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Research Article

A Note on Constant Mean Curvature Foliations of Noncompact Riemannian Manifolds

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We aimed to study constant mean curvature foliations of noncompact Riemannian manifolds, satisfying some geometric constraints. As a byproduct, we answer a question by M. P. do Carmo (see Introduction) about the leaves of such foliations.

1. Introduction

Consider a codimension-one foliation of a Riemannian manifold, whose leaves have a constant mean curvature. When the ambient manifold is compact, there are a bunch of results stating that such a foliation is totally geodesic, provided some geometric assumption is satisfied [1, 2]. This kind of phenomenon is sometimes true in the noncompact case as well. Meeks in [3] proved that any codimension-one constant mean curvature foliation of the three-dimensional Euclidean space is totally geodesic. Oshikiri [4] proved the analogous result in a Riemannian manifold with nonnegative Ricci curvature, provided the leaves have quadratic volume growth.

In this article, we prove that the leaves of any codimension-one constant mean curvature foliation of a Riemannian manifold with nonnegative Ricci curvature are totally geodesic, provided they are parabolic (Theorem 4). Theorem 4 generalizes Oshikiri's result, as quadratic volume growth implies they are parabolic.

Then, we consider a Riemannian manifold $\mathscr{N}$ with zero volume entropy (Section 2), and we prove that, if $\mathscr{F}$ is a codimension-one foliation of N such that any leaf L has constant mean curvature H_L, then $\inf|H_L| = 0$ (Theorem 2). Notice that having zero volume entropy is a weaker assumption than having nonnegative Ricci curvature, by Bishop's comparison theorem [5] (Corollary 2.1.1).

Finally, we point out that if the leaves of a foliation $\mathscr{F}$ have the same constant mean curvature, then the leaves are stable as shown in [6], Proposition 3, and Section 3, where a sketch of the proof is given.

We recall that do Carmo in [7] asked the following question: is a noncompact, complete, stable, constant mean curvature hypersurface of $\mathbb{R}^{n+1}$, $n \geq 3$, necessarily minimal?

Our result yields a positive answer to do Carmo's question, provided the hypersurface is a leaf of a foliation such that the leaves have the same constant mean curvature.

The answer was already known to be positive for $n = 2$ [8] ([9], when the ambient manifold is the hyperbolic space). Later, the answer was proved to be positive for $n = 3$ and 4 by Elbert et al. [10] and independently by Cheng [11], using a Bonnet–Myers' type method. Moreover, the authors gave a positive answer to do Carmo's question in the following cases: (1) a hypersurface with zero volume entropyof a space form of any dimension [12] (Corollary 8). (2) a hypersurface of $\mathbb{R}^{n+1}$, $\mathbb{H}^{n+1}$, $n \leq 5$ with total curvature with polynomial growth [13] (Corollary 6.3).

2. Basic Notions

Let N be a complete, noncompact Riemannian manifold. We define the entropy associated with the volume of geodesic balls in N [14, 15].

Definition 1. Let $B_\sigma^{\mathcal{N}}(R)$ be a geodesic ball in N of radius R, centered at a fixed point $\sigma \in N$, and denoted by $|B_\sigma^{\mathcal{N}}(R)|$as its volume. The volume entropy of N is

$$\mu_{\mathcal{N}} := \limsup_{R\longrightarrow\infty}\left(\frac{\ln\left|B_\sigma^{\mathrm{N}}(R)\right|}{R}\right). \quad (1)$$

The notion of volume entropy does not depend on the center σ of the balls. It is worthwhile to notice that $\mu_{\mathrm{N}} = 0$ is equivalent to

$$\limsup_{r\longrightarrow\infty}\frac{\left|B_\sigma^{\mathrm{N}}(R)\right|}{e^{\alpha r}} = 0, \quad \forall\alpha > 0. \quad (2)$$

Then, it is natural to say that NN has subexponential growth if $\mu_N = 0$ and exponential growth if $\mu_N > 0$.

We observe that having subexponential growth is a weaker assumption than being bounded by a polynomial of any degree. For example, if $|B_\sigma^{\mathrm{N}}(R)| = e^{r^\beta}$ and $\beta < 1$, then N has subexponential growth.

