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Research Article

On the Determinants of the Square-Type Stirling Matrix and Bell Matrix

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We study determinants of the square-type Stirling matrix S^* and the square-type Bell matrix B^*. For this purpose, we prove that S^* and B^* have LU factorizations $S^* = L_S U_S$ and $B^* = L_B U_B$ where the diagonal entries of U_S are k^{k-1}, while those of U_B are $k!$ $(k \geq 1)$.

1. Introduction

The Stirling numbers $s_{m,n}$ $(m, n \geq 0)$ of second kind count the number of ways to partition an m element set into n subsets. The Stirling matrix $\widetilde{S} = [s_{m,n}]$ satisfies a recurrence rule $s_{m+1,n+1} = s_{m,n} + (n+1)s_{m,n+1}$ [1]. The sum $\sum_{k=0}^{m} s_{m,k}$ of the m^{th} row of $\widetilde{S}$ is called the m^{th} Bell number $B(m)$, so $\{B(m) \mid m \geq 0\} = \{1, 1, 2, 5, 15, \ldots\}$. A triangular matrix $B = [b_{i,j}]$ having Bell numbers on both border and holding $b_{i+1,j+1} = b_{i,j} + b_{i+1,j}$ $(i, j \geq 1)$ is called the Bell matrix [2]. Since every entries in the first row and column of $\widetilde{S}$ are zeros except $s_{0,0} = 1$, we denote by S the Stirling matrix deleted in the first row and column from $\widetilde{S}$.

$$\widetilde{S} = \begin{bmatrix} 1 & & & & \\ 0 & 1 & & & \\ 0 & 1 & 1 & & \\ 0 & 1 & 3 & 1 & \\ 0 & 1 & 7 & 6 & 1 \\ & & \cdots & & \end{bmatrix},$$

$$S = \left[s_{i,j}\right] = \begin{bmatrix} 1 & & & & \\ 1 & 1 & & & \\ 1 & 3 & 1 & & \\ 1 & 7 & 6 & 1 & \\ 1 & 15 & 25 & 10 & 1 \\ & & \cdots & & \end{bmatrix},$$
$$B = \left[b_{i,j}\right] = \begin{bmatrix} 1 & & & & \\ 1 & 2 & & & \\ 2 & 3 & 5 & & \\ 5 & 7 & 10 & 15 & \\ 15 & 20 & 27 & 37 & 52 \\ & & \cdots & & \end{bmatrix}, \quad i, j \geq 1. \tag{1}$$

Let

$$S^* = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & 6 & 10 \\ 1 & 7 & 25 & 65 \\ 1 & 15 & 90 & 350 \\ & & & \cdots \end{bmatrix}, \quad B^* = \begin{bmatrix} 1 & 2 & 5 & 15 \\ 1 & 3 & 10 & 37 \\ 2 & 7 & 27 & 114 \\ 5 & 20 & 87 & 409 \\ & & & \cdots \end{bmatrix}, \tag{2}$$

be a square-type Stirling matrix and a square-type Bell matrix. In the work, we study the determinants of S^* and B^* by finding their LU factorizations. In fact, we prove that S^* has an LU factorization $S^* = L_S U_S$, where $L_S = S$ and U_S has diagonal entries $1, 2, 3^2, 4^3, \ldots$ (Theorem 2), and B^* has an LU factorization $B^* = L_B U_B$ where $L_B = \widetilde{S}P$ along with the Pascal matrix P and U_B has diagonal entries $1, 1, 2!, 3!, \ldots$ (Theorem 12). This consideration is motivated by the square-type Pascal matrix

$$P^* = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \\ & & \cdots \end{bmatrix}, \tag{3}$$

where its LU factorization is $P^* = PP^T$ [3] so $\det P^* = 1$. Note that $\det B_n^*$ was discussed in [4] by means of Hankel transformation. Our feature in the work is to study $\det B_n^*$ by recurrence rules of Bell numbers over an LU factorization of B_n^*. For our notations, with a matrix M, M_k denotes the size $k \times k$, and let $r_i(M_k)$ and $c_j(M_k)$ be the i^{th} row and j^{th} column of M_k. Write $(x, \ldots, x, a, b, \ldots)$ and $(a, b, \ldots, x, \ldots, x)$ simply by $(\overline{x}^t; a, b, \ldots)$ and $(a, b, \ldots; \overline{x}^t)$ with the t copies $\overline{x}^t$ of x. Therefore, $(\overline{x}^t; r_i(M_k))$ means a row matrix having t x's followed by a row matrix $r_i(M_k)$, and similarly, $(\overline{x}^t; c_j(M_k))$ means a column matrix $\begin{bmatrix} x \\ \vdots \\ x \\ c_j(M_k) \end{bmatrix}$. Let $\text{di}[a, b, \ldots]$ be a diagonal matrix having diagonal entries $a, b, \ldots$.

2. Square-Type Stirling Matrix

For $i, j \geq 1$, let $r_i(S)$ and $c_j(S)$ (resp., $r_i(S^{-1})$ and $c_j(S^{-1})$) be the i^{th} row and j^{th} column of S (resp., S^{-1}). Since S and S^{-1} are lower triangular matrices, $r_i(S)$ and $r_i(S^{-1})$ can be considered as of size $1 \times i$, while $c_j(S)$ and $c_j(S^{-1})$ are of size $\infty \times 1$. But, if necessary, like the case of multiplication $r_i(S)c_j(S)$, we may regard $r_i(S)$ filled with infinitely many zeros after the first i entries.

Lemma 1. *Let $S = [s_{i,j}]$, $S^{-1} = [s^{\vee}_{i,j}]$, and $S^* = [s^*_{i,j}]$ for $i, j \geq 1$. Let $r_i(S^*)$ and $c_j(S^*)$ be the i^{th} row and j^{th} column of S^*.*

(1) In S, $s_{i+1,j+1} = s_{i,j} + (j+1)s_{i,j+1}$ and $c_{j+1}(S) = (0; c_j(S)) + (j+1)(0; c_{j+1}(S))$.

(2) In S^{-1}, $s^{\vee}_{i+1,j+1} = s^{\vee}_{i,j} - i s^{\vee}_{i,j+1}$. Thus, $s^{\vee}_{i+1,1} = (-1)^i i!$ and $\sum_{j=1}^{i} s^{\vee}_{i,j} = 0$ for $i \geq 2$.

(3) In S^*, $s^*_{i,j} = s_{i+j-1,j}$ and $s^*_{i,j} = s^*_{i,j-1} + j s^*_{i-1,j}$. And, $c_j(S^*) = c_{j-1}(S^*) + j(0; c_j(S^*)) = \sum_{t=0}^{k-1} j^t (\overline{0}^t; c_{j-1}(S^*)) + j^k (\overline{0}^k; c_j(S^*)) = \sum_{t=0}^{\infty} j^t (\overline{0}^t; c_{j-1}(S^*))$.

