

Research Article

On a Predator-Prey Model Involving Age and Spatial Structure

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In this paper, we study the mathematical analysis of a nonlinear age-dependent predator-prey system with diffusion in a bounded domain with a non-standard functional response. Using the fixed point theorem, we first show a global existence result for the problem with spatial variable. Other results of existence concerning the spatial homogeneous problem and the stationary system are discussed. At last, numerical simulations are performed by using finite difference method to validate the results.

1. Introduction and Assumptions

The study of predator-prey systems has attracted the attention of many mathematicians in the last century. Pioneers like Volterra and Lotka were the first to mathematically model the interaction between predators and preys. The standard model of Lotka and Volterra is:

$$\begin{cases} x'(t) = rx(t) - p(x(t))y(t), \\ y'(t) = \eta p(x(t))y(t) - \sigma y(t), & t > 0, \\ x(0) > 0, \quad y(0) > 0, \end{cases} \quad (1)$$

where x and y denote preys and predators density respectively; r , η and σ are positive constants stand respectively for the prey intrinsic growth rate, the coefficient for the conversion that predator intake to per capital prey and the predator mortality rate. And, $p(x)$ is the functional response which can take many forms [1].

In order to be closer to biological reality, the Lotka-Volterra model (1) has been improved. Models taking into account both mortality and fertility of species, the carrying capacity of prey of the environment, environmental configuration, migratory movements (diffusion coefficients) have emerged. In [2], B. Ainseba, F. Heiser, and M. Langlais establish the existence of a solution of a predator-prey system in a highly heterogeneous environment but without

the age variable and with a holling II type functional response. Note that in their model, prey dynamics is governed by logistic growth. In [3], the authors discuss a diffuse predator-prey system with a mutually interfering predator and a nonlinear harvest in the predator with a Crowley-Martin functional response. They analyze the existence and uniqueness of the solution of the system using the C_0 semi-group. They show that the upper bound on the predator harvest rate for species coexistence can be guaranteed. Furthermore, they establish the existence and non-existence of a positive non-constant steady state. They give explicit conditions on predator harvesting for local and global stability of the interior equilibrium, as well as for the existence and non-existence of a non-constant steady state solution. In [4] the authors propose a diffusive prey-predator system with mutual interference between the predator (Crowley-Martin functional response) and the prey pool. In particular, they develop and analyze both a spatially homogeneous model based on ordinary differential equations and a reaction-diffusion model. The authors mainly study the global existence and limit of the positive solution, the stability properties of the homogeneous steady state, the non-existence of the non-constant positive steady state, the Turing instability and the Hopf bifurcation conditions of the diffusive system analytically. The classical stability properties of the non-spatial counterpart of the system are also studied.

The analysis ensures that the prey pool leaves a stabilizing effect on the stability of the time system. A model of predator-prey interaction with Beddington-DeAngelis functional response and incorporating the cost of fear in prey reproduction is proposed and analyzed in [5]. The authors study the stability and existence of transcritical bifurcations. For the spatial system, the Hopf bifurcation around the inner equilibrium, the stability of the homogeneous steady state, the direction and stability of spatially homogeneous periodic orbits have been established. Using the normal form of the steady state bifurcation, they establish the possibility of a pitchfork bifurcation. A Leslie-Gower type prey-predator system with feedback is constructed in [6]. The authors systematically analyze the effects of feedback controls on ecosystem dynamics. In this study, they examine the global dynamics of non-autonomous and autonomous systems based on the Leslie-Gower type model using the Beddington-DeAngelis functional response with time independent and time dependent model parameters. The global stability of the unique positive equilibrium solution of the autonomous model is determined by defining an appropriate Lyapunov function. The autonomous system exhibits complex dynamics via bifurcation scenarios, such as the period doubling bifurcation. They then prove the existence of a globally stable quasiperiodic solution of the associated non-autonomous model. For the mathematical and

qualitative study of some prey-predator models, the reader can also consult the following articles [7–14].

In this article, we will mainly study the existence of solutions of a predator-prey system structured in age, time and space with a functional response F subject to the K-lipschitz condition.

Our motivation arose from the fact that, there is no existence result concerning predator-prey systems simultaneously structuring in age, time and space with a non-standard functional response. But there is works on population dynamics systems that take these three variables into account according to our best knowledge. For example in [15], an existence result is proved by Ainseba where the system models the transmission of an epidemic to holy individuals by carrier individuals. A result of existence and uniqueness and positivity of solution is also proved in [16] by Traoré et al. where their system models the population dynamics of *Callosobruchus Maculatus*. Traoré et al. have proved an existence result in [17] where the system models a nonlinear age and two-sex population dynamics.

Let us denote by $p(a, t, x)$ and $q(a, t, x)$ respectively the distribution of preys and predators of age a being at time $t > 0$ and location x over a bounded domain Ω .

We consider in this paper the following nonlinear age-dependent population dynamics diffusive system:

$$\left\{ \begin{array}{l} \partial_t p + \partial_a p - \Delta p + \mu_1(a)p = f_1 - \int_0^{A_2} F(p(a, t, x), q(\alpha, t, x)) d\alpha \text{ in } Q_1, \\ \partial_t q + \partial_a q - \Delta q + \mu_2(a)q = f_2 \text{ in } Q_2, \\ p(0, t, x) = \int_0^{A_1} \beta_1(a)p(a, t, x) da \text{ in } Q_T, \\ q(0, t, x) = \int_0^{A_2} \beta_2(a)q(a, t, x) da + \int_0^{A_1} \int_0^{A_2} b(x, a, \alpha)F(p(a, t, x), q(\alpha, t, x)) d\alpha da \text{ in } Q_T, \\ p(a, 0, x) = p_0(a, x) \text{ in } Q_{A_1}, \\ q(a, 0, x) = q_0(a, x) \text{ in } Q_{A_2}, \\ \partial_\eta p = 0 \text{ on } \Sigma_1, \\ \partial_\eta q = 0 \text{ on } \Sigma_2, \end{array} \right. \quad (2)$$

where T is a positive number, and Ω a bounded open subset of \mathbb{R}^n which boundary is assumed to be of class C^2 .

Actually, $Q_i = (0, A_i) \times (0, T) \times \Omega$, $Q_{A_i} = (0, A_i) \times \Omega$, $\Sigma_i = (0, A_i) \times (0, T) \times \partial\Omega$ for $i = 1, 2$ and $Q_T = (0, T) \times \Omega$.

We denote by A_1 (resp. A_2) the maximum life expectancy of prey (resp. predators).

In (2), $\beta_1(a)$ and $\beta_2(a)$ are respectively the natural birth-rate at age a of preys and predators, μ_1 and μ_2 are the functions describing the mortality rate respectively of preys and predators that depends on a , η is the external normal derivative on $\partial\Omega$.

In this model, predators and prey live on the same domain and any movement across the boundaries is impossible.

Our model is much more general because it simultaneously involves the notion of time, age and space. Moreover, the dynamics of prey and predators are governed by partial differential equations and not by the usual exponential or logistic growth laws. It is also a realistic model because in this model the prey is not the only source of food for the predators. Since in nature, it is almost impossible to find predators that feed exclusively on a single prey. Here, prey is not the only food source for predators. External food sources are also available. Not all of the prey that is consumed by predators is converted into predator energy (biomass), only a fraction is used.

Consumption of prey directly affects prey density (direct decrease in prey numbers) but indirectly affects predator

density through an increase in predator fertility (predator numbers do not increase immediately after consumption but over time predator density increases).

We have denoted by $F(p(a, t, x), q(\alpha, t, x))$ the functional response to predation, that is the capture rate of prey having age a per predator of age α or the average number of prey having age a captured by predators of age α at times $t > 0$, and location $x \in \Omega$.

Thus, $\int_0^{A_2} F(p(a, t, x), q(\alpha, t, x))d\alpha$ is the amount of prey of age a consumed by predator at time $t > 0$ and location $x \in \Omega$.

The function $b(x, a, \alpha)$ is the conversion rate of the biomass of captured prey having age a by predators of age α into predator offspring at location $x \in \Omega$.

Thus, the biomass is transformed and influences the birth process through the quantity

$$\int_0^{A_1} \int_0^{A_2} b(x, a, \alpha)F(p(a, t, x), q(\alpha, t, x))dad\alpha, \quad (3)$$

which is the supply.

The function $f_1(a, t, x)$ (resp. $f_2(a, t, x)$) is the external supply for prey persistence (resp. for predator persistence) having age a at time $t > 0$ and location $x \in \Omega$.

Our main goal in this paper is to answer some ecological questions:

Is the cohabitation of predators and prey modeled by the model (2) possible?

The answer to this question will bring us back to the notion of a well posedness problem or to the notion of the existence of a solution in suitably chosen spaces.

Does the biomass b influence the size of the two populations?

Will the predators succeed in consuming all the prey? Can predators or prey disappear into the environment?

To answer these last questions, we will use numerical simulations by varying the values of b , that is to say we will take small and large values of b to observe the behavior of the two populations.

Our work will be structured as follows:

In Section 2, we give a global existence result of solution of system (2) with the space variable in appropriate spaces. We will also study the existence of solutions of the spatially homogeneous problem in Section 3. The Section 4 is devoted to the analysis of the stationary problem. Results of numerical simulations are given in Section 5

and we will end in Section 6 with a conclusion and some perspectives.

Before starting, let

$$\pi_i(a) = \exp\left\{-\int_0^a \mu_i(\sigma)d\sigma\right\}, \quad a \in [0; A_i], i \in \{1, 2\}. \quad (4)$$

Which is the probability for a newborn to survive to age a and

$$R_i = \int_0^{A_i} \beta_i(a)\pi_i(a)da, \quad i \in \{1, 2\}. \quad (5)$$

And assume that the following hypotheses hold:

- (A₁) $\forall i \in \{1, 2\}, \beta_i \in L^\infty(0, A_i), \beta_i(a) \geq 0 \text{ a.e } a \in (0, A_i)$
- (A₂) $\forall i \in \{1, 2\}, \mu_i \in L^1_{loc}(0, A_i), \mu_i(a) \geq 0 \text{ a.e } a \in (0, A_i)$
- (A₃) $\int_0^{A_i} \mu_i(a)da = +\infty, i = 1, 2.$
- (A₄) F is a positive and mesurable function on $[(0; \infty)]^2$ and satisfies the usual locally boundedness and Lipschitz continuity conditions with respect to the pair variable, that is

$$\begin{aligned} \exists K > 0 \forall (p_1, q_1), (p_2, q_2) \in [(0; \infty)]^2, \\ |F(p_1, q_1) - F(p_2, q_2)| \leq K(|p_1 - p_2| + |q_1 - q_2|). \end{aligned} \quad (6)$$

And $F(0, 0) = 0$.

For the biological meanings of assumptions (A₁), (A₂) and (A₃), the functions μ_i, β_i, π_i and the constants R_i , we refer the reader to books such [18, 19].

2. Spatially Heterogeneous Solutions

Let us make the following assumptions:

- (H₁) $(p_0, q_0) \in L^\infty(Q_{A_1}) \times L^\infty(Q_{A_2}), p_0 \geq 0 \text{ a.e in } Q_{A_1}$
and. $q_0 \geq 0 \text{ a.e in } Q_{A_2}$
- (H₂) $(f_1, f_2) \in L^\infty(Q_1) \times L^\infty(Q_2), f_1 \geq 0 \text{ a.e in } Q_1$
and. $f_2 \geq 0 \text{ a.e in } Q_2$

We have the following result:

Theorem 1. *Under the hypotheses (A₁)—(A₄), (H₁), (H₂) the system (2) admits a unique solution in $L^2(Q_1) \times L^2(Q_2)$. Moreover, there exist a constant C depending on T such that*

$$\|p\|_{L^2(Q_1)}^2, \|p\|_{L^2(Q_2)}^2 \leq C \left(\|p_0\|_{L^2(Q_{A_1})}^2 + \|q_0\|_{L^2(Q_{A_2})}^2 + \|f_1\|_{L^2(Q_1)}^2 + \|f_2\|_{L^2(Q_2)}^2 \right). \quad (7)$$

Proof. Let us fix $T > 0$. The proof of the theorem is based on the method used in [20].

Set $u = e^{-\lambda_0 t} p$ in Q_1 and $v = e^{-\lambda_0 t} q$ in Q_2 where λ_0 is a positive parameter that we will be fixed later.

