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Research Article

Shadow Price Approximation for the Fractional Black Scholes Model

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In this work, we used Tran Hung Thao's approximation of fractional Brownian motion to approximate the shadow price of the fractional Black Scholes model. In the case to maximize expectation of the utility function in a portfolio optimization problem under transaction cost, the shadow price is approximated by a Markovian process and semimartingale.

1. Introduction

Let us consider a financial market without frictions and a portfolio consisting of a risky asset S_t and a nonrisky asset S_t^0 such that the dynamics of the evolution of the assets is respectively given by the equations $dS_t = \mu S_t dt + \sigma S_t dW_t$ and $dS_t^0 = rS_t^0 dt$ where r, μ, and σ are real constants. Let ϕ_t^0 be the proportion of the nonrisky asset in the portfolio and ϕ_t^1 the proportion of the risky asset in the portfolio. For any pair $\Theta(t) = (\phi_t^0, \phi_t^1)_{0 \le t \le T} (T \in \mathbb{R})$, the value of the portfolio at time t is given by $V_\Theta(t) = \phi_t^0 S_t^0 + \phi_t^1 S_t$. A portfolio optimization consists in determining an optimal allocation $\Theta(t) = (\phi_t^0, \phi_t^1)_{0 \le t \le T}$ of the portfolio which maximizes the expectation of the utility function under terminal wealth $X_{x,T} = x + \int_0^T \phi_t^1 dS_t$, i.e., find $\Theta(t) = (\phi_t^0, \phi_t^1)$ which maximizes

$$E\left[U\left(X_{x,t}\right)\right] = E\left[U\left(x + \int_0^T \phi_t^1 dS_t\right)\right]. \tag{1}$$

ϕ_t^0 is given by the following relation:

$$\phi_t^0 = x + \int_0^t \phi_t^1 \mathrm{d}S_t - \phi_t^1 S_t, \tag{2}$$

where $U(x)$ is the economic function which accounts for the risk aversion of an economic agent with initial wealth x. Robert Merton first dealt with this problem in the frictionless market case; in [1], optimal control methods is used to solve the (1). The utility function is assumed logarithmic and he proved that the optimal strategy consists to keep a constant portion of risky assets in the portfolio, which is also proportional to the sharp ratio μ/σ^2. In [2], it is proved that this result remains valid when μ_t and σ_t are bounded predictable processes. The following relation holds,

$$\frac{\phi_t^1 S_t}{\phi_t^0 + \phi_t^1 S_t} = \frac{\mu_t}{\sigma_t^2}, \tag{3}$$

where μ_t and σ_t are bounded predictable processes.

In [3], Magill studies the case of the hedging with transaction cost. The hedging of the risky asset is done under transaction cost λ (with $\lambda \in [0;1]$) proportional to the risky asset, i.e., the investor buys the asset at price S_t but receives the amount $(1-\lambda)S_t$ at the time of sale. In this case, the terminal wealth is replaced by $X_{x,T}^\lambda = x + \int_0^T \phi_u^1 dS_u - \lambda \int_0^T S_u d|\phi^1|_u$.

To solve the problem (1), Micharl used the stochastic optimal control theory, which linked in particular the solution of partial differential equations of Hamilton Jacobi–Bellemann type in the Markovian framework, see [4, 5] for the details. An alternative approach called convex duality martingale method has been developed to take into account non-Markovian models, see [6, 7].

This method makes use of the results of convex analysis and martingales. If we consider the equation the maximization of the function $E[U(x+\int_0^T \phi_u^1 dS_u - \lambda \int_0^T S_u d|\phi^1|_u)]$ as the primal of the optimization problem, the convex duality method allows to reduce the problem to the form of the problem (1). The method of convex duality is used to pass from a model with transaction cost to a model without transaction cost, in particular the existence of a new process $\widehat{S}_t$ which is a semimartingale called shadow price such that the optimal hedging strategy of the model with transaction cost coincides with the model without transaction cost.

The existence of the shadow is theoretically proved by the duality methods for portfolio optimization (see [6, 7]). Thanks to the work of Bender and Guasoni (see [8, 9]) on arbitrage, Christoph Czichowsky et al. proved in [2, 7] that the shadow price can exist for a non-semimartingale model under certain conditions.

Thus, the existence of the shadow price has been proved when the price of the risky asset $S_t^{(1)}$ in the portfolio follows a fractional Brownian motion:

$$S_t^{(1)} = S_0 \exp\left(\mu t + \sigma B_t^H\right), \tag{4}$$

where

$$\begin{aligned} B_t^H = C &\int_{-\infty}^0 \left[(t-s)^{H-1/2} - (-s)^{H-1/2}\right] dW_s \\ &+ \int_0^t (t-s)^{H-1/2} dW_s, \end{aligned} \tag{5}$$

and $S_0 > 0$, the process value at $t=0$. We extended the result to

$$S_t^{(2)} = S_0 \exp\left(\mu t + \sigma B_t^H - \frac{1}{2}\sigma^2 t^{2H}\right), \tag{6}$$

and thanks to the work of Tran Dung and Thao [10] on the approximation of processes, it is proposed that an approximation of the shadow price which is a semimartingale process is of the form

$$d\widetilde{S}_t = \widetilde{\mu}\widetilde{S}_t dt + \widetilde{\sigma}\widetilde{S}_t dW_t. \tag{7}$$

The paper is structured as follows: in Section 1, we state some basic facts about the shadow price and its application to the case of a problem driven by $S_t^{(1)} = S_0 \exp(\mu t + \sigma B_t^H)$ and by extension to $S_t^{(2)} = S_0 \exp(\mu t + \sigma B_t^H - 1/2\sigma^2 t^{2H})$ which is a generalization of the classical Black scholes model in the fractional case.

Section 2 is devoted to the recall of some results on fractional Brownian motion, and Section 3 is devoted to our main approximation results.

