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*Research Article*

# Catalan Transform of $k$-Balancing Sequences

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In this work, the Catalan transformation (CT) of $k$-balancing sequences, $\{B_{k,n}\}_{n\geq 0}$, is introduced. Furthermore, the obtained Catalan transformation $\{CB_{k,n}\}_{n\geq 0}$ was shown as the product of lower triangular matrices called Catalan matrices and the matrix of $k$-balancing sequences, $\{B_{k,n}\}_{n\geq 0}$, which is an $n\times 1$ matrix. Apart from that, the Hankel transform is applied further to calculate the determinant of the matrices formed from $\{CB_{k,n}\}_{n\geq 0}$.

## 1. Introduction

A balancing number $n$ is a positive integer that satisfies the Diophantine equation as follows:

$$1+2+\cdots+(n-1)=(n+1)+\cdots+(n+r), \tag{1}$$

for some positive integer $r$, called the balancer corresponding to $n$ [1]. The $n$-th balancing number is denoted by $B_n$. For each $n$, $8B_n^2+1$ is a perfect square, and its positive square root is called a Lucas-balancing number [2]. The balancing numbers are solutions of Pell's equation $y^2-8x^2=1$, and its Binet formula is given by

$$B_n=\frac{\alpha^{2n}-\beta^{2n}}{\alpha^2-\beta^2}, \tag{2}$$

where $\alpha=1+\sqrt{2}$ and $\beta=1-\sqrt{2}$. Furthermore, the balancing numbers $B_n$ satisfy the recurrence relation as follows:

$$B_{n+1}=6B_n-B_{n-1}, \tag{3}$$

with initial terms $B_0=0$ and $B_1=1$ [2–10].

There are several transformations that operate on various integer sequences. Some of them are Binomial Transform [11], Discrete Cosine Transform (DCT) [12], Laguerre Transform [13], Lah Transform [14], Discrete Wavelet Transform (DWT) [15], Discrete Fourier Transform (DFT), and Catalan Transform [16]. In the real life processes, these transformations are basically utilized in various Steganographic schemes and techniques [12–14]. In addition, there are numerous other usages of the above displayed transforms but despite that certain shortcomings of some of the transform techniques are still observed. For instance, the coefficients of the DCT are real-valued which makes the calculation very much time-consuming. Furthermore, the coefficients of DFT are, in general, complex numbers which generate complex value as output. In addition, the computational complexity of DFT is $O(n^2)$ which also suggests that the transform is cumbersome and time-consuming. To overcome the shortcomings discussed regarding the different transformations, the Catalan transformation, introduced by Paul Barry [16] in 2005, proves to be a superior alternative. The reason behind CT being an appropriate substitute is due to the fact that unlike the earlier mentioned transformations, the overall calculation of Catalan transform is integer-based and not based on floating-point values which makes it very faster and reliable method of transformation. Furthermore, the computational complexity of Catalan transform is $O(n\log(n))$ which also indicates a substantial improvement as we go from $O(n^2)$ of DFT. Apart from the real life phenomena, the beneficial property of the Catalan transform makes it quite suitable to be applied for the derivation of important results relating to the properties of several integer sequences [17–19]. Several classical core sequences, like the Fibonacci sequence $\{F_n\}$ [20], Pell sequence $\{P_n\}$ [21], and Jacobsthal sequence $\{J_n\}$ [17], can be paired by means of this transformation.

There are some research studies where the Catalan transform has been applied to the $k$-Fibonacci sequences [22]. Furthermore, the Catalan transform was applied to the $k$-Jacobsthal sequence in [23]. In [24], the Catalan transformation was applied to the $k$-Lucas sequence. In addition, a new sequence realizing the known Lucas numbers has been discussed by Özkan et al. [25]. In application point of view, Mukhopadhyay et al. [26] dealt with the secured image steganography via CT. In addition, the Lah transform has been applied for security data in the field of telecommunication in [14]. The past literature depicts the Catalan transform of various sequences but it was observed that the balancing sequence was never been tackled via the Catalan transform which illustrates the novelty of the present work. Hence, in the present work, deriving motivation from the literature survey, we employed the Catalan transformation in case of the $k$-balancing sequence $\{B_{k,n}\}_{n\geq 0}$, discussed in the later section, and inspected the properties of the sequences. Then, we apply the Hankel transform to the Catalan transform of $\{B_{k,n}\}$.