Hence the system (2) admits a unique solution if and only the following system admits a unique solution.

$$\left\{ \begin{array}{l} \partial_t u + \partial_a u - \Delta u + (\lambda_0 + \mu_1(a))u = e^{-\lambda_0 t} f_1 - e^{-\lambda_0 t} \int_0^{A_2} F(e^{\lambda_0 t} u(a, t, x), e^{\lambda_0 t} v(\alpha, t, x)) d\alpha \text{ in } Q_1, \\ \partial_t v + \partial_a v - \Delta v + (\lambda_0 + \mu_2(a))v = e^{-\lambda_0 t} f_2 \text{ in } Q_2, \\ u(0, t, x) = \int_0^{A_1} \beta_1(a) u(t, a, x) da \text{ in } Q_T, \\ v(0, t, x) = \int_0^{A_2} \beta_2(a) v(a, t, x) da + e^{-\lambda_0 t} \int_0^{A_1} \int_0^{A_2} b(x, a, \alpha) F(e^{\lambda_0 t} u(a, t, x), e^{\lambda_0 t} v(\alpha, t, x)) d\alpha da \text{ in } Q_T, \\ u(a, 0, x) = p_0(a, x) \text{ in } Q_{A_1}, \\ v(a, 0, x) = q_0(a, x) \text{ in } Q_{A_2}, \\ \partial_\eta u = 0 \text{ on } \Sigma_1, \\ \partial_\eta v = 0 \text{ on } \Sigma_2. \end{array} \right. \quad (8)$$

For any nonnegative $h \in L^2(Q_1)$, we introduce the following cascade system:

$$\left\{ \begin{array}{l} \partial_t u + \partial_a u - \Delta u + (\lambda_0 + \mu_1(a))u = e^{-\lambda_0 t} f_1 - e^{-\lambda_0 t} \int_0^{A_2} F(e^{\lambda_0 t} u(a, t, x), e^{\lambda_0 t} v(\alpha, t, x)) d\alpha \text{ in } Q_1, \\ \partial_t v + \partial_a v - \Delta v + (\lambda_0 + \mu_2(a))v = e^{-\lambda_0 t} f_2 \text{ in } Q_2, \\ u(0, t, x) = \int_0^{A_1} \beta_1(a) u(t, a, x) da \text{ in } Q_T, \\ v(0, t, x) = \int_0^{A_2} \beta_2(a) v(a, t, x) da + e^{-\lambda_0 t} \int_0^{A_1} \int_0^{A_2} b(x, a, \alpha) F(e^{\lambda_0 t} u(a, t, x), e^{\lambda_0 t} v(\alpha, t, x)) d\alpha da \text{ in } Q_T, \\ u(a, 0, x) = p_0(a, x) \text{ in } Q_{A_1}, \\ v(a, 0, x) = q_0(a, x) \text{ in } Q_{A_2}, \\ \partial_\eta u = 0 \text{ on } \Sigma_1, \\ \partial_\eta v = 0 \text{ on } \Sigma_2. \end{array} \right. \quad (9)$$

Using the Fubini's theorem, the function v solves the following system:

$$\left\{ \begin{array}{l} \partial_t v + \partial_a v - \Delta v + \widehat{\mu}_2(a)v = e^{-\lambda_0 t} f_2 \text{ in } Q_2, \\ v(0, t, x) = \int_0^{A_2} \widehat{\beta}_2(a, t, x, v(a, t, x)) v(a, t, x) da \text{ in } Q_T, \\ v(a, 0, x) = q_0(a, x) \text{ in } Q_{A_2}, \\ \partial_\eta v = 0 \text{ on } \Sigma_2. \end{array} \right. \quad (10)$$

With $\widehat{\mu}_2(a) = \lambda_0 + \mu_2(a)$ and $\widehat{\beta}_2(a, t, x, v(a, t, x)) = \beta_2(a) + e^{-\lambda_0 t} \int_0^{A_1} b(x, \alpha, a) F(e^{\lambda_0 t} h(\alpha, t, x), e^{\lambda_0 t} v(a, t, x)) d\alpha$.

As λ_0 is fixed, $e^{-\lambda_0 t} f_2 \in L^\infty(Q_2)$, $b \in L^\infty(A_1 \times A_2 \times \Omega)$, F is bounded then $\widehat{\mu}_2 \in L^1_{loc}(0, A_2)$ and $\widehat{\beta}_2$ is bounded.

Hence, (10) admits a unique nonnegative solution v_h in $L^2(Q_2)$ (see [20]).

Multiplying (10) by v_h and integrating over Q_2 , we get

$$\begin{aligned} \lambda_0 \int_{Q_2} v_h(a, t, x)^2 da dt dx &\leq \frac{1}{2} \|q_0\|_{L^2(Q_{A_2})}^2 + \frac{1}{2} \|f_2\|_{L^2(Q_2)}^2 \\ &\quad + \frac{1}{2} \int_{Q_2} v_h^2(a, t, x) da dt dx \quad (11) \\ &\quad + \frac{1}{2} \int_{Q_2} v_h^2(0, t, x) dt dx. \end{aligned}$$

Using the Young and Cauchy Schwarz's inequalities, we obtain

$$\begin{aligned} \lambda_0 \int_{Q_2} v_h(a, t, x)^2 da dt dx &\leq \frac{1}{2} \|q_0\|_{L^2(Q_{A_2})}^2 + \frac{1}{2} \|f_2\|_{L^2(Q_2)}^2 + \frac{1}{2} \int_{Q_2} v_h^2(a, t, x) da dt dx + \\ &\|\beta_2\|_{\infty}^2 A_2 \int_{Q_2} v_h(a, t, x)^2 da dt dx + A_1 A_2^2 K^2 \|b\|_{\infty}^2 \int_{Q_1} h^2(a, t, x) da dt dx + \\ &A_1^2 A_2 K^2 \|b\|_{\infty}^2 \int_{Q_1} v_h^2(a, t, x) da dt dx. \end{aligned} \tag{12}$$

So, we have

$$\begin{aligned} \left(\lambda_0 - \|\beta_2\|_{\infty}^2 A_2 - A_1^2 A_2 K^2 \|b\|_{\infty}^2 - \frac{1}{2}\right) \int_{Q_2} v_h^2(a, t, x) da dt dx &\leq \frac{1}{2} \|q_0\|_{L^2(Q_{A_2})}^2 + \frac{1}{2} \|f_2\|_{L^2(Q_2)}^2 + \\ &A_1 A_2^2 K^2 \|b\|_{\infty}^2 \int_{Q_1} h^2(a, t, x) da dt dx. \end{aligned} \tag{13}$$

That is to say

$$\|v^h\|_{L^2(Q_2)}^2 \leq C_1 \left(\|q_0\|_{L^2(Q_{A_2})}^2 + \|f_2\|_{L^2(Q_2)}^2 + \|h\|_{L^2(Q_1)}^2 \right), \tag{14}$$

where

$$\begin{aligned} C_1 &= \frac{1}{\lambda_0 - \|\beta_2\|_{\infty}^2 A_2 - A_1^2 A_2 K^2 \|b\|_{\infty}^2 - (1/2)} \\ &\times \max\left\{ \frac{1}{2}, A_1 A_2^2 K^2 \|b\|_{\infty}^2 \right\}. \end{aligned} \tag{15}$$

For any $h_1, h_2 \in L^2_+(Q_1)$, set $w = e^{-\lambda_0 t} (v_{h_1} - v_{h_2}) a.e$ in Q_2 . So, w solves the following system

$$\begin{cases} \partial_t w + \partial_a w - \Delta w + (\lambda_0 + \mu_2(a))w = 0 \text{ in } Q_2, \\ w(0, t, x) = \int_0^{A_2} \beta_2(a) w(a, t, x) da + \int_0^{A_1} \int_0^{A_2} e^{-\lambda_0 t} b(x, a, \alpha) \times F(e^{\lambda_0 t} h_1(a, t, x), e^{\lambda_0 t} v_{h_1}(\alpha, t, x)) da d\alpha \\ - \int_0^{A_1} \int_0^{A_2} e^{-\lambda_0 t} b(x, a, \alpha) F(e^{\lambda_0 t} h_2(a, t, x), e^{\lambda_0 t} v_{h_2}(\alpha, t, x)) da d\alpha \text{ in } Q_T, \\ w(a, 0, x) = 0 \text{ in } Q_{A_2}, \\ \partial_\eta w = 0 \text{ on } \Sigma_2. \end{cases} \tag{16}$$

Multiplying (16) by w and integrating over $(0, A) \times (0, t) \times \Omega$, and following the same calculations as before, one has

$$\begin{aligned} \left(\lambda_0 - \|\beta_2\|_{\infty}^2 A_2 - A_2 A_1^2 K^2 \|b\|_{\infty}^2\right) \int_0^t \|w(a, s, x)\|_{L^2(Q_2)}^2 ds \\ \leq A_1 A_2^2 K^2 \|b\|_{\infty}^2 \int_0^t \|(h_1 - h_2)(a, s, x)\|_{L^2(Q_1)}^2 ds. \end{aligned} \tag{17}$$

Therefore, we have

$$\begin{aligned} \int_0^t \|(v_{h_1} - v_{h_2})(a, s, x)\|_{L^2((0, A_2) \times \Omega)}^2 ds \\ \leq C_2 \int_0^t \|(h_1 - h_2)(a, s, x)\|_{L^2((0, A_2) \times \Omega)}^2 ds, \end{aligned} \tag{18}$$

where

$$C_2 = \frac{1}{\lambda_0 - \|\beta_2\|_{\infty}^2 A_2 - A_2 A_1^2 K^2 \|b\|_{\infty}^2} \times A_1 A_2^2 K^2 \|b\|_{\infty}^2. \tag{19}$$

Now, as h and v_h are known, the remainder of (9) can be rewritten as following

$$\begin{cases} \partial_t u + \partial_a u - \Delta u + \hat{\mu}_1(a)u = g(a, t, x) \text{ in } Q_1, \\ u(0, t, x) = \int_0^{A_1} \beta_1(a) u(t, a, x) da \text{ in } Q_T, \\ u(a, 0, x) = p_0(a, x) \text{ in } Q_{A_1}, \\ \partial_\eta u = 0 \text{ on } \Sigma_1, \end{cases} \tag{20}$$

where $\hat{\mu}_1(a) = \lambda_0 + \mu_1(a)$ and $g(a, t, x) = e^{-\lambda_0 t} f_1 - e^{-\lambda_0 t} \int_0^{A_2} F(e^{\lambda_0 t} u(a, t, x), e^{\lambda_0 t} v(\alpha, t, x)) da$.

As λ_0 is fixed, $\hat{\mu}_1 \in L^1_{loc}(0, A_1)$ and $g \in L^\infty(Q_1)$ because $f_1 \in L^\infty(Q_1)$ and F is bounded.

According to the results in [20], (20) has a unique nonnegative solution \tilde{u}_h in $L^2(Q_1)$.

Multiplying (20) by \tilde{u}_h , integrating over Q_1 and using Young's inequality, we get

$$\lambda_0 \int_{Q_1} \tilde{u}_h^2(a, t, x) \, da \, dt \, dx \leq \frac{1}{2} \|p_0\|_{L^2(Q_{A_1})}^2 + \frac{1}{2} \|f_1\|_{L^2(Q_1)}^2 + \int_{Q_1} \tilde{u}_h^2(a, t, x) \, da \, dt \, dx + \frac{1}{2} \int_0^T \int_{Q_1} \tilde{u}_h^2(0, t, x) \, da \, dt \, dx + \frac{1}{2} \int_{Q_2} \left(\int_0^{A_2} F(e^{\lambda_0 t} h(a, t, x), e^{\lambda_0 t} v_h(a, t, x)) \, d\alpha \right)^2 \, da \, dt \, dx. \tag{21}$$

Using now Cauchy Schwarz's inequality, $\lambda_0 \int_{Q_1} \tilde{u}_h^2(a, t, x) \, da \, dt \, dx \leq 1/2 \|p_0\|_{L^2(Q_{A_1})}^2 + 1/2 \|f_1\|_{L^2(Q_1)}^2 + \int_{Q_1} \tilde{u}_h^2(a, t, x) \, da \, dt \, dx + 1/2 \int_0^T \int_{Q_1} \tilde{u}_h^2(0, t, x) \, da \, dt \, dx + 1/2 e^{-2\lambda_0 t} \int_{Q_1} \left(\int_0^{A_2} F(e^{\lambda_0 t} h(a, t, x), e^{\lambda_0 t} v_h(a, t, x)) \, d\alpha \right)^2 \, da \, dt \, dx.$ we obtain

$$\left(\lambda_0 - \frac{1}{2} A_1 \|\beta_1\|_{\infty}^2 - 1 \right) \int_{Q_1} \tilde{u}_h^2(a, t, x) \, da \, dt \, dx \leq \frac{1}{2} \|p_0\|_{L^2(Q_{A_1})}^2 + \frac{1}{2} \|f_1\|_{L^2(Q_1)}^2 + \frac{1}{2} A_2^2 K^2 \|b\|_{\infty}^2 \int_{Q_1} h^2(a, t, x) \, da \, dt \, dx + \frac{1}{2} A_1 A_2 K^2 \|b\|_{\infty}^2 \int_{Q_2} |v_h|^2(a, t, x) \, da \, dt \, dx. \tag{22}$$

Then, one gets

$$\int_{Q_1} \tilde{u}_h^2(a, t, x) \, da \, dt \, dx \leq C_3 \left(\|p_0\|_{L^2(Q_{A_1})}^2 + \|q_0\|_{L^2(Q_{A_2})}^2 + \|f_1\|_{L^2(Q_1)}^2 + \|f_2\|_{L^2(Q_2)}^2 + \|h\|_{L^2(Q_1)}^2 \right), \tag{23}$$

where

$$C_3 = \frac{1}{\lambda_0 - (1/2) A_1 \|\beta_1\|_{\infty}^2 - 1} \times \max \left\{ \frac{1}{2}, \left(\frac{1}{2} A_2^2 K^2 \|b\|_{\infty}^2 + \frac{1}{2} A_1 A_2 K^2 \|b\|_{\infty}^2 C_1 \right) \right\}. \tag{24}$$

Denote by $\Lambda: L_+^2(Q_1) \rightarrow L_+^2(Q_1)$, the application given by

$$\Lambda h = \tilde{u}_h, \tag{25}$$

where \tilde{u}_h is the unique solution of (20). For any $h_1, h_2 \in L_+^2(Q_1)$, we denote by $V = e^{-\gamma_0 t} (\tilde{u}_{h_1} - \tilde{u}_{h_2})$ in Q_1 , V satisfies the following equations

$$\begin{cases} \partial_t V + \partial_a V - \Delta V + (\gamma_0 + \mu_1(a))V = -e^{-\gamma_0 t} \int_0^{A_2} F(e^{\lambda_0 t} h_1(a, t, x), e^{\lambda_0 t} v_{h_1}(a, t, x)) \, d\alpha + e^{-\gamma_0 t} \int_0^{A_2} F(e^{\lambda_0 t} h_2(a, t, x), e^{\lambda_0 t} v_{h_2}(a, t, x)) \, d\alpha & \text{in } Q_1, \\ V(0, t, x) = \int_0^{A_1} \beta_1(a) V(t, a, x) \, da & \text{in } Q_T, \\ V(a, 0, x) = 0 & \text{in } Q_{A_1}, \\ \partial_{\eta} V = 0 & \text{on } \Sigma_1, \end{cases} \tag{26}$$

where $v_{h_i}, i = 1, 2$ is solution of

$$\begin{cases} \partial_t v_{h_i} + \partial_a v_{h_i} - \Delta v_{h_i} + (\lambda_0 + \mu_2(a))v_{h_i} = e^{-\lambda_0 t} f_2 \text{ in } Q_2, \\ v_{h_i}(0, t, x) = \int_0^{A_2} \beta_2(a)v_{h_i}(a, t, x)da + \\ e^{-\lambda_0 t} \int_0^{A_1} \int_0^{A_2} b(x, a, \alpha)F(e^{\lambda_0 t} h_i(a, t, x), e^{\lambda_0 t} v_{h_i}(\alpha, t, x))d\alpha da \text{ in } Q_T, \\ v_{h_i}(a, 0, x) = q_0(a, x) \text{ in } Q_{A_2}, \\ \partial_\eta v_{h_i} = 0 \text{ on } \Sigma_2. \end{cases} \tag{27}$$