2. Preliminary

2.1. Existence of the Shadow Price for a Fractional Black Scholes Financial Model. Consider a financial portfolio consisting of a nonrisky asset $B_t = e^{rt}$ and a risky asset S_t defined on a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbf{P})$ having the following dynamics:

$$dS_t = \mu S_t dt + \sigma S_t dB_t^H, \tag{8}$$

where B_t^H denotes fractional Brownian motion and $t \in [0;T]$ with $T < \infty$; this equation is known as the fractional Black scholes equation. Portfolio optimization under transaction cost λ proportional to S_t consists to find an admissible and optimal strategy which maximizes the function $E[U(X_{x,T}(\Theta_t))]$. The optimization problem can be presented in the following form: how to find $\Theta_t = (\phi_t^0, \phi_t^1)$ which maximizes

$$E\left[U\left(X_{x,T}(\Theta_t)\right)\right] = E\left[U\left(x + \int_0^T \phi_u^1 dS_u - \lambda \int_0^T S_u d|\phi^1|_u\right)\right]. \tag{9}$$

Definition 1. Let S_t be a continuous process on a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbf{P})$; the process $\widehat{S}_t$ is called shadow price for the problem (9) if:

(1) $\widehat{S}_t \in [(1-\lambda)S_t; S_t]$.

(2) The solution Ψ_t of the problem of maximization without transaction cost of the utility expectation

$$E[U(V_T(\Psi))] = E\left[U\left(x + \int_0^T \Psi_s d\widehat{S}_s\right)\right], \tag{10}$$

exists and the optimal solution $\widehat{\Psi}_t$ of (10) coincides with the solution of the equation (9) $\widehat{\Theta}$ under transaction transaction cost.

Definition 2. (Bender): let $X = (X_t)_{0 \le t \le T}$ be a real-valued continuous stochastic process. For a finite stopping time τ, $\tau_+ := \inf\{t > \tau\colon X_t - X_\tau > 0\}$ and $\tau_- := \inf\{t > \tau\colon X_t - X_\tau < 0\}$.

X verifies the TWC (two-way crossing) condition of crossing if $\tau_+ = \tau_-$, for all finite stopping times τ.

The existence of the shadow price (see [7]) is related to the following conditions:

(1) S_t is continuous and satisfies (TWC) for U: $[0; +\infty] \longrightarrow \mathbb{R}$

(2) S_t is continuous and sticky for U: $\mathbb{R} \longrightarrow \mathbb{R}$

Czichowsky and Schachermayer (see [2, 11]) use duality results to prove the existence of the shadow price when $S_t^{(1)} = S_0 \exp(\mu t + \sigma B_t^H)$ with $U(x) = \ln(x)$.

Guasoni in [9] shows that if $X_t = X_0 \exp(f_t + \sigma B_t^H)$ with $f_t\colon \mathbb{R}_+ \longrightarrow \mathbb{R}$ a continuous function and if small transaction costs $(\lambda S_t, \lambda \in 0, 1)$ are taken into account, then X_t is sticky and there is no arbitrage in the portfolio. The existence results can be extended $S_t^{(2)} = S_t^{(1)} \exp(-1/2\sigma^2 t^{2H})$.

2.2. Stochastic Calculus for Fractional Brownian Motion and Application to the Fractional Black Scholes Model. In this section, we recall some results on fractional Brownian motion.

Definition 3. We call fractional Brownian motion a Gaussian process B_t^H, $t \in \mathbb{R}_+$ almost surely with continuous trajectories such that $B_0 = 0$ and $B_{t+s}^H - B_t^H$ is independent of $\sigma(B_s^H, s \leq t, s \in T)$ of normal distribution $N(0, t)$ for all $t \geq 0$. In particular, $B_t^{1/2}$ is the standard Brownian motion.

In [12], Benoit and John show that $B_t^H = C\int_{\mathbb{R}}[(t-s)_+^{H-1/2} - (-s)_+^{H-1/2}]dW_s$ with $C = (1/2H + \int_0^\infty [(1+s)^{H-1/2} - s^{H-1/2}]^2 ds)^{-1/2}$, H $\in$]0; 1[et $x_+ = \max(x; 0)$.

If $H \neq 0$, then B_t^H is neither a Markov process, nor a semimartingale with respect to $\mathscr{F}_t$ (see [12, 13] for details and proofs). This process is not semimartingale, and the classical Itô lemma can not be applied, thus we make use of other types of integration theories (so-called Malliavin calculus, Wick-Itô calculus approach, and pathwise calculus), see [12, 14] for more details. In this article, we will use the Wick-Itô formulation which is closer to the Itô calculus and we will try to find some results of the classical Black Scholes model when H tends to 1/2; we will only study the case $H \in 1/2; 1$. Benoit and John [13] show that B_t^H is not differentiable. Let $\widetilde{B}_t^H = C\int_{-\infty}^t [(t+\tau-s)^{H-1/2} - (-s)_+^{H-1/2}]dW_s$, for $\tau > 0$.

$\widetilde{B}_t^H$ is infinitely differentiable (mean square) and we can give a meaning to the derivative of $\widetilde{B}_t^H$.

$$\frac{\widetilde{\mathrm{d}}B_t^H}{\mathrm{d}\tau} = \frac{\widetilde{\mathrm{d}}B_t^H(t+\tau,w)}{\mathrm{d}\tau} = C\left(H-\frac{1}{2}\right)\int_{-\infty}^t (t+\tau-s)^{H-3/2}dW_s. \tag{11}$$

Let S_t be a process such that $dS_t = \mu S_t dt + \sigma S_t dB_t^H$ and $f(t,x)$ a function of class $C^{1,2}$; in [15], we show that

$$\begin{aligned} f(T,S_T) &= f(t,S_t) + \int_t^T \frac{\partial f(u,S_u)}{\partial u}du \\ &+ \int_t^T \mu S_u \frac{\partial f(u,S_u)}{\partial x}du \\ &+ \sigma\int_t^T S_u\frac{\partial f(u,S_u)}{\partial x}dB_u^H \\ &+ H\sigma^2\int_t^T u^{2H-1}S_u^2\frac{\partial^2 f(u,S_u)}{\partial x^2}du. \end{aligned} \tag{12}$$

The differential form of the Wick-Ito lemma for geometric Brownian motion can be written as

$$df(t,S_t) = \left[\frac{\partial f(t,S_t)}{\partial t} + \mu S_t\frac{\partial f(t,S_t)}{\partial S_t} + H\sigma^2 t^{2H-1}S_t^2\frac{\partial^2 f(t,S_t)}{\partial S_t^2}\right]dt + \sigma S_t\frac{\partial f(t,S_t)}{\partial S_t}dB_t^H. \tag{13}$$