The paper has been arranged in the following manner: first the preliminaries related to the $k$-balancing sequence, Catalan numbers, Hankel transform, and Catalan transform are presented in Section 2. Furthermore, Section 3 is devoted towards the evaluation of Catalan transformation of the $k$-balancing sequence. Apart from that, the generating function is calculated for the CT of $k$-balancing sequence in Section 4. The next section deals with the Hankel transform of CT of $k$-balancing numbers. The concluding remarks are mentioned in the final section.

## 2. Preliminaries

*2.1. k-Balancing Numbers.* The $k$-balancing numbers, denoted by $\{B_{k,n}\}_{n=0}^{\infty}$, for any positive number $k$, are defined by the recursive relation as follows [27]:

$$B_{k,n+1} = 6kB_{k,n} - B_{k,n-1}, \quad n \geq 1, \tag{4}$$

with the initial values $B_{k,0} = 0$ and $B_{k,1} = 1$. Furthermore, the Binet formula for $k$-balancing numbers is given by

$$B_{k,n} = \frac{\alpha_1^n - \alpha_2^n}{\alpha_1 - \alpha_2}, \tag{5}$$

where $\alpha_1 = 3k + \sqrt{9k^2 - 1}$ and $\alpha_2 = 3k - \sqrt{9k^2 - 1}$. Hence, the first few $k$-balancing numbers are given by $B_{k,0} = 0$, $B_{k,1} = 1$, $B_{k,2} = 6k$, $B_{k,3} = 36k^2 - 1$, $B_{k,4} = 216k^3 - 12k$, $B_{k,5} = 1296k^4 - 108k^2 + 1$, and so on. The $k$-balancing numbers are tabulated in Table 1.

Table 1: Some $k$-balancing numbers.

| $n$ | $B_{k,n}$ |
|---|---|
| 0 | 0 |
| 1 | 1 |
| 2 | $6k$ |
| 3 | $36k^2 - 1$ |
| 4 | $216k^3 - 12k$ |
| 5 | $1296k^4 - 108k^2 + 1$ |
| 6 | $7776k^5 - 864k^3 + 18k$ |

*2.2. Catalan Numbers.* The Catalan numbers [28], termed after the French-Belgian mathematician named Eugene Charles Catalan, are defined as a sequence of natural numbers that are encountered in several counting problems and specifically occur where the items are defined recursively. They are denoted by $C_n$ and are further defined by [16], in terms of binomial coefficients, as

$$C_n = \frac{1}{n+1}\binom{2n}{n} = \frac{(2n)!}{(n+1)!n!}. \tag{6}$$

Furthermore, by virtue of [29], we have the recurrence relation of the Catalan numbers $C_n$ as

$$C_{n+1} = \frac{2(2n+1)}{n+2}C_n, \tag{7}$$

with $C_0 = 1$, and its ordinary generating function is given by [29]

$$c(x) = \frac{1-\sqrt{1-4x}}{2x}, \tag{8}$$

$$= 1 + x + 2x^2 + 5x^3 + 14x^4 + \cdots. \tag{9}$$

Furthermore, for some $n$, the first few Catalan numbers [28] are given by (sequence A000108 in [30]):

$$\{1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, \ldots\}. \tag{10}$$

*2.3. The Hankel Transform.* By definition, the Hankel matrix of a sequence of real numbers $\{g_0, g_1, g_2, \ldots\}$ is given by the infinite matrix as follows [31]:

$$\mathscr{H}_n(g) = \begin{bmatrix} g_0 & g_1 & g_2 & g_3 & g_4 & g_5 & \cdots \\ g_1 & g_2 & g_3 & g_4 & g_5 & g_6 & \cdots \\ g_2 & g_3 & g_4 & g_5 & g_6 & g_7 & \cdots \\ g_3 & g_4 & g_5 & g_6 & g_7 & g_8 & \cdots \\ g_4 & g_5 & g_6 & g_7 & g_8 & g_9 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}. \tag{11}$$