Multiplying (26) by V and integrating over $(0, A_1) \times (0, t) \times \Omega$, on gets

$$\begin{aligned} & \lambda_0 \int_0^t \|V(a, s, x)\|_{L^2(Q_{A_1})}^2 ds dx \leq \\ & \frac{1}{2} \|\beta_1\|_\infty^2 A_1 \left(\int_0^t \int_0^{A_1} \int_\Omega V(a, s, x)^2 dt dx \right) + \frac{1}{2} \int_0^t \int_0^{A_1} \int_\Omega V^2(a, s, x) da ds dx + \\ & \int_0^t \int_0^{A_1} \int_\Omega \left(A_2^2 \|b\|_\infty^2 K^2 |(h_1 - h_2)(a, s, x)|^2 + A_2 \|b\|_\infty^2 K^2 \int_0^{A_2} |(v_{h_1} - v_{h_2})(\alpha, s, x)|^2 d\alpha \right) da ds dx. \end{aligned} \tag{28}$$

And we also have

$$\begin{aligned} & \left(\lambda_0 - \left(\frac{1}{2} A_1 \|\beta_1\|_\infty^2 + \frac{1}{2} \right) \right) \int_0^t \|V(a, s, x)\|_{L^2(Q_{A_1})}^2 ds \\ & \leq A_2^2 \|b\|_\infty^2 K^2 \int_0^t \|(h_1 - h_2)(a, s, x)\|_{L^2(Q_{A_1})}^2 ds \\ & + A_2 A_1 \|b\|_\infty^2 K^2 \int_0^t \|(v_{h_1} - v_{h_2})(\alpha, s, x)\|_{L^2(Q_2)}^2 ds. \end{aligned} \tag{29}$$

By recalling the inequality (18) in (29), we deduce that, there exists a constant C_4 , such that

$$\begin{aligned} & \int_0^t \|\tilde{u}_{h_1} - \tilde{u}_{h_2}(a, s, x)\|_{L^2(Q_{A_1})}^2 ds \\ & \leq C_4 \int_0^t \|(h_1 - h_2)(a, s, x)\|_{L^2(Q_{A_1})}^2 ds, \end{aligned} \tag{30}$$

where

$$\begin{aligned} C_4 &= \frac{1}{\lambda_0 - \left((1/2) A_1 \|\beta_1\|_\infty^2 - (1/2) \right)} \\ & \times \left(A_2^2 \|b\|_\infty^2 K^2 + A_2 A_1 \|b\|_\infty^2 K^2 C_2 \right). \end{aligned} \tag{31}$$

Let us define on $L^2_+(Q)$ the metric d by: for any $h_1, h_2 \in L^2_+(Q_1)$,

$$d(h_1, h_2) = \left(\int_0^T \|(h_1 - h_2)(t)\|_{L^2(Q_{A_1})}^2 \exp\{-2C_4 t\} dt \right)^{(1/2)}. \tag{32}$$

So,

$$\begin{aligned} [d(\Lambda h_1, \Lambda h_2)]^2 &= \int_0^T \|\tilde{u}_{h_1} - \tilde{u}_{h_2}(t)\|_{L^2(Q_{A_1})}^2 \exp\{-2C_4 t\} dt \\ &\leq C_4 \int_0^T \exp\{-2C_4 t\} \int_0^t \|(h_1 - h_2)(s)\|_{L^2(Q_{A_1})}^2 ds dt. \end{aligned} \tag{33}$$

Using the Fubini's theorem, we conclude that:

$$\begin{aligned} [d(\Lambda h_1, \Lambda h_2)]^2 &\leq C_4 \int_0^T \exp\{-2C_4 t\} \int_0^t \|(h_1 - h_2)(s)\|_{L^2(Q_{A_1})}^2 ds dt \\ &= \int_0^T \int_s^T C_4 \exp\{-2C_4 t\} \|(h_1 - h_2)(s)\|_{L^2(Q_{A_1})}^2 dt ds \\ &= \left(\int_0^T \|(h_1 - h_2)(s)\|_{L^2(Q_{A_1})}^2 \int_s^T C_4 \exp\{-2C_4 t\} dt ds \right). \end{aligned} \tag{34}$$

From here we have:

$$d(\Lambda h_1, \Lambda h_2) \leq \frac{1}{\sqrt{2}} d(h_1, h_2). \tag{35}$$

Then, Λ is a contraction on the complete metric space $L^2_+(Q_1)$ and using Banach's fixed point theorem, we conclude the existence of a unique fixed point u_h nonnegative

for Λ , so the unique couple (u_h, v_h) is the unique solution of (9). Hence, we deduce that the couple $(p, q) = (e^{\lambda_0 t} u_h, e^{\lambda_0 t} v_h)$ is the unique solution to the problem (2). From the explicit expression of the constant C_3 , we see that it is always possible to choose λ_0 so that $C_3 < 1$. Replace h by u_h in (14) and in (23), summing the inequalities (14) and (17), we conclude that there exists a constant C independent on T such that

$$\|u_h\|_{L^2(Q_1)}^2, \|v_h\|_{L^2(Q_2)}^2 \leq C \left(\|p_0\|_{L^2(Q_{A_1})}^2 + \|q_0\|_{L^2(Q_{A_2})}^2 + \|f_1\|_{L^2(Q_1)}^2 + \|f_2\|_{L^2(Q_2)}^2 \right). \tag{36}$$

And the inequalities (7) follow clearly. □

3. Spatially Homogeneous Solutions

Let consider the following spatial homogeneous system deduced from (2):

$$\begin{cases} \partial_t p + \partial_a p + \mu_1(a)p = - \int_0^{A_2} F(p(a, t), q(\alpha, t)) d\alpha \text{ in } Q, \\ \partial_t q + \partial_a q + \mu_2(a)q = f \text{ in } Q, \\ p(0, t) = \int_0^{A_1} \beta_1(a)p(t, a) da \text{ in } (0, T), \\ q(0, t) = \int_0^{A_2} \beta_2(a)q(a, t) da + \int_0^{A_1} \int_0^{A_2} b(a, \alpha) F(p(a, t), q(\alpha, t)) d\alpha da \text{ in } (0, T), \\ p(a, 0) = p_0(a) \text{ in } (0, A), \\ q(a, 0) = q_0(a) \text{ in } (0, A). \end{cases} \tag{37}$$

Theorem 2. Under the assumptions (A_1) - (A_4) , for all $(p_0, q_0) \in L^1(0, A)^2$ and $f \in C([0, T], L^1(0, A_1))$, the system (37) admits a unique solution in $C([0, T], L^1(0, A_1)) \times C([0, T], L^1(0, A_2))$.

Proof. To simplify the calculations and without losing sight of the generality, we set $A = \max\{A_1, A_2\}$. Let set $Y = C([0, T], L^1(0, A))$ endowed with the norm $\|h\|_Y = \sup_{t \in [0, T]} e^{-\lambda t} \|h\|_{L^1(0, A)}$ where λ is a positive constant that will be choosed later. We fix \bar{p} in Y . Consider now the following system

$$\begin{cases} \partial_t p + \partial_a p + \mu_1(a)p = - \int_0^{A_2} F(\bar{p}(a, t), q(\alpha, t)) d\alpha \text{ in } Q, \\ \partial_t q + \partial_a q + \mu_2(a)q = f \text{ in } Q, \\ p(0, t) = \int_0^{A_1} \beta_1(a)p(t, a) da \text{ in } (0, T), \\ q(0, t) = \int_0^{A_2} \beta_2(a)q(a, t) da + \int_0^{A_1} \int_0^{A_2} b(a, \alpha) F(\bar{p}(a, t), q(\alpha, t)) d\alpha da \text{ in } (0, T), \\ p(a, 0) = p_0(a) \text{ in } (0, A), \\ q(a, 0) = q_0(a) \text{ in } (0, A). \end{cases} \tag{38}$$

Integrating the system (38) along the characteristic curves $a - t = c$, we obtain implicit formulas for the solutions of (2) stated below:

$$p(a, t) = \frac{\prod_1(a)}{\prod_1(a-t)p_0(a-t) - \int_0^t \frac{\prod_1(a)}{\prod_1(s+a-t)} \int_0^A F(\bar{p}(s+a-t, s), q(\alpha, s)) d\alpha ds, \quad a \geq t, p(a, t) = \prod_1(a) \int_0^A \beta_1(s) p(s, t-a) ds - \int_0^a \frac{\prod_1(a)}{\prod_1(s)} \int_0^A F(\bar{p}(s, s+t-a), q(\alpha, s+t-a)) d\alpha ds, \quad a < t, \tag{39}$$

$$q(a, t) = \frac{\prod_2(a)}{\prod_2(a-t)q_0(a-t) + \int_0^t \frac{\prod_1(a)}{\prod_1(s+a-t)} \int_0^A f(s+a-t, s) ds, \quad a \geq t, q(a, t) = \prod_2(a) \left[\int_0^A \beta_2(s) q(s, t-a) ds + \int_0^a \int_0^A b(s, \alpha) F(\bar{p}(s, t-a), q(\alpha, t-a)) d\alpha ds \right] + \int_0^a \frac{\prod_1(a)}{\prod_1(s)} \int_0^A f(s, t-a+s) ds, \quad a < t. \tag{40}$$

Let us fixed $\bar{q} \in Y$ and define the mapping $G: Y \rightarrow Y$ by:
for every $\bar{q} \in Y$ and for all $(t, a) \in Q$,

$$G(\bar{q})(a, t) = q(a, t) = \begin{cases} \frac{\prod_2(a)}{\prod_2(a-t)q_0(a-t) + \int_0^t \frac{\prod_1(a)}{\prod_1(s+a-t)} \int_0^A f(s+a-t, s) ds, & a \geq t \\ \prod_2(a) \left[\int_0^A \beta_2(s) q(s, t-a) ds + \int_0^a \int_0^A b(s, \alpha) F(\bar{p}(s, t-a), \bar{q}(\alpha, t-a)) d\alpha ds \right] + \int_0^a \frac{\prod_1(a)}{\prod_1(s)} \int_0^A f(s, t-a+s) ds, & a < t. \end{cases} \tag{41}$$

For all $(a, t) \in Q$ such that $a \geq t$,

$$e^{-\lambda t} \int_t^A |G(\bar{q})(a, t)| da \leq \|q_0\|_{L^1(0,A)} + \frac{1}{\lambda} \|f\|_Y, \tag{42}$$

And for $a < t$,

$$e^{-\lambda t} \int_0^t |G(\bar{q})(a, t)| da \leq \|\beta_2\|_\infty e^{-\lambda t} \int_0^t e^{\lambda l} \cdot e^{-\lambda l} \|G(\bar{q})(\cdot, l)\|_{L^1(0,A)} dl + (AK\|b\|_\infty\|\bar{p}\|_Y + AK\|b\|_\infty\|\bar{q}\|_Y) \frac{1}{\lambda}. \tag{43}$$

Summing of the previous inequalities, we get

$$e^{-\lambda t} \|G(\bar{q})(\cdot, t)\|_{L^1(0,A)} \leq \|\beta_2\|_\infty e^{-\lambda t} \int_0^t e^{\lambda l} \cdot e^{-\lambda l} \|G(\bar{q})(\cdot, l)\|_{L^1(0,A)} dl + (AK\|b\|_\infty\|\bar{p}\|_Y + AK\|b\|_\infty\|\bar{q}\|_Y) \frac{1}{\lambda} + \|q_0\|_{L^1(0,A)} + \frac{1}{\lambda} \|f\|_Y. \tag{44}$$

So, we have

$$\|G(\bar{q})\|_Y \leq \frac{\lambda \|q_0\|_{L^1(0,A)} + AK\|b\|_\infty\|\bar{p}\|_Y + AK\|b\|_\infty\|\bar{q}\|_Y + \|f\|_Y}{\lambda - \|\beta_2\|_\infty} < \infty, \tag{45}$$

For λ large enough, thus $G(\bar{q}) \in Y$.