Proof. The recurrence in (1) is well known [5]. The column $c_{j+1}(S)$ is

$$\begin{aligned} c_{j+1}(S) &= \left(0, \ldots, 0, s_{j+1,j+1}, s_{j+2,j+1}, s_{j+3,j+1}, \ldots\right)^T \\ &= \left(\overline{0}^j; s_{j,j}, s_{j+1,j} + (j+1)s_{j+1,j+1}, s_{j+2,j} + (j+1)s_{j+2,j+1}, \ldots\right)^T \\ &= \left(\overline{0}^j; s_{j,j}, s_{j+1,j}, s_{j+2,j}, \ldots\right)^T + (j+1)\left(\overline{0}^j; 0, s_{j+1,j+1}, s_{j+2,j+1}, \ldots\right)^T \\ &= \left(0; c_j(S)\right) + (j+1)\left(0; c_{j+1}(S)\right). \end{aligned} \tag{4}$$

The inverse S^{-1} is the signed Stirling matrix of first kind [6] satisfying the recurrence $s^{\vee}_{i+1,j+1} = s^{\vee}_{i,j} - i s^{\vee}_{i,j+1}$. From $I = S_n^{-1} S_n = [r_i(S_n^{-1}) c_j(S_n)]$, we have $0 = r_i(S^{-1}) c_1(S) = \sum_{j=1}^{i} s^{\vee}_{i,j}$ for $i \geq 2$, since $c_1(S)$ is composed of all 1s. Moreover, simple computation of S^{-1} shows $s^{\vee}_{i,1}$ $(1 \leq i \leq 5)$ equals $1, -1, 2, -3!, 4!$, respectively. Hence, if we assume $s^{\vee}_{i+1,1} = (-1)^i i!$ for some i, then $s^{\vee}_{i+2,1} = s^{\vee}_{i+1,0} - (i+1)s^{\vee}_{i+1,1} = -(i+1)(-1)^i i! = (-1)^{i+1}(i+1)!$.

Comparing

$$S^* = \left[s^*_{i,j}\right] = \begin{bmatrix} s_{1,1} & s_{2,2} & s_{3,3} \\ s_{2,1} & s_{3,2} & s_{4,3} \\ s_{3,1} & s_{4,2} & s_{5,3} \\ & & \cdots \end{bmatrix} \tag{5}$$

with $S = [s_{i,j}]$, it is easy to see $s^*_{i,j} = s_{i+j-1,j}$ and $s^*_{i,j} = s^*_{i,j-1} + j s^*_{i-1,j}$. Now, for the j^{th} column $c_j(S^*)$, we have

$$\begin{aligned} c_j(S^*) &= c_{j-1}(S^*) + j\left(0; c_j(S^*)\right) \\ &= c_{j-1}(S^*) + j\left(0; c_{j-1}(S^*) + j\left(0; c_j(S^*)\right)\right) \\ &= c_{j-1}(S^*) + j\left(0; c_{j-1}(S^*)\right) + j^2\left(\overline{0}^2; c_{j-1}(S^*)\right) + j^3\left(\overline{0}^3; c_j(S^*)\right) \\ &= \cdots = \sum_{t=0}^{k-1} j^t\left(\overline{0}^t; c_{j-1}(S^*)\right) + j^k\left(\overline{0}^k; c_j(S^*)\right) \\ &= \cdots = \sum_{t=0}^{\infty} j^t\left(\overline{0}^t; c_{j-1}(S^*)\right). \end{aligned} \tag{6}$$

□

Theorem 1. *$r_i(S^{-1})(\overline{0}^{j-t}; c_j(S^*)) = 0$ for all $0 \le t < j < i$.*

Proof. Note that $c_j(S^*) = (s^*_{1,j}, s^*_{2,j}, s^*_{3,j}, \ldots)^T = (s_{j,j}, s_{j+1,j}, s_{j+2,j}, \ldots)^T$ and $c_j(S) = (\overline{0}^{j-1}; c_j(S^*))$. Hence, we have $r_i(S^{-1})(\overline{0}^{j-1}; c_j(S^*)) = r_i(S^{-1})c_j(S) = 0$ from $S^{-1}S = I$. When $t = 2$, by Lemma 1 (3), we have

$$\begin{aligned} r_i\left(S^{-1}\right)\left(\overline{0}^{j-2}; c_j(S^*)\right) &= r_i\left(S^{-1}\right)\left(\overline{0}^{j-2}; c_{j-1}(S^*) + j\left(0; c_j(S^*)\right)\right) \\ &= r_i\left(S^{-1}\right)\left(\overline{0}^{j-2}; c_{j-1}(S^*)\right) + jr_i\left(S^{-1}\right)\left(\overline{0}^{j-1}; c_j(S^*)\right) \\ &= r_i\left(S^{-1}\right)c_{j-1}(S) + jr_i\left(S^{-1}\right)c_j(S) = 0. \end{aligned} \tag{7}$$

Thus, by assuming $r_i(S^{-1})(\overline{0}^{j-t}; c_j(S^*)) = 0$ for $1 \le t \le j-2$, we have $r_i(S^{-1})(\overline{0}; c_j(S^*)) = r_i(S^{-1})(\overline{0}; c_{j-1}(S^*) + j(0; c_j(S^*))) = r_i(S^{-1})(\overline{0}; c_{j-1}(S^*)) + jr_i(S^{-1})(\overline{0}^2; c_j(S^*)) = 0$. □

Theorem 2. *S^* has an LU factorization SX, where X is an upper triangular matrix having diagonal entries i^{i-1} $(i \ge 1)$. Therefore, $\det S^*_k = \prod_{i=1}^{k} i^{i-1}$.*

Proof. Let $[x_{i,j}] = X = S^{-1}S^*$. Then, Lemma 1 (2) shows $x_{i,1} = r_i(S^{-1})c_1(S^*) = \sum_{t=1}^{i} s^{\vee}_{i,t} = 0$ for all $i > 1$, since all entries in $c_1(S^*)$ are 1. And,

$$\begin{aligned} x_{i,2} &= r_i\left(S^{-1}\right)c_2(S^*) = r_i\left(S^{-1}\right)\left(c_1(S^*) + 2\left(0; c_2(S^*)\right)\right) \\ &= x_{i,1} + 2r_i\left(S^{-1}\right)\left(0; c_2(S^*)\right) = 0, \end{aligned} \tag{8}$$

by Theorem 1. Thus, by assuming $x_{i,i-2} = 0$, we have

$$\begin{aligned} x_{i,i-1} &= r_i\left(S^{-1}\right)c_{i-1}(S^*) = r_i\left(S^{-1}\right)\left(c_{i-2}(S^*) + (i-1)\left(0; c_{i-1}(S^*)\right)\right) \\ &= x_{i,i-2} + (i-1)r_i\left(S^{-1}\right)\left(0; c_{i-2}(S^*)\right)) = 0, \end{aligned} \tag{9}$$

which shows X is an upper triangular matrix. Now, for $x_{i,i}$, we have

$$\begin{aligned} x_{i,i} &= r_i\left(S^{-1}\right)c_i(S^*) \\ &= r_i\left(S^{-1}\right)\left(c_{i-1}(S^*) + i\left(0; c_{i-1}(S^*)\right) + \cdots + i^{i-2}\left(\overline{0}^{i-2}; c_{i-1}(S^*)\right) + i^{i-1}\left(\overline{0}^{i-1}; c_i(S^*)\right)\right) \\ &= r_i\left(S^{-1}\right)c_{i-1}(S^*) + ir_i\left(S^{-1}\right)\left(0; c_{i-1}(S^*)\right) + \cdots + i^{i-2}r_i\left(S^{-1}\right)\left(\overline{0}^{i-2}; c_{i-1}(S^*)\right) + i^{i-1}r_i\left(S^{-1}\right)\left(\overline{0}^{i-1}; c_i(S^*)\right) \\ &= i^{i-1}r_i\left(S^{-1}\right)\left(\overline{0}^{i-1}; 1\right) = i^{i-1}. \end{aligned} \tag{10}$$

□

Indeed,

$$S^* = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & 6 & 10 \\ 1 & 7 & 25 & 65 \\ 1 & 15 & 90 & 350 \end{bmatrix} = \begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ 1 & 3 & 1 & \\ 1 & 7 & 6 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ & 2 & 5 & 9 \\ & & 3^2 & 37 \\ & & & 4^3 \end{bmatrix}. \quad (11)$$

3. Bell Matrix with the Pascal Matrix

Let $r_i(P)$ and $c_j(P)$ be the i^{th} row and j^{th} column of the Pascal matrix $P = [p_{i,j}]$ $(i, j \geq 1)$. Well-known recurrence rules of $r_i(P)$ and $c_j(P)$ are

$$\begin{aligned} r_{i+1}(P) &= (r_i(P); 0) + (0; r_i(P)), \\ c_{j+1}(P) &= (0; c_j(P)) + (0; c_{j+1}(P)). \end{aligned} \quad (12)$$

Theorem 3. *For any* $1 \leq j \leq k$, $di[0, 1, \ldots, k-1]$ $(0; c_j(P_{k-1})) = j c_{j+1}(P_k)$ *and* $r_k(P) di[2^{k-1}, \ldots, 2, 1] = r_k(P) P_k$.

