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Research Article

Multivalent Functions Related with an Integral Operator

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In this present paper, we introduce and explore certain new classes of uniformly convex and starlike functions related to the Liu–Owa integral operator. We explore various properties and characteristics, such as coefficient estimates, rate of growth, distortion result, radii of close-to-convexity, starlikeness, convexity, and Hadamard product. It is important to mention that our results are a generalization of the number of existing results in the literature.

1. Introduction

Let $\mathbb{C}$ denote the complex plane and assume that A_p denotes the class of p-valent function of the form

$$\lambda(\omega) = \omega^p + \sum_{t=1}^{\infty} a_{t+p}\omega^{t+p} \quad (p \in \mathbb{N} = \{1, 2, \ldots\}), \tag{1}$$

which are analytic in the open unit disc $\mathbb{U} = \{\omega\colon \omega \in \mathbb{C}$ and $|\omega| < 1$. Specially, for $p = 1$, we denote $A=A_1$.

By U, K, and S, the subclasses of A_p consist of all univalent, convex, and starlike functions $S(\alpha)$. We also denote $S(\alpha)$, the class of starlike function of order α, $\alpha \in [0 \;\; 1)$. In 1991, Goodman [1, 2] introduced the classes UST and UCV of uniformly starlike and uniformly convex functions, respectively. A function λ is uniformly starlike (uniformly convex) in $\mathbb{U}$ if λ is in UST (UCV) and has the property that, for every circular arc γ contained in $\mathbb{U}$, with center ζ also in $\mathbb{U}$, the arc $\lambda(\gamma)$ is starlike (convex) with respect to $\lambda(\zeta)$. A more useful representation of UST and UCV was given in [3]; see [4, 5], for details:

$$\lambda \in \text{UST} \Leftrightarrow \left|\frac{\omega\lambda'(\omega)}{\lambda(\omega)} - 1\right| \le \Re\left(\frac{\omega\lambda'(\omega)}{\lambda(\omega)}\right), \quad (\omega \in \mathbb{U}), \tag{2}$$

and

$$\lambda \in \text{UCV} \Leftrightarrow \left|\frac{\omega\lambda''(\omega)}{\lambda'(\omega)}\right| \le \Re\left(1 + \frac{\omega\lambda''(\omega)}{\lambda'(\omega)}\right), \quad (\omega \in \mathbb{U}). \tag{3}$$

In 1999, for $k \ge 0$, Kanas and Wisniowska [6] introduced the classes $k-\text{UST}$ and $k-\text{UCV}$ of k-uniformly convex and k–uniformly starlike functions, respectively, see also [7–10].

Let $k-\text{UST}(\alpha, \beta)$ denote the subclass of A_p consisting of functions of form (1) and satisfy the following inequality:

$$\Re\left(\frac{\omega\lambda'(\omega)}{\lambda(\omega)} - \alpha\right) > k\left|\frac{\omega\lambda'(\omega)}{\lambda(\omega)} - \beta\right|, \quad (0 \le \alpha < \beta \le 1; k(1-\beta) < (1-\alpha);\ \omega \in \mathbb{U}). \tag{4}$$

Also, let $k-\mathrm{UCV}(\alpha,\beta)$ denote the subclass of A_p consisting of functions of form (1) and satisfy the following inequality:

$$\Re\left(1+\frac{\omega\lambda''(\omega)}{\lambda'(\omega)}-\alpha\right)>k\left|1+\frac{\omega\lambda''(\omega)}{\lambda'(\omega)}-\beta\right|,\quad (0\le\alpha<\beta\le 1; k(1-\beta)<(1-\alpha);\ \omega\in\mathbb{U}). \tag{5}$$

It follows from (4) and (5) that

$$k-\mathrm{UCV}(\alpha,\beta)\Leftrightarrow\omega\lambda'\in k-\mathrm{UST}(\alpha,\beta). \tag{6}$$

Notice that, $k-\mathrm{UST}(\alpha,0)=S(\alpha)$ and $k-\mathrm{UCV}(\alpha,0)=K(\alpha)$, for $k=0$. The convolution (Hadamard product) for two functions λ, $\delta\in A_p$, is defined by

$$\lambda(\omega)*\delta(\omega)=\omega^p+\sum_{t=0}^{\infty}a_{t+p}b_{t+p}\omega^{t+p}, \tag{7}$$

where λ is given in (1) and $\delta(\omega)=\omega^p+\sum_{t=0}^{\infty}b_{t+p}\omega^{t+p}$.

Taking from the above cited work and using Liu–Owa integral operator, we introduce the following class of p-valent analytic function. In 2004, Liu and Owa [11] (see also [12–14]) introduced the integral operator $Q^a_{b,p}$: $A_p\longrightarrow A_p$ as follows:

$$Q^a_{b,p}\lambda(\omega)=\begin{pmatrix}p+a+b-1\\ p+b-1\end{pmatrix}\frac{a}{\omega^b}\int_0^{\omega}\left(1-\frac{x}{\omega}\right)^{a-1}x^{b-1}\lambda(x)\mathrm{d}x\quad (a>0; b>-1; p\in\mathbb{N}) \tag{8}$$

and

$$Q^0_{b,p}\lambda(\omega)=\lambda(\omega)\quad (a=0; b>-1). \tag{9}$$

For $\lambda\in A_p$, given by (1), and using properties of gamma function, we have

$$Q^a_{b,p}\lambda(\omega)=\omega^p+\frac{\Gamma(a+b+p)}{\Gamma(b+p)}\sum_{t=1}^{\infty}\frac{\Gamma(b+p+t)}{\Gamma(a+b+p+t)}a_{t+p}\omega^{t+p}\quad (a\ge 0; b>-1; p\in\mathbb{N}). \tag{10}$$

Definition 1. For $0\le\alpha<\beta\le 1$, $0\le\nu<1$, $k(1-\beta)<(1-\alpha)$, and $0\le\mu<1$, a function $\lambda\in A_p$ is in class $k-U(a,b,p,\alpha,\beta,\mu,\nu)$ if and only if

$$\Re\left[\frac{(1-\nu)\omega\left(Q^a_{b,p}\lambda(\omega)\right)'+\nu\left\{\begin{array}{c}\omega\left(Q^a_{b,p}\lambda(\omega)\right)'+(1+2\mu)\omega^2\left(Q^a_{b,p}\lambda(\omega)\right)''\\ +\mu\omega^3\left(Q^a_{b,p}\lambda(\omega)\right)'''\end{array}\right\}}{(1-\nu)\left(Q^a_{b,p}\lambda(\omega)\right)+\nu\left\{\omega\left(Q^a_{b,p}\lambda(\omega)\right)'+\mu\omega^2\left(Q^a_{b,p}\lambda(\omega)\right)''\right\}}-\alpha\right]$$
$$\ge k\left|\frac{\begin{array}{c}(1-\nu)\omega\left(Q^a_{b,p}\lambda(\omega)\right)'+\nu\omega\left(Q^a_{b,p}\lambda(\omega)\right)'+(1+2\mu)\omega^2\left(Q^a_{b,p}\lambda(\omega)\right)''\\ +\mu\omega^3\left(Q^a_{b,p}\lambda(\omega)\right)'''\end{array}}{(1-\nu)\left(Q^a_{b,p}\lambda(\omega)\right)+\nu\left\{\left(\omega\left(Q^a_{b,p}\lambda(\omega)\right)\right)'+\mu\omega^2\left(Q^a_{b,p}\lambda(\omega)\right)''\right\}}-\beta\right|. \tag{11}$$