Now, for all $\bar{q}_1, \bar{q}_2 \in Y$, for all $(a, t) \in Q$ such that $a < t$; one has

$$\begin{aligned} \int_0^t |(G(\bar{q}_1) - G(\bar{q}_2))(a, t)| da &\leq \|\beta_2\|_\infty \int_0^t \int_0^A |q_1(s, t-a) - q_2(s, t-a)| ds da \\ &\quad + \|b\|_\infty \int_0^t \int_0^A \int_0^A |F(\bar{p}(s, t-a), \bar{q}_1(\alpha, t-a)) - F(\bar{p}(s, t-a), \bar{q}_2(\alpha, t-a))| d\alpha ds da \\ &\leq \|\beta_2\|_\infty \int_0^t e^{\lambda l} e^{-\lambda l} \int_0^A |q_1(s, l) - q_2(s, l)| ds dl + AK\|b\|_\infty \int_0^t e^{\lambda l} e^{-\lambda l} \int_0^A |\bar{q}_1(\alpha, l) - \bar{q}_2(\alpha, l)| d\alpha dl. \end{aligned} \tag{46}$$

Finally, it follows that

$$\int_0^t |(G(\bar{q}_1) - G(\bar{q}_2))(a, t)| da \leq (\|\beta_2\|_\infty \|q_1 - q_2\|_Y + K\|b\|_\infty A \|\bar{q}_1 - \bar{q}_2\|_Y) \times \frac{1 - e^{-\lambda t}}{\lambda}. \tag{47}$$

Multiplying the inequality (47) by $e^{-\lambda t}$, we get

$$e^{-\lambda t} \int_0^t |(G(\bar{q}_1) - G(\bar{q}_2))(a, t)| da \leq \frac{1}{\lambda} (\|\beta_2\|_\infty \|q_1 - q_2\|_Y + K\|b\|_\infty A \|\bar{q}_1 - \bar{q}_2\|_Y). \tag{48}$$

It is obvious that for all $(a, t) \in Q$ such that $a \geq t$, we have

$$e^{-\lambda t} \int_t^A |(G(\bar{q}_1) - G(\bar{q}_2))(a, t)| da \leq \frac{1}{\lambda} (\|\beta_2\|_\infty \|q_1 - q_2\|_Y + K\|b\|_\infty A \|\bar{q}_1 - \bar{q}_2\|_Y). \tag{49}$$

Combining the inequalities (46) and (47) it follows that

$$\|G(\bar{q}_1) - G(\bar{q}_2)\|_Y \leq \frac{2AK\|b\|_\infty}{\lambda - 2\|\beta_2\|_\infty} \|\bar{q}_1 - \bar{q}_2\|_Y. \tag{50}$$

For λ large enough, it is clear that G is a contraction in Y and the (40) have a unique solution q in Y .

Now, define the mapping $\Lambda: Y \rightarrow Y$ by: for every $\bar{p} \in Y$ and for all $(t, a) \in Q$,

$$\Lambda(\bar{p})(a, t) = p(a, t)$$

$$= \begin{cases} \frac{\Pi_1(a)}{\Pi_1(a-t)p_0(a-t) - \int_0^t \frac{\Pi_1(a)}{\Pi_1(s+a-t) \int_0^A F(\bar{p}(s+a-t, s), q(a', s)) da' ds}, & a \geq t, \Pi_1(a) \int_0^A \beta_1(s)p(s, t-a) ds - \int_0^a \frac{\Pi_1(a)}{\Pi_1(s) \int_0^A F(\bar{p}(s, s+t-a), q(a', s+t-a)) da' ds}, & a < t. \end{cases} \tag{51}$$

For every $(a, t) \in Q$ such that $a \geq t$, one has

$$e^{-\lambda t} \int_t^A |\Lambda(\bar{p})(a, t)| da \leq \|p_0\|_{L^1(0,A)} + \frac{KA}{\lambda} (\|\bar{p}\|_Y + \|q\|_Y), \tag{52}$$

And for $a < t$, we have

$$e^{-\lambda t} \int_0^t |\Lambda(\bar{p})(a, t)| da \leq \|\beta_1\|_\infty e^{-\lambda t} \int_0^t e^{\lambda l} e^{-\lambda l} \|\Lambda(\bar{p})(\cdot, l)\|_{L^1(0,A)} dl + \frac{AK}{\lambda} (\|\bar{p}\|_Y + \|q\|_Y). \tag{53}$$

By adding the inequalities (52) and (53) and using (45), one gets

$$\|\Lambda(\bar{p})\|_Y \leq \frac{\lambda \|p_0\|_{L^1(0,A)} + 2AK(\|\bar{p}\|_Y + \|q\|_Y)}{\lambda - \|\beta_1\|_\infty} \leq \frac{\lambda \|p_0\|_{L^1(0,A)} + 2AK\|\bar{p}\|_Y}{\lambda - \|\beta_1\|_\infty} + \frac{2KA(\lambda \|q_0\|_{L^1(0,A)} + AK\|b\|_\infty \|\bar{p}\|_Y + \|f\|_Y)}{(\lambda - \|\beta_1\|_\infty)(\lambda - (\|\beta_2\|_\infty + AK\|b\|_\infty))} < \infty. \tag{54}$$

Therefore, $\Lambda(\bar{p}) \in Y$ for \bar{p} fixed in Y .

For all $\bar{p}_1, \bar{p}_2 \in Y$, for all $(a, t) \in Q$ such that $a \geq t$; we have

$$\int_t^A |(\Lambda(\bar{p}_1) - \Lambda(\bar{p}_2))(a, t)| da \leq \int_t^A \int_0^t \int_0^A |(\bar{p}_1(s+a-t, s), q_1(\alpha, s)) - F(\bar{p}_2(s+a-t, s), q_2(\alpha, s))| dadsd\alpha \leq KA \int_0^t \int_s^{A+s-t} |\bar{p}_1(l, s) - \bar{p}_2(l, s)| dl ds + K \int_0^t \int_t^A \int_0^A |q_1(\alpha, s) - q_2(\alpha, s)| dad\alpha ds \leq KA \int_0^t e^{\lambda s} e^{-\lambda s} \int_0^A |\bar{p}_1(l, s) - \bar{p}_2(l, s)| dl ds + KA \int_0^t e^{\lambda s} e^{-\lambda s} \int_0^A |q_1(\alpha, s) - q_2(\alpha, s)| dad\alpha. \tag{55}$$

It is clear that

$$\int_t^A |(\Lambda(\bar{p}_1) - \Lambda(\bar{p}_2))(a, t)| da \leq KA(\|\bar{p}_1 - \bar{p}_2\|_Y + \|q_1 - q_2\|_Y) \frac{1 - e^{-\lambda t}}{\lambda}. \tag{56}$$

Multiplying the inequality (56) by $e^{-\lambda t}$, we get

$$e^{-\lambda t} \int_t^A |(\Lambda(\overline{p}_1) - \Lambda(\overline{p}_2))(a, t)| da \leq \frac{KA}{\lambda} (\|\overline{p}_1 - \overline{p}_2\|_Y + \|q_1 - q_2\|_Y). \tag{57}$$

And for $a < t$, we have

$$\begin{aligned} \int_0^t |(\Lambda(\overline{p}_1) - \Lambda(\overline{p}_2))(a, t)| da &\leq \|\beta_1\|_\infty \int_0^t \int_0^A |p_1(s, l) - p_2(s, l)| ds dl + \\ &AK \int_0^t \int_0^a |\overline{p}_1(s, s+t-a) - \overline{p}_2(s, s+t-a)| ds da \\ &+ K \int_0^t \int_0^a \int_0^A |q_1(\alpha, s+t-a) - q_2(\alpha, s+t-a)| d\alpha ds da. \end{aligned} \tag{58}$$

In other hand, one has

$$\begin{aligned} \int_0^t |(\Lambda(\overline{p}_1) - \Lambda(\overline{p}_2))(a, t)| da &\leq (\|\beta_1\|_\infty \|p_1 - p_2\|_Y + AK \|\overline{p}_1 - \overline{p}_2\|_Y + AK \|q_1 - q_2\|_Y) \frac{1 - e^{-\lambda t}}{\lambda}. \end{aligned} \tag{59}$$

Multiplying the inequality (59) by $e^{-\lambda t}$, we get

$$e^{-\lambda t} \int_0^t |(\Lambda(\overline{p}_1) - \Lambda(\overline{p}_2))(a, t)| da \leq \frac{1}{\lambda} (\|\beta_1\|_\infty \|p_1 - p_2\|_Y + AK \|\overline{p}_1 - \overline{p}_2\|_Y + AK \|q_1 - q_2\|_Y). \tag{60}$$

Combining the inequalities (57) and (59), we deduce that

$$\|\Lambda(\overline{p}_1) - \Lambda(\overline{p}_2)\|_Y \leq \frac{2KA}{\lambda - \|\beta_1\|_\infty} (\|\overline{p}_1 - \overline{p}_2\|_Y + \|q_1 - q_2\|_Y), \tag{61}$$

where q_1 and q_2 are solutions of (39) and (40) associated respectively to \overline{p}_1 and \overline{p}_2 .

Then, we have for all $(a, t) \in Q$ such that $a < t$,

$$\begin{aligned} \int_0^t |(q_1 - q_2)(a, t)| da &\leq \|\beta_2\|_\infty \int_0^t \int_0^A |(q_1 - q_2)(s, l)| ds dl + KA \|b\|_\infty \int_0^t \int_0^A |(\overline{p}_1 - \overline{p}_2)(s, l)| ds dl + \\ &KA \|b\|_\infty \int_0^t \int_0^A |q_1 - q_2|(\alpha, l) d\alpha dl \\ &\leq (\|\beta_2\|_\infty \|q_1 - q_2\|_Y + K \|b\|_\infty A \|\overline{p}_1 - \overline{p}_2\|_Y + K \|b\|_\infty A \|q_1 - q_2\|_Y) \frac{1 - e^{-\lambda t}}{\lambda}. \end{aligned} \tag{62}$$

So, it follows that

$$\begin{aligned} e^{-\lambda t} \int_0^t |(q_1 - q_2)(a, t)| da &\leq \frac{1}{\lambda} (\|\beta_2\|_\infty \|q_1 - q_2\|_Y + K \|b\|_\infty A \|\overline{p}_1 - \overline{p}_2\|_Y \\ &+ K \|b\|_\infty A \|q_1 - q_2\|_Y). \end{aligned} \tag{63}$$

We deduce from (63) that

$$\begin{aligned} \|q_1 - q_2\|_Y &\leq \frac{2}{\lambda} (\|\beta_2\|_\infty \|q_1 - q_2\|_Y + K \|b\|_\infty A \|\overline{p}_1 - \overline{p}_2\|_Y \\ &+ K \|b\|_\infty A \|q_1 - q_2\|_Y). \end{aligned} \tag{64}$$

From (64), we obtain

$$\|q_1 - q_2\|_Y \leq \frac{2AK \|b\|_\infty}{\lambda - 2(\|\beta_2\|_\infty + AK \|b\|_\infty)} \|\overline{p}_1 - \overline{p}_2\|_Y. \tag{65}$$

By combining the inequalities (61) and (65), we get

$$\|\Lambda(\overline{p_2}) - \Lambda(\overline{p_2})\|_Y \leq \frac{2KA(\lambda - 2\|\beta_2\|_\infty)}{(\lambda - \|\beta_1\|_\infty)(\lambda - 2(\|\beta_2\|_\infty + AK\|b\|_\infty))} \|\overline{p_1} - \overline{p_2}\|_Y. \tag{66}$$

For λ large enough such that

$$\lambda > \frac{1}{2}(\|\beta_1\|_\infty + 2\|\beta_2\|_\infty + 2AK\|b\|_\infty + 2AK) +$$

$$\frac{1}{2} \left[(\|\beta_1\|_\infty - 2\|\beta_2\|_\infty - 2AK\|b\|_\infty + 2AK)^2 + 16A^2K^2\|b\|_\infty \right]^{(1/2)}, \tag{67}$$

where K is a Lipschitz constant, one gets clearly that Λ is a contraction in Y . Therefore, the (37) and (38) have a unique

solution p in Y . The couple (p, q) is the unique solution to the system (35) because the problem (35) is equivalent to solve the equations (39)–(42). (see [21])

We have also the following result: □

Theorem 3. *Under the assumptions (A_1) – (A_4) , for all $(p_0, q_0) \in L^2(0, A_1) \times L^2(0, A_2)$ and $f \in L^2((0, A_2) \times (0, T))$, the system (37) admits a unique solution in $L^2((0, A_1) \times (0, T)) \times L^2((0, A_2) \times (0, T))$. Moreover, there exists a positive constant $C = C(\|\beta_1\|_\infty, \|\beta_2\|_\infty, \|b\|_\infty, A_1, A_2, K, T)$, such that*

$$\|p\|_{L^2((0, A_1) \times (0, T))}^2, \|q\|_{L^2((0, A_2) \times (0, T))}^2 \leq C \left(\|p_0\|_{L^2(0, A_1)}^2 + \|q_0\|_{L^2(0, A_2)}^2 + \|f\|_{L^2((0, A_2) \times (0, T))}^2 \right). \tag{68}$$

Proof. Let $u = e^{-\lambda t}p$ and $v = e^{-\lambda t}q$. The system becomes

$$\begin{cases} \partial_t u + \partial_a u + (\lambda + \mu_1(a))u = -e^{-\lambda t} \int_0^{A_2} F(e^{\lambda t} u(a, t), e^{\lambda t} v(\alpha, t)) d\alpha \text{ in } Q_1, \\ \partial_t v + \partial_a v + (\lambda + \mu_2(a))v = e^{-\lambda t} f \text{ in } Q_2, \\ u(0, t) = \int_0^{A_1} \beta_1(a)u(t, a) da \text{ in } (0, T), \\ v(0, t) = \int_0^{A_2} \beta_2(a)v(a, t) da + e^{-\lambda t} \int_0^{A_1} \int_0^{A_2} b(a, \alpha)F(e^{\lambda t} u(a, t), e^{\lambda t} v(\alpha, t)) d\alpha da \text{ in } (0, T), \\ u(a, 0) = p_0(a) \text{ in } (0, A_1), \\ v(a, 0) = q_0(a) \text{ in } (0, A_2). \end{cases} \tag{69}$$

Fix h in $L^2((0, A_1) \times (0, T))$ and consider the following system:

$$\begin{cases} \partial_t u + \partial_a u + (\lambda + \mu_1(a))u = -e^{-\lambda t} \int_0^{A_2} F(e^{\lambda t} h(a, t), e^{\lambda t} v(\alpha, t)) d\alpha \text{ in } Q_1, \\ \partial_t v + \partial_a v + (\lambda + \mu_2(a))v = e^{-\lambda t} f \text{ in } Q_2, \\ u(0, t) = \int_0^{A_1} \beta_1(a)u(t, a) da \text{ in } (0, T), \\ v(0, t) = \int_0^{A_2} \beta_2(a)v(a, t) da + e^{-\lambda t} \int_0^{A_1} \int_0^{A_2} b(a, \alpha)F(e^{\lambda t} h(a, t), e^{\lambda t} v(\alpha, t)) d\alpha da \text{ in } (0, T), \\ u(a, 0) = p_0(a) \text{ in } (0, A_1), \\ v(a, 0) = q_0(a) \text{ in } (0, A_2). \end{cases} \tag{70}$$