Using Tran Dung and Thao's approximation of fractional Brownian motion [10], we have

$$E\left(S_{t,\zeta} - S_t\right)^2 \leq M\zeta^{2\alpha}, \quad \alpha = H - \frac{1}{2}, M > 0, \quad \zeta \in (0,1), \tag{14}$$

and

$$E\left(B_{t,\zeta}^H - B_t^H\right)^2 \leq T\zeta^{2\alpha}, \tag{15}$$

where

$$B_{t,\zeta}^H = C\int_{-\infty}^0 \left[(t-s)^{H-1/2} - (-s)^{H-1/2}\right]dW_s + \int_0^t (t-s+\zeta)^{H-1/2}dW_s, \tag{16}$$

and

$$S_{t,\zeta} = S_0\exp\left(\mu t + \sigma B_{t,\zeta}^H - \frac{1}{2}\sigma^2 t^{2H}\right). \tag{17}$$

We used a method of approximating B_t^H by a semimartingale $B_{t,\zeta}^H$ to write $df(t,S_{t,\zeta})$ in the form

$$df\left(t,S_{t,\zeta}\right) = g\left(t,f\left(t,S_{t,\zeta}\right)\right)dt + h\left(t,f\left(t,S_{t,\zeta}\right)\right)dW_t. \tag{18}$$

3. Approximation Results

We make the following assumptions:

Hypothesis 1. By definition, the shadow price $\widehat{S}_t$ is a semimartingale process which takes its values in the interval $[(1-\lambda)S_t; S_t]$; we will suppose that the shadow price $\widehat{S}_t$ can be written as $\widehat{S}_t = f(t,S_t)$ and that $\widehat{S}_{t,\zeta} \in [(1-\lambda)S_{t,\zeta}; S_{t,\zeta}]$ with $\zeta \in [0;1]$.

Hypothesis 2. Let f: $[0;T]\times\mathbb{R}_+ \longrightarrow \mathbb{R}_+$ be a function of class $C^{1,3}$ such that $\forall t \in [0;T]$, x, $y \in \mathbb{R}_+^*$,

(H1) $|\partial f(t,x)/\partial t - \partial f(t,y)/\partial t| \leq M_1|x-y|$

(H2) $|\partial f(t,x)/\partial x| \leq M_2$

(H3) $|\partial^2 f(t,x)/\partial x^2| \leq M_3$

(H4) $|\partial^3 f(t,x)/\partial x^3| \leq M_4$

where M_1, M_2, M_3, M_4 are real positive constants.

The following lemma establishes the convergence result of the process $f(t,S_{t,\zeta})$ to the process $f(t,S_t)$.

Lemma 1. *Let $S_{t,\zeta}$ and S_t be the respective solutions of the equations $dS_{t,\zeta} = \mu S_{t,\zeta}dt + \sigma S_{t,\zeta}dB_{t,\zeta}^H$ and $dS_t = \mu S_t dt + \sigma S_t dB_t^H$, t∈ [0, T]. Let $V(t, f(t,S_{t,\zeta})) = \phi_t^0 S_t^0 + \phi_t^1 f(t,S_{t,\zeta})$ the wealth process and f a function defined on $[0;T]\times\mathbb{R}_+$ and satisfying the assumptions* ***(H1)**, **(H2)**, **(H3)**, **(H4)**.*

(1) $f(t,S_{t,\zeta})$ is a semimartingale which converges uniformly to $f(t,S_t)$ in $L^2(\Omega)$ when ζ tends to 0

(2) *The wealth process* $V(t, f(t, S_{t,\zeta}))$ *is a semimartingale which converges uniformly to* $V(t, f(t, S_t))$ *in* $L^2(\Omega)$ *when* ζ *tends to 0*

Proof of Lemma 1. Let f be a function of class $C^{1,3}$, using the differential form of de $B^H_{s,\zeta}$, in (11) and the Wick-Ito lemma, we have

$$dB^H_{t,\zeta} = C\left(H - \frac{1}{2}\right)\left[\int_{-\infty}^{t} (t-s+\zeta)^{H-3/2} dW_s\right]dt + C\zeta^{H-1/2} dW_t = A(t)dt + C\zeta^{H-1/2} dW_t, \quad (19)$$

with $A(t) = C(H-1/2)[\int_{-\infty}^{t} (t-s+\zeta)^{H-3/2} dW_s]$ and

$$\begin{aligned} f\left(t, S_{t,\zeta}\right) &= f\left(0, S_{0,\zeta}\right) + \int_0^t \frac{\partial f}{\partial s}\left(s, S_{s,\zeta}\right)ds + \int_0^t \mu S_{s,\zeta} \frac{\partial f}{\partial x}\left(s, S_{s,\zeta}\right)ds + \sigma \int_0^t \mu S_{s,\zeta} \frac{\partial f}{\partial x}\left(s, S_{s,\zeta}\right)dB^H_{s,\zeta} + H\sigma^2 \int_0^t s^{2H-1} S^2_{s,\zeta} \frac{\partial^2 f}{\partial x^2}\left(s, S_{s,\zeta}\right)ds \\ &= f\left(0, S_{0,\zeta}\right) + \int_0^t \left[\frac{\partial f}{\partial s}\left(s, S_{s,\zeta}\right) + \mu S_{s,\zeta} \frac{\partial f}{\partial x}\left(s, S_{s,\zeta}\right) + H\sigma^2 s^{2H-1} S^2_{s,\zeta} \frac{\partial^2 f}{\partial x^2}\left(s, S_{s,\zeta}\right) + \sigma\mu A(t) S_{s,\zeta} \frac{\partial f}{\partial x}\left(s, S_{s,\zeta}\right)\right]dt, \end{aligned} \quad (20)$$

$+C\zeta^{H-1/2}\sigma\mu S_{t,\zeta} \partial f/\partial x(t, S_{t,\zeta}) dW_t$, so $f(t, S_{t,\zeta})$ is a semimartingale.