Proof. Due to the binomial identity $p\binom{p-1}{q-1} = q\binom{p}{q}$ for $p, q \geq 1$, we have

$$\begin{aligned} j c_{j+1}(P_k) &= j\left(\bar{0}^j, \binom{j}{j}, \binom{j+1}{j}, \binom{j+2}{j}, \ldots, \binom{k-1}{j}\right)^T \\ &= \left(\bar{0}^j, j\binom{j-1}{j-1}, (j+1)\binom{j}{j-1}, (j+2)\binom{j+1}{j-1}, \ldots, (k-1)\binom{k-2}{j-1}\right)^T \\ &= \mathrm{di}[0, \ldots, j-1; j, \ldots, k-1]\left(0, \bar{0}^{j-1}, \binom{j-1}{j-1}, \binom{j}{j-1}, \binom{j+1}{j-1}, \ldots, \binom{k-2}{j-1}\right)^T \\ &= \mathrm{di}[0, 1, \ldots, k-1](0; c_j(P_{k-1})). \end{aligned} \quad (13)$$

Clearly, $r_3(P)\mathrm{di}[2^2, 2, 1] = (1, 2, 1)P_3$. Assume $r_k(P)P_k = r_k(P)\mathrm{di}[2^{k-1}, \ldots, 2, 1]$ for some k. Note that $r_i(P)$ is the set of coefficients of $(x+1)^{i-1}$ and $r_i(P)P_i$ equals $r_i(P^2)$ which is the set of coefficients of $(2x+1)^{i-1}$ expanded in descending order. Thus, (12) with $(2x+1)^k = 2x(2x+1)^{k-1} + (2x+1)^{k-1}$ implies

$$\begin{aligned} r_{k+1}(P)P_{k+1} &= \text{the set of coeff. of } (2x+1)^k \\ &= \text{the set of coeff. of } 2x(2x+1)^{k-1} + \text{the set of coeff. of } (2x+1)^{k-1} \\ &= 2(r_k(P)P_k; 0) + (0; r_k(P)P_k) \\ &= \left(2r_k(P)\mathrm{di}\left[2^{k-1}, \ldots, 1\right]; 0\right) + \left(0; r_k(P)\mathrm{di}\left[2^{k-1}, \ldots, 1\right]\right) \\ &= (r_k(P); 0)\begin{bmatrix} 2^k & & \\ & \ddots 2 & \\ & & 1 \end{bmatrix} + (0; r_k(P))\begin{bmatrix} 2^k & & \\ & 2^{k-1} & \\ & & \ddots 1 \end{bmatrix} \\ &= ((r_k(P) \colon 0) + (0; r_k(P)))\mathrm{di}\left[2^k, \ldots, 1\right] \\ &= r_{k+1}(P)\mathrm{di}\left[2^k, \ldots, 1\right]. \end{aligned} \quad (14)$$

□

A matrix $P_k^F = [p_{i,j}^F]$ is called a flipped matrix of $P_k = [p_{i,j}]$ if it is horizontally flipped sideways of P_k. Hence,

$$P_k^F = \begin{bmatrix} p_{1,k} & \cdots & p_{1,1} \\ p_{2,k} & \cdots & p_{2,1} \\ & \cdots & \\ p_{k,k} & \cdots & p_{k,1} \end{bmatrix}, \quad p_{i,j}^F = p_{i,k-j+1}, \quad j \le k. \tag{15}$$

Theorem 4. *Let $r_i(P_k^F P_k)$ be the i^{th} row of $P_k^F P_k$ for $1 \le i \le k$. Then,*

(1) $r_1(P_k^F P_k) = r_k(P_k)$ and $r_k(P_k^F P_k) = r_k(P_k) P_k$

(2) $r_i(P_k^F P_k) = r_{i-1}(P_k^F P_k) + (r_{i-1}(P_{k-1}^F P_{k-1}); 0) = r_k(P_k) + \sum_{t=1}^{i-1}(r_t(P_{k-1}^F P_{k-1}); 0)$

Proof. Clearly, $r_1(P_k^F P_k) = r_1(P_k^F)P_k = (\overline{0}^{k-1}; 1)P_k = r_k(P_k)$ and $r_k(P_k^F P_k) = r_k(P_k^F)P_k = r_k(P_k)P_k$. And, for $1 \le i < k$, we have

$$\begin{aligned} r_i(P_k^F P_k) + (r_i(P_{k-1}^F P_{k-1}); 0) &= (r_i(P_k^F)) + (r_i(P_{k-1}^F); 0)P_k \\ &= r_{i+1}(P_k^F P_k). \end{aligned} \tag{16}$$

Thus, it follows immediately that

$$\begin{aligned} r_i(P_k^F P_k) &= r_{i-1}(P_k^F P_k) + (r_{i-1}(P_{k-1}^F P_{k-1}); 0) \\ &= r_{i-2}(P_k^F P_k) + (r_{i-2}(P_{k-1}^F P_{k-1}); 0) + (r_{i-1}(P_{k-1}^F P_{k-1}); 0) \\ &= r_{i-3}(P_k^F P_k) + (r_{i-3}(P_{k-1}^F P_{k-1}); 0) + (r_{i-2}(P_{k-1}^F P_{k-1}); 0) + (r_{i-1}(P_{k-1}^F P_{k-1}); 0) \\ &= \cdots = r_1(P_k^F P_k) + (r_1(P_{k-1}^F P_{k-1}); 0) + (r_2(P_{k-1}^F P_{k-1}); 0) + \cdots + (r_{i-1}(P_{k-1}^F P_{k-1}); 0) \\ &= r_k(P_k) + \sum_{t=1}^{i-1}(r_t(P_{k-1}^F P_{k-1}); 0). \end{aligned} \tag{17}$$

□

We now develop some interrelations of the Bell matrix $B = [b_{i,j}]$, square-type Bell matrix $B^* = [b_{i,j}^*]$, and Pascal matrix P.