Multiplying the second equation of (70) by v , integrating over $(0, A_2) \times (0, T)$ and using Young and Cauchy-Schwarz's inequalities, we get

$$\begin{aligned}
 \lambda \int_0^T \int_0^{A_2} |v(a, t)|^2 da dt &\leq \frac{1}{2} \int_0^{A_2} v^2(a, 0) da + \frac{1}{2} \int_0^T v^2(0, t) dt + \frac{1}{2} \int_0^T \int_0^{A_2} f^2(a, t) da dt + \frac{1}{2} \int_0^T \int_0^{A_2} v^2(a, t) da dt + \\
 &\int_0^T \left(\int_0^{A_2} \beta_2(a) v(a, t) da \right)^2 + \int_0^T e^{-2\lambda t} \|b\|_{\infty}^2 A_1 A_2 \int_0^{A_1} \int_0^{A_2} |F e^{\lambda t} h(a, t), e^{\lambda t} v(a, t)|^2 da' da dt \\
 &\leq \frac{1}{2} \|q_0\|_{L^2(0, A_2)}^2 + \frac{1}{2} \|f\|_{L^2((0, A_2) \times (0, T))}^2 + \frac{1}{2} \int_0^T \int_0^{A_2} v^2(a, t) da dt + \|\beta_2\|_{\infty}^2 A_2 \int_0^T \int_0^{A_2} v^2(a, t) da dt \\
 &\quad + \|b\|_{\infty}^2 A_1^2 A_2 K^2 \int_0^T \int_0^{A_2} v^2(a, t) da dt + \|b\|_{\infty}^2 A_1 A_2^2 K^2 \int_0^T \int_0^{A_1} h^2(a, t) da dt.
 \end{aligned} \tag{71}$$

Therefore, we deduce that

$$\begin{aligned}
 &\int_0^T \int_0^{A_2} v^2(a, t) da dt \\
 &\leq \frac{C \left(\|q_0\|_{L^2(0, A_2)}^2 + \|f\|_{L^2((0, A_2) \times (0, T))}^2 + \|h\|_{L^2((0, A_1) \times (0, T))}^2 \right)}{\lambda - \left(\|\beta_2\|_{\infty}^2 A_2 + \|b\|_{\infty}^2 A_1^2 A_2 K^2 + (1/2) \right)} < \infty,
 \end{aligned} \tag{72}$$

where C is a constant that does not depend on λ .

For λ large enough, we showed that $v \in L^2((0, A_2) \times (0, T))$.

For every \bar{v} in $L^2((0, A_2) \times (0, T))$, we define the mapping $\Psi: L^2((0, A_2) \times (0, T)) \rightarrow L^2((0, A_2) \times (0, T))$ by $\Psi(\bar{v}) = v$, where v satisfies the following equations:

$$\begin{cases} \partial_t v + \partial_a v + (\lambda + \mu_2(a))v = -e^{-\lambda t} f \text{ in } Q_2, \\ v(0, t) = \int_0^{A_2} \beta_2(a) v(a, t) da + \\ e^{-\lambda t} \int_0^{A_1} \int_0^{A_2} b(a, a') [F(e^{\lambda t} h(a, t), e^{\lambda t} \bar{v}(a, t))] da da \text{ in } (0, T), \\ v(a, 0) = q_0(a, 0) \text{ in } (0, A_2). \end{cases} \tag{73}$$

Set $w = v_1 - v_2$, then w solves the system:

$$\begin{cases} \partial_t w + \partial_a w + (\lambda + \mu_1(a))w = -e^{-\lambda t} f \text{ in } Q_2, \\ w(0, t) = \int_0^{A_1} \beta_2(a) w(a, t) da + \\ e^{-\lambda t} \int_0^{A_1} \int_0^{A_2} b(a, \alpha) [F(e^{\lambda t} h(a, t), e^{\lambda t} v_1(\alpha, t)) - F(e^{\lambda t} h(a, t), e^{\lambda t} v_2(\alpha, t))] da da \text{ in } (0, T), \\ w(a, 0) = 0 \text{ in } (0, A_2). \end{cases} \tag{74}$$

Multiplying (74) by w , integrating over $(0, A_2) \times (0, T)$ and using Young and Cauchy-Schwarz's inequalities, we get

$$\begin{aligned}
 \lambda \int_0^T \int_0^{A_2} w^2(a, t) da dt &\leq \int_0^T \left(\int_0^{A_2} \beta_2(a) w(a, t) da \right)^2 + \\
 &\int_0^T \left(e^{-\lambda t} \int_0^{A_1} \int_0^{A_2} b(a, \alpha) [F(e^{\lambda t} h(a, t), e^{\lambda t} \bar{v}_1(\alpha, t)) - F(e^{\lambda t} h(a, t), e^{\lambda t} \bar{v}_2(\alpha, t))] da da \right)^2 dt \\
 &\leq A_2 \|\beta_2\|_{\infty}^2 \int_0^T \int_0^{A_2} w^2(a, t) da dt + K^2 \|b\|_{\infty}^2 A_1^2 A_2 \int_0^T \int_0^{A_2} |\bar{v}_1(\alpha, t) - \bar{v}_2(\alpha, t)|^2 da dt.
 \end{aligned} \tag{75}$$

That is,

$$\begin{aligned}
 &\|\Psi(\bar{v}_1) - \Psi(\bar{v}_2)\|_{L^2((0, A_2) \times (0, T))}^2 \\
 &\leq \frac{K^2 A_1^2 A_2 \|b\|_{\infty}^2}{\lambda - A_2 \|\beta_2\|_{\infty}^2} \|(\bar{v}_1) - (\bar{v}_2)\|_{L^2((0, A_2) \times (0, T))}^2.
 \end{aligned} \tag{76}$$

Hence, for λ large enough, Ψ is a contraction in $L^2((0, A_2) \times (0, T))$ and using Banach's fixed point theorem, Ψ has a unique fixed point v which is a unique solution to the system (73).

Now, v and h are being known. So, the first equation of (70) has a unique solution in $L^2((0, A_1) \times (0, T))$.

Multiplying the first equation of (70), integrating over $(0, A_1) \times (0, T)$ and using Cauchy–Schwarz and Young's inequalities, one gets

$$\begin{aligned} \lambda \int_0^T \int_0^{A_1} u^2(a, t) da dt \leq & \frac{1}{2} \|p_0\|_{L^2(0, A_1)}^2 + \frac{1}{2} \|\beta_1\|_{\infty}^2 A_1 \int_0^T \int_0^{A_1} u^2(a, t) da dt + \\ & \frac{1}{2} \int_0^T \int_0^{A_1} u^2(a, t) da dt + \frac{1}{2} A_2^2 K^2 \int_0^T \int_0^{A_1} h^2(a, t) da dt + \frac{1}{2} A_2 A_1 K^2 \int_0^T \int_0^{A_1} v^2(a, t) da dt. \end{aligned} \quad (77)$$

Thus,

$$\int_0^T \int_0^{A_1} u^2(a, t) da dt \leq \frac{C}{\lambda - \left((1/2) \|\beta_1\|_{\infty}^2 A_1 + (1/2) \right)} \left(\|p_0\|_{L^2(0, A_1)}^2 + \int_0^T \int_0^{A_1} h^2(a, t) da dt + \int_0^T \int_0^{A_1} v^2(a, t) da dt \right) < \infty, \quad (78)$$

where C is also a constant that does not depend on λ .

Now, let us define the mapping $\Phi: L^2((0, A_1) \times (0, T)) \rightarrow L^2((0, A_1) \times (0, T))$ by:

For every $h \in L^2((0, A_1) \times (0, T))$, $\Phi(h) = u$ where u is a unique solution of the system

$$\begin{cases} \partial_t u + \partial_a u + (\lambda + \mu_1(a))u = -e^{-\lambda t} \int_0^{A_2} F(e^{\lambda t} h(a, t), e^{\lambda t} v(a', t)) da' & \text{in } Q_1, \\ u(0, t) = \int_0^{A_1} \beta_1(a) u(t, a) da & \text{in } (0, T), \\ u(a, 0) = p_0(a) & \text{in } (0, A_1). \end{cases} \quad (79)$$

For every h_1 and h_2 in $L^2((0, A_1) \times (0, T))$ such that $\Phi(h_1) = u_1$ and $\Phi(h_2) = u_2$, we set again $w = u_1 - u_2$ where

u_1 and u_2 are the solutions of (79) corresponding respectively to h_1 and h_2 .

So, w solves the following system:

$$\begin{cases} \partial_t w + \partial_a w + (\lambda + \mu_1(a))w = e^{-\lambda t} \int_0^{A_2} [F(e^{\lambda t} h_2(a, t), e^{\lambda t} v_2(a, t)) - F(e^{\lambda t} h_1(a, t), e^{\lambda t} v_1(a, t))] da & \text{in } Q_1, \\ w(0, t) = \int_0^{A_1} \beta_1(a) w(t, a) da & \text{in } (0, T), \\ w(a, 0) = 0 & \text{in } (0, A_1), \end{cases} \quad (80)$$

where v_1 and v_2 are solutions of (73) corresponding respectively to h_1 and h_2 .

Multiplying the (80) by w , integrating over $(0, A_1) \times (0, T)$ and using Young and Cauchy-Schwarz's inequalities, we deduce that

$$\begin{aligned} \lambda \int_0^T \int_0^{A_1} w^2(a, t) da dt &\leq \frac{1}{2} \int_0^T w^2(0, t) da dt + \frac{1}{2} \int_0^T \int_0^{A_1} w^2(a, t) da dt + \\ &\frac{1}{2} \int_0^T \int_0^{A_1} e^{-2\lambda t} \left(\int_0^{A_2} \left| F(e^{\lambda t} h_2(a, t), e^{\lambda t} v_2(\alpha, t)) - F(e^{\lambda t} h_1(a, t), e^{\lambda t} v_1(\alpha, t)) \right| d\alpha \right)^2 da dt \\ &\leq \frac{1}{2} A_1 \|\beta_1\|_\infty^2 \int_0^T \int_0^{A_1} w^2(a, t) da dt + \frac{1}{2} \int_0^T \int_0^{A_1} w^2(a, t) da dt + \\ &K^2 A_2^2 \int_0^T \int_0^{A_1} |h_1(a, t) - h_2(a, t)|^2(a, t) da dt + K^2 A_2 A_1 \int_0^T \int_0^{A_2} |v_1(\alpha, t) - v_2(\alpha, t)|^2 da dt. \end{aligned} \tag{81}$$

We also have

Finally, we obtain

$$\begin{aligned} &\left(\lambda - \left(\frac{1}{2} \|\beta_1\|_\infty^2 A_1 + \frac{1}{2} \right) \right) \int_0^T \int_0^{A_1} |w(a, t)|^2(a, t) da dt \\ &\leq K^2 A_2^2 \int_0^T \int_0^{A_1} |h_1(a, t) - h_2(a, t)|^2(a, t) da dt + \\ &K^2 A_2 A_1 \int_0^T \int_0^{A_2} |v_1(\alpha, t) - v_2(\alpha, t)|^2 da dt. \end{aligned} \tag{82}$$

$$\begin{aligned} &\lambda \int_0^T \int_0^{A_2} |(v_1 - v_2)(a, t)|^2(a, t) da dt \\ &\leq \int_0^T \left(\int_0^{A_2} \beta_2(a) w(a, t) da \right)^2 dt \\ &\quad + \int_0^T \left(e^{-\lambda t} \int_0^{A_1} \int_0^{A_2} b(a, \alpha) \left[F(e^{\lambda t} h(a, t), e^{\lambda t} v_1(\alpha, t)) - F(e^{\lambda t} h(a, t), e^{\lambda t} v_2(\alpha, t)) \right] d\alpha da \right)^2 dt \\ &\leq A_2 \|\beta_2\|_\infty^2 \int_0^T \int_0^{A_2} w^2(a, t) da dt + K^2 \|b\|_\infty^2 A_1^2 A_2 \int_0^T \int_0^{A_2} |v_1(\alpha, t) - v_2(\alpha, t)|^2 da dt \\ &\quad + K^2 \|b\|_\infty^2 A_2^2 \int_0^T \int_0^{A_1} |h_1(a, t) - h_2(a, t)|^2 da dt. \end{aligned} \tag{83}$$

So, one has

$$\begin{aligned} &\left(\lambda - \left(\|\beta_2\|_\infty^2 A_2 + A_1^2 A_2 K^2 \|b\|_\infty^2 \right) \right) \int_0^T \int_0^{A_2} |v_1(a, t) - v_2(a, t)|^2(a, t) da dt \\ &\leq K^2 \|b\|_\infty^2 A_2^2 \int_0^T \int_0^{A_1} |h_1(a, t) - h_2(a, t)|^2 da dt. \end{aligned} \tag{84}$$

Combining the inequalities (83) and (84), we get

$$\|\Phi(h_1) - \Phi(h_2)\|_{L^2((0, A_1) \times (0, T))}^2 \leq C(\lambda) \|h_1 - h_2\|_{L^2((0, A_1) \times (0, T))}^2, \tag{85}$$

where

$$C(\lambda) = \frac{K^2 A_2^2 \left[\lambda - \left(\|\beta_2\|_\infty^2 A_2 + A_1^2 A_2 K^2 \|b\|_\infty^2 \right) \right] + K^4 A_2^3 A_1 \|b\|_\infty^2}{\left[\lambda - \left((1/2) \|\beta_1\|_\infty^2 (A_1 + (1/2)) \right) \right] \lambda - \|\beta_2\|_\infty^2 A_2 + A_1^2 A_2 K^2 \|b\|_\infty^2}. \tag{86}$$

And, it is clear that for λ large enough, that is $\lambda > \lambda_0$ with

$$\lambda_0 = \frac{1}{2} \left(\frac{1}{2} \|\beta_1\|_\infty^2 A_1 + \|\beta_2\|_\infty^2 A_2 + A_1^2 A_2 K^2 \|b\|_\infty^2 + K^2 A_2^2 + \frac{1}{2} \right) + \frac{1}{2} \left[\left(\frac{1}{2} \|\beta_1\|_\infty^2 A_1 - \|\beta_2\|_\infty^2 A_2 - A_1^2 A_2 K^2 \|b\|_\infty^2 + K^2 A_2^2 + \frac{1}{2} \right)^2 + 4K^2 A_2^3 A_1 \|b\|_\infty^2 \right]^{(1/2)}, \tag{87}$$

$C(\lambda) < 1$. So, Φ is a contraction in $L^2((0, A_1) \times (0, T))$, hence Φ has a unique fixed point u . So, the couple $(u, v) \in L^2((0, A_1) \times (0, T)) \times L^2((0, A_1) \times (0, T))$ is a unique solution of the system (82).