We also denote $k-£U_\eta(a,b,p,\alpha,\beta,\mu,\nu)=k-U(a,b,p,\alpha,\beta,\mu,\nu)\cap£_\eta$, where $£_\eta$ the class of functions $\lambda\in A_p$ of form (1) for which $\arg(a_t)=\pi+(n-1)\eta$. For more details, see [15–20].

1.1. Special Cases. Specializing parameters, $a, b, p, \alpha, \beta, \mu$, and ν, we obtain the following subclasses studied by various authors:

(1) $k - U(0, b, p, \alpha, \beta, \mu, 1) = k - U(\alpha, \beta, \mu)$ [21]
(2) $k - £U_0(0, b, p, \alpha, \beta, \mu, 1) = k - £U_\eta(\alpha, \beta, \mu, \nu)$ [21]
(3) $0 - £U_0(0, b, p, \alpha, 1, 0, 1) = \mathrm{CV}(\alpha)$ [22]
(4) $k - £U_0(0, b, p, \alpha, 1, 0, 1) = k - \mathrm{UCV}(\alpha)$ [23]
(5) $1 - U(0, b, p, \alpha, 1, 0, 1) = \mathrm{UCV}(\alpha)$ [24]
(6) $k - U(0, b, p, \alpha, \beta, 0, 0) = k - \mathrm{UST}(\alpha, \beta)$ [25]

2. Main Results for the Class $k - U(a, b, p, \alpha, \beta, \mu, \nu)$

2.1. Coefficient Estimates. In this section, we obtain a necessary and sufficient condition for functions $\lambda(\omega)$ in the classes $k - £U_\eta(a, b, p, \alpha, \beta, \mu, \nu)$.

Theorem 1. *A function $\lambda(\omega)$ given by (1) is in the class $k - U(a, b, p, \alpha, \beta, \mu, \nu)$ if*

$$\frac{\Gamma(a+b+p)}{\Gamma(b+p)}\sum_{t=1}^{\infty}\left[D_{t+p}(1+k) - C_{t+p}(\alpha+k\beta)\right]\frac{\Gamma(b+p+t)}{\Gamma(a+b+p+t)}\left|a_{t+p}\right| \le C_p(1+k) - D_p(\alpha+k\beta), \tag{12}$$

where

$$C_{t+p} = 1 - \nu + \nu(t+p)(1+\mu(t+p-1)), \tag{13}$$

$$D_{t+p} = t + p + \nu(t+p)(t+p-1)(1+\mu(t+p)), \tag{14}$$

$$C_p = p + \nu p(p-1)(1+\mu p), \tag{15}$$

$$D_p = 1 - \nu + \nu p(1+\mu(p-1)), \tag{16}$$

and

$$-1 \le \alpha < \beta \le 1, 0 \le \mu < 1, k(1-\beta) < 1-\alpha, a \ge 0, b > -1, \quad p \in \mathbb{N} \text{ and } \omega \in \mathbb{U}. \tag{17}$$

Proof. It suffices to show that inequality (11) holds true. As we know,

$$\mathfrak{R}(\varpi) > k|\varpi - \beta| + \alpha \Leftrightarrow \mathfrak{R}\left[\left(1+ke^{i\theta}\right)\varpi - \beta k e^{i\theta}\right] > \alpha. \tag{18}$$

Then, inequality (11) may be written as

$$\mathfrak{R}\left[\begin{array}{c}\left(1+ke^{i\theta}\right)\left\{\dfrac{(1-\nu)\omega\left(Q^a_{b,p}\lambda(\omega)\right)' + \nu\omega\left(Q^a_{b,p}\lambda(\omega)\right)' + (1+2\mu)\omega^2\left(Q^a_{b,p}\lambda(\omega)\right)'' + \mu\omega^3\left(Q^a_{b,p}\lambda(\omega)\right)'''}{\left((1-\nu\left(Q^a_{b,p}\lambda(\omega)\right) + \nu\left(\omega\left(Q^a_{b,p}\lambda(\omega)\right)' + \mu\omega^2\left(Q^a_{b,p}\lambda(\omega)\right)''\right)\right.}\right\} \\ -\beta k e^{i\theta}\end{array}\right] \ge \alpha, \tag{19}$$

which can be written as $\mathfrak{R}(A(\omega)/B(\omega)) \ge \alpha$, where

$$A(\omega) = \left(1+ke^{i\theta}\right)\left\{\begin{array}{c}(1-\nu)\omega\left(Q^a_{b,p}\lambda(\omega)\right)' + \nu\omega\left(Q^a_{b,p}\lambda(\omega)\right)' \\ +(1+2\mu)\omega^2\left(Q^a_{b,p}\lambda(\omega)\right)'' + \mu\omega^3\left(Q^a_{b,p}\lambda(\omega)\right)'''\end{array}\right\} - \beta k e^{i\theta}\left[(1-\nu)Q^a_{b,p}\lambda(\omega) + \nu\left\{\omega\left(Q^a_{b,p}\lambda(\omega)\right)' + \mu\omega^2\left(Q^a_{b,p}\lambda(\omega)\right)''\right\}\right] \tag{20}$$

and

$$B(\omega) = (1-\nu)\left(Q^a_{b,p}\lambda(\omega)\right) + \nu\left\{\omega\left(Q^a_{b,p}\lambda(\omega)\right)' + \mu\omega^2\left(Q^a_{b,p}\lambda(\omega)\right)''\right\}. \tag{21}$$

Then, we have

$$|A(\omega) + (1-\alpha)B(\omega)| - |A(\omega) - (1+\alpha)B(\omega)| \ge 0. \tag{22}$$