Furthermore, the Hankel transform [29] of a sequence of real numbers $\{g_0, g_1, \ldots\}$ is defined as the sequence of determinants as follows:

$$H_n = \mathrm{Det}(\mathscr{H}_n(g)), \tag{12}$$

where $\mathscr{H}_n(g)$ is the Hankel matrix given by $\mathscr{H}_n(g) = (g_{i+j-2})_{0<i,j\leq n+1}$. Alternatively, it can be said that

the Hankel transform produces the sequence of determinants of the Hankel matrices formed from the known or given sequence. Furthermore, the Hankel transform of the Catalan sequence is given by the sequence $\{1, 1, 1, \ldots\}$ as given in [30]. In addition, the Hankel transform of the sum of consecutive generalized Catalan numbers is the bisection of the classical Fibonacci sequence [32].

**Lemma 1.** *The Hankel transform of a sequence is invariant under the binomial transform of that sequence. In other words, if we write the binomial transform of a given sequence $x_n$ as*

$$y_n = \sum_{k=0}^{n} \binom{n}{k} x_k, \tag{13}$$

*then*

$$\mathrm{Det} = \left(x_{i+j-2}\right)_{1\le i,j\le n+1} = \left(y_{i+j-2}\right)_{1\le i,j\le n+1}. \tag{14}$$

*Proof.* The proof of the lemma can be referred from [31]. □

*2.4. The Catalan Transformation*

*Definition 1.* The Catalan transformation, as introduced in [16], is a sequence transform which is defined as follows.

Let us consider $\{d_n\}_{n\ge0}$ to be a sequence with generating function as follows:

$$G(x) = d_0 + d_1 x + d_2 x^2 + \cdots. \tag{15}$$

Then, the Catalan transformation of $\{d_n\}$ is defined to be the sequence whose ordinary generating function (o.g.f.) is given by $G(xc(x))$, where $c(x)$ is the series as defined in (8) [16]. The Catalan transformation is also linked to various known transforms, particularly the binomial transformation.

**Lemma 2.** *The Catalan transformation $b_n$ of a given sequence $a_n$ is given by*

$$b_n = \sum_{i=0}^{n} \frac{i}{2n-i} \binom{2n-i}{n-i} a_i. \tag{16}$$

*Proof.* The proof of the lemma can be referred from [16]. □

## 3. Catalan Transformation of the $k$-Balancing Sequence

By virtue of Lemma 2, we define the Catalan transformation of the $k$-balancing sequence, $\{B_{k,n}\}_{n\ge0}$, as

$$CB_{k,n} = \sum_{i=0}^{n} \frac{i}{2n-i} \binom{2n-i}{n-i} B_{k,i}, \tag{17}$$

with $CB_{k,0} = 0$. Hence, it can be observed that the members of Catalan transform of the first $k$-balancing numbers are the polynomials in $k$ given by

$$\begin{aligned}
CB_{k,1} &= \sum_{i=0}^{1} \frac{i}{2-i} \binom{2-i}{1-i} B_{k,i} \\
&= 1, \\
CB_{k,2} &= \sum_{i=0}^{2} \frac{i}{4-i} \binom{4-i}{2-i} B_{k,i} \\
&= 1 + 6k, \\
CB_{k,3} &= \sum_{i=0}^{3} \frac{i}{6-i} \binom{6-i}{3-i} B_{k,i} \\
&= 1 + 12k + 36k^2, \\
CB_{k,4} &= \sum_{i=0}^{4} \frac{i}{8-i} \binom{8-i}{4-i} B_{k,i} \\
&= 2 + 18k + 108k^2 + 216k^3 \\
&= 5 + 5(6k) + 3\left(36k^2 - 1\right) + \left(216k^3 - 12k\right), \\
CB_{k,5} &= \sum_{i=0}^{5} \frac{i}{10-i} \binom{10-i}{5-i} B_{k,i} \\
&= 6 + 36k + 216k^2 + 864k^3 + 1296k^4 \\
&= 14 + 14(6k) + 9\left(36k^2 - 1\right) + 4\left(216k^3 - 12k\right) \\
&\quad + \left(1296k^4 - 108k^2 + 1\right), \\
CB_{k,6} &= \sum_{i=0}^{6} \frac{i}{12-i} \binom{12-i}{6-i} B_{k,i} \\
&= 19 + 102k + 468k^2 + 2160k^3 + 6480k^4 + 7776k^5 \\
&= 42 + 42(6k) + 28\left(36k^2 - 1\right) + 14\left(216k^3 - 12k\right) \\
&\quad + 5\left(1296k^4 - 108k^2 + 1\right) \\
&\quad + \left(18k - 864k^3 + 7776k^5\right).
\end{aligned} \tag{18}$$