Replacing h by u in (85) and in (78) and summing the inequalities (85) and (90), we deduce (81). \square

4. Spatially Homogeneous Stationary Solutions

We now consider problem (2) and we look for spatial homogeneous stationary solutions i.e for solutions that are constant in time and space. The System (2) becomes:

$$\begin{cases} p_a + \mu_1 p(a) = - \int_0^{A_2} F(p(a); q(\alpha)) d\alpha \text{ in } Q_{A_1}, \\ q_a + \mu_2 q(a) = 0 \text{ in } Q_{A_2}, \\ p(0) = \int_0^{A_1} \beta_1(a) p(a) da, \\ q(0) = \int_0^{A_2} \beta_2(\alpha) q(\alpha) d\alpha + \int_0^{A_1} \left(\int_0^{A_2} b(a, \alpha) F(p(a); q(\alpha)) d\alpha \right) da. \end{cases} \tag{88}$$

Theorem 4. Under the hypothesis $(A_1) - (A_4)$, the system (88), admits at least one non-trivial positive solution in $L^1_+(0, A_1) \times L^1_+(0, A_2)$.

Proof. Let $(\bar{p}, \bar{q}) \in L^1_+(0, A_1) \times L^1_+(0, A_2) \longrightarrow L^1_+(0, A_1) \times L^1_+(0, A_2)$ and denote by $\Gamma: L^1_+(0, A_1) \times L^1_+(0, A_2) \longrightarrow L^1_+(0, A_1) \times L^1_+(0, A_2)$, the application given by:

$$\Gamma(\bar{p}, \bar{q}) = (p, q), \tag{89}$$

where $(p; q)$ is a solution of the following uncoupled system

$$\begin{cases} p_a + \mu_1 p(a) = - \int_0^{A_2} F(\bar{p}(a); \bar{q}(\alpha)) d\alpha \text{ in } Q_{A_1}, \\ q_a + \mu_2 q(a) = 0 \text{ in } Q_{A_2}, \\ p(0) = \int_0^{A_1} \beta_1(a) p(a) da, \\ q(0) = \int_0^{A_2} \beta_2(\alpha) q(\alpha) d\alpha + \int_0^{A_1} \left(\int_0^{A_2} b(a, \alpha) F(\bar{p}(a); \bar{q}(\alpha)) d\alpha \right) da. \end{cases} \tag{90}$$

The comparison theorem (see [20]) imply that, for any $(\bar{p}, \bar{q}) \in L^1_+(0, A_1) \times L^1_+(0, A_2)$,

$$0 \leq p \leq \bar{p} \text{ and } 0 \leq q \leq \bar{q}, \tag{91}$$

where \bar{p} and \bar{q} satisfy respectively

$$\begin{cases} \bar{p}_a + \mu_1 \bar{p}(a) = 0 \text{ in } Q_{A_1}, \\ \bar{p}(0) = \|\beta_1\|_\infty \int_0^{A_1} \bar{p}(a) da, \end{cases} \tag{92}$$

And

$$\begin{cases} \bar{q}_a + \mu_2 \bar{q}(a) = 0 \text{ in } Q_{A_2}, \\ \bar{q}(0) = (\|\beta_2\|_\infty + A_1 \|b\|_\infty \|F\|_\infty) \int_0^{A_2} \bar{q}(a) da. \end{cases} \tag{93}$$

Consider now the set $\mathfrak{X} = \{(\bar{p}, \bar{q}) \in L^1_+(0, A_1) \times L^1_+(0, A_2) \text{ such that } 0 \leq \bar{p} \leq \bar{p} \text{ and } 0 \leq \bar{q} \leq \bar{q}\}$.

It clear that \mathfrak{X} is a closed convex set and $\Gamma(\mathfrak{X}) \subset \mathfrak{X}$ because $\Gamma(\bar{p}, \bar{q}) \in \mathfrak{X}$ for every $(\bar{p}, \bar{q}) \in \mathfrak{X}$.

We fix $(\bar{\eta}, \bar{\zeta}) \in L^1_+(0, A_1) \times L^1_+(0, A_2)$ and set (η, ζ) the corresponding solution of (90). Let (\bar{p}_n) and (\bar{q}_n) be two convergent sequences respectively in $L^1_+(0, A_1)$ s and in $L^1_+(0, A_2)$ such that $\bar{p}_n \longrightarrow \bar{\eta}$ in $L^1_+(0, A_1)$ and $\bar{q}_n \longrightarrow \bar{\zeta}$ in

$L^1(0, A_2)$. Set (p_n, q_n) the corresponding solution of (90). Then (p_n, q_n) solves

$$\begin{cases} p'_n(a) + \mu_1 p_n(a) = -f_n(a) \text{ in } Q_{A_1}, \\ q'_n(a) + \mu_2 p_n(a) = 0 \text{ in } Q_{A_2}, \\ p(0) = \int_0^{A_1} \beta_1(a) p_n(a) da, \\ q(0) = \int_0^{A_2} \beta_2(a) q_n(a) da + g_n, \end{cases} \quad (94)$$

where $f_n(a) = \int_0^{A_2} F(\bar{p}_n(a); \bar{q}_n(a)) d\alpha$ a.e in $(0, A_1)$ and $g_n = \int_0^{A_1} \int_0^{A_2} b(a, \alpha) F(\bar{p}_n(a); \bar{q}_n(a)) d\alpha da$.

Denote by $f(a) = \int_0^{A_2} F(\bar{p}(a); \bar{q}(a)) d\alpha$ a.e in $(0, A_1)$ and $g = \int_0^{A_1} \int_0^{A_2} b(a, \alpha) F(\bar{p}(a); \bar{q}(a)) d\alpha da$.

$$\begin{aligned} \|f_n - f\|_{L^1(0, A_1)} &\leq A_2 K \|\bar{p}_n - \bar{p}\|_{L^1(0, A_1)} \\ &\quad + A_1 K \|\bar{q}_n - \bar{q}\|_{L^1(0, A_2)} \longrightarrow 0, \end{aligned} \quad (95)$$

And

$$\begin{aligned} |g_n - g| &\leq K A_2 \|b\|_{\infty} \|\bar{p}_n - \bar{p}\|_{L^1(0, A_1)} \\ &\quad + K A_1 \|b\|_{\infty} \|\bar{q}_n - \bar{q}\|_{L^1(0, A_2)} \longrightarrow 0. \end{aligned} \quad (96)$$

So, $f_n \rightarrow f$ in $L^1(0, A_1)$ and $g_n \rightarrow g$ in \mathbb{R}^+ .

By elementary calculations, the first equation of (94) yields:

$$\begin{aligned} p_n(a) &= \pi_1(a) \int_0^{A_1} \beta_1(a) p_n(a) da \\ &\quad - \int_0^a f_n(\tau) \exp\left\{-\int_{\tau}^a \mu_1(\theta) d\theta\right\} d\tau, \quad a \in (a; A_1). \end{aligned} \quad (97)$$

Multiplying (93) by β_1 and integrating over $(0, A_1)$, we get

$$\begin{aligned} \left(1 - \int_0^{A_1} \pi_1(s) \beta_1(s) ds\right) \int_0^{A_1} \beta_1(a) p_n(a) da \\ = - \int_0^{A_1} \beta_1(s) \int_0^s f_n(\tau) \exp\left\{-\int_{\tau}^s \mu_1(\theta) d\theta\right\} d\tau. \end{aligned} \quad (98)$$

The sequence $j_n = \int_0^s f_n(\tau) \exp\left\{-\int_{\tau}^s \mu_1(\theta) d\theta\right\} d\tau$ converges to $j = \int_0^s f(\tau) \exp\left\{-\int_{\tau}^s \mu_1(\theta) d\theta\right\} d\tau$ in \mathbb{R}^+ because $|j_n - j| \leq \int_0^s |f_n - f|(\tau) \exp\left\{-\int_{\tau}^s \mu_1(\theta) d\theta\right\} d\tau \leq \|f_n - f\|_{L^1(0, A_1)} \int_0^s \beta_1(a) p_n(a) da \rightarrow 0$. Thus, $p_n(a) \rightarrow p(a)$ a.e $(0, A_1)$ and

$$\begin{aligned} \gamma(a) &= \pi_1(a) l - \int_0^a f(\tau) \exp\left\{-\int_{\tau}^a \mu_1(\theta) d\theta\right\} d\tau, \\ a &\in (a; A_1). \end{aligned} \quad (99)$$

So, we see that γ solves

$$\begin{cases} \gamma'(a) + \mu_1(a) \gamma = -f(a) \text{ in } Q_{A_1}, \\ \gamma(0) = l = \int_0^{A_1} \beta_1(a) \gamma(a) da. \end{cases} \quad (100)$$

The uniqueness properties of the solutions of (100) give that $\gamma = \eta$.

The solution of the second equation in (92) can be rewritten as

$$q_n(a) = \pi_2(a) \left(\int_0^{A_2} \beta_2(a) q_n(a) da + g_n \right), \quad a \in (0, A_2). \quad (101)$$

As before, we multiply (97) by β_2 and integrate over $(0, A_2)$. We get

$$\begin{aligned} \left(1 - \int_0^{A_2} \beta_2(s) \pi_2(s) ds\right) \int_0^{A_2} \beta_2(a) q_n(a) da \\ = g_n \int_0^{A_2} \beta_2(a) \pi_2(a) da. \end{aligned} \quad (102)$$

Then, $\left(\int_0^{A_2} \beta_2(a) q_n(a) da\right)$ converges to h . So, (97) implies that $(q_n(a))$ converges to $\varphi(a)$. And, one has

$$\varphi(a) = \pi_2(a) h + g \pi_2(a). \quad (103)$$

By derivation of (105), we see that φ satisfies

$$\begin{cases} \varphi'(a) + \mu_2(a) \varphi = 0 \text{ in } Q_{A_2}, \\ \varphi(0) = h = \int_0^{A_2} \beta_2(a) \varphi(a) da + g. \end{cases} \quad (104)$$

Using again uniqueness properties, we conclude that $\varphi = \zeta$.

We have shown that Γ is a continuous application in $L^1_+(0, A_1) \times L^1_+(0, A_2)$.

Integrating the first equation of (94) over $(0, A_1)$, we obtain

$$\begin{aligned} \int_0^{A_1} \mu_1(a) p_n(a) da = p_n(A_1) - \int_0^{A_1} \beta_1(a) p_n(a) da, \\ - \int_0^{A_1} f_n(a) da. \end{aligned} \quad (105)$$

The right hand term of this equality is bounded then the left hand one is bounded too. This also implies that $(\mu_1 p_n)$ is bounded. Therefore, by the first equation of (94), we deduce that (p'_n) is bounded. Since (p_n) and (p'_n) are both bounded. Then, (p_n) is bounded in $\in W^{1,1}(0, A_1)$. Similarly, we show also that (q_n) is bounded in $W^{1,1}(0, A_2)$.

From the compact injections of $W^{1,1}(0, A_1)$ into $L^1(0, A_1)$ and $W^{1,1}(0, A_2)$ into $L^1(0, A_2)$, we deduce that $\Gamma(\mathfrak{Q})$ is relatively compact. By invoking Schauder's theorem [22], Γ has at least one fixed point (p, q) which is the solution of (90). \square

5. Numericals Simulations

We use the finite difference method to approximate the solution of problem (35) ([23, 24]). Let us denote by $u_i(t)$ and $v_i(t)$ the approximations of $u(a_i, t)$ and $v(a_i, t)$ respectively, where $a_i = i\Delta a$, $0 \leq i \leq N$, $\Delta a = (A/N)$.

We consider that $F(p, q) \equiv p + q$ and $b(a, \alpha) \equiv b_0$. And we use the discrete approximation

$$\frac{\partial u_i(t)}{\partial a} \cong \frac{u_i(t) - u_{i-1}(t)}{\Delta a},$$

and

$$(106)$$

$$\frac{\partial v_i(t)}{\partial a} \cong \frac{v_i(t) - v_{i-1}(t)}{\Delta a}; \quad i = 1, 2, \dots, N.$$

We approximate the boundary condition by replacing the integrals with the series [25]. This yields

$$\int_0^A \beta_1(a)u(a,t)da \cong \frac{A}{N} \sum_{k=1}^N \beta_1^k u_k(t) + \frac{\beta_1(0)u(0,t) + \beta_1(A)u(A,t)}{2}$$

and

$$\int_0^A \beta_2(a)v(a,t)da + \int_0^A \int_0^A b(a,\alpha)(u(a,t) + v(\alpha,t))d\alpha da$$

$$\cong \frac{A}{N} \sum_{k=1}^N \left(\beta_2^k + \frac{Ab_0}{N} \right) v_k(t) + \frac{A^2 b_0}{N} \sum_{k=1}^N u_k(t) + \frac{\beta_2(0)v(0,t) + \beta_2(A)v(A,t)}{2}.$$

$$(107)$$

Finally, the initial conditions are replaced by

$$u_i(0) = u^0(a_i),$$

$$v_i(0) = v^0(a_i) \text{ for } i = 1, 2, \dots, N.$$

$$(108)$$

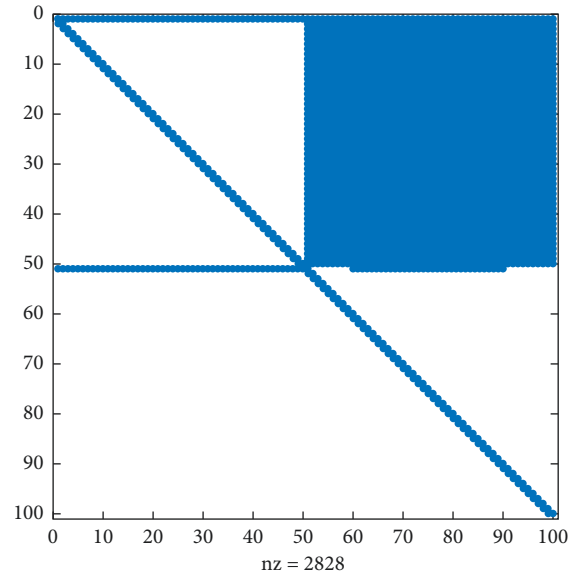


FIGURE 1: Matrix G.