Now,

$$\begin{aligned}
&|A(\omega)+(1-\alpha)B(\omega)|\Bigg|\left[C_p+(1-\alpha)D_p+ke^{i\theta}\left(C_p-\beta D_p\right)\right]\omega^p-\frac{\Gamma(a+b+p)}{\Gamma(b+p)}\\
&\times\sum_{t=1}^{\infty}\left[ke^{i\theta}\left(\beta C_{t+p}-D_{t+p}\right)-D_{t+p}-(1-\alpha)C_{t+p}\right]\times\frac{\Gamma(b+p+t)}{\Gamma(a+b+p+t)}a_{t+p}\omega^{t+p}\Bigg|\\
&\geq\left[ke^{i\theta}\left(\beta D_p-C_p\right)-C_p-(1-\alpha)D_p\right]\omega^p-\frac{\Gamma(a+b+p)}{\Gamma(b+p)}\\
&\times\sum_{t=1}^{\infty}\left[ke^{i\theta}\left(\beta C_{t+p}-D_{t+p}\right)-D_{t+p}-(1-\alpha)C_{t+p}\right]\times\frac{\Gamma(b+p+t)}{\Gamma(a+b+p+t)}\left|a_{t+p}\right|\left|\omega^{t+p}\right|.
\end{aligned}\tag{23}$$

Also,

$$\begin{aligned}
|A(\omega)-(1+\alpha)B(\omega)|&=\Bigg|\left[C_p-(1+\alpha)D_p+ke^{i\theta}\left(C_p-\beta D_p\right)\right]\omega^p+\frac{\Gamma a+b+p}{\Gamma b+p}\\
&\times\sum_{t=1}^{\infty}\left[ke^{i\theta}\left(\beta C_{t+p}-D_{t+p}\right)-D_{t+p}+(1+\alpha)C_{t+p}\right]\times\frac{\Gamma b+p+t}{\Gamma a+b+p+t}a_{t+p}\omega^{t+p}\Bigg|\\
&\leq\left[ke^{i\theta}\left(C_p-\beta D_p\right)+C_p-(1+\alpha)D_p\right]\omega^p-\frac{\Gamma a+b+p}{\Gamma b+p}\\
&\times\sum_{t=1}^{\infty}\left[ke^{i\theta}\left(\beta C_{t+p}-D_{t+p}\right)-D_{t+p}+(1+\alpha)C_{t+p}\right]\times\frac{\Gamma b+p+t}{\Gamma a+b+p+t}\left|a_{t+p}\right|\left|\omega^{t+p}\right|.
\end{aligned}\tag{24}$$

Using (23) and (24), then we can obtain the following inequality:

$$\begin{aligned}
|A(\omega)+(1-\alpha)B(\omega)|-|A(\omega)-(1+\alpha)B(\omega)|&\geq\left[ke^{i\theta}\left\{\left(\beta D_p-C_p\right)-\left(C_p-\beta D_p\right)\right\}\right.\\
&\left.-C_p-(1-\alpha)D_p-C_p+(1+\alpha)D_p\right]\omega^p-\frac{\Gamma(a+b+p)}{\Gamma(b+p)}\\
&\times\sum_{t=1}^{\infty}\left[ke^{i\theta}\left\{\left(\beta C_{t+p}-D_{t+p}\right)-\left(\beta C_{t+p}-D_{t+p}\right)\right\}\right.\\
&\left.-D_{t+p}-(1-\alpha)C_{t+p}+D_{t+p}-(1+\alpha)C_{t+p}\right]\times\frac{\Gamma(b+p+t)}{\Gamma(a+b+p+t)}\left|a_{t+p}\right|\left|\omega^{t+p}\right|.
\end{aligned}\tag{25}$$

The last expression is bounded below by 0 if

$$\frac{\Gamma(a+b+p)}{\Gamma(b+p)}\sum_{t=1}^{\infty}\left[D_{t+p}(1+k)-C_{t+p}(\alpha+k\beta)\right]\frac{\Gamma(b+p+t)}{\Gamma(a+b+p+t)}\left|a_{t+p}\right|\leq C_p(1+k)-D_p(\alpha+k\beta),\tag{26}$$

which complete the proof. □

Theorem 2. *Let $\lambda(\omega)$ be given by (1) and in $£_\eta$; then, $\lambda \in k - £U_\eta(a, b, p, \alpha, \beta, \mu, \nu)$ if and only if*

$$\frac{\Gamma(a+b+p)}{\Gamma(b+p)}\sum_{t=1}^{\infty}\left[D_{t+p}(1+k) - C_{t+p}(\alpha+k\beta)\right]$$
$$\frac{\Gamma(b+p+t)}{\Gamma(a+b+p+t)}\left|a_{t+p}\right| \le C_p(1+k) - D_p(\alpha+k\beta), \tag{27}$$

where C_{t+p}, D_{t+p}, C_p, and D_p are given by (13)–(16), respectively.

Proof. In view of Theorem 2, we need only to show that $\lambda \in k - £U_\eta(a, b, p, \alpha, \beta, \mu, \nu)$ satisfies coefficient inequality (27). If $\lambda \in k - £U_\eta(a, b, p, \alpha, \beta, \mu, \nu)$, then, by definition, we have

$$\Re\left[\frac{\left(C_p - \alpha D_p\right) + \Gamma(a+b+p)/\Gamma(b+p)\sum_{t=1}^{\infty}\left(D_{t+p} - \alpha C_{t+p}\right)\Gamma(b+p+t)/\Gamma(a+b+p+t)\left|a_{t+p}\right|\omega^{t+p-1}}{1 + \Gamma(a+b+p)/\Gamma(b+p)\sum_{t=1}^{\infty}C_{t+p}\Gamma(b+p+t)/\Gamma(a+b+p+t)\left|a_{t+p}\right|\omega^{t+p-1}}\right]$$
$$\ge k\left|\frac{\left(C_p - \beta D_p\right) + \Gamma(a+b+p)/\Gamma(b+p)\sum_{t=1}^{\infty}\left(D_{t+p} - \beta C_{t+p}\right)\Gamma(b+p+t)/\Gamma(a+b+p+t)\left|a_{t+p}\right|\omega^{t+p-1}}{1 + \Gamma(a+b+p)/\Gamma(b+p)\sum_{t=1}^{\infty}C_{t+p}\Gamma(b+p+t)/\Gamma(a+b+p+t)\left|a_{t+p}\right|\omega^{t+p-1}}\right|. \tag{28}$$