Theorem 5. *Let $c_j(B^*)$ be the j^{th} column of B^* for $i, j \le k$. Then,*

$$\begin{aligned} &(1)\ r_{k+1}(P)\begin{bmatrix} b_{i,j} \\ b_{i+1,j} \\ \cdots \\ b_{i+k,j} \end{bmatrix} = b_{i+k,j+k}, \text{ so } P\begin{bmatrix} b_{i,j} \\ b_{i+1,j} \\ b_{i+2,j} \\ \cdots \end{bmatrix} = \begin{bmatrix} b_{i,j} \\ b_{i+1,j+1} \\ b_{i+2,j+2} \\ \cdots \end{bmatrix}. \\ &(2)\ P_k c_j(B^*) = \begin{bmatrix} b_{j+1,1} \\ b_{j+2,1} \\ \cdots \\ b_{j+k,1} \end{bmatrix}, \text{ so } r_i(P_k)c_j(B^*) = b_{j+i,1}. \\ &(3)\ P_{k+1}^F\begin{bmatrix} b_{m,1} \\ b_{m+1,1} \\ \cdots \\ b_{m+k,1} \end{bmatrix} = \begin{bmatrix} b_{m+k,1} \\ b_{m+k,2} \\ \cdots \\ b_{m+k,k+1} \end{bmatrix}, \text{ so } r_i(P_{k+1}^F)\begin{bmatrix} b_{m,1} \\ b_{m+1,1} \\ \cdots \\ b_{m+k,1} \end{bmatrix} = b_{m+k,i}. \end{aligned} \tag{18}$$

Proof

$$b_{i+1,j+1} = r_2(P)\begin{bmatrix} b_{i,j} \\ b_{i+1,j} \end{bmatrix},$$

$$b_{i+2,j+2} = b_{i,j} + 2b_{i+1,j} + b_{i+2,j} = r_3(P)\begin{bmatrix} b_{i,j} \\ b_{i+1,j} \\ b_{i+2,j} \end{bmatrix}. \tag{19}$$

So if we assume

$$r_k(P)\begin{bmatrix} b_{i,j} \\ b_{i+1,j} \\ \cdots \\ b_{i+(k-1),j} \end{bmatrix} = b_{i+(k-1),j+(k-1)}, \tag{20}$$

for some k, then

$$\begin{aligned} r_{k+1}(P)\begin{bmatrix} b_{i,j} \\ b_{i+1,j} \\ \cdots \\ b_{i+k,j} \end{bmatrix} &= r_k(P)\begin{bmatrix} b_{i,j} \\ \cdots \\ b_{i+(k-1),j} \end{bmatrix} + r_k(P)\begin{bmatrix} b_{i+1,j} \\ \cdots \\ b_{i+k,j} \end{bmatrix} \\ &= b_{i+(k-1),j+(k-1)} + b_{i+k,j+(k-1)} = b_{i+k,j+k}, \end{aligned} \tag{21}$$

by recurrence (12). Comparing $B^* = [b^*_{i,j}]$ with $B = [b_{i,j}]$, we have

$$\begin{aligned} b^*_{i,j} &= b_{i+j-1,j}, \\ b^*_{1,j} &= b^*_{j+1,1}, \\ b^*_{i,j} &= b^*_{i,j-1} + b^*_{i+1,j-1}. \end{aligned} \tag{22}$$

Thus, with $c_j(B^*) = \begin{bmatrix} b^*_{1,j} \\ b^*_{2,j} \\ b^*_{3,j} \\ \cdots \end{bmatrix} = \begin{bmatrix} b_{j,j} \\ b_{j+1,j} \\ b_{j+2,j} \\ \cdots \end{bmatrix}$, (1) implies

$$P_k c_j(B^*) = P_k\begin{bmatrix} b_{j,j} \\ b_{j+1,j} \\ \cdots \\ b_{j+k-1,j} \end{bmatrix} = \begin{bmatrix} b_{j,j} \\ b_{j+1,j+1} \\ \cdots \\ b_{j+k-1,j+k-1} \end{bmatrix} = \begin{bmatrix} b_{j+1,1} \\ b_{j+2,1} \\ \cdots \\ b_{j+k,1} \end{bmatrix}. \tag{23}$$

Moreover, (1) gives rise to (3) such that

$$\begin{aligned} P^F_{k+1}\begin{bmatrix} b_{m,1} \\ b_{m+1,1} \\ \cdots \\ b_{m+k,1} \end{bmatrix} &= \begin{bmatrix} & & & & 1 \\ & & & 1 & 1 \\ & & \cdots & & \\ 1 & k & \cdots & k & 1 \end{bmatrix}\begin{bmatrix} b_{m,1} \\ b_{m+1,1} \\ \cdots \\ b_{m+k,1} \end{bmatrix} \\ &= \left(b_{m+k,1}, r_2(P)\begin{bmatrix} b_{m+k-1,1} \\ b_{m+k,1} \end{bmatrix}, r_3(P)\begin{bmatrix} b_{m+k-2,1} \\ b_{m+k-1,1} \\ b_{m+k,1} \end{bmatrix}, \dots, r_{k+1}(P)\begin{bmatrix} b_{m,1} \\ \cdots \\ b_{m+k,1} \end{bmatrix} \right)^T \\ &= \left(b_{m+k,1}, b_{m+k,2}, b_{m+k,3}, \dots, b_{m+k,k+1}\right)^T. \end{aligned} \tag{24}$$

□

Theorem 6. $b^*_{i,j} = \begin{cases} b^*_{i,j-1} + r_{j-1}(P^F_{i-1}P_{i-1})c_j(B^*), & \text{if } j>1, \\ r_{i-1}(P)c_1(B^*), & \text{if } j=1. \end{cases}$

Proof. Clearly, $b^*_{i,1} = b_{i,1} = r_{i-1}(P)c_1(B^*)$ by Theorem 5 (2). And, observe $b^*_{3,j}$ $(1<j\le 3)$ from $B^*_3 = \begin{bmatrix} 1 & 2 & 5 \\ 1 & 3 & 10 \\ 2 & 7 & 27 \end{bmatrix}$ such that $b^*_{3,2} = 7 = 2 + (1,1)(2,3)^T = b^*_{3,1} + r_1(P^F_2P_2)(b^*_{1,2}, b^*_{2,2})^T$ and $b^*_{3,3} = 27 = 7 + (2,1)(5,10)^T = b^*_{3,2} + r_2(P^F_2P_2)(b^*_{1,3}, b^*_{2,3})^T$.

From $B^*_4 = \begin{bmatrix} & & & 15 \\ & B^*_3 & & 37 \\ & & & 114 \\ 5, & 20, & 87 & 409 \end{bmatrix}$, $b^*_{4,j}$ $(1<j\le 4)$ satisfy $b^*_{4,2} = 20 = 5 + (1,2,1)(2,3,7)^T = b^*_{4,1} + r_1(P^F_3P_3)(b^*_{1,2}, b^*_{2,2}, b^*_{3,2})^T$, $b^*_{4,3} = 87 = 20 + (2,3,1)(5,10,27)^T = b^*_{4,2} + r_2(P^F_3P_3)(b^*_{1,3}, b^*_{2,3}, b^*_{3,3})^T$, and $b^*_{4,4} = 409 = 87 + (4,4,1)(15,37,114)^T = b^*_{4,3} + r_3(P^F_3P_3)(b^*_{1,4}, b^*_{2,4}, b^*_{3,4})^T$.