Substituting (106)–(109) into (37) and replacing the approximate equalities with equalities, yields,

$$\left\{ \begin{array}{l} \frac{du_i(t)}{dt} (t) = \frac{1}{\Delta a} [u_{i-1}(t) - u_i(t)] - \mu_1^i u_i(t) - Au_i(t) - \frac{A}{N} \sum_{k=1}^N v_k(t), \\ \frac{dv_i(t)}{dt} (t) = \frac{1}{\Delta a} [v_{i-1}(t) - v_i(t)] - \mu_2^i v_i(t), \\ u_0(t) = \frac{A}{N} \sum_{k=1}^N \beta_1^k u_k(t), \quad i = 1, \dots, N, \\ v_0(t) = \frac{A}{N} \sum_{k=1}^N \left(\beta_2^k + \frac{Ab_0}{N} \right) v_k(t) + \frac{A^2 b_0}{N} \sum_{k=1}^N u_k(t), \\ u_i(0) = u^0(a_i), \\ v_i(0) = v^0(a_i), \end{array} \right. \quad (109)$$

where we set $\mu_1^i = \mu_1(a_i)$, $\beta_1^i = \beta_1(a_i)$, $\mu_2^i = \mu_2(a_i)$, $\beta_2^i = \beta_2(a_i)$ for $i = 1, 2, \dots, N$.

Set $U(t) = (u_1(t), \dots, u_N(t), v_1(t), \dots, v_N(t))^T$ and $U(0) = U_0 = (u_1(0), \dots, u_N(0), v_1(0), \dots, v_N(0))^T$; the linear system (5) becomes

$$\left\{ \begin{array}{l} \frac{dU(t)}{dt} = GU(t), \quad i = 1, \dots, N, \\ U(0) = U_0, \end{array} \right. \quad (110)$$

where $G = \begin{pmatrix} C & D \\ E & F \end{pmatrix}$, see (Figure 1).

The matrix C, D, E and F are square matrix of order N with

$$\begin{aligned}
 C &= (1/\Delta a) \begin{pmatrix} (A/N)\beta_1^1 + k_1 & (A/N)\beta_1^2 & \dots & (A/N)\beta_1^{N-1} & (A/N)\beta_1^N \\ 1 & k_2 & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & 1 & k_{N-1} & 0 \\ 0 & \dots & 0 & 1 & K_N \end{pmatrix}, \\
 D &= -(A/N) \begin{pmatrix} 1 & \dots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \dots & 1 \end{pmatrix}, \\
 E &= \begin{pmatrix} ((A^2b_0)/N) & \dots & \dots & ((A^2b_0)/N) \\ 0 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & 0 \end{pmatrix}, \\
 F &= (1/\Delta a) \begin{pmatrix} d_1 + c_1 & c_2 & \dots & c_{N-1} & c_N \\ 1 & d_2 & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & 1 & d_{N-1} & 0 \\ 0 & \dots & 0 & 1 & d_N \end{pmatrix}.
 \end{aligned} \tag{111}$$

And $k_i = -\Delta a(\mu_1^i + A) - 1$; $c_i = (A/N)\beta_2^i + b_0(A^2/N)$, $d_i = -\Delta a\mu_2^i - 1$ for $i = 1, 2, \dots, N$.

We visualize in Matlab the general form of the matrix G (see Figure 1).

All the numerical tests run on Matlab.

For numerical simulations, we take the functions

$$\begin{aligned}
 \mu_1(a) &= 2(A - a)^{-1}, \\
 \mu_2(a) &= (A - a)^{-1}, \\
 \beta_1(a) &= 10^5 a^2 (A - a)^2 e^{-3(a-(A/2))^2}, \\
 \beta_2(a) &= \begin{cases} (100a - 5A)^4 e^{-0.52(100a - 5A)^2}, & 0.4 \leq a \leq 1.7, \\ 0, & \text{otherwise.} \end{cases}
 \end{aligned} \tag{112}$$

With the initial conditions $u_0(a) = 10e^{-2(a-(A/3))^2}$ and $v_0(a) = 20e^{-0.3(2a-A)^2}$.

At $t = 0$, we have a large prey population, in particular the prey which has an age between $a = 0.2$ and $a = 1.1$ (yellow zone, see (Figure 2)). This population generates significant births especially between the instants $t = 0.03$ and $t = 0.8$ (yellow zone).

No birth is observed in the population of predators between the instants $t = 0$ and $t = 0.3$. The consumption of prey under the action of biomass increases the fertility of predators so there are births between the instants $t = 0.3$ and $t = 0.6$. These births are important at times $t > 0.7$. The population of predators then increases from $t > 0.8$ and at the same time leads to a decrease in that of preys.

We also take the following functions

$$\begin{aligned}
 \mu_1(a) &= 0.5(A - a)^{-1}; \\
 \mu_2(a) &= 1.05(A - a)^{-2}; \\
 \beta_1(a) &= 10^5 a^2 (A - a)^2 e^{-3(a-(A/3))^2}; \\
 \beta_2(a) &= \begin{cases} \left(\frac{(100a - 5A)^4 e^{-0.52(100a - 5A)^2}}{\Gamma(5)} \right), & 0.4 \leq a \leq 1.7, \\ 0, & \text{otherwise.} \end{cases}
 \end{aligned} \tag{113}$$

With the initial conditions $u_0(a) = 10e^{-2(a-(A/3))^2}$ and $v_0(a) = 10e^{-0.2(2a-A)^2}$.

Preys are rare so the predators population does not develop. As soon as the preys population began to be abundant from $t \geq 3$, that of predators also becomes important from $t \geq 4$ (see Figure 3).

To account for the effect of predation on the evolution of the two populations, we present cases where the transformation of the biomass is more or less important ($b_0 = 1.25$ or $b_0 = 0.01$):

The transformation of biomass is important, preys consumed benefits predators by considerably increasing their fertility, which increases their births (see Figure 4). So the population of predators increases but on the other hand that of prey decreases. The biomass is important, but the predators do not live long so they do not have the time to procreate which leads to their decrease therefore the prey population increases with many births (see Figure 5).

In Figure 6, the biomass is low so the prey consumed does not influence the fertility of predators so births are very low. So the predator population is decreasing and that of the prey too because the prey does not live long enough to procreate.

An example of code under Matlab to get Figure 6.

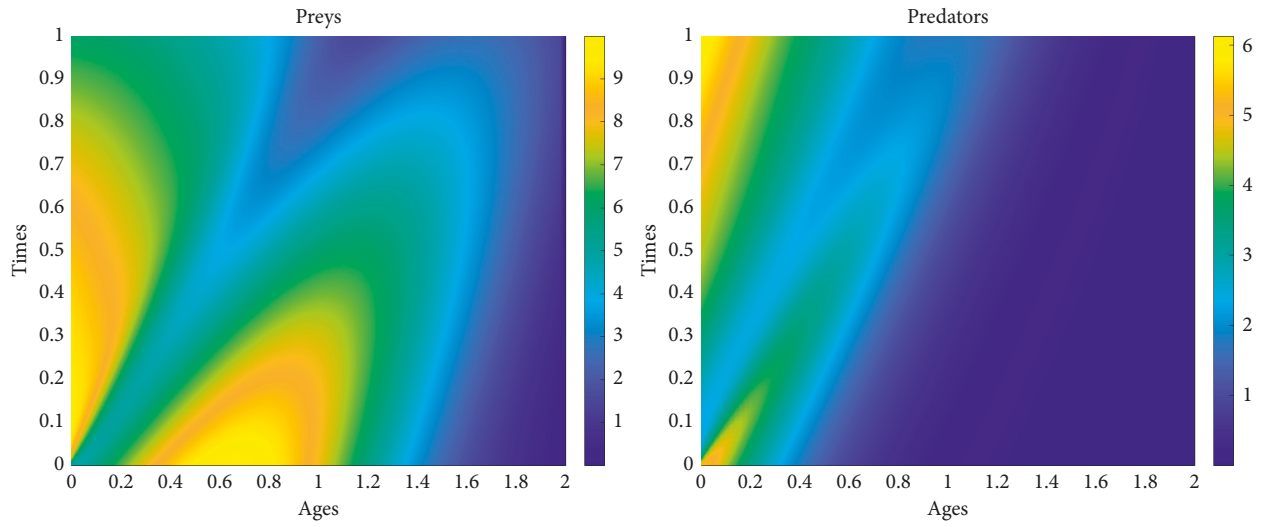


FIGURE 2: Approximate solution of the problem (37) with $T = 1$, $A = 2$.

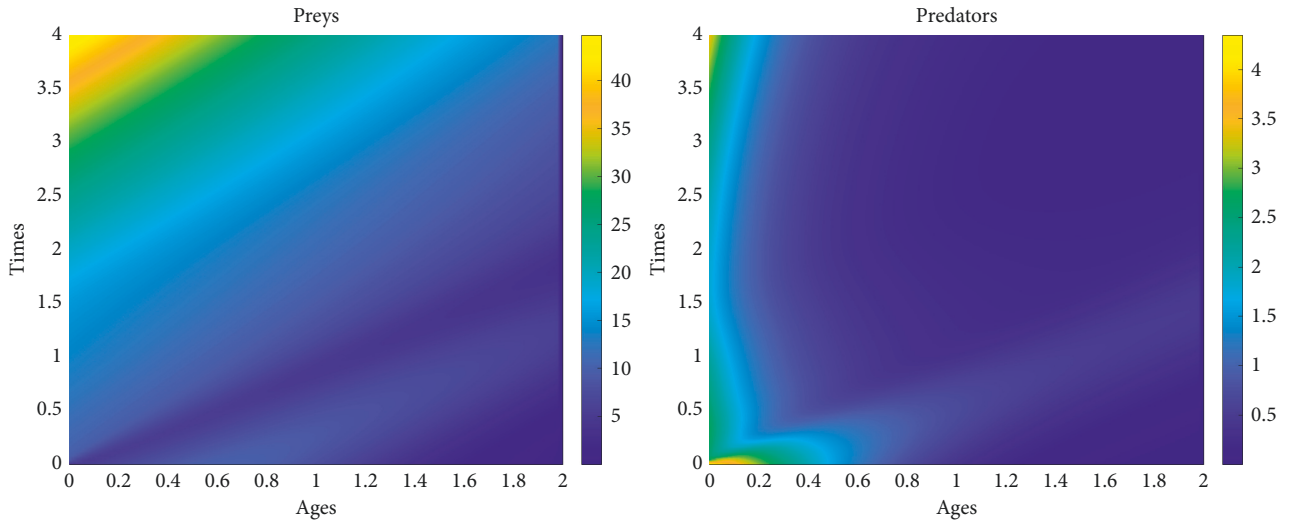


FIGURE 3: Approximate solution of the problem (37) with $T = 4$ and $A = 2$.

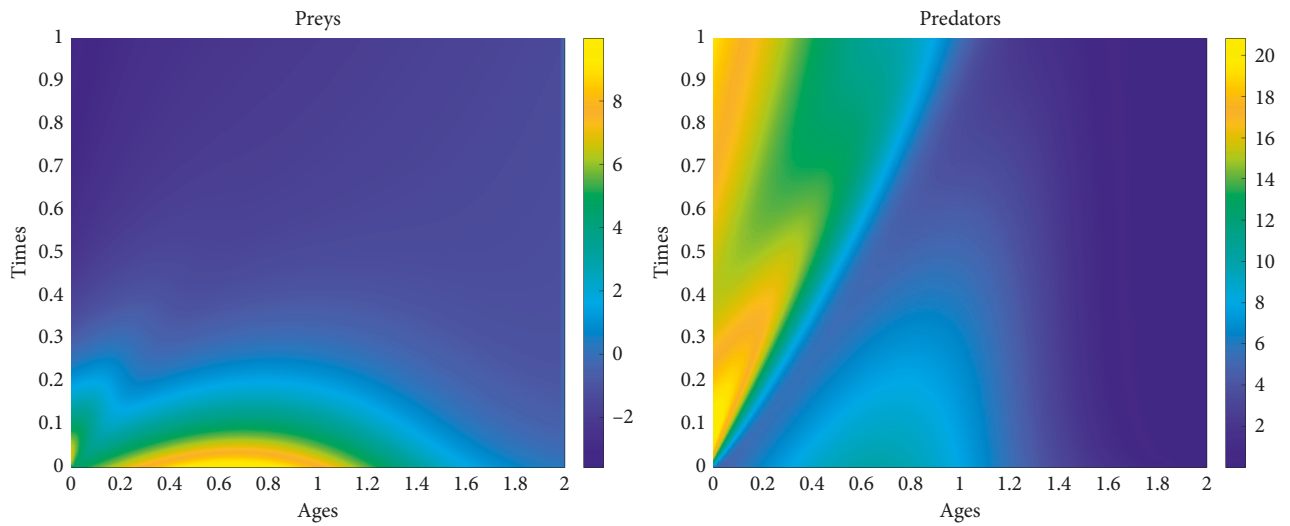


FIGURE 4: Approximate solution of the problem (37) with $T = 1$ and $A = 2$ and $b_0 = 1.25$.