Since λ is a function of form (1) with the argument properties given in the class $£_\eta$ and setting $\omega = re^{i\eta}$ in the above inequality, we have

$$\Re\left[\frac{\left(C_p - \alpha D_p\right) - \Gamma(a+b+p)/\Gamma(b+p)\sum_{t=1}^{\infty}\left(D_{t+p} - \alpha C_{t+p}\right)\Gamma(b+p+t)/\Gamma(a+b+p+t)\left|a_{t+p}\right|\omega^{t+p-1}}{1 - \Gamma(a+b+p)/\Gamma(b+p)\sum_{t=1}^{\infty}C_{t+p}\Gamma(b+p+t)/\Gamma(a+b+p+t)\left|a_{t+p}\right|\omega^{t+p-1}}\right]$$
$$\ge k\left|\frac{\left(C_p - \beta D_p\right) + \Gamma(a+b+p)/\Gamma(b+p)\sum_{t=1}^{\infty}\left(D_{t+p} - \beta C_{t+p}\right)\left(\Gamma(b+p+t)/\Gamma(a+b+p+t)\right)\left|a_{t+p}\right|\omega^{t+p-1}}{1 - \Gamma(a+b+p)/\Gamma(b+p)\sum_{t=1}^{\infty}C_{t+p}\left(\Gamma(b+p+t)/\Gamma(a+b+p+t)\right)\left|a_{t+p}\right|\omega^{t+p-1}}\right|. \tag{29}$$

Letting $r \longrightarrow 1^-$ (29) leads to the desired inequality:

$$\frac{\Gamma(a+b+p)}{\Gamma(b+p)}\sum_{t=1}^{\infty}\left[D_{t+p}(1+k) - C_{t+p}(\alpha+k\beta)\right]$$
$$\frac{\Gamma(b+p+t)}{\Gamma(a+b+p+t)}\left|a_{t+p}\right| \le C_p(1+k) - D_p(\alpha+k\beta). \tag{30}$$

The function,

$$\lambda_{t.\eta}(\omega) = \omega^p - \left(\frac{\left[C_p(1+k) - D_p(\alpha+k\beta)\right]e^{i(1-t)\eta}}{\left(\Gamma(a+b+p)/\Gamma(b+p)\right)\left[D_{t+p}(1+k) - C_{t+p}(\alpha+k\beta)\right]\left(\Gamma(b+p+t)/\Gamma(a+b+p+t)\right)}\right)\omega^{t+p}, \tag{31}$$
$$0 \le \eta \le 2\pi, t \ge 1,$$

is an external function for (27). □

Corollary 1. *Let the function $\lambda(\omega)$ defined by (1) be in the class $k-£U_{\eta}(a,b,p,\alpha,\beta,\mu,\nu)$; then,*

$$\left|a_{t+p}\right| \leq \frac{C_p(1+k)-D_p(\alpha+k\beta)}{(\Gamma(a+b+p)/\Gamma(b+p))\left[D_{t+p}(1+k)-C_{t+p}(\alpha+k\beta)\right](\Gamma(b+p+t)/\Gamma(a+b+p+t))}\ (t \in N), \tag{32}$$

with equality in (32), is attained for the function $\lambda_{t.\eta}(\omega)$ given by (31).

Theorem 3. *Let the function $\lambda \in k-£U_{\eta}(a,b,p,\alpha,\beta,\mu,\nu)$ with argument property as in class $£_{\eta}$. Define $\lambda_j(\omega)=\omega$ and*

$$\lambda_{t.\eta}(\omega)=\omega^p-\left(\frac{\left[C_p(1+k)-D_p(\alpha+k\beta)\right]e^{i(1-t)\eta}}{(\Gamma(a+b+p)/\Gamma(b+p))\left[D_{t+p}(1+k)-C_{t+p}(\alpha+k\beta)\right](\Gamma(b+p+t)/\Gamma(a+b+p+t))}\right)\omega^{t+p}, \tag{33}$$

where $0 \leq \eta \leq 2\pi$ and $t \geq 1$. Then, $\lambda(\omega)$ is in the class $k-£U_{\eta}(a,b,p,\alpha,\beta,\mu,\nu)$ if and only if it can be expressed as

$$\lambda(\omega)=\sum_{t=0}^{\infty}\vartheta_t\lambda_{t.\eta}, \tag{34}$$

where $\vartheta_t \geq 0$ $(t \geq 0)$ and $\sum_{t=0}^{\infty}\vartheta_t=1$.

Proof. Assume that

$$\begin{aligned}\lambda(\omega) &= \vartheta_0\lambda_0(\omega)+\sum_{t=1}^{\infty}\vartheta_t\left[\omega^p-\left(\frac{\left[C_p(1+k)-D_p(\alpha+k\beta)\right]e^{i(1-t)\eta}}{(\Gamma(a+b+p)/\Gamma(b+p))\left[D_{t+p}(1+k)-C_{t+p}(\alpha+k\beta)\right]\Gamma(b+p+t)/\Gamma(a+b+p+t)}\right)\omega^{t+p}\right]\\ &= \sum_{t=1}^{\infty}\vartheta_t\omega^p-\sum_{t=1}^{\infty}\left(\frac{\left[C_p(1+k)-D_p(\alpha+k\beta)\right]e^{i(1-t)\eta}}{(\Gamma(a+b+p)/\Gamma(b+p))\left[D_{t+p}(1+k)-C_{t+p}(\alpha+k\beta)\right]\Gamma(b+p+t)/\Gamma(a+b+p+t)}\right)\vartheta_t\omega^{t+p}.\end{aligned} \tag{35}$$

Then, by Theorem 2, $\lambda \in k-£U_{\eta}(a,b,p,\alpha,\beta,\mu,\nu)$. It follows that

$$\begin{aligned}\lambda(\omega) &= \sum_{t=1}^{\infty}\left|\frac{\left[C_p(1+k)-D_p(\alpha+k\beta)\right]e^{i(1-t)\eta}}{(\Gamma(a+b+p)/\Gamma(b+p))\left[D_{t+p}(1+k)-C_{t+p}(\alpha+k\beta)\right](\Gamma(b+p+t)/\Gamma(a+b+p+t))}\right|\vartheta_t\\ &\quad\times\left[\frac{\Gamma(a+b+p)}{\Gamma(b+p)}\left[D_{t+p}(1+k)-C_{t+p}(\alpha+k\beta)\right]\frac{\Gamma(b+p+t)}{\Gamma(a+b+p+t)}\right]\\ &= \sum_{t=1}^{\infty}\left[C_p(1+k)-D_p(\alpha+k\beta)\right]\vartheta_t \leq (1-\vartheta_1)\left[C_p(1+k)-D_p(\alpha+k\beta)\right] \leq \left[C_p(1+k)-D_p(\alpha+k\beta)\right].\end{aligned} \tag{36}$$