We have

$$\begin{aligned} f\left(t, S_{t,\zeta}\right) - f\left(t, S_t\right) &= f\left(0, S_{0,\zeta}\right) - f\left(0, S_0\right) + \int_0^t \left[\frac{\partial f}{\partial s}\left(s, S_{s,\zeta}\right) - \frac{\partial f}{\partial s}\left(s, S_s\right)\right]ds + \int_0^t \left[\mu S_{s,\zeta} \frac{\partial f}{\partial x}\left(s, S_{s,\zeta}\right) - \mu S_s \frac{\partial f}{\partial x}\left(s, S_s\right)\right]ds \\ &+ H\sigma^2 \int_0^t \left[s^{2H-1} S^2_{s,\zeta} \frac{\partial^2 f\left(s, S_{s,\zeta}\right)}{\partial x^2} - s^{2H-1} S^2_s \frac{\partial^2 f\left(s, S_s\right)}{\partial x^2}\right]ds + \sigma \int_0^t \left[S_{s,\zeta} \frac{\partial f}{\partial x}\left(s, S_{s,\zeta}\right)dB^H_{s,\zeta} - S_s \frac{\partial f}{\partial x}\left(s, S_s\right)dB^H_s\right]. \end{aligned} \quad (21)$$

In the relation (7), we have $S_{0,\zeta} = S_0 \exp(\sigma B^H_{0,\zeta}) = S_0$ so $f(S_{0,\zeta}) = f(S_0)$.

Using the inequality $(a+b+c+d)^2 \le 4(a^2+b^2+c^2+d^2)$, we get

$$\begin{aligned} E\left[f\left(t, S_{t,\zeta}\right) - f\left(t, S_t\right)\right]^2 &\le 4\int_0^t E\left[\frac{\partial f}{\partial s}\left(s, S_{s,\zeta}\right) - \frac{\partial f}{\partial s}\left(s, S_s\right)\right]^2 ds + 4\int_0^t E\left[\mu S_{s,\zeta} \frac{\partial f}{\partial x}\left(s, S_{s,\zeta}\right) - \mu S_s \frac{\partial f}{\partial x}\left(s, S_s\right)\right]^2 ds \\ &+ 4H^2\sigma^4 \int_0^t E\left[s^{2H-1} S^2_{s,\zeta} \frac{\partial^2 f\left(s, S_{s,\zeta}\right)}{\partial x^2} - s^{2H-1} S^2_s \frac{\partial^2 f\left(s, S_s\right)}{\partial x^2}\right]^2 ds \\ &+ 4\sigma^2 \int_0^t E\left[S_{s,\zeta} \frac{\partial f}{\partial x}\left(s, S_{s,\zeta}\right)dB^H_{s,\zeta} - S_s \frac{\partial f}{\partial x}\left(s, S_s\right)dB^H_s\right]^2. \end{aligned} \quad (22)$$

Let $C1 = \int_0^t E[\partial f/\partial s(s, S_{s,\zeta}) - \partial f/\partial s(s, S_s)]^2 ds$,

$$C2 = \int_0^t E\left[\mu S_{s,\zeta}\frac{\partial f}{\partial x}(s, S_{s,\zeta}) - \mu S_s \frac{\partial f}{\partial x}(s, S_s)\right]^2 ds,$$

$$C3 = H^2\sigma^4 \int_0^t E\left[s^{2H-1}S_{s,\zeta}^2 \frac{\partial^2 f(s, S_{s,\zeta})}{\partial x^2} - s^{2H-1}S_s^2 \frac{\partial^2 f(s, S_s)}{\partial x^2}\right]^2 ds, \quad (23)$$

$$C4 = \sigma^2 \int_0^t E\left[S_{s,\zeta}\frac{\partial f}{\partial x}(s, S_{s,\zeta})dB_{s,\zeta}^H - S_s \frac{\partial f}{\partial x}(s, S_s)dB_s^H\right]^2.$$

Using the relations (14), (15), and the Hypotheses **(H1)**, **(H2)**, **(H3)**, **(H4)**, we get

$$C1 = \int_0^t E\left[\frac{\partial f}{\partial s}(s, S_{s,\zeta}) - \frac{\partial f}{\partial s}(s, S_s)\right]^2 ds$$

$$\leq M_1^2 \int_0^t E(S_{s,\zeta} - S_s)^2 ds \leq M_1^2 \int_0^t M\zeta^{2\alpha} ds \leq tM_1^2 M\zeta^{2\alpha},$$

$$C2 = \int_0^t E\left[\mu S_{s,\zeta}\frac{\partial f}{\partial x}(s, S_{s,\zeta}) - \mu S_s \frac{\partial f}{\partial x}(s, S_s)\right]^2 ds$$

$$= \mu^2 \int_0^t E\left[S_{s,\zeta}\frac{\partial f}{\partial x}(s, S_{s,\zeta}) - S_s \frac{\partial f}{\partial x}(s, S_s)\right]^2 ds$$

$$\leq 2\mu^2 \int_0^t E\left(\frac{\partial f}{\partial x}(s, S_{s,\zeta})\right)^2 E(S_{s,\zeta} - S_s)^2 ds$$

$$+ 2\mu^2 \int_0^t E(S_s^2) E\left(\frac{\partial f}{\partial x}(s, S_{s,\zeta}) - \frac{\partial f}{\partial x}(s, S_s)\right)^2 ds$$

$$\leq 2\mu^2 M\zeta^{2\alpha} M_2^2 t + 2\mu^2 M_3^2 M\zeta^{2\alpha} \int_0^t E(S_s^2) ds, \quad (24)$$

where $E(S_s^2) = S_0 \exp(\mu t - 1/2\sigma^2 t^{2H}) E(\exp\ (B_t^H)^2)$ which is finite.

In fact, $\forall t \in [0, T]$, $E((B_t^H)^2) = t^{2H}$ thus $\int_0^t E(S_s^2)ds$ is finite.

Let $A = \int_0^t E(S_s^2)ds$,

We have

$$C2 \leq 2\mu^2 M\zeta^{2\alpha} M_2^2 T + 2\mu^2 M_3^2 M\zeta^{2\alpha} A$$

$$\leq 2(\mu^2 M M_2^2 T + \mu^2 M_3^2 M A)\zeta^{2\alpha},$$

$$C3 = H^2\sigma^4 \int_0^t E\left[s^{2H-1}S_{s,\zeta}^2 \frac{\partial^2 f(s, S_{s,\zeta})}{\partial x^2} - s^{2H-1}S_s^2 \frac{\partial^2 f(s, S_s)}{\partial x^2}\right]^2 ds$$

$$\leq 2H^2\sigma^4 \int_0^t s^{4H-2} E\left[\left(\frac{\partial^2 f(s, S_{s,\zeta})}{\partial x^2}\right)(S_{s,\zeta}^2 - S_s^2)\right]^2 ds$$