Furthermore, we can write the above equations as the product of a lower triangular matrix say $\mathscr{C}$ and a $n \times 1$ matrix $B_k$ which is depicted as

$$\begin{bmatrix} CB_{k,1} \\ CB_{k,2} \\ CB_{k,3} \\ CB_{k,4} \\ CB_{k,5} \\ . \\ . \\ . \end{bmatrix} = \begin{bmatrix} 1 & & & & & \\ 1 & 1 & & & & \\ 2 & 2 & 1 & & & \\ 5 & 5 & 3 & 1 & & \\ 14 & 14 & 9 & 4 & 1 & \\ 42 & 42 & 28 & 14 & 5 & 1 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \end{bmatrix} \begin{bmatrix} B_{k,1} \\ B_{k,2} \\ B_{k,3} \\ B_{k,4} \\ B_{k,5} \\ . \\ . \\ . \end{bmatrix}. \quad (19)$$

The elements of matrix $\mathscr{C}$ satisfy the recursive relation given by

$$\mathscr{C}_{a,b} = \sum_{k=b-1}^{a-1} \mathscr{C}_{a-1,k}. \quad (20)$$

Furthermore, the first and second column of $\mathscr{C}$ are equal for $a > 1$, denoting the Catalan numbers. In addition to it, the lower triangular matrix denoted by $\mathscr{C}_{n,n-a}$ is known as the Catalan triangle and its entries satisfy the relation as follows [22]:

$$\mathscr{C}_{n,n-a} = \frac{(a+1)(2n-a)!}{(n+1)!(n-a)!}, \quad 0 \le a \le n. \quad (21)$$

Hence, the Catalan triangle is obtained via the coefficients of the Catalan transform of the $k$-balancing sequence and is given by the figure in Table 2.

In addition, we obtain the first few terms of the series of Catalan transform of the $k$-balancing sequence as follows:

$$\begin{aligned} CB_1 &= \{0, 1, 7, 49, 344, 2418, 17005, 119617 \ldots\}, \\ CB_2 &= \{0, 1, 13, 169, 2198, 28590, 371887, 4837381, \ldots\}, \\ CB_3 &= \{0, 1, 19, 361, 6860, 130362, 2477305, 47076937, \ldots\}, \\ CB_4 &= \{0, 1, 25, 625, 15626, 390678, 9767659, 244209229, \ldots\}, \\ CB_5 &= \{0, 1, 31, 961, 29792, 923586, 28632229, 887632081, \ldots\}, \\ CB_6 &= \{0, 1, 37, 1369, 50654, 1874238, 69348295, 2565942037, \ldots\}. \end{aligned} \quad (22)$$

Table 2: Catalan triangle of the $k$-balancing sequence.

| | | | | | | |
|---|---|---|---|---|---|---|
| $CB_1$ | 1 | | | | | |
| $CB_2$ | 6 | 1 | | | | |
| $CB_3$ | 36 | 12 | 1 | | | |
| $CB_4$ | 216 | 108 | 18 | 2 | | |
| $CB_5$ | 1296 | 864 | 216 | 36 | 6 | |
| $CB_6$ | 7776 | 6480 | 2160 | 468 | 102 | 19 |
| ... | ... | ... | ... | ... | ... | ... |