Conversely, assume that the function $\lambda(\omega)$ defined by (1) belongs to the class $k-£U_{\eta}(a,b,p,\alpha,\beta,\mu,\nu)$; then,

$$\left|a_{t+p}\right| \leq \frac{C_p(1+k)-D_p(\alpha+k\beta)}{\Gamma(a+b+p)/\Gamma(b+p)\left[D_{t+p}(1+k)-C_{t+p}(\alpha+k\beta)\right](\Gamma(b+p+t)/\Gamma(a+b+p+t))},\quad (t \in N). \tag{37}$$

Setting $\vartheta_t = ((\Gamma(a+b+p)/\Gamma(b+p))[D_{t+p}(1+k)-C_{t+p}(\alpha+k\beta)](\Gamma(b+p+t)/\Gamma(a+b+p+t))/C_p(1+k)-D_p(\alpha+k\beta))|a_{t+p}|$, $(t\geq 1)$ and $\vartheta_1 = 1-\sum_{t=1}^{\infty}\vartheta_t$, then $\lambda(\omega)=\sum_{t=0}^{\infty}\vartheta_t\lambda_{t.\eta}$ and this completes the proof. □

2.2. Growth and Distortion Result. In this section, we find a growth and distortion bound for functions in the classes $k-£U_\eta(a,b,p,\alpha,\beta,\mu,\nu)$.

Theorem 4. *Let the function $\lambda(\omega)$ be defined by (1) in the class $k-£U_\eta(a,b,p,\alpha,\beta,\mu,\nu)$; then, for $|\omega|=r<1$,*

$$r^p-\left(\frac{C_p(1+k)-D_p(\alpha+k\beta)}{(b+p/a+b+p)\left[D_{p+1}(1+k)-C_{p+1}(\alpha+k\beta)\right]}\right)r^{p+1}\leq|\lambda(\omega)|\leq r^p+\left(\frac{C_p(1+k)-D_p(\alpha+k\beta)}{(b+p/a+b+p)\left[D_{p+1}(1+k)-C_{p+1}(\alpha+k\beta)\right]}\right)r^{p+1}\ (|\omega|=r) \tag{38}$$

and

$$\mathrm{p}r^{p-1}-\left(\frac{(p+1)\left[C_p(1+k)-D_p(\alpha+k\beta)\right]}{(b+p/a+b+p)\left[D_{p+1}(1+k)-C_{p+1}(\alpha+k\beta)\right]}\right)r^{p}\leq\left|\lambda'(\omega)\right|\leq \mathrm{p}r^{p-1}+\left(\frac{(p+1)\left[C_p(1+k)-D_p(\alpha+k\beta)\right]}{(b+p/a+b+p)\left[D_{p+1}(1+k)-C_{p+1}(\alpha+k\beta)\right]}\right)r^{p}\ (|\omega|=r), \tag{39}$$

where equalities (38) and (39) hold for the function $\lambda(\omega)$ given by (27), for $\omega=\pm r$.

Proof. From Theorem 2, we have

$$\frac{(b+p)}{(a+b+p)}\left[D_{p+1}(1+k)-C_{p+1}(\alpha+k\beta)\right]\sum_{t=1}^{\infty}\left|a_{t+p}\right|\leq\frac{\Gamma(a+b+p)}{\Gamma(b+p)}\sum_{t=1}^{\infty}\left[D_{t+p}(1+k)-C_{t+p}(\alpha+k\beta)\right]\frac{\Gamma(b+p+t)}{\Gamma(a+b+p+t)}\left|a_{t+p}\right|\leq C_p(1+k)-D_p(\alpha+k\beta). \tag{40}$$

The last inequality follows from Theorem 2 Thus,

$$|\lambda(\omega)|\leq|\omega|^p+\sum_{t=1}^{\infty}\left|a_{t+p}\right||\omega|^{t+p}\leq r^p+r^{p+1}\sum_{t=1}^{\infty}\left|a_{t+p}\right|\leq r^p+\left(\frac{C_p(1+k)-D_p(\alpha+k\beta)}{(b+p/a+b+p)\left[D_{p+1}(1+k)-C_{p+1}(\alpha+k\beta)\right]}\right)r^{p+1}. \tag{41}$$

Similarly,

$$|\lambda(\omega)|\geq|\omega|^p-\sum_{t=1}^{\infty}\left|a_{t+p}\right||\omega|^{t+p}\geq r^p-r^{p+1}\sum_{t=1}^{\infty}\left|a_{t+p}\right|\geq r^p-\left(\frac{C_p(1+k)-D_p(\alpha+k\beta)}{(b+p/a+b+p)\left[D_{p+1}(1+k)-C_{p+1}(\alpha+k\beta)\right]}\right)r^{p+1}. \tag{42}$$

Now, by differentiating (1), we obtain

$$\left|\lambda'(\omega)\right|\leq p|\omega|^{p-1}+\sum_{t=1}^{\infty}(t+p)\left|a_{t+p}\right||\omega|^{t+p-1}\leq \mathrm{p}r^{p-1}+r^p\sum_{t=1}^{\infty}(t+p)\left|a_{t+p}\right|\leq \mathrm{p}r^{p-1}+\left(\frac{(p+1)\left[C_p(1+k)-D_p(\alpha+k\beta)\right]}{(b+p/a+b+p)\left[D_{p+1}(1+k)-C_{p+1}(\alpha+k\beta)\right]}\right)r^{p} \tag{43}$$

and

$$|\lambda'(\omega)| \ge |\omega|^p - \sum_{t=1}^{\infty} |a_{t+p}||\omega|^{t+p} \ge \mathrm{pr}^{p-1} - r^p \sum_{t=1}^{\infty} (t+p)|a_{t+p}| \ge \mathrm{pr}^{p-1} - \left(\frac{(p+1)\left[C_p(1+k) - D_p(\alpha + k\beta)\right]}{(b+p/a+b+p)\left[D_{p+1}(1+k) - C_{p+1}(\alpha+k\beta)\right]} \right) r^p. \quad (44)$$

Using Theorem 2 in (44), we have

$$\frac{(b+p)}{(a+b+p)} \frac{\left[D_{p+1}(1+k) - C_{p+1}(\alpha+k\beta)\right]}{2} \sum_{t=1}^{\infty} (t+p)|a_{t+p}| \le \frac{\Gamma(a+b+p)}{\Gamma(b+p)} \cdot \sum_{t=1}^{\infty} \left[D_{t+p}(1+k) - C_{t+p}(\alpha+k\beta)\right] \frac{\Gamma(b+p+t)}{\Gamma(a+b+p+t)} |a_{t+p}| \le C_p(1+k) - D_p(\alpha+k\beta), \quad (45)$$

or, equivalently

$$\sum_{t=1}^{\infty} (t+p)|a_{t+p}| \le \frac{2(a+b+p)\left[C_p(1+k) - D_p(\alpha+k\beta)\right]}{(b+p)\left[D_{p+1}(1+k) - C_{p+1}(\alpha+k\beta)\right]}. \quad (46)$$

Using (46) into (43) and (44) yields inequality (39). □

2.3. Radii of Close-to-Convexity, Starlikeness, and Convexity. In this section, we obtain the radii of close-to-convexity, starlikeness, and convexity for functions $\lambda(\omega)$ in the classes $k - £U_\eta(a, b, p, \alpha, \beta, \mu, \nu)$.