$$+ 2H^2\sigma^4 \int_0^t s^{4H-2} E(S_s^2) E$$

$$\cdot \left[\left(\frac{\partial^2 f(s, S_{s,\zeta})}{\partial x^2} - \frac{\partial^2 f(s, S_s)}{\partial x^2}\right)\right]^2 ds$$

$$\leq 2H^2\sigma^4 \int_0^t s^{4H-2} E\left[\left(\frac{\partial^2 f(s, S_{s,\zeta})}{\partial x^2}\right)^2\right] E(S_{s,\zeta}^2 - S_s^2)^2 ds$$

$$+ 2H^2\sigma^4 M_4 \int_0^t s^{4H-2} E(S_{s,\zeta} - S_s)^2 E(S_s^2) ds$$

$$\leq 2H^2\sigma^4 M_3^2 M\zeta^{2\alpha} \int_0^t s^{4H-2} E(S_{s,\zeta} + S_s)^2 ds$$

$$+ 2M_4 H^2\sigma^4 M\zeta^{2\alpha} \int_0^t s^{4H-2} E(S_s^2) ds. \quad (25)$$

As $E(S_s^2)$ is finite, $E(S_{s,\zeta}+S_s)^2$ is also finite.

Let $B=E(S_s^2)$ and $C=E(S_{s,\zeta}+S_s)^2$, then we have

$$
\begin{aligned}
C3 &\le 2H^2\sigma^4 M_3^2 M\zeta^{2\alpha} C\int_0^t s^{4H-2}ds + 2M_4H^2\sigma^4 M\zeta^{2\alpha} B\int_0^t s^{4H-2}ds \le 2\left(H^2\sigma^4M_3^2M\zeta^{2\alpha}C + M_4H^2\sigma^4M\zeta^{2\alpha}B\right)\left(\frac{t^{4H-1}}{4H-1}\right)\\
&\le 2\left(\frac{H^2\sigma^4\left(M_3^2MC+M_4MB\right)}{4H-1}\right)T^{4H-1}\zeta^{2\alpha},\\
C4 &= \sigma^2\int_0^t E\left[S_{s},\varsigma\frac{\partial f}{\partial x}(s,S_s,\varsigma)dB^H_{s,\varsigma} - S_s\frac{\partial f}{\partial x}(s,S_s)dB^H_s\right]^2\\
&\le 2\sigma^2\int_0^t E\left[\frac{\partial f}{\partial x}(s,S_{s,\varsigma})(S_{s,\varsigma}-S_s)dB^H_{s,\varsigma}\right]^2 + 2\sigma^2\int_0^t E(S_s^2)E\left[\frac{\partial f}{\partial x}(s,S_{s,\varsigma})dB^H_{s,\varsigma} - \frac{\partial f}{\partial x}(s,S_s)dB^H_s\right]^2\\
&\le 2\sigma^2\int_0^t E(S_{s,\varsigma}-S_s)^2E\left[\frac{\partial f}{\partial x}(s,S_{s,\varsigma})dB^H_{s,\varsigma}\right]^2 + 4\sigma^2\int_0^t E(S_s^2)E\left[\frac{\partial f}{\partial x}(s,S_{s,\varsigma})dB^H_{s,\varsigma} - \frac{\partial f}{\partial x}(s,S_s)dB^H_{s,\varsigma}\right]^2\\
&\quad + 4\sigma^2\int_0^t E(S_s^2)E\left[\frac{\partial f}{\partial x}(s,S_s)dB^H_{s,\varsigma} - \frac{\partial f}{\partial x}(s,S_s)dB^H_s\right]^2\\
&\le 2\sigma^2M\varsigma^{2\alpha}\int_0^t E\left(\frac{\partial f}{\partial x}(s,S_{s,\varsigma})\right)^2E\left(dB^H_{s,\varsigma}\right)^2 + 4\sigma^2M_3^2B\int_0^t E(S_{s,\varsigma}-S_s)^2E\left(dB^H_{s,\varsigma}\right)^2 + 4\sigma^2B\int_0^t E\left(\frac{\partial f}{\partial x}(s,S_s)\right)^2E\left(dB^H_{s,\varsigma}-dB^H_s\right)^2\\
&\le 2\sigma^2M\varsigma^{2\alpha}M_2^2E\left(B^H_{t,\varsigma}\right)^2 + 4\sigma^2M_3^2BM\varsigma^{2\alpha}E\left(B^H_{t,\varsigma}\right)^2 + 4\sigma^2BM_2^2E\left(B^H_{t,\varsigma}-B^H_t\right)^2\\
&\le \varsigma^{2\alpha}\left(2\sigma^2MT^{2H}M_2^2 + 4\sigma^2M_3^2BMT^{2H} + 4\sigma^2BM_2^2T\right).
\end{aligned}
\tag{26}
$$

We conclude that $E[f(t,S_{t,\zeta})-f(t,S_t)]^2\le D\zeta^{2\alpha}$ where D a constant. We deduce that $f(t,S_{t,\zeta})$ converges in mean square to $f(t,S_t)$ when ζ tends to 0.

$V(t,f(t,S_{t,\zeta}))=\phi_t^0S_t^0+\phi_t^1f(t,S_{t,\zeta})$ is also a semimartingale because $f(t,S_{t,\zeta})$ is a semimartingale and $\phi_t^0S_t^0$ is a finite variation process.

Similarly,

$$
\begin{gathered}
V\left(t,f\left(t,S_{t,\zeta}\right)\right)-V\left(t,f\left(t,S_t\right)\right)=\phi_t^1\left(f\left(t,S_{t,\zeta}\right)-f\left(t,S_t\right)\right),\\
E\left(V\left(t,f\left(t,S_{t,\zeta}\right)\right)-V\left(t,f\left(t,S_t\right)\right)\right)^2=\left(\phi_t^1\right)^2E\left(f\left(t,S_{t,\zeta}\right)-f\left(t,S_t\right)\right)^2\le\left(\phi_t^1\right)^2D\zeta^{2\alpha}.
\end{gathered}
\tag{27}
$$

Hence, the convergence of $V(t,f(t,S_{t,\zeta}))$ to $V(t,f(t,S_t))$ in $L^2(\Omega)$. □