Now, for some i, we assume $b^*_{i,j} = b^*_{i,j-1} + r_{j-1}(P^F_{i-1}P_{i-1})c_j(B^*)$ for all $1 < j < i$. Then, $b^*_{i+1,j} = b^*_{i,j+1} - b^*_{i,j}$ equals

$$b^*_{i+1,j} = \left(b^*_{i,j} + r_j\left(P^F_{i-1}P_{i-1}\right)c_{j+1}(B^*)\right) - \left(b^*_{i,j-1} + r_{j-1}\left(P^F_{i-1}P_{i-1}\right)c_j(B^*)\right)$$
$$= \left(b^*_{i,j} - b^*_{i,j-1}\right) + \left(r_j\left(P^F_{i-1}P_{i-1}\right)c_{j+1}(B^*) - r_{j-1}\left(P^F_{i-1}P_{i-1}\right)c_j(B^*)\right). \quad (25)$$

But, since

$$r_j\left(P^F_{i-1}P_{i-1}\right)c_{j+1}(B^*) = r_j\left(P^F_{i-1}\right)P_{i-1}c_{j+1}(B^*) = r_j\left(P^F_{i-1}\right)\begin{bmatrix} b_{j+2,1} \\ b_{j+3,1} \\ \cdots \\ b_{j+i,1} \end{bmatrix} = b_{i+j,j},$$
$$r_{j-1}\left(P^F_{i-1}P_{i-1}\right)c_j(B^*) = b_{i+j-1,j-1}, \quad (26)$$

by Theorem 5, we have

$$b^*_{i+1,j} = \left(b^*_{i,j} - b^*_{i,j-1}\right) + \left(b_{i+j,j} - b_{i+j-1,j-1}\right)$$
$$= b^*_{i+1,j-1} + b_{i+j,j-1} = b^*_{i+1,j-1} + r_{j-1}\left(P^F_i\right)\begin{bmatrix} b_{j,j} \\ b_{j+1,j+1} \\ b_{j+2,j+2} \\ \cdots \\ b_{j+i-1,j+i-1} \end{bmatrix}$$
$$= b^*_{i+1,j-1} + r_{j-1}\left(P^F_i\right)P_i c_j(B^*) = b^*_{i+1,j-1} + r_{j-1}\left(P^F_i P_i\right)c_j(B^*). \quad (27)$$

□

4. LU Factorization of the Square-Type Bell Matrix

We are ready to have an LU factorization of $B^*_k = [b^*_{i,j}]$ with diagonal entries.

Theorem 7. $B^*_k = L_k U_k$ $(1 \le k \le 5)$ *where the lower triangular matrix* $L_k = \widetilde{S}_k P_k$ *and the upper triangular matrix* U_k *has diagonal entries* $\{1, 1, 2!, 3!, 4!\}$.

Proof. Let $U_k = [u_{i,j}]$ $(1 \le i, j \le k)$ be with

$$u_{1,j} = b^*_{1,j},$$
$$u_{2,j} = b^*_{2,j} - u_{1,j}, \text{ for all } j \ge 1. \quad (28)$$

Then, $B^*_2 = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$ yields $U_2 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$. And, $B^*_2 U_2^{-1} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \widetilde{S}_2 P_2$ is a lower triangular matrix; denote it by L_2.

From $B^*_3 = \begin{bmatrix} 1 & 2 & 5 \\ 1 & 3 & 10 \\ 2 & 7 & 27 \end{bmatrix}$, clearly $u_{1,3} = b^*_{1,3}$ and $u_{2,3} = b^*_{2,3} - u_{1,3} = 5$ by (28). Let

$$u_{3,j} = \left[b^*_{3,j} - r_2(P)\left(b^*_{1,j}, b^*_{2,j}\right)^T\right] - 2u_{2,j}, \text{ for all } j \ge 1. \quad (29)$$

Then, the identity $b^*_{k+2,1} = r_{k+1}(P)c_1(B^*)$ in Theorem 5 (2) implies $b^*_{3,1} = r_2(P)(b^*_{1,1}, b^*_{2,1})^T$, so $u_{3,1} = 0$ because $u_{2,1} = 0$. Similarly, $u_{3,2} = [b^*_{3,2} - r_2(P)(b^*_{1,2}, b^*_{2,2})^T] - 2u_{2,2} = 2 - 2 \cdot 1 = 0$, $u_{3,3} = [b^*_{3,3} - r_2(P)(b^*_{1,3}, b^*_{2,3})^T] - 2u_{2,3} = 12 - 2 \cdot 5 = 2$, so

$$U_3 = \begin{bmatrix} U_2 & u_{1,3} \\ & u_{2,3} \\ u_{3,1}, u_{3,2} & u_{3,3} \end{bmatrix} = \begin{bmatrix} 1 & 2 & 5 \\ & 1 & 5 \\ & & 2 \end{bmatrix},$$
$$B^*_3 U_3^{-1} = \begin{bmatrix} 1 & & \\ 1 & 1 & \\ 2 & 3 & 1 \end{bmatrix} = \widetilde{S}_3 P_3, \quad (30)$$

is a lower triangular matrix and denote it by L_3.

From $B_4^* = \begin{bmatrix} B_3^* & \begin{matrix}15\\37\\114\end{matrix} \\ 5,20,87 & 409 \end{bmatrix}$, (28), and (29), we have $u_{1,4} = b_{1,4}^*$, $u_{2,4} = b_{2,4}^* - u_{1,4} = 22$, and $u_{3,4} = [b_{3,4}^* - r_2(P)(b_{1,4}^*, b_{2,4}^*)^T] - 2u_{2,4} = 18$. Now, let

$$u_{4,j} = \left[b_{4,j}^* - r_3(P)c_j(B^*)\right] - (5,5)\left(u_{2,j}, u_{3,j}\right)^T, \text{ for all } j \ge 1. \tag{31}$$

Since $b_{4,1}^* = r_3(P)c_1(B^*)$ and $u_{2,1} = u_{3,1} = 0$, we have $u_{4,1} = 0$, $u_{4,2} = [b_{4,2}^* - r_3(P)c_2(B^*)] - (5,5)\ (u_{2,2}, u_{3,2})^T = 5 - (5,5)(1,0)^T = 0$, $u_{4,3} = [b_{4,3}^* - r_3(P)c_3(B^*)] - (5,5)\ (u_{2,3}, u_{3,3})^T = 35 - (5,5)(5,2)^T = 0$, and $u_{4,4} = [b_{3,1}^* - r_3(P)c_4(B^*)] - (5,5)(u_{2,4}, u_{3,4})^T = 6$. Therefore,

$$U_4 = \begin{bmatrix} U_3 & \begin{matrix}15\\22\\18\end{matrix} \\ \vec{0}^3 & 6 \end{bmatrix},$$

$$B_4^* U_4^{-1} = \begin{bmatrix} L_3 & 0 \\ 5,10,6 & 1 \end{bmatrix} = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & 1 & 1 & \\ & 1 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ 1 & 2 & 1 & \\ 1 & 3 & 3 & 1 \end{bmatrix} = \widetilde{S}_4 P_4. \tag{32}$$

Denote it by L_4.

From

$$B_5^* = \begin{bmatrix} B_4^* & \begin{matrix}52\\151\\523\\2066\end{matrix} \\ 15,67,322,1657 & 9089 \end{bmatrix}, \tag{33}$$

(28), (29), and (31), give $u_{2,5} = b_{2,5}^* - u_{1,5} = 99$, $u_{3,5} = [b_{3,5}^* - r_2(P)c_5(B^*)] - 2u_{2,5} = 122$, and $u_{4,5} = [b_{4,5}^* - r_3(P)c_5(B^*)] - (5,5)(u_{2,5}, u_{3,5})^T = 84$. Now, let

$$W_{5,j} = \left[b_{5,j}^* - r_4(P)c_j(B^*)\right] - 3\left[b_{4,j}^* - r_3(P)c_j(B^*)\right], \tag{34}$$

and let $u_{5,j} = W_{5,j} - (7,6)(u_{3,j}, u_{4,j})^T$ for $j \ge 1$.