Theorem 5. *Let $\lambda \in k - £U_\eta(a, b, p, \alpha, \beta, \mu, \nu)$; then,*

(i) $\lambda(\omega)$ is starlike of order $\varkappa$ $(0 \le \varkappa < 1)$ in the disc $|\omega| < r_1$, where

$$r_1 = \inf \left[\left(\frac{2-p-\varkappa}{t+p-\varkappa} \right) \frac{(\Gamma(a+b+p)/\Gamma(b+p))\left[D_{t+p}(1+k) - C_{t+p}(\alpha+k\beta)\right]\Gamma(b+p+t)/\Gamma(a+b+p+t)}{\left[C_p(1+k) - D_p(\alpha+k\beta)\right]} \right]^{1/t}, \quad t \ge 1. \quad (47)$$

(ii) $\lambda(\omega)$ is convex of order $\varkappa$ $(0 \le \varkappa < 1)$ in the disc $|\omega| < r_2$, where

$$r_2 = \inf \left[\left(\frac{p(2-p-\varkappa)}{(t+p)(t+p-\varkappa)} \right) \frac{\Gamma(a+b+p)/\Gamma(b+p)\left[D_{t+p}(1+k) - C_{t+p}(\alpha+k\beta)\right]\Gamma(b+p+t)/\Gamma(a+b+p+t)}{\left[C_p(1+k) - D_p(\alpha+k\beta)\right]} \right]^{1/t}, \quad t \ge 1. \quad (48)$$

These results are sharp for the extremal function $\lambda(\omega)$ given by (31).

Proof

(i) Given $\lambda \in £_\eta$ and λ is starlike of order $\varkappa$, we have

$$\left| \frac{\omega\lambda'(\omega)}{\lambda(\omega)} - 1 \right| < 1 - \varkappa. \quad (49)$$

For the left-hand side of (49), we have

$$\left| \frac{\omega\lambda'(\omega)}{\lambda(\omega)} - 1 \right| \le \frac{p - 1 + \sum_{t=1}^{\infty} k + p - 1|a_{t+p}||\omega|^t}{1 - \sum_{t=1}^{\infty} |a_{t+p}||\omega|^t}. \quad (50)$$

The last expression is less than $1-\varkappa$ if

$$\sum_{t=1}^{\infty}\left(\frac{k+p-\varkappa}{2-p-\varkappa}\right)\left|a_{t+p}\right||\omega|^{t}<1. \tag{51}$$

Use the fact that $\lambda \in k-£U_{\eta}(a,b,p,\alpha,\beta,\mu,\nu)$ if and only if

$$\sum_{t=1}^{\infty}\frac{(\Gamma(a+b+p)/\Gamma(b+p))\left[D_{t+p}(1+k)-C_{t+p}(\alpha+k\beta)\right]\Gamma(b+p+t)/\Gamma(a+b+p+t)}{C_{p}(1+k)-D_{p}(\alpha+k\beta)}\left|a_{t+p}\right|\leq 1. \tag{52}$$

We can say (49) is true if

$$\left(\frac{k+p-\varkappa}{2-p-\varkappa}\right)|\omega|^{t}=\frac{\Gamma(a+b+p)/\Gamma(b+p)\left[D_{t+p}(1+k)-C_{t+p}(\alpha+k\beta)\right]\Gamma(b+p+t)/\Gamma(a+b+p+t)}{C_{p}(1+k)-D_{p}(\alpha+k\beta)}. \tag{53}$$

or, equivalently,

$$|\omega|^{t}=\frac{\Gamma(a+b+p)/\Gamma(b+p)(2-p-\varkappa)\left[D_{t+p}(1+k)-C_{t+p}(\alpha+k\beta)\right]\Gamma(b+p+t)/\Gamma(a+b+p+t)}{(k+p-\varkappa)\left[C_{p}(1+k)-D_{p}(\alpha+k\beta)\right]}, \tag{54}$$

which is required.

(ii) Using the fact that λ is convex if and only if $\omega\lambda'(\omega)$ is starlike, we can prove (ii) on similar lines to the proof of (i). □

2.4. Modified Hadamard Product. Let the function $\lambda_{j}(\omega)\quad(j=1,2)$ be defined by

$$\lambda_{j}(\omega)=\omega^{p}+\sum_{t=1}^{\infty}a_{t+p,i}\omega^{t+p},\quad a_{t+p,i}\geq 0, i\in\mathbb{N}. \tag{55}$$

Then, we define the modified Hadamard product of $\lambda_1(\omega)$ and $\lambda_2(\omega)$ by

$$(\lambda_1 * \lambda_2)(\omega)=\omega^{p}-\sum_{t=0}^{\infty}a_{t+p,1}a_{t+p,2}\omega^{t+p}. \tag{56}$$

Now, we prove the following.

Theorem 6. *Let $\lambda_{j}(\omega)\quad(j=1,2,\ldots)$ given by (55) be in the class $k-£U_{\eta}(a,b,p,\alpha,\beta,\mu,\nu)$; then, $(\lambda_1 * \lambda_2)\in k-£U_{\eta}(a,b,p,\Phi_1,\mu,\nu)$ for*

$$\Phi_1=\frac{\left[C_p(1+k)-D_p(k\beta)\right]\left(\Gamma(a+b+p)/\Gamma(b+p)\left[D_{t+p}(1+k)-C_{t+p}(\alpha+k\beta)\right]\Gamma(b+p+t)/\Gamma(a+b+p+t)\right)^{2}-\left[D_{t+p}(1+k)-C_{t+p}(k\beta)\right]\left[C_p(1+k)-D_p(\alpha+k\beta)\right]^{2}}{D_p\left((\Gamma(a+b+p)/\Gamma(b+p))\left[D_{t+p}(1+k)-C_{t+p}(\alpha+k\beta)\right]\Gamma(b+p+t)/\Gamma(a+b+p+t)\right)^{2}-C_{t+p}\left[C_p(1+k)-D_p(\alpha+k\beta)\right]^{2}}. \tag{57}$$