The volume entropy of a manifold N is strictly related to the bottom of its essential spectrum and to the Cheeger constant. Let us be more precise about this.

Let Δ be the Laplacian on N, then the bottom of the spectrum $\sigma(\mathrm{N})$ of $-\Delta$ is

$$\lambda_0(N) = \inf\{\sigma(N)\} = \inf_{\substack{f\in C_0^\infty(N)\\ f\neq 0}}\left(\frac{\int_{\mathrm{N}}|\nabla f|^2}{\int_{\mathrm{N}} f^2}\right). \quad (3)$$

The bottom of the essential spectrum $\sigma_{ess}(\mathcal{N})$ of $-\Delta$ is

$$\lambda_0^{\mathrm{ess}}(N) = \inf\{\sigma_{ess}(N)\} = \sup_K \lambda_0\left(\frac{N}{K}\right), \quad (4)$$

where K runs through all compact subsets of N.

The Cheeger constant h_N of a Riemannian manifold N is defined as $h_N = \inf_\Omega|\partial\Omega|/|\Omega|$, where Ω runs over all compact domains of N with piecewise smooth boundary $\partial\Omega$.

Cheeger [16] and Brooks [14] proved the following important comparison result between the bottom of the essential spectrum, the volume entropy, and the Cheeger constant.

Theorem 1 (Brooks–Cheeger's Theorem). *If N has infinite volume, then*

$$\frac{h_{\mathrm{N}}^2}{4} \le \lambda_0(N) \le \lambda_0\left(\frac{N}{K}\right) \le \lambda_0^{ess}(N) \le \frac{\mu_{\mathrm{N}}^2}{4}, \quad (5)$$

where K is any compact subset of N.

We finally recall that a Riemannian manifold is called parabolic if it does not admit a nonconstant positive superharmonic function. Parabolicity is strictly related to the volume growth of a manifold. In fact, quadratic area growth implies parabolicity [17] (Corollary 7.4) [6].

3. Foliations of Manifolds with Zero Volume Entropy

In this section, we study foliations, with constant mean curvature leaves, of manifolds with zero volume entropy. In the article, $\mathcal{F}$ will be a C^3 codimension-one foliation of a manifold N and N will be a C^3 unit vector field of N normal to the leaves of $\mathcal{F}$.

We have the following result.

Theorem 2. *Let N be a manifold with zero volume entropy. Let $\mathcal{F}$ be a codimension-one C^3 foliation of N such that any leaf L has constant mean curvature $H_L \geq 0$. Then, $\inf H_L = 0$.*

Before doing the proof of Theorem 2, let us state a consequence in the case of a constant mean curvature foliation.

Corollary 1. *Let N be a manifold with zero volume entropy. Let $\mathcal{F}$ be a codimension-one C^3 foliation of NN by leaves of constant mean curvature H. Then, $H = 0$.*

As we remarked in Introduction, the leaves of a constant mean curvature foliation are stable, provided all the leaves have the same mean curvature ([2], Proposition 3).

We give a sketch of the proof of the latter for the sake of completeness.

By [2] (Proposition 1), one has

$$Ric(N,N) + |A|^2 + |\theta|^2 = \mathrm{div}_L(\nabla_N N), \quad (6)$$

where N is a unit vector field defined in N, perpendicular to $\mathcal{F}$ at any point, A is the second fundamental form of L, θ is defined by $\theta(X) = \langle\nabla_N N, X\rangle$ for every vector field X tangent to L, and $|\theta|^2 = \sum_{i=1}^n \theta(E_i)^2$, where $\{E_1,\ldots,E_n\}$ is a local orthonormal base of the tangent space to L.

Recall that the second variation of the volume of a leaf L is

$$V''(0) = \int_L |\nabla f|^2 - f^2\left(\mathrm{Ric}(N,N) + |A|^2\right), \quad (7)$$

where f is any smooth function on N with compact support.

