Research Article

Primitive Zero-Symmetric Sign Pattern Matrices with Zero Diagonal Attaining the Maximum Base

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The base set of primitive zero-symmetric sign pattern matrices with zero diagonal is $\{1, 2, \ldots, 2n - 1\}$. In this paper, the primitive zero-symmetric sign pattern matrices with zero diagonal attaining the maximal base $2n - 1$ are characterized.

1. Introduction

A sign pattern matrix (or sign pattern) A is a matrix whose entries are from the set $\{1, -1, 0\}$. Notice that for a square sign pattern matrix A, in the computation of (the signs of) the entries of the power A^k, an ambiguous sign may arise when a positive sign is added to a negative sign. So a new symbol # was introduced in [1] to denote such an ambiguous sign. The powers of a square sign pattern have been investigated to some extent, see, for example, [1–12]. In [1], the set $\Gamma = \{1, -1, 0, \#\}$ is defined as the generalized sign set and the matrices with entries in the set Γ are called generalized sign pattern matrices, and the addition and multiplication involving the symbol # are defined as follows:

$$(-1) + 1 = 1 + (-1) = \#; \quad a + \# = \# \quad (\forall a \in \Gamma),$$
$$0 \cdot \# = \# \cdot = 0; \quad b \cdot \# = \# \cdot b = \# \quad (\forall b \in \Gamma \setminus \{0\}).$$

(1.1)

From now on, we assume that all the matrix operations considered in this paper are operations of the matrices over the set Γ.

Definition 1.1. A square generalized sign pattern matrix A is called powerful if each power of A contains no # entry.

In [1], Li et al. introduced the concepts of base and period for (powerful) sign pattern matrices which are the generalizations of the concepts of "index of convergence" and period for square nonnegative matrices. These concepts are extended from (powerful) sign pattern matrices to (square) generalized sign pattern matrices by You et al. in [12] as follows.

Definition 1.2 (see [12]). Let A be a square generalized sign pattern matrix of order n and $A, A^2, A^3, \ldots$ the sequence of powers of A. (Since there are only 4^{n^2} different generalized sign patterns of order n, there must be repetitions in the sequence.) Suppose A^l is the first power that is repeated in the sequence, that is, l is the least positive integer such that $A^l = A^{l+p}$ holds for some positive integer p. Then l is called the generalized base (or simply base) of A, and is denoted by $l(A)$. The least positive integer p such that $A^l = A^{l+p}$ holds for $l = l(A)$ is called the generalized period (or simply period) of A and is denoted by $p(A)$.

For a sign pattern matrix A, we use $|A|$ to denote the $(0,1)$ matrix obtained from A by replacing each nonzero entry by 1.

A nonnegative square matrix A is primitive if some power $A^k > 0$. The least such as k is called the primitive exponent (or simply exponent) of A, denoted by $\exp(A)$. For convenience, a square sign pattern matrix A is called primitive if $|A|$ is primitive, and in this case we define $\exp(A) = \exp(|A|)$.

Definition 1.3 (see [3]). A square sign pattern matrix $A = (a_{ij})_{n \times n}$ is called zero-pattern symmetric (abbreviated zero-symmetric, or simply ZS) if $|A|$ is symmetric.

It is well known that graph-theoretical methods are often useful in the study of the powers of square matrices, so we now introduce some graph-theoretical concepts.

Let D be a digraph with vertex set V and arc set E (which permits loops but no multiple arcs). By assigning a sign of 1 or -1 to each arc of the digraph D, we obtain a *signed digraph* S. By a walk W in the digraph D (or the signed digraph S), we mean a sequence of vertices $(v_0, v_1, v_2, \ldots, v_k)$ such that $e_i = (v_{i-1}, v_i)$ is an arc of D for $i = 1, \ldots, k$. The number k is called the length of the walk W, denoted by $l(W)$. If the vertices $v_0, v_1, \ldots, v_{k-1}$ are distinct, the walk W is called a *path*, if $v_k = v_0$, the path W is called a *cycle* in D (and in S). The sign of the walk W in S, denoted by $\text{sgn}(W)$, is defined to be $\prod_{i=1}^{k} \text{sgn}(e_i)$ where $\text{sgn}(e_i)$ is the sign of the arc e_i.

Let $A = (a_{ij})$ be a sign pattern matrix of order n. Then the associated digraph $D(A)$ of A is defined to be the digraph with vertex set $V = \{1, 2, \ldots, n\}$ and arc set $E = \{(v_i, v_j) \mid a_{ij} \neq 0\}$. The associated signed digraph $S(A)$ is obtained from $D(A)$ by assigning the sign of a_{ij} to each arc (v_i, v_j) in $D(A)$.

Definition 1.4 (see [10]). Let D be a digraph (permitting loops but no multiple arcs). Digraph D is called primitive if there is a positive integer k such that for all ordered pairs of vertices v_i and v_j (not necessarily distinct) in D, there exists a walk of length k from v_i to v_j. The least such k is called the primitive exponent of D, denoted by $\exp(D)$.

It is well known that a digraph D is primitive if and only if D is strongly connected and the greatest common divisor (gcd for short) of the lengths of all the cycles of D is 1 (see [13]).
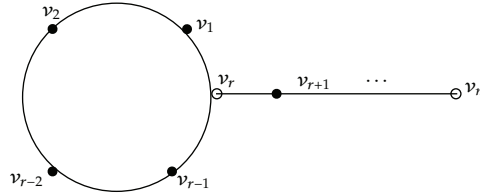
Figure 1: The graph G. (All 2-cycles in G are negative, r is odd, $3 \leq r \leq n$).