Proof. We need to prove the largest Φ_1, such that

$$\frac{\left[D_{t+p}(1+k)-C_{t+p}(\Phi_1+k\beta)\right]}{C_p(1+k)-D_p(\Phi_1+k\beta)}\left|a_{t+p,1}\right|\left|a_{t+p,2}\right|\leq 1. \tag{58}$$

From Theorem 2, we have

$$\begin{aligned}&\sum_{t=1}^{\infty}\left[\frac{\Gamma(a+b+p)/\Gamma(b+p)\left[D_{t+p}(1+k)-C_{t+p}(\alpha+k\beta)\right]\Gamma(b+p+t)/\Gamma(a+b+p+t)}{C_p(1+k)-D_p(\alpha+k\beta)}\right]\left|a_{t+p,1}\right|\leq 1,\\&\sum_{t=1}^{\infty}\left[\frac{\Gamma(a+b+p)/\Gamma(b+p)\left[D_{t+p}(1+k)-C_{t+p}(\alpha+k\beta)\right]\Gamma(b+p+t)/\Gamma(a+b+p+t)}{C_p(1+k)-D_p(\alpha+k\beta)}\right]\left|a_{t+p,2}\right|\leq 1.\end{aligned} \tag{59}$$

By Cauchy–Schwarz inequality, we have

$$\sum_{t=1}^{\infty}\left[\frac{\Gamma(a+b+p)/\Gamma(b+p)\left[D_{t+p}(1+k)-C_{t+p}(\alpha+k\beta)\right]\Gamma(b+p+t)/\Gamma(a+b+p+t)}{C_p(1+k)-D_p(\alpha+k\beta)}\right]\sqrt{a_{t+p,1}\,a_{t+p,2}}\leq 1. \tag{60}$$

Thus, it is sufficient to show that

$$\begin{aligned}&\frac{\left[D_{t+p}(1+k)-C_{t+p}(\Phi_1+k\beta)\right]}{C_p(1+k)-D_p(\Phi_1+k\beta)}\left|a_{t+p,1}\right|\left|a_{t+p,2}\right|\\&\quad\leq\left[\frac{\Gamma(a+b+p)/\Gamma(b+p)\left[D_{t+p}(1+k)-C_{t+p}(\alpha+k\beta)\right]\Gamma(b+p+t)/\Gamma(a+b+p+t)}{C_p(1+k)-D_p(\alpha+k\beta)}\right]\sqrt{a_{t+p,1}\,a_{t+p,2}}.\end{aligned} \tag{61}$$

That is,

$$\sqrt{a_{t+p,1}\,a_{t+p,2}}\leq\left[\frac{\left[C_p(1+k)-D_p(\Phi_1+k\beta)\right]\Gamma(a+b+p)/\Gamma(b+p)\left[D_{t+p}(1+k)-C_{t+p}(\alpha+k\beta)\right]\Gamma(b+p+t)/\Gamma(a+b+p+t)}{C_p(1+k)-D_p(\alpha+k\beta)\left[D_{t+p}(1+k)-C_{t+p}(\Phi_1+k\beta)\right]}\right]. \tag{62}$$

Note that

$$\sqrt{a_{t+p,1}\,a_{t+p,2}}\leq\frac{C_p(1+k)-D_p(\alpha+k\beta)}{\Gamma(a+b+p)/\Gamma(b+p)\left[D_{t+p}(1+k)-C_{t+p}(\alpha+k\beta)\right]\Gamma(b+p+t)/\Gamma(a+b+p+t)}. \tag{63}$$

Consequently, from (62) and (63), we obtain

$$\begin{aligned}&\frac{C_p(1+k)-D_p(\alpha+k\beta)}{\Gamma(a+b+p)/\Gamma(b+p)\left[D_{t+p}(1+k)-C_{t+p}(\alpha+k\beta)\right]\Gamma(b+p+t)/\Gamma(a+b+p+t)}\\&\leq\left[\frac{\left[C_p(1+k)-D_p(\Phi_1+k\beta)\right]\Gamma(a+b+p)/\Gamma(b+p)\left[D_{t+p}(1+k)-C_{t+p}(\alpha+k\beta)\right]\Gamma(b+p+t)/\Gamma(a+b+p+t)}{\left[D_{t+p}(1+k)-C_{t+p}(\Phi_1+k\beta)\right]\left[C_p(1+k)-D_p(\alpha+k\beta)\right]}\right],\end{aligned} \tag{64}$$

or, equivalently,

$$\Phi_1\leq\frac{\left[C_p(1+k)-D_p(k\beta)\right]\left(\Gamma(a+b+p)/\Gamma(b+p)\left[D_{t+p}(1+k)-C_{t+p}(\alpha+k\beta)\right]\Gamma(b+p+t)/\Gamma(a+b+p+t)\right)^2-\left[D_{t+p}(1+k)-C_{t+p}(k\beta)\right]\left[C_p(1+k)-D_p(\alpha+k\beta)\right]^2}{D_p\left(\Gamma(a+b+p)/\Gamma(b+p)\left[D_{t+p}(1+k)-C_{t+p}(\alpha+k\beta)\right]\Gamma(b+p+t)/\Gamma(a+b+p+t)\right)^2-C_{t+p}\left[C_p(1+k)-D_p(\alpha+k\beta)\right]^2}=\chi(t). \tag{65}$$

Since $\chi_1(t)$ is an increasing function for $t\geq 1$, letting $t=1$ in (65), we obtain

$$\Phi_1 \leq \chi_1(1) = \frac{\left[C_p(1+k) - D_p(k\beta)\right]\left((b+p/a+b+p)\left[D_{p+1}(1+k) - C_{p+1}(\alpha+k\beta)\right]\right)^2 - \left[D_{p+1}(1+k) - C_{p+1}(k\beta)\right]\left[C_p(1+k) - D_p(\alpha+k\beta)\right]^2}{D_p\left((b+p/a+b+p)\left[D_{p+1}(1+k) - C_{p+1}(\alpha+k\beta)\right]\right)^2 - C_{P+1}\left[C_p(1+k) - D_p(\alpha+k\beta)\right]^2}. \tag{66}$$