Then, by using equality (6) multiplied by f, one gets

$$\begin{aligned} V''(0) &= \int_L |\nabla f|^2 - f^2\left(Ric(N,N) + |A|^2\right) = \int_L |\nabla f|^2 + f^2|\theta|^2 - f^2\mathrm{div}_L(\nabla_N N) \\ &= \int_L |\nabla f|^2 + f^2|\theta|^2 - \mathrm{div}_L\left(f^2\nabla_N N\right) + \nabla_N N\left(f^2\right) \\ &= \int_L |\nabla f + f\theta|^2 \geq 0. \end{aligned} \quad (8)$$

Then, $V''(0) \geq 0$; this means that the leaves are stable.

The stability of the leaves of such a foliation suggests to inquire if we can answer Carmo's question [7]: is a noncompact, complete, stable, constant mean curvature hypersurface of $\mathbb{R}^n$, $n \geq 3$, necessarily minimal?

Corollary 1 yields a positive answer to Carmo's question, provided the hypersurface is a leaf of a foliation with constant mean curvature leaves.

Proof of Theorem 2. If there is a minimal leaf L, we have nothing to prove. Then, we may assume that $\inf H_L > 0$, and let N be the unit vector field normal to the leaves of $\mathcal{F}$ given by the normalized mean curvature vector. Then, there exists $c > 0$ such that $H_L \geq c$ for every leaf L. By straightforward computation, we obtain the following:

$$nH(p) = \mathrm{div}_{\mathrm{N}} N(p), \tag{9}$$

where H is the mean curvature of the leaf passing through p.

Then, we integrate (9) on a ball of radius R of N, say B_R,

$$\int_{B_R} nH(p) = \int_{B_R} \mathrm{div}_N N(p) = \int_{\partial B_R} \langle N, \nu\rangle \leq \mathrm{vol}(\partial B_R), \tag{10}$$

where the last equality is by the divergence theorem. Keeping into account that $H(p) \geq c$ at any point $p \in \mathrm{N}$, (10) yields

$$nc \leq \frac{\mathrm{vol}(\partial B_R)}{\mathrm{vol}(B_R)}. \tag{11}$$

In particular,

$$nc \leq \inf_R \frac{\mathrm{vol}(\partial B_R)}{\mathrm{vol}(B_R)} = h_{\mathrm{N}}. \tag{12}$$

Then, the Brooks–Cheeger's theorem stated in Section 2 yields $\mu_{\mathrm{N}} \geq h_{\mathrm{N}} > 0$ that is a contradiction.

Using the Gauss formula, one can prove the analogous of Theorem 2 for foliations of constant scalar curvature. □

Theorem 3. *Let N be a manifold with zero volume entropy and nonpositive sectional curvatures. Let $\mathcal{F}$ be a codimension-one C^3 foliation of N such that any leaf L has constant scalar curvature $S_L \geq 0$. Then,* $\inf S_L = 0$.

Proof. By the Gauss formula, one has

$$S_L = \sum_{i<j} \overline{K}(e_i, e_j) + H_L^2 - |A|^2, \tag{13}$$

where $e_1, \ldots, e_n$ is a base of the tangent space to a leaf L, $\overline{K}(e_i, e_j)$ is the sectional curvature of $N\mathrm{N}$ along $\mathrm{span}\{e_i, e_j\}$, $|A|$ is the norm of the second fundamental form of L, and H_L is its mean curvature.

Then, being the sectional curvatures of $N\mathcal{N}$ nonpositive,one has that $S_L < H_L^2$; hence, $S_L > c$ implies $H_L > \sqrt{c}$, and one can apply Theorem 2.

An immediate consequence of Theorem 2 is the following result. □

Corollary 2. *Let N be a manifold with zero volume entropy and nonpositive sectional curvatures. Let $\mathcal{F}$ be a codimension-one C^3 foliation of $N\mathrm{N}$ by leaves of constant scalar curvature S. Then,* $S = 0$.

Remark 1. It is worthwhile noticing that all the results of this section hold for foliations given by complete graphs of constant mean or scalar curvature, when they exist.