Theorem 1. *Let $S_{t,\zeta}$ be a continuous semimartingale defined on a probability space $(\Omega,\mathcal{F},\mathbf{P})$ and f a function of class $C^{1,3}$ then under certain assumptions there exist two processes $\widehat{S}_{t,\zeta}=f(t,S_{t,\zeta})$ and $V(t,f(t,S_{t,\zeta}))$ such that*

(1) $\widehat{S}_{t,\zeta}$ is the unique solution of the equation

$$
\frac{d\widehat{S}_{t,\zeta}}{\widehat{S}_{t,\zeta}}=g\left(t,f\left(t,S_{t,\zeta}\right)\right)dt+h\left(t,f\left(t,S_{t,\zeta}\right)\right)dW_t,
\tag{28}
$$

and verifies the relation

$$
\frac{\sigma^2C^2S_{t,\zeta}^2\zeta^{2H-1}\left(\partial f\left(t,S_{t,\zeta}\right)/\partial S_{t,\zeta}\right)^2}{\beta+f\left(t,S_{t,\zeta}\right)}=\frac{\partial f\left(t,S_{t,\zeta}\right)}{\partial t}+\frac{\partial f\left(t,S_{t,\zeta}\right)}{\partial S_{t,\zeta}}(\mu+\alpha_t)S_{t,\zeta}+H\sigma^2t^{2H-1}\frac{\partial^2 f\left(t,S_{t,\zeta}\right)}{\partial S_{t,\zeta}^2}S_{t,\zeta}^2,
\tag{29}
$$

with

$$g\left(t, f\left(t, S_{t,\zeta}\right)\right) = \frac{1}{f\left(t, S_{t,\zeta}\right)}\left[\frac{\partial f\left(t, S_{t,\zeta}\right)}{\partial t} + \frac{\partial f\left(t, S_{t,\zeta}\right)}{\partial S_{t,\zeta}}(\mu + \alpha_t)S_{t,\zeta} + H\sigma^2 t^{2H-1}\frac{\partial^2 f\left(t, S_{t,\zeta}\right)}{\partial S_{t,\zeta}^2}S_{t,\zeta}^2\right], \tag{30}$$

and

$$h\left(t, f\left(t, S_{t,\zeta}\right)\right) = \frac{1}{f\left(t, S_{t,\zeta}\right)} C\sigma S_{t,\zeta}\frac{\partial f\left(t, S_{t,\zeta}\right)}{\partial S_{t,\zeta}}\Bigg)\zeta^{H-1/2}, \tag{31}$$

two bounded predictable processes and $\alpha_t = C\sigma(H - 1/2)\int_{-\infty}^{t}(t - s + \zeta)^{H-3/2}dW_s$.

(2) The processes $f(t, S_{t,\zeta})$ and $V(t, f(t, S_{t,\zeta}))$ are Markovian processes.

Proof of Theorem 1.

(1) Let f be a function of class $C^{1,3}$ and the process S_t defined in the relation (5); according to the Wick-ito, we have:

$$df(t, S_t) = \left[\frac{\partial f(t, S_t)}{\partial t} + \frac{\partial f(t, S_t)}{\partial S_t}\mu S_t + H\sigma^2 t^{2H-1}\frac{\partial^2 f(t, S_t)}{\partial S_t^2}S_t^2\right]dt + \sigma S(t)\frac{\partial f(t, S_t)}{\partial S_t}dB_t^H. \tag{32}$$

We approximate B_t^H by $B_{t,\zeta}^H$, and we have

$$C^{-1}B_t^H = \int_{-\infty}^{0}\left[(t - s)^{H-1/2} - (-s)^{H-1/2}\right]dW_s + \int_0^t (t - s)^{H-1/2}dW_s, \tag{33}$$

and the integral

$$\int_{-\infty}^{0}\left[(t - s)^{H-1/2} - (-s)^{H-1/2}\right]dW_s, \tag{34}$$

is a process with absolutely continuous trajectories, and by taking $\tau = 0$ in the relation (11), we can give a meaning to the derivative

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\int_{-\infty}^{0}(t - s)^{H-1/2}dW_s\right]. \tag{35}$$

Using the fractional Brownian motion approximation of Tran Hung Thao, we have

$$C^{-1}dB_{t,\zeta}^H = \left[\frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^{0}(t - s + \zeta)^{H-1/2}dW_s + \left(H - \frac{1}{2}\right)\int_0^t (t - s + \zeta)^{H-3/2}dW_s\right]dt + \zeta^{H-1/2}dW_t. \tag{36}$$

Therefore,

$$dB_{t,\zeta}^H = C\left(H - \frac{1}{2}\right)\left[\int_{-\infty}^{t}(t - s + \zeta)^{H-3/2}dW_s\right]dt + C\zeta^{H-1/2}dW_t. \tag{37}$$

Replacing dB_t^H by $dB_{t,\zeta}^H$, S_t by $S_{t,\zeta}$ and $f(t, S_t)$ by $f(t, S_{t,\zeta})$ in (32), we obtain

$$\frac{d\widehat{S}_{t,\zeta}}{\widehat{S}_{t,\zeta}} = \frac{1}{f\left(t, S_{t,\zeta}\right)}\left[\frac{\partial f\left(t, S_{t,\zeta}\right)}{\partial t} + \frac{\partial f\left(t, S_{t,\zeta}\right)}{\partial S_{t,\zeta}}(\mu + \alpha_t)S_{t,\zeta} + H\sigma^2 t^{2H-1}\frac{\partial^2 f\left(t, S_{t,\zeta}\right)}{\partial S_{t,\zeta}^2}S_{t,\zeta}^2\right]dt + \frac{1}{f\left(t, S_{t,\zeta}\right)}C\sigma S_{t,\zeta}\frac{\partial f\left(t, S_{t,\zeta}\right)}{\partial S_{t,\zeta}}\Bigg)\zeta^{H-1/2}dW_s. \tag{38}$$

Let

$$g\left(t, f\left(t, S_{t,\zeta}\right)\right) = \frac{1}{f\left(t, S_{t,\zeta}\right)}\left[\frac{\partial f\left(t, S_{t,\zeta}\right)}{\partial t} + \frac{\partial f\left(t, S_{t,\zeta}\right)}{\partial S_{t,\zeta}}(\mu + \alpha_t)S_{t,\zeta} + H\sigma^2 t^{2H-1}\frac{\partial^2 f\left(t, S_{t,\zeta}\right)}{\partial S_{t,\zeta}^2}S_{t,\zeta}^2\right], \tag{39}$$

and

$$h\left(t, f\left(t, S_{t,\zeta}\right)\right) = \frac{1}{f\left(t, S_{t,\zeta}\right)} C\sigma S_{t,\zeta} \frac{\partial f\left(t, S_{t,\zeta}\right)}{\partial S_{t,\zeta}} \Bigg) \zeta^{H-1/2}. \tag{40}$$