Since $u_{3,1} = u_{4,1} = 0$, $b_{4,1}^* = r_3(P)c_1(B^*)$, and $b_{5,1}^* = r_4(P)c_1(B^*)$, we have $u_{5,1} = 0$. Also, $u_{3,2} = u_{4,2} = u_{4,3} = 0$ and $u_{3,3} = 2$ imply $u_{5,2} = W_{5,2} - (7,6)(u_{3,2}, u_{4,2})^T = 15 - 3\cdot 5 = 0$, $u_{5,3} = W_{5,3} - (7,6)\ (u_{3,3}, u_{4,3})^T = 0 = W_{5,4} - (7,6)\ (18,6)^T = u_{5,4}$, and $u_{5,5} = W_{5,5} - (7,6)(122,84)^T = 4!$. Hence,

$$U_5 = \begin{bmatrix} U_4 & \begin{matrix}52\\99\\122\\84\end{matrix} \\ \vec{0}^4 & 4! \end{bmatrix},$$
$$B_5^* U_5^{-1} = \begin{bmatrix} L_4 & 0 \\ 15,37,31,10 & 1 \end{bmatrix} = \widetilde{S}_5 P_5. \tag{35}$$

Denote it by L_5. □

Note $u_{j,j} = (j-1)!$ for $1 \le j \le 5$. From (34), we let

$$W_{t,j} = \left[b_{t,j}^* - r_{t-1}(P)c_j(B^*)\right] - 3\left[b_{t-1,j}^* - r_{t-2}(P)c_j(B^*)\right], \quad \text{for all } j \ge 1. \tag{36}$$

Theorem 8. *Assume the matrix U_k in Theorem 7 further satisfies $u_{6,j} = W_{6,j} - \Gamma_4(u_{2,j}, \dots, u_{5,j})^T$ and $u_{7,j} = W_{7,j} - \Gamma_5(u_{2,j}, \dots, u_{6,j})^T$ with row matrices $\Gamma_4 = (7,33,34,11)$ and $\Gamma_5 = (47,174,202,93,17)$. Then, for $1 \le k \le 7$, U_k is an upper triangular matrix having $u_{k,k} = (k-1)!$ and $L_k = B_k^* U_k^{-1}$ is a lower triangular matrix such that $L_k = \widetilde{S}_k P_k$.*

Proof. From

$$B_6^* = \begin{bmatrix} B_5^* & \begin{matrix}203\\674\\2589\\11155\\52922\end{matrix} \\ 52,255,1335,7432,43833 & 272947 \end{bmatrix}, \tag{37}$$

(28), …, (34), show $u_{1,6} = b_{1,6}^*$, $u_{2,6} = 471$, $u_{3,6} = 770$, $u_{4,6} = 810$, and $u_{5,6} = 480$. Let

$$u_{6,j} = W_{6,j} - (t_1, t_2, t_3, t_4)\left(u_{2,j}, u_{3,j}, u_{4,j}, u_{5,j}\right)^T, \quad \text{for some } t_i \in \mathbb{Z}. \tag{38}$$

Note from Theorem 5 that $u_{2,1} = u_{3,1} = u_{4,1} = u_{5,1} = 0$, $b_{5,1}^* = r_4(P)c_1(B^*)$, and $b_{6,1}^* = r_5(P)c_1(B^*)$. Thus, $u_{6,1} = 0$ by (38). And, we also observe $u_{6,2} = W_{6,2} - (t_1, t_2, t_3, t_4)(u_{2,2}, \dots, u_{5,2})^T = 52 - 3\cdot 15 - (t_1, t_2, t_3, t_4)(1; \vec{0}^3)^T$, so $u_{6,2} = 0$ if $t_1 = 7$. Similarly, since $W_{6,3} = 458 - 3\cdot 119$, $W_{6,4} = 3292 - 3\cdot 780$, and $W_{6,5} = 22686 - 3\cdot 4949$ from (36), in order to be $u_{6,3}, u_{6,4}$, and $u_{6,5}$ all zeros, the identities

$$\begin{aligned} u_{6,3} &= W_{6,3} - (t_1,\dots,t_4)(u_{2,3},\dots,u_{5,3})^T = 101 - (7,t_2,t_3,t_4)(5,2;\overline{0}^2)^T, \\ u_{6,4} &= W_{6,4} - (t_1,\dots,t_4)(u_{2,4},\dots,u_{5,4})^T = 952 - (7,t_2,t_3,t_4)(22,18,6,0)^T, \\ u_{6,5} &= W_{6,5} - (t_1,\dots,t_4)(u_{2,5},\dots,u_{5,5})^T = 7839 - (7,t_2,t_3,t_4)(99,122,84,24)^T, \end{aligned} \tag{39}$$

yield $t_2 = 33$, $t_3 = 34$, and $t_4 = 11$. Thus, with $W_{6,6} = 156972 - 3 \cdot 31775$ in (36), we have $u_{6,6} = W_{6,6} - (7,33,34,11)(u_{2,6},\dots,u_{5,6})^T = 5!$.

Hence,

$$U_6 = \begin{bmatrix} & 203 \\ & 471 \\ U_5 & 770 \\ & 810 \\ & 480 \\ \overline{0}^5 & 5! \end{bmatrix},$$

$$L_6 = B_6^* U_6^{-1} = \begin{bmatrix} L_5 & 0 \\ 52,151,160,75,15 & 1 \end{bmatrix} = \widetilde{S}_6 P_6. \tag{40}$$

Similarly, from

$$B_7^* = \begin{bmatrix} & 877 \\ & 3263 \\ B_6^* & 13744 \\ & 64077 \\ & 325869 \\ & 1788850 \\ 203,1080,6097,36401,229114,1515903 & 10515147 \end{bmatrix}, \tag{41}$$

we get $(u_{1,7},\dots,u_{6,7}) = (877,2386,4832,6840,6240,3240)$ by (28), …, (38). Now, let

$$u_{7,j} = W_{7,j} - (t_1,\dots,t_5)(u_{2,j},\dots,u_{6,j})^T, \quad \text{for some } t_i \in \mathbb{Z}. \tag{42}$$

Since $u_{i,1} = 0$ $(1 \le i \le 6)$, $b_{7,1}^* = r_6(P)c_1(B^*)$, and $b_{6,1}^* = r_5(P)c_1(B^*)$, we also have $u_{7,1} = 0$. In order to get $u_{7,j} = 0$ $(2 \le j \le 6)$, the integers t_i $(1 \le i \le 5)$ are determined as follows: Note that $W_{7,2} = 203 - 3 \cdot 52$ from (36), so $0 = u_{7,2} = W_{7,2} - (t_1,\dots,t_5)(1;\overline{0}^4)^T$ implies $t_1 = 47$.