## 4. Generating Function of the Catalan Transform of $k$-Balancing Sequence

The generating functions (g.f.s), initially introduced by Abraham de Moivre, are a step to encode an infinite sequence of numbers via treating them as the coefficients of a power series. In other words, the problems about sequences can be transformed into problems about functions by means of the generating functions. Now, in this case, we find the generating function of the Catalan transformation of the $k$-balancing sequence and denote it by $CB_k(x)$. By virtue of (8), we have the generating function of the Catalan numbers as $c(x)$. Furthermore, by virtue of [16], it can be clearly observed that $Z(c(x))$ is the generating function of the Catalan transform of any sequence $\{z_n\}$. Hence, by means of the g.f. of the $k$-balancing sequence in [27], we have

$$\begin{aligned} CB_k(x) &= B_k(x.c(x)) \\ &= \frac{x.c(x)}{1 - 6kx.c(x) - (x.c(x))^2} \\ &= \frac{1 - \sqrt{1-4x}}{1 + 2x - 6k + (6k+1)\sqrt{1-4x}}. \end{aligned} \quad (23)$$

## 5. Hankel Transform of Catalan Transform of $k$-Balancing Numbers

In the present section, we find the Hankel transform of the Catalan transform of $k$-balancing numbers in the form of determinants of Hankel matrices. By considering the Catalan transform of the $k$-balancing sequence of the preceding section, we obtain the following:

$$\begin{aligned} HCB_1 &= \text{Det}\left[CB_{k,1}\right] = \text{Det}[1] = 1 = 1, \\ HCB_2 &= \begin{vmatrix} CB_{k,1} & CB_{k,2} \\ CB_{k,2} & CB_{k,3} \end{vmatrix} \\ &= \begin{vmatrix} 1 & 1+6k \\ 1+6k & 1+12k+36k^2 \end{vmatrix} \\ &= 0, \end{aligned}$$

$$
\begin{aligned}
HCB_3 &= \begin{vmatrix} CB_{k,1} & CB_{k,2} & CB_{k,3} \\ CB_{k,2} & CB_{k,3} & CB_{k,4} \\ CB_{k,3} & CB_{k,4} & CB_{k,5} \end{vmatrix} \\
&= \begin{vmatrix} 1 & 1+6k & 1+12k+36k^2 \\ 1+6k & 1+12k+36k^2 & 2+18k+108k^2+216k^3 \\ 1+12k+36k^2 & 2+18k+108k^2+216k^3 & 6+36k+216k^2+864k^3+1296k^4 \end{vmatrix} \\
&= -1, \\
HCB_4 &= \begin{vmatrix} CB_{k,1} & CB_{k,2} & CB_{k,3} & CB_{k,4} \\ CB_{k,2} & CB_{k,3} & CB_{k,4} & CB_{k,5} \\ CB_{k,3} & CB_{k,4} & CB_{k,5} & CB_{k,6} \\ CB_{k,4} & CB_{k,5} & CB_{k,6} & CB_{k,7} \end{vmatrix} \\
&= \begin{vmatrix} 1 & 1+6k & 1+12k+36k^2 & 7+6k+108k^2+216k^3 \\ 1+6k & 1+12k+36k^2 & 2+18k+108k^2+216k^3 & 6+36k+216k^2+864k^3+1296k^4 \\ 1+12k+36k^2 & 2+18k+108k^2+216k^3 & 6+36k+216k^2+864k^3+1296k^4 & 19+102k+468k^2+2160k^3+6480k^4+7776k^5 \end{vmatrix} \\
&= -1.
\end{aligned}
\tag{24}
$$

## 6. Conclusion

In the present study, the Catalan transformation is applied to the $k$-balancing sequence and some identities are obtained. The identities were further represented in matrix form, and the terms of the CT of $k$-balancing sequence are displayed. Apart from that, the Catalan triangle is obtained by means of the coefficients of the Catalan transform of the $k$-balancing sequence. In addition, the generating function of CT of the sequence is manifested. Finally, the Hankel transform is utilized to the Catalan transformation of the known $k$-balancing sequence.

## Data Availability

No data were used to support this study.

## Conflicts of Interest

The authors declare that they have no conflicts of interest.

## Authors' Contributions

Asim Patra actualized the study, prepared the initial draft, developed methodology, validated the study, and edited the manuscript. Mohammed K. A. Kaabar actualized the study, validated the study, was responsible for resources, supervised the original draft, and edited the manuscript. All authors have taken equal part in this research, and they have read and approved the final manuscript.

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