We have

$$\frac{d\widehat{S}_{t,\zeta}}{\widehat{S}_{t,\zeta}} = \frac{d f\left(t, S_{t,\zeta}\right)}{f\left(t, S_{t,\zeta}\right)} = g\left(t, f\left(t, S_{t,\zeta}\right)\right)dt + h\left(t, f\left(t, S_{t,\zeta}\right)\right)dW_t. \tag{41}$$

From hypothesis (H1) and (H2), we have

$$\begin{aligned} &(1-\lambda)S_{t,\zeta} \leq f\left(t, S_{t,\zeta}\right) \leq S_{t,\zeta}, \\ &\left|h\left(t, f\left(t, S_{t,\zeta}\right)\right| = \left|\frac{1}{f\left(t, S_{t,\zeta}\right)} C\sigma S_{t,\zeta} \frac{\partial f\left(t, S_{t,\zeta}\right)}{\partial x} \zeta^{H-1/2}\right| \leq \frac{1}{1-\lambda} C\sigma M_2 \zeta^{H-1/2}. \end{aligned} \tag{42}$$

So, $\int_0^t (h(s, f(s, S_{s,\zeta}))^2 ds < \int_0^t ((1/1-\lambda)C\sigma M_2 \zeta^{H-1/2})^2 ds < ((1/1-\lambda)C\sigma M_2 \zeta^{H-1/2})^2 T < +\infty$.

$$\left|g\left(t, f\left(t, S_{t,\zeta}\right)\right)\right| = \frac{1}{f\left(t, S_{t,\zeta}\right)} \left[\frac{\partial f\left(t, S_{t,\zeta}\right)}{\partial t} + \frac{\partial f\left(t, S_{t,\zeta}\right)}{\partial S_{t,\zeta}} (\mu + \alpha_t) S_{t,\zeta} + H\sigma^2 t^{2H-1} \frac{\partial^2 f\left(t, S_{t,\zeta}\right)}{\partial S_{t,\zeta}^2} S_{t,\zeta}^2\right]. \tag{43}$$

From Hypothesis (H1) 1 and the hypothesis **(H2)**, (H3),

$$\begin{aligned} &\left|g\left(t, f\left(t, S_{t,\zeta}\right)\right)\right| \leq \frac{1}{(1-\lambda)S_{t,\zeta}} \left[\frac{\partial f\left(t, S_{t,\zeta}\right)}{\partial t} + M_2 (\mu + \alpha_t) S_{t,\zeta} + H\sigma^2 t^{2H-1} M_3 S_{t,\zeta}^2\right] \\ &\left|g\left(t, f\left(t, S_{t,\zeta}\right)\right)\right| \leq Sup_{t\in[0,T]} \left|\frac{1}{(1-\lambda)S_{t,\zeta}} \frac{\partial f\left(t, S_{t,\zeta}\right)}{\partial t} + \frac{M_2 (\mu + \alpha_t)}{1-\lambda}\right| + \frac{H\sigma^2 T^{2H-1} M_3}{1-\lambda} \mathrm{Sup}_{t\in[0,T]}\left(S_{t,\zeta}\right). \end{aligned} \tag{44}$$

Therefore,

$$\int_0^t \left|g\left(u, f\left(u, S_{u,\zeta}\right)\right)\right| du < +\infty. \tag{45}$$

From Assumption 1 and the hypothesis **(H3)**, we have

$$\begin{aligned} \left|h(t, f(t, x_1)) - h(t, f(t, x_2))\right| &= \left|\frac{1}{f(t, x_1)} C\sigma x_1 \frac{\partial f(t, x_1)}{\partial x} \zeta^{H-1/2} - \frac{1}{f(t, x_2)} C\sigma x_2 \frac{\partial f(t, x_2)}{\partial x} \zeta^{H-1/2}\right| \\ &\leq \frac{\sigma C \zeta^{H-1/2}}{1-\lambda} \left|\frac{\partial f(t, x_1)}{\partial x} - \frac{\partial f(t, x_2)}{\partial x}\right| \leq \frac{\sigma C \zeta^{H-1/2}}{1-\lambda} M_3 |x_1 - x_2|, \\ k_t = |\mu(t, 0)| + |\sigma(t, 0)| &= \left|\frac{1}{f(t, 0)} \left(\frac{\partial f(t, 0)}{\partial t} + \frac{\partial f(t, 0)}{\partial x} + C\right)\right|. \end{aligned} \tag{46}$$

The functions $f(t, 0)$, $\partial f(t, 0)/\partial x$, $\partial f(t, 0)/\partial t$ are continuous and bounded functions for $t \in [0; T]$ so $E[\int_0^t |k_u|^2 du] < +\infty$.

Assume that f is monotonic, i.e., $f(t, x_2) > f(t, x_1)$ and $x_2 > x_1$ (where $f(t, x_2) > f(t, x_1)$ and $x_2 < x_1$) and $N = \mathrm{Sup}_{t\in[0,T]} (1/f(t, x_1))$,

$$\begin{aligned}|g(t,x_1)-g(t,x_2)| \leq & \left|\frac{1}{f(t,x_1)}\left(\frac{\partial f(t,x_1)}{\partial t}-\frac{\partial f(t,x_2)}{\partial t}\right)\right| \\ & +\left|\frac{1}{f(t,x_1)}\left(\frac{\partial f(t,x_1)}{\partial x}(\mu+\alpha_t)x_1-\frac{\partial f(t,x_2)}{\partial x}\right)(\mu+\alpha_t)x_2\right| \\ & +H\sigma^2 t^{2H-1}\left|\frac{1}{f(t,x_1)}\left(\frac{\partial^2 f(t,x_1)}{\partial x^2}x_1^2-\frac{\partial^2 f(t,x_2)}{\partial x^2}x_2^2\right)\right|.\end{aligned} \tag{47}$$