Analogously, with $W_{7,3} = 1957 - 3 \cdot 458$, $W_{7,4} = 15254 - 3 \cdot 3292$, $W_{7,5} = 113139 - 3 \cdot 22686$, and $W_{7,6} = 837333 - 3 \cdot 156972$ from (36), we also have

$$\begin{aligned} u_{7,3} &= W_{7,3} - (47,174,t_3,t_4,t_5)(5,2;\overline{0}^3)^T = 0 \\ u_{7,4} &= W_{7,4} - (47,174,202,t_4,t_5)(22,18,6;\overline{0}^2)^T = 0 \\ u_{7,5} &= W_{7,5} - (47,174,202,93,t_5)(99,122,84,24,0)^T = 0 \\ u_{7,6} &= W_{7,6} - (47,174,202,93,17)(471,770,810,480,120)^T = 0. \end{aligned} \tag{43}$$

Thus, with $(t_1,t_2,t_3,t_4,t_5) = (47,174,202,93,17)$ and $W_{7,7} = 6301550 - 3 \cdot 1110280$ from (36), we have $u_{7,7} = W_{7,7} - (t_1,t_2,t_3,t_4,t_5)(u_{2,7},u_{3,7},\dots,u_{6,7})^T = 6!$. Therefore,

$$U_7 = \begin{bmatrix} & 877 \\ & 2386 \\ U_6 & 4832 \\ & 6840 \\ & 6240 \\ & 3240 \\ \vec{0}^6 & 6! \end{bmatrix},$$
$$L_7 = \begin{bmatrix} L_6 & 0 \\ & \\ 203, 674, 856, 520, 155, 21 & 1 \end{bmatrix} = \widetilde{S}_7 P_7. \quad (44)$$

□

Theorem 9. *The above upper triangular matrix $U_k = [u_{i,j}]$ $(j \geq i)$ satisfies*

$$\begin{aligned} u_{1,j} &= m_{1,1} b^*_{1,j} \text{ with } m_{1,1} = 1, \\ u_{2,j} &= (m_{2,1}, m_{2,2}) c_j(B^*_2), \text{ with } \{m_{2,t}\} = (-1, 1), \\ u_{3,j} &= (m_{3,1}, m_{3,2}, m_{3,3}) c_j(B^*_3) \text{ with } \{m_{3,t}\} = (1, -3, 1), \\ u_{4,j} &= (m_{4,1}, \ldots, m_{4,4}) c_j(B^*_4) \text{ with } \{m_{4,t}\} = (-1, 8, -6, 1), \\ u_{5,j} &= (m_{5,1}, \ldots, m_{5,5}) c_j(B^*_5) \text{ with } \{m_{5,t}\} = (1, -24, 29, -10, 1), \\ u_{6,j} &= (m_{6,1}, \ldots, m_{6,6}) c_j(B^*_6) \text{ with } \{m_{6,t}\} = (-1, 89, -145, 75, -15, 1), \\ u_{7,j} &= (m_{7,1}, \ldots, m_{7,7}) c_j(B^*_7) \text{ with } \{m_{7,t}\} = (1, -415, 814, -545, 160, -21, 1). \end{aligned} \quad (45)$$

Proof. $u_{1,j} = b^*_{1,j}$ and $u_{2,j} = b^*_{2,j} - u_{1,j} = -b^*_{1,j} + b^*_{2,j} = (-1, 1) c_j(B^*_2)$ by Theorems 7 and 8. Similarly, we have

$$\begin{aligned} u_{3,j} &= \left[b^*_{3,j} - r_2(P)\left(b^*_{1,j}, b^*_{2,j}\right)^T \right] - 2u_{2,j} \\ &= \left[b^*_{3,j} - (1, 1)\left(b^*_{1,j}, b^*_{2,j}\right)^T \right] - 2(-1, 1)\left(b^*_{1,j}, b^*_{2,j}\right)^T \\ &= (1, -3, 1)\left(b^*_{1,j}, b^*_{2,j}, b^*_{3,j}\right)^T = (1, -3, 1) c_j(B^*_3) \\ u_{4,j} &= \left[b^*_{4,j} - r_3(P)\left(b^*_{1,j}, b^*_{2,j}, b^*_{3,j}\right)^T \right] - (5, 5)\left(u_{2,j}, u_{3,j}\right)^T = (-1, 8, -6, 1) c_j(B^*_4). \end{aligned} \quad (46)$$

Moreover, with $c_j(B^*) = (b^*_{1,j}, b^*_{2,j}, \ldots)^T$, we also have

$$\begin{aligned} u_{5,j} &= \left[b^*_{5,j} - r_4(P) c_j(B^*_4) \right] - 3\left[b^*_{4,j} - r_3(P) c_j(B^*_3) \right] - (7, 6)\left(u_{3,j}, u_{4,j}\right)^T \\ &= (1, -24, 29, -10, 1) c_j(B^*_5), \end{aligned} \quad (47)$$

and, similarly, the rest $u_{6,j}$ and $u_{7,j}$ follow immediately. □

With all $m_{i,t}$'s in Theorem 9, write a lower triangular matrix

$$M_7 = [m_{i,t}] = \begin{bmatrix} 1 \\ -1 & 1 \\ 1 & -3 & 1 \\ -1 & 8 & -6 & 1 \\ 1 & -24 & 29 & -10 & 1 \\ -1 & 89 & -145 & 75 & -15 & 1 \\ 1 & -415 & 814 & -545 & 160 & -21 & 1 \end{bmatrix}. \tag{48}$$

Then,

$$U_7 = M_7 B_7^* = \begin{bmatrix} 1 & 2 & 5 & 15 & 52 & 203 & 877 \\ & 1 & 5 & 22 & 99 & 471 & 2386 \\ & & 2 & 18 & 122 & 770 & 4832 \\ & & & 6 & 84 & 810 & 6840 \\ & & & & 24 & 480 & 6240 \\ & & & & & 120 & 3240 \\ & & & & & & 720 \end{bmatrix}, \tag{49}$$

gives an LU factorization $B_7^* = M_7^{-1} U_7$, where

$$L_7 = M_7^{-1} = \begin{bmatrix} 1 \\ 1 & 1 \\ 2 & 3 & 1 \\ 5 & 10 & 6 & 1 \\ 15 & 37 & 31 & 10 & 1 \\ 52 & 151 & 160 & 75 & 15 & 1 \\ 203 & 674 & 856 & 520 & 155 & 21 & 1 \end{bmatrix} = \widetilde{S}_7 P_7. \tag{50}$$

Observe that the matrix $M_7 = [m_{i,t}]$ is the exponential Riordan array (without signs) (refer to [7]) satisfying a recurrence rule

$$m_{i,t} = \left(m_{i-1,t+1}, m_{i,t+1}, m_{i+1,t+1}\right)(i-1, i, 1)^T. \tag{51}$$

M is also known as the coefficients of the Charlier polynomial [8], and we may refer Table 3 in [9] for $M^{-1} = L$.

Theorem 10. *Let $M = [m_{i,t}]$ be a matrix satisfying recurrence (51) with $m_{i,1} = (-1)^{i-1}$ and $M_3 = \begin{bmatrix} 1 \\ -1 & 1 \\ 1 & -3 & 1 \end{bmatrix}$. Then, the diagonal entries of the upper triangular matrix $U_k = M_k B_k^*$ are $(k-1)!$ for all $k > 1$.*

Proof. Let $U = [u_{i,j}]$. Then, $U_7 = M_7 B_7^*$ has diagonal entries $u_{i,i}$ $(1 \le i \le 7)$ as $1, 1, 2!, \ldots, 6!$ due to Theorem 9. We note the following identities.