4. Minimal Parabolic Foliations of Manifolds with Nonnegative Ricci Curvature

In this section, we prove a kind of Bernstein's theorem for the leaves of a minimal foliation of a Riemannian manifold with nonnegative Ricci curvature, provided the leaves are parabolic.

Theorem 4. *Let N be a Riemannian manifold with nonnegative Ricci curvature and let $\mathcal{F}$ be a codimension-one foliation of N by minimal leaves. If the leaves of the foliation are parabolic, then they are totally geodesic.*

Theorem 4 is a generalization of Theorem 1 in [4]. In fact, quadratic area growth implies parabolicity ([17], Corollary 7.4, and [6]). When the ambient space is $\mathbb{R}^{n+1}$, for $n = 2$, the leaves of a minimal foliation must be totally geodesic [3], while for $n \geq 4$, there is no parabolic complete minimal hypersurface ([18], Proposition 2.3).

Proof of Theorem 4. We first need a definition. We say that ν is an exhaustion function on L if ν is continuous on L and such that all the sets $\mathcal{B}_r := \{p \in L : \nu(x) \leq r\}$ are precompact. Notice that the latter is equivalent to say that $\nu(p) \longrightarrow \infty$ as $p \longrightarrow \infty$ (that is, p leaves every compact).

By [12] (Theorem 7.6), a manifold L is parabolic if and only if there exists a smooth exhaustion function ν on L such that

$$\int^{\infty} \frac{dr}{\mathrm{Flux}_{\partial\mathcal{B}_r} \nu} = \infty, \tag{14}$$

where $\mathrm{Flux}_{\partial\mathcal{B}_r} \nu = \int_{\partial\mathcal{B}_r} \langle \nabla\nu, \nu\rangle$, being the outward unit normal to $\partial\mathcal{B}_r$.

Equality (6) in Section 3 holds for the minimal leaf L, and integrating it on $\mathcal{B}_r$ yields

$$\int_{\mathcal{B}_r} Ric(N,N) + |A|^2 + |\theta|^2 = \int_{\mathcal{B}_r} \mathrm{div}_L(\nabla_N N) = \int_{\partial\mathcal{B}_r} \theta(\nu), \tag{15}$$

and for the last equality, we used the divergence theorem.

As $\mathrm{Ric}(N,N)$ and $|A|^2$ are nonnegative, we have

$$\int_{\mathcal{B}_r} |\theta|^2 \leq \int_{\partial\mathcal{B}_r} \theta(\nu) \leq \left(\int_{\partial\mathcal{B}_r} \frac{|\theta|^2}{|\nabla\nu|}\right)^{1/2} \left(\int_{\partial\mathcal{B}_r} |\nabla\nu|\right)^{1/2}, \tag{16}$$

where the last inequality is by the Cauchy–Schwarz inequality.

By defining $f(r) = \int_{\mathcal{B}_r} |\theta|^2$, the coarea formula yields $f(r) = \int_0^r \int_{\partial\mathcal{B}_s} |\theta|^2/|\nabla\nu| ds$. Then, $f'(r) = \int_{\partial\mathcal{B}_s} |\theta|^2/|\nabla\nu| ds$.

Assume that $f(r) \neq 0$. With this notation, the square of (16) is written as

$$\frac{1}{\mathrm{Flux}_{\partial\mathscr{B}_r}\nu} \leq \frac{f'(r)}{f^2(r)}. \tag{17}$$

We integrate inequality (17) between a fixed r_0 and R (where f is nonzero), and we get

$$\int_{r_0}^{R} \frac{dr}{\mathrm{Flux}_{\partial\mathscr{B}_r}\nu} \leq \frac{1}{f(r_0)} - \frac{1}{f(R)}. \tag{18}$$

By letting R go to ∞, inequality (18) gives a contradiction. In fact, as f is a nondecreasing function, then the right-hand side is bounded, while by hypothesis, the left-hand side tends to infinity.

Then, $f \equiv 0$, that is $\nabla_N N \equiv 0$, on the leave L, and equality (6) yields $R(N,N) + |A|^2 \equiv 0$. As the Ricci curvature is nonnegative, we get $|A|^2 \equiv 0$, i.e., L is totally geodesic. □

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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