Under the Hypotheses **(H1)**, **(H3)**, and **(H4)**, we have

$$\begin{aligned}|g(t,x_1)-g(t,x_2))| &\leq NM_1|x_1-x_2|+M_3N|x_1||\mu+\alpha_t||x_1-x_2|+H\sigma^2t^{2H-1}M_4N|x_1^2||x_1-x_2| \\ &\leq \mathrm{Sup}_{t\in[0,T]}\left(\left(M_1N+M_3N|x_1||\mu+\alpha_t|+NH\sigma^2t^{2H-1}M_4|x_1^2|\right)\right)|x_1-x_2|.\end{aligned} \tag{48}$$

We have

$$\begin{aligned}&|h(t,x_1)-h(t,x_2)|+|g(t,x_1)-g(t,x_2)| \\ &\leq |x_1-x_2|\left(\mathrm{Sup}_{t\in[0,T]}\left(M_1N+M_3N|x_1||\mu+\alpha_t|+H\sigma^2t^{2H-1}M_4N|x_1^2|\right)+\frac{\sigma C\zeta^{H-1/2}}{1-\lambda}M_3\right).\end{aligned} \tag{49}$$

Using Hypothesis 1, we have $\widehat{S}_{t,\zeta}\in[(1-\lambda)S_{t,\zeta};S_{t,\zeta}]$,

$$g\left(t,f\left(t,S_{t,\zeta}\right)\right)=\frac{1}{f\left(t,S_{t,\zeta}\right)}\left[\frac{\partial f\left(t,S_{t,\zeta}\right)}{\partial t}+\frac{\partial f\left(t,S_{t,\zeta}\right)}{\partial S_{t,\zeta}}(\mu+\alpha_t)S_{t,\zeta}+H\sigma^2t^{2H-1}\frac{\partial^2 f\left(t,S_{t,\zeta}\right)}{\partial S_{t,\zeta}^2}S_{t,\zeta}^2\right], \tag{50}$$

and

$$h\left(t,f\left(t,S_{t,\zeta}\right)\right)=\frac{1}{f\left(t,S_{t,\zeta}\right)}C\sigma S_{t,\zeta}\frac{\partial f\left(t,S_{t,\zeta}\right)}{\partial S_{t,\zeta}}\Bigg)\zeta^{H-1/2}. \tag{51}$$

According to the relation (3),

$$\frac{\phi_t^1\widehat{S}_{t,\zeta}}{\phi_t^0+\phi_t^1\widehat{S}_{t,\zeta}}=\frac{\mu_t}{\sigma_t^2}. \tag{52}$$

We have

$$\begin{aligned}\frac{\phi_t^1 f\left(t,S_{t,\zeta}\right)}{\phi_t^0+\phi_t^1 f\left(t,S_{t,\zeta}\right)} &= \frac{1/f\left(t,S_{t,\zeta}\right)\left[\partial f\left(t,S_{t,\zeta}\right)/\partial t+\partial f\left(t,S_{t,\zeta}\right)/\partial S_{t,\zeta}(\mu+\alpha_t)S_{t,\zeta}+H\sigma^2t^{2H-1}\partial^2 f\left(t,S_{t,\zeta}\right)/\partial S_{t,\zeta}^2S_{t,\zeta}^2\right]}{C^2/\left(f\left(t,S_{t,\zeta}\right)\right)^2\sigma^2S_{t,\zeta}^2\zeta^{2H-1}\left(\partial f\left(t,S_{t,\zeta}\right)/\partial S_{t,\zeta}\right)^2}, \\ \frac{f\left(t,S_{t,\zeta}\right)}{\beta+f\left(t,S_{t,\zeta}\right)} &= \frac{f\left(t,S_{t,\zeta}\right)\left(\partial f\left(t,S_{t,\zeta}\right)/\partial t+\partial f\left(t,S_{t,\zeta}\right)/\partial S_{t,\zeta}(\mu+\alpha_t)S_{t,\zeta}+H\sigma^2t^{2H-1}\partial^2 f\left(t,S_{t,\zeta}\right)/\partial S_{t,\zeta}^2S_{t,\zeta}^2\right)}{C^2\sigma^2S_{t,\zeta}^2\zeta^{2H-1}\left(\partial f\left(t,S_{t,\zeta}\right)/\partial S_{t,\zeta}\right)^2},\end{aligned} \tag{53}$$

with $\beta=\phi_t^0/\phi_t^1$,

$$\begin{aligned}\frac{1}{\beta+f\left(t,S_{t,\zeta}\right)} &= \frac{\partial f\left(t,S_{t,\zeta}\right)/\partial t+\partial f\left(t,S_{t,\zeta}\right)/\partial S_{t,\zeta}(\mu+\alpha_t)S_{t,\zeta}+H\sigma^2t^{2H-1}\partial^2 f\left(t,S_{t,\zeta}\right)/\partial S_{t,\zeta}^2S_{t,\zeta}^2}{\sigma^2C^2S_{t,\zeta}^2\zeta^{2H-1}\left(\partial f\left(t,S_{t,\zeta}\right)/\partial S_{t,\zeta}\right)^2}, \\ \frac{\sigma^2C^2S_{t,\zeta}^2\zeta^{2H-1}\left(\partial f\left(t,S_{t,\zeta}\right)/\partial S_{t,\zeta}\right)^2}{\beta+f\left(t,S_{t,\zeta}\right)} &= \frac{\partial f\left(t,S_{t,\zeta}\right)}{\partial t}+\frac{\partial f\left(t,S_{t,\zeta}\right)}{\partial S_{t,\zeta}}(\mu+\alpha_t)S_{t,\zeta}+H\sigma^2t^{2H-1}\frac{\partial^2 f\left(t,S_{t,\zeta}\right)}{\partial S_{t,\zeta}^2}S_{t,\zeta}^2.\end{aligned} \tag{54}$$

(2) The process $f(t, S_{t,\zeta})$ is a solution of (12); thus, it is Markovian process. We deduce that $V(t, f(t, S_{t,\zeta})) = \phi_t^0 S_t^0 + \phi_t^1 f(t, S_{t,\zeta})$ is also a Markovian process. □

4. Conclusion

In this work, we have proposed a shadow price approximation method for the fractional Black model in the sense of Wick-Itô in the context of an optimization problem under transaction costs. We obtained (54) whose resolution gives a candidate process for the shadow price approximation. The problem is thus reduced to a frictionless optimization problem in the Markovian framework which could be solved by Hamilton–Jacobi type equations.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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