Since $U = MB^*$ is an upper triangular matrix, we have

$$r_i(M) c_j(B^*) = u_{i,j} = 0, \quad \text{for all } 1 \le j < i. \tag{52}$$

Recurrence (51) gives

$$m_{i,t} = m_{i-1,t-1} - (i-2) m_{i-2,t} - (i-1) m_{i-1,t}, \tag{53}$$

over M. Hence, with (52), the i th row $r_i(M) = (m_{i,1}, \ldots, m_{i,i})$ satisfies

$$\begin{aligned} r_i(M) &= \left(0, m_{i-1,1}, \ldots, m_{i-1,i-1}\right) - (i-2)\left(m_{i-2,1}, \ldots, m_{i-2,i-2}, 0, 0\right) - (i-1)\left(m_{i-1,1}, \ldots, m_{i-1,i-1}, 0\right) \\ &= \left(0; r_{i-1}(M)\right) - (i-2)\left(r_{i-2}(M); \overline{0}^2\right) - (i-1)\left(r_{i-1}(M); 0\right). \end{aligned} \tag{54}$$

On the other hand, by (22), the i th column $c_i(B^*)$ of B^* satisfies

$$c_i(B^*) = \begin{bmatrix} b_{1,i}^* \\ b_{2,i}^* \\ b_{3,i}^* \\ \cdots \end{bmatrix} = \begin{bmatrix} b_{1,i-1}^* + b_{2,i-1}^* \\ b_{2,i-1}^* + b_{3,i-1}^* \\ b_{3,i-1}^* + b_{4,i-1}^* \\ \cdots \end{bmatrix} = c_{i-1}(B^*) + \begin{bmatrix} b_{2,i-1}^* \\ b_{3,i-1}^* \\ b_{4,i-1}^* \\ \cdots \end{bmatrix}. \tag{55}$$

Thus, (52) and (55) together show

$$\begin{aligned} r_i(M) \begin{bmatrix} b_{2,i-1}^* \\ b_{3,i-1}^* \\ \cdots \\ b_{i+1,i-1}^* \end{bmatrix} &= r_i(M) c_i(B^*) - r_i(M) c_{i-1}(B^*) \\ &= u_{i,i} - u_{i,i-1} = u_{i,i}, \end{aligned} \tag{56}$$

and similarly,

$$r_i(M)\begin{bmatrix} b^*_{2,i} \\ b^*_{3,i} \\ \cdots \\ b^*_{i,i} \end{bmatrix} = r_i(M)\left(c_{i+1}(B^*) - c_i(B^*)\right) = u_{i,i+1} - u_{i,i}. \tag{57}$$

Moreover, by (28), (55), and (56), $u_{i,i} = r_i(M)c_i(B^*)$ equals

$$\begin{aligned} u_{i,i} &= (0; r_{i-1}(M))c_{i-1}(B^*) - (i-2)r_{i-2}(M)c_{i-1}(B^*) \\ &\quad - (i-1)r_{i-1}(M)c_{i-1}(B^*) + r_i(M)\begin{bmatrix} b^*_{2,i-1} \\ b^*_{3,i-1} \\ \cdots \\ b^*_{i+1,i-1} \end{bmatrix} \\ &= \left(u_{i-1,i} - u_{i-1,i-1}\right) - (i-2)u_{i-2,i-1} - (i-1)u_{i-1,i-1} + u_{i,i}, \end{aligned} \tag{58}$$

so we have $(u_{i-1,i} - u_{i-1,i-1}) - (i-2)u_{i-2,i-1} = (i-1)u_{i-1,i-1}$. Hence, if we assume $u_{k,k} = (k-1)!$ for all $k \le i$, then $(u_{i-1,i} - u_{i-1,i-1}) - (i-2)u_{i-2,i-1} = (i-1)! = u_{i,i}$, so the induction hypothesis with (58) yields

$$\begin{aligned} u_{i+1,i+1} &= r_{i+1}(M)c_{i+1}(B^*) \\ &= (0; r_i(M))c_i(B^*) - (i-1)\left(r_{i-1}(M); \bar{0}^2\right)c_i(B^*) - i\left(r_i(M); 0\right)c_i(B^*) + r_{i+1}(M)\begin{bmatrix} b^*_{2,i} \\ b^*_{3,i} \\ b^*_{4,i} \\ \cdots \end{bmatrix} \\ &= (0; r_i(M))c_i(B^*) - (i-1)r_{i-1}(M)c_i(B^*) - ir_i(M)c_i(B^*) + r_{i+1}(M)\begin{bmatrix} b^*_{2,i} \\ b^*_{3,i} \\ b^*_{4,i} \\ \cdots \end{bmatrix} \\ &= i! - (i-1)i! + i! = i!. \end{aligned} \tag{59}$$

□

Theorem 11. *The lower triangular matrix $L_k = [l_{i,j}] = \widetilde{S}_k P_k$ in Theorem 8 satisfies $l_{i+1,j+1} = (l_{i,j}, l_{i,j+1}, l_{i,j+2})(1, j+1, j+1)^T$.*

Proof. Since $L_k = \widetilde{S}_k P_k$, for any $1 \le i, j \le k$, Theorem 3 shows

$$\begin{aligned} l_{i+1,j+1} &= r_{i+1}(\widetilde{S})c_{j+1}(P) = \left(\left(0; r_i(\widetilde{S})\right) + r_i(\widetilde{S})\mathrm{di}[0,1,\ldots,i-1]\right)\left(\left(0; c_j(P)\right) + \left(0; c_{j+1}(P)\right)\right) \\ &= \left(0; r_i(\widetilde{S})\right)\left(0; c_j(P)\right) + \left(0; r_i(\widetilde{S})\right)\left(0; c_{j+1}(P)\right) \\ &\quad + \left(r_i(\widetilde{S})\mathrm{di}[0,\ldots,i-1]\right)\left(0; c_j(P)\right) + \left(r_i(\widetilde{S})\mathrm{di}[0,\ldots,i-1]\right)\left(0; c_{j+1}(P)\right) \\ &= l_{i,j} + l_{i,j+1} + r_i(\widetilde{S})\left(\mathrm{di}[0,1,\ldots]\left(0; c_j(P)\right)\right) + r_i(\widetilde{S})\left(\mathrm{di}[0,1,\ldots]\left(0; c_{j+1}(P)\right)\right) \\ &= l_{i,j} + l_{i,j+1} + jr_i(\widetilde{S})c_{j+1}(P) + (j+1)r_i(\widetilde{S})c_{j+2}(P) \\ &= l_{i,j} + l_{i,j+1} + jl_{i,j+1} + (j+1)l_{i,j+2} = \left(l_{i,j}, l_{i,j+1}, l_{i,j+2}\right)(1, j+1, j+1)^T. \end{aligned} \tag{60}$$

□

The L_k and U_k are obtained by the recurrence in Theorems 10 and 11. In fact, from M_7 and L_7, we get

$$L_9 = \begin{bmatrix} & & & \widetilde{S}_7 P_7 & & & & 0 & \\ & & & & & & & & \\ 877 & 3263 & 4802 & 3556 & 1400 & 287 & 28 & 1 & \\ 4140 & 17007 & 28337 & 24626 & 11991 & 3290 & 490 & 36 & 1 \end{bmatrix},$$
$$M_9 = \begin{bmatrix} & & & M_7 & & & & 0 & \\ & & & & & & & & \\ -1 & 2372 & -5243 & 4179 & -1575 & 301 & -28 & 1 & \\ 1 & -16072 & 38618 & -34860 & 15659 & -3836 & 518 & -36 & 1 \end{bmatrix}, \quad (61)$$

so

$$M_9 B_9^* = \begin{bmatrix} U_7 & \ddots & \\ & & \\ 0 & 7!, & 221760 \\ & 0, & 8! \end{bmatrix} = U_9, \quad (62)$$

in which all diagonal entries are $(k-1)!$.

Theorem 12. $detB_k^* = \prod_{i=1}^{k-1} i! = (k-1)!\, detB_{k-1}^*$.

Clearly, $\det B_k^* = \det L_k \det U_k = \det U_k = \prod_{i=1}^{k-1} i! = (k-1)!\ \det B_{k-1}^*$.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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