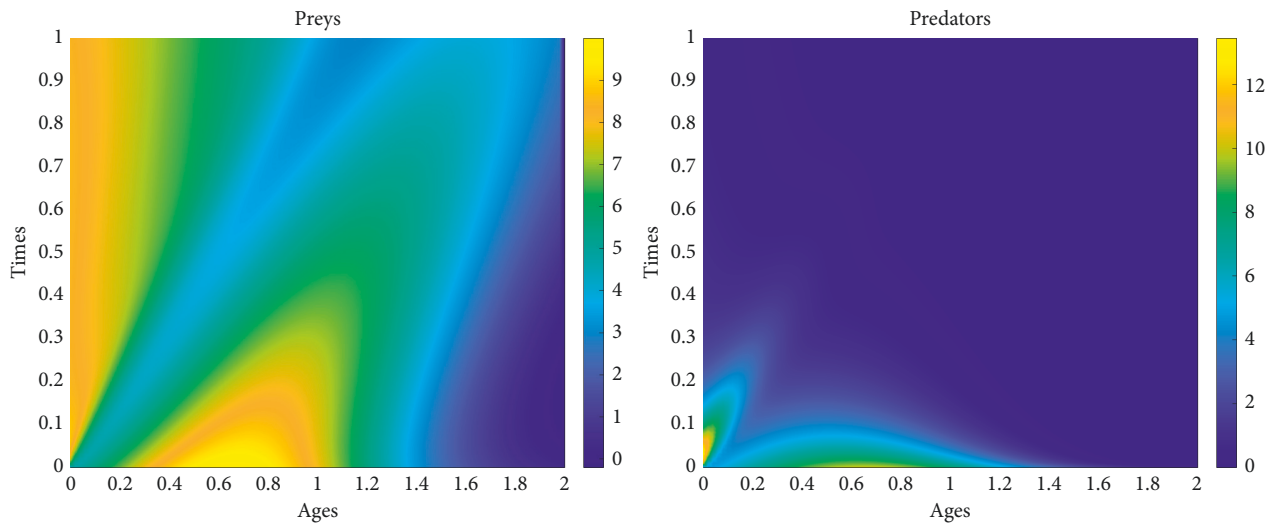


FIGURE 5: Approximate solution of the problem (37) with $T = 1$ and $A = 2$ and $b_0 = 1.25$.

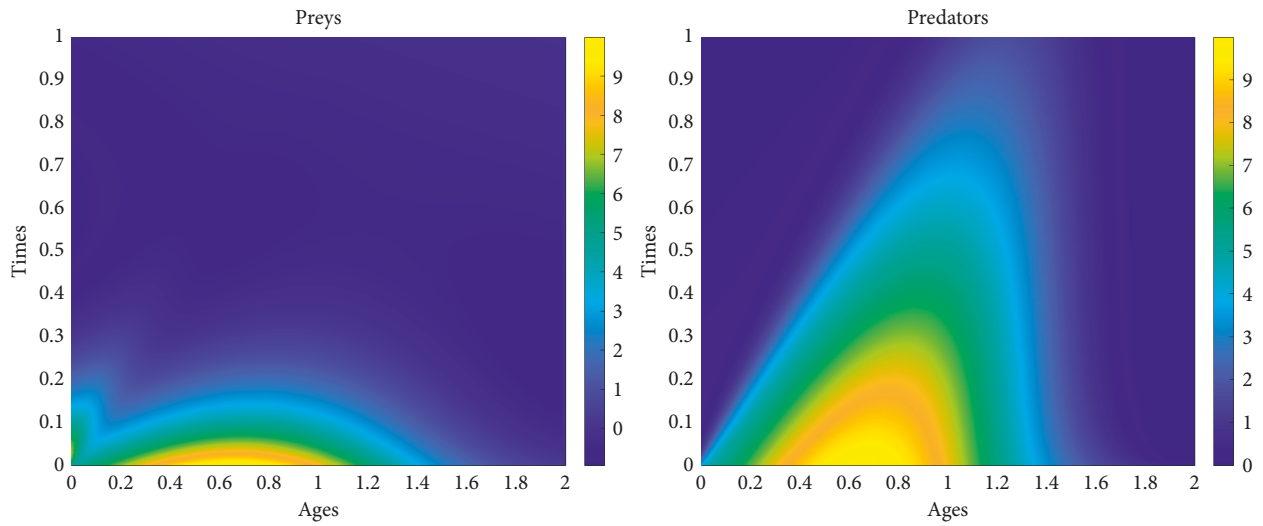


FIGURE 6: Approximate solution of the problem (37) with $b_0 = 0.01$.

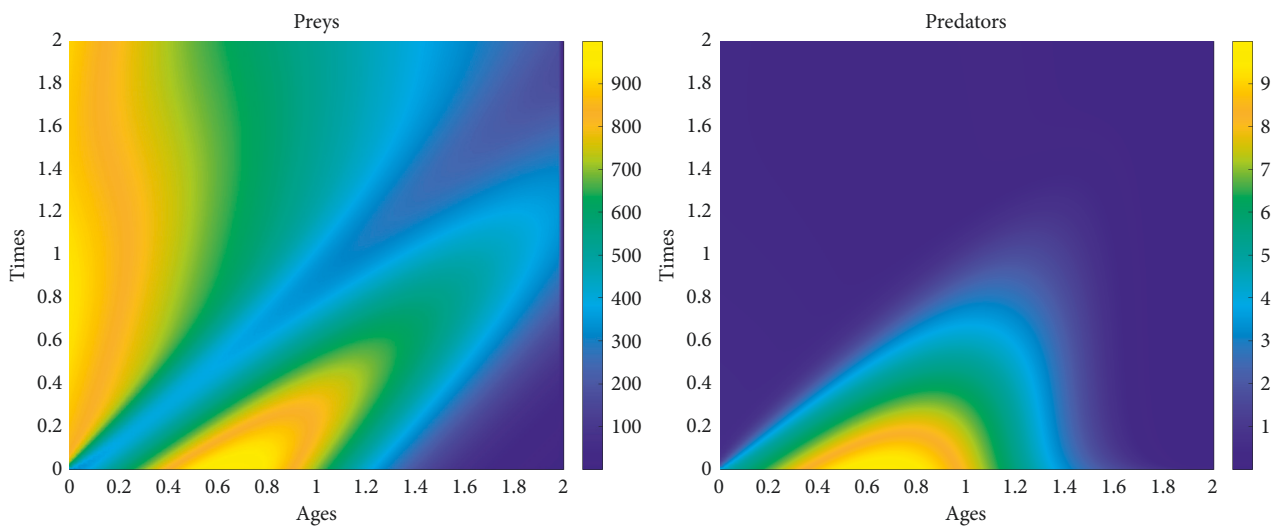


FIGURE 7: Approximate solution of the problem (37) with $b_0 = 0.01$.

The biomass is almost zero, so the preys are eaten without contribution on the fertility of the predators so the births are very low. The predator population is decreasing and the prey population is increasing see (Figure 7).

6. Conclusion and Perspectives

We have analyzed in this work existence results of a predator-prey model. Existence results already exist on predator-prey models but these models do not simultaneously take into account the variables of space, time and age and use classical functional response functions (see [1, 2]). Thus, we have proposed the model that we consider much more complete with a more general functional response subject to the condition of K -lipschitz. This model has been analyzed in the different previous sections under these different variants proving that the cohabitation of predators and prey in our model is possible.

The numerical simulation section confirms the theoretical results and shows that the quantity of prey and predators present also depends strongly on the biomass conversion rate b : Indeed, a high biomass conversion rate increases the fertility of predators by the amount $\int_0^{A_1} (b(a, x, t)F(p; q)(a; x, t))da$ therefore leads to a large population of predators that consume almost all prey from at a given time see (Figure 4). And if the biomass is very low then the consumption of prey does not benefit the birth rate of predators so their number does not increase and end up disappearing under the effect of mortality see (Figure 7).

In practice, it will be difficult to control the behavior of these two populations by acting on the biomass conversion rate since it is an intrinsic and biological factor of predators. So the investigation of (2) is not yet complete. We believe that it is possible to control the model (2) through the external functions f_1 and f_2 . In other words, by taking the functions f_1 and f_2 as controls in a bounded domain of Ω , it is possible to have the extinction either the population of prey or that of predators or both simultaneously from of a time T as we did in [17].

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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References

- [1] S. Ruan and D. Xiao, "Global analysis in a predator-prey system with nonmonotonic functional response," *SIAM*

- Journal on Applied Mathematics*, vol. 61, no. 4, pp. 1445–1472, 2001.
- [2] B. Ainseba, F. Heiser, and M. Langlais, "A mathematical analysis of a predator-prey system in a highly heterogeneous environment," *Differential and Integral Equations*, vol. 15, no. 4, pp. 385–404, 2002.
- [3] V. Tiwari, J. P. Tripathi, S. Abbas, J. S. Wang, G. Q. Sun, and Z. Jin, "Qualitative analysis of a diffusive Crowley–Martin predator–prey model: the role of nonlinear predator harvesting," *Nonlinear Dynamics*, vol. 98, no. 2, pp. 1169–1189, 2019.
- [4] J. P. Tripathi, S. Abbas, G. Q. Sun, D. Jana, and C. H. Wang, "Interaction between prey and mutually interfering predator in prey reserve habitat: pattern formation and the Turing–Hopf bifurcation," *Journal of the Franklin Institute*, vol. 355, no. 15, pp. 7466–7489, 2018.
- [5] V. Tiwari, J. P. Tripathi, S. Mishra, and R. K. Upadhyay, "Modeling the fear effect and stability of non-equilibrium patterns in mutually interfering predator–prey systems," *Applied Mathematics and Computation*, vol. 371, Article ID 124948, 2020.
- [6] V. Tiwari, J. P. Tripathi, R. K. Upadhyay, Y. P. Wu, J. S. Wang, and G. Q. Sun, "Predator–prey interaction system with mutually interfering predator: role of feedback control," *Applied Mathematical Modelling*, vol. 87, pp. 222–244, 2020.
- [7] X. Zhanga and Z. Liua, "Hopf bifurcation of the Michaelis–Menten type ratio-dependent predator-prey model with age structure," 2017, <https://arxiv.org/abs/1711.01599>.
- [8] R. Xu, M. A. J. Chaplain, and F. A. Davidson, "Global Stability of a Lotka–Volterra Type Predator–Prey Model with Stage Structure and Time Delay," *Applied Mathematics and Computation*, vol. 159, no. 3, pp. 863–880, 2004.
- [9] E. N. Bodine and A. E. Yust, "Predator–prey Dynamics with Intraspecific Competition and an Allee Effect in the Predator Population," *Letters in Biomathematics*, vol. 4, no. 1, pp. 23–38, 2017.
- [10] J. Promprak, G. C. Wake, and C. Rattanukul, "Predator–prey model with age structure," *Journal of Mathematical Biology*, vol. 14, no. 2, pp. 231–250, 1982.
- [11] M. Sarkar, T. Das, and R. N. Mukherjee, "Bifurcation and stability of prey-predator model with beddington–DeAngelis functional response," *Applications and Applied Mathematics: An International Journal*, vol. 12, no. 1, pp. 350–366, 2017.
- [12] K. Das, M. N. Srinivas, V. Madhusudanan, and S. Pinelas, "Mathematical Analysis of a Prey–Predator System: An Adaptive Back-Stepping Control and Stochastic Approach," *Mathematical and Computational Applications*, vol. 24, no. 1, p. 22, 2019.
- [13] Y. Lu and S. Liu, "Threshold dynamics of a predator–prey model with age-structured prey," *Advances in Difference Equations*, vol. 2018, no. 1, p. 164, 2018.
- [14] K. Kuto and Y. Yamada, "Coexistence problem for a prey-predator model with density-dependent diffusion," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 71, no. 12, pp. e2223–e2232, 2009.
- [15] B. Ainseba, "Age-dependent population dynamics diffusive systems," *Discrete & Continuous Dynamical Systems - B*, vol. 4, no. 4, pp. 1233–1247, 2004.
- [16] A. Traoré, B. Ainseba, and O. Traoré, "On the existence of solution of a four-stage and age-structured population dynamics model," *Journal of Mathematical Analysis and Applications*, vol. 495, no. 1, Article ID 124699, 2021.
- [17] A. Traoré, O. S. Sougué, Y. Simporté, and O. Traoré, "Null controllability of a nonlinear age and two-sex population

- dynamics structured model,” 2020, <https://arxiv.org/abs/2009.05380>.
- [18] H. Inaba, *Age-structured Population Dynamics in Demography and Epidemiology*, Springer Science+Business Media, Singapore, 2017.
- [19] M. Iannelli and F. Milner, “The basic approach to age-structured population dynamics,” *Models Methods and Numerics*, vol. 10, pp. 978–994, 2017.
- [20] S. Anita, *Analysis and Control of Age-dependent Population Dynamics*, Kluwer Academic Publishers, Dordrecht, 2000.
- [21] K. Kunisch, W. Schappacher, and G. F. Webb, “Nonlinear age-dependent population dynamics with random diffusion,” *Computers & Mathematics with Applications*, vol. 11, no. 1-3, pp. 155–173, 1985.
- [22] E. Zeidler, *Nonlinear Functional Analysis and its Applications: I: Fixed Point-theorems*, p. 56, Springer-Verlag, Berlin, Germany, 1986.
- [23] J. Koko, *Approximation Numérique Avec MATLAB: Programmation Vectorisée, Équations Aux Dérivées Partielles*, Ellipses, California, U. S, 2009.
- [24] J. P. Grenier, *Débuter en algorithmique avec MATLAB et SCILAB*, Ellipses, California, U. S, 2007.
- [25] S. Yacouba, *Null Controllability of a Nonlinear Population Dynamics with Age Structuring and Spatial Diffusion*, Springer, Berlin, Germany, 2020.