The proof of our theorem is now completed. □

Theorem 7. *Let $\lambda_j(\omega)$ $(j = 1, 2)$ given by (55) be in the class $k - £U_\eta(a, b, p, \alpha, \beta, \mu, \nu)$. If the sequence $\{\Gamma(a+b+p)/\Gamma(b+p)$ $[D_{t+p}(1+k) - C_{t+p}(\alpha+k\beta)]\Gamma(b+p+t)/\Gamma(a+b+p+t)\}$ is nondecreasing, then function*

$$h(\omega) = \omega^p - \sum_{t=1}^{\infty}\left(a_{t+p,1}^2 + a_{t+p,2}^2\right)\omega^{t+p} \tag{67}$$

belongs to the class $k - £U_\eta(a, b, p, \Phi_2, \mu, \nu)$, where

$$\Phi_2 \leq \frac{\left[\Gamma(a+b+p)/\Gamma(b+p)\left[D_{t+p}(1+k) - C_{t+p}(\alpha+k\beta)\right]\Gamma(b+p+t)/\Gamma(a+b+p+t)\right]^2\left(C_p(1+k) - D_p k\beta\right) - 2\left[C_p(1+k) - D_p(\alpha+k\beta)\right]^2\left[D_{t+p}(1+k) - C_{t+p}(k\beta)\right]}{D_p\left[\Gamma(a+b+p)/\Gamma(b+p)\left[D_{t+p}(1+k) - C_{t+p}(\alpha+k\beta)\right]\Gamma(b+p+t)/\Gamma(a+b+p+t)\right]^2 - 2C_{t+p}\left[C_p(1+k) - D_p(\alpha+k\beta)\right]^2}. \tag{68}$$

Proof. We need to prove the largest Φ_2.

From Theorem 2, we have

$$\begin{aligned}&\sum_{t=1}^{\infty}\left[\frac{\Gamma(a+b+p)/\Gamma(b+p)\left[D_{t+p}(1+k) - C_{t+p}(\alpha+k\beta)\right]\Gamma(b+p+t)/\Gamma(a+b+p+t)}{C_p(1+k) - D_p(\alpha+k\beta)}\right]^2 a_{t+p,1}^2 \\ &\leq \sum_{t=1}^{\infty}\left[\frac{\Gamma(a+b+p)/\Gamma(b+p)\left[D_{t+p}(1+k) - C_{t+p}(\alpha+k\beta)\right]\Gamma(b+p+t)/\Gamma(a+b+p+t)}{C_p(1+k) - D_p(\alpha+k\beta)} a_{t+p,1}\right]^2 \leq 1\end{aligned} \tag{69}$$

and

$$\begin{aligned}&\sum_{t=1}^{\infty}\left[\frac{\Gamma(a+b+p)/\Gamma(b+p)\left[D_{t+p}(1+k) - C_{t+p}(\alpha+k\beta)\right]\Gamma(b+p+t)/\Gamma(a+b+p+t)}{C_p(1+k) - D_p(\alpha+k\beta)}\right]^2 a_{t+p,2}^2 \\ &\leq \sum_{t=1}^{\infty}\left[\frac{\Gamma(a+b+p)/\Gamma(b+p)\left[D_{t+p}(1+k) - C_{t+p}(\alpha+k\beta)\right]\Gamma(b+p+t)/\Gamma(a+b+p+t)}{C_p(1+k) - D_p(\alpha+k\beta)} a_{t+p,2}\right]^2 \leq 1.\end{aligned} \tag{70}$$

It follows from (69) and (70) that

$$\begin{aligned}&\sum_{t=1}^{\infty}\frac{1}{2}\left[\frac{\Gamma(a+b+p)/\Gamma(b+p)\left[D_{t+p}(1+k) - C_{t+p}(\alpha+k\beta)\right]\Gamma(b+p+t)/\Gamma(a+b+p+t)}{C_p(1+k) - D_p(\alpha+k\beta)}\right]^2 \\ &\left(a_{t+p,1}^2 + a_{t+p,2}^2\right) \leq 1.\end{aligned} \tag{71}$$

Therefore, we need to find the largest Φ_2, such that

$$\frac{\left[D_{t+p}(1+k) - C_{t+p}(\Phi_2 + k\beta)\right]}{C_p(1+k) - D_p(\Phi_2 + k\beta)} \leq \frac{1}{2}\left[\frac{\Gamma(a+b+p)/\Gamma(b+p)\left[D_{t+p}(1+k) - C_{t+p}(\alpha + k\beta)\right]\Gamma(b+p+t)/\Gamma(a+b+p+t)}{C_p(1+k) - D_p(\alpha + k\beta)}\right]^2, \tag{72}$$

That is,

$$\Phi_2 \leq \frac{\left[\Gamma(a+b+p)/\Gamma(b+p)\left[D_{t+p}(1+k) - C_{t+p}(\alpha+k\beta)\right]\Gamma(b+p+t)/\Gamma(a+b+p+t)\right]^2\left(C_p(1+k) - D_p k\beta\right) - 2\left[C_p(1+k) - D_p(\alpha+k\beta)\right]^2\left[D_{t+p}(1+k) - C_{t+p}(k\beta)\right]}{D_p\left[\Gamma(a+b+p)/\Gamma(b+p)\left[D_{t+p}(1+k) - C_{t+p}(\alpha+k\beta)\right]\Gamma(b+p+t)/\Gamma(a+b+p+t)\right]^2 - 2C_{t+p}\left[C_p(1+k) - D_p(\alpha+k\beta)\right]^2}. \tag{73}$$

Since $\chi_2(t)$ is an increasing function for $t \geq 1$, letting $t = 1$ in (73), we readily have

$$\Phi_2 \leq \chi_2(1) = \frac{\left[\Gamma(a+b+p)/\Gamma(b+p)\left[D_{p+1}(1+k) - C_{p+1}(\alpha+k\beta)\right]\Gamma(b+p+t)/\Gamma(a+b+p+t)\right]^2\left(C_p(1+k) - D_p k\beta\right) - 2\left[C_p(1+k) - D_p(\alpha+k\beta)\right]^2\left[D_{p+1}(1+k) - C_{p+1}(k\beta)\right]}{D_p\left[\Gamma(a+b+p)/\Gamma(b+p)\left[D_{p+1}(1+k) - C_{p+1}(\alpha+k\beta)\right]\Gamma(b+p+t)/\Gamma(a+b+p+t)\right]^2 - 2C_{p+1}\left[C_p(1+k) - D_p(\alpha+k\beta)\right]^2}. \tag{74}$$

The proof of our theorem is now completed. □

3. Conclusion

In our current investigation, we have presented and studied thoroughly some new subclasses of p−valent functions related with uniformly convex and starlike functions, in connection with the Liu–Owa integral operator $Q^{a}_{b,p}\lambda(\omega)$ given by (8). We have obtained sufficient and necessary conditions in relation to these classes, including growth, distortion theorem, and radius problem. Some special cases have been discussed as applications of our main results. The techniques and ideas of this paper may stimulate for further research in this area of knowledge.

Data Availability

No data were used to support the findings of the study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

SH came with the main thoughts and helped to draft the manuscript, SGAS and IA proved the main theorems, and SN and MD revised the paper. All authors read and approved the final manuscript.

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