + c_{12}\right), \\
{}^2\beta_r &= \left(-\frac{2s}{3\sqrt{6}}\sin\left(\sqrt{2}\ln\frac{s}{2}\right) + \frac{s}{6\sqrt{3}}\cos\left(\sqrt{2}\ln\frac{s}{2}\right) + c_{13}, \frac{s}{6\sqrt{3}}\sin\left(\sqrt{2}\ln\frac{s}{2}\right) + \frac{2s}{3\sqrt{6}}\cos\left(\sqrt{2}\ln\frac{s}{2}\right) + c_{14}, \frac{\sqrt{3}\,s}{2} + c_{15}\right), \\
{}^2\beta_n &= \left(\frac{-s}{2\sqrt{3}}\cos\left(\sqrt{2}\ln\frac{s}{2}\right) + c_{16}, \frac{-s}{2\sqrt{3}}\sin\left(\sqrt{2}\ln\frac{s}{2}\right) + c_{17}, \frac{\sqrt{3}\,s}{2} + c_{18}\right),
\end{aligned} \tag{89}$$

where c_i, $i = 10, 11, \ldots, 18$, is a real constant.

Data Availability

No data were used to support the study.

Conflicts of Interest

The authors declare no conflicts of interest.

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