Definition 1.5 (see [3]). Let S be a signed digraph of order n. Then there is a sign pattern matrix A of order n whose associated signed digraph $S(A)$ is S. We say that S is powerful if A is powerful. Also we define $l(S) = l(A)$.

We say that a sign pattern matrix $A = (a_{ij})$ has *zero diagonal* if $a_{ii} = 0$ for all i. A digraph D with vertex set $\{v_1, \ldots, v_n\}$ and arc set E is called *symmetric* provided that $(v_i, v_j) \in E$ iff $(v_j, v_i) \in E$ for all i, j. It is clear that a sign pattern matrix A is ZS iff its associated digraph $D(A)$ is symmetric. For simplicity, we represent a symmetric (signed) digraph by its underlying graph.

The base set of primitive ZS sign pattern matrices and the base set of primitive ZS sign pattern matrices with zero diagonal are given, respectively, in [3, 10]. In [2], Cheng and Liu characterized the primitive ZS sign pattern matrices with the maximum base.

In this paper, we characterize the primitive sign pattern matrices with zero diagonal attaining the maximum base. Our main result is given in the following theorem.

Theorem 1.6. *Let $A = (a_{ij})$ be an $n \times n$ primitive zero-symmetric sign pattern matrix with zero diagonal. Then*

$$l(A) \leq 2n - 1 \tag{1.2}$$

and the equality holds if and only if A is nonpowerful and skew symmetric, namely, $a_{ij} = -a_{ji}$ for all $1 \leq i \leq j \leq n$, and the associated digraph $D(A)$ is isomorphic to G (see Figure 1).

The proof of Theorem 1.6 will be given in Section 3.

2. Preliminary Results

In this section, we introduce some theorems, definitions, and lemmas which we need to use in the proof of our main result in Section 3.

In [1], Li et al. showed that if an irreducible sign pattern matrix A is powerful, then $l(A) = l(|A|)$. That is to say the study of the base for a primitive powerful sign pattern matrix is essentially the study of the base (i.e., exponent) for primitive $(0, 1)$ matrices. Therefore, for a primitive powerful ZS sign pattern matrix with zero diagonal, Theorem 2.1 gives the base.

Theorem 2.1 (see [8]). *Let A be an $n \times n$ primitive symmetric $(0, 1)$ matrix with zero diagonal. Then $\exp(A) \leq 2n - 4$ and the primitive exponent set of $n \times n$ primitive symmetric $(0, 1)$ matrix with zero diagonal is $\{2, 3, \ldots, 2n - 4\} \setminus D$, where D is the set of odd numbers in $\{n - 2, n - 1, \ldots, 2n - 5\}$.*

Theorem 2.2 (see [10]). *Let A be an $n \times n$ primitive ZS sign pattern matrix with zero diagonal. Then $l(A) \leq 2n - 1$.*

By Theorems 2.1 and 2.2, the sign pattern matrices with zero diagonal attaining this upper bound $l(A) = 2n - 1$ must be nonpowerful. So it remains to consider nonpowerful sign pattern matrices with zero diagonal.

Definition 2.3. Two walks W_1 and W_2 in a signed digraph are called a pair of SSSD walks, if they have the same initial vertex, same terminal vertex, and same length, but they have different signs.

Lemma 2.4 (see [8]). *Let D be a symmetric digraph. Then D is primitive if and only if D is strongly connected and there exists an odd cycle in D.*

Lemma 2.5 (see [1, 12]). *If S is a primitive signed digraph, then S is nonpowerful if and only if S contains a pair of cycles C_1 and C_2, with lengths p_1 and p_2, respectively, satisfying one of the following conditions:*

(A_1) *p_1 is odd and p_2 is even and $\text{sgn}(C_2) = -1$;*

(A_2) *both p_1 and p_2 are odd and $\text{sgn}(C_1) = -\text{sgn}(C_2)$.*

Lemma 2.6 (see [12]). *Let S be a primitive, nonpowerful signed digraph. Then we have the following.*

(1) *There is an integer k such that there exists a pair of SSSD walks of length k from each vertex x to each vertex y in S.*

(2) *If there exists a pair of SSSD walks of length k from each vertex x to each vertex y, then there also exists a pair of SSSD walks of length $k + 1$ from each vertex u to each vertex v in S.*

(3) *The minimal such k (as in (1)) is just $l(S)$, the base of S.*

Lemma 2.7 (see [7]). *Suppose that an $n \times n$ sign pattern matrix $A = (a_{ij})$ is skew symmetric. Let $S(A)$ be the associated signed digraph of A. Let r be an odd integer with $3 \leq r \leq n$ $(n \geq 3)$. Then $l(S(A)) = 2n - 1$ if and only if $S(A)$ is isomorphic to G (see Figure 1).*

3. Main Results

For an undirected walk W of graph G and two vertices x, y on W, we denote by $Q_W(x \to y)$ a shortest path from x to y on W and by $Q(x \to y)$ a shortest path from x to y on G. For a cycle C of G, if x and y are two (not necessarily distinct) vertices on C and P is a path from x to y along C, then $C \setminus P$ denotes the path or cycle from x to y along C obtained by deleting the edges of P.

Lemma 3.1. *Let A be an $n \times n$ primitive nonpowerful ZS sign pattern matrix with zero diagonal. If all the 2 cycles in $S(A)$ are positive, then $l(A) \leq 2n - 2$.*

Proof. Since A is primitive, it follows from **Lemma 2.4** that $S(A)$ is strongly connected and there is an odd cycle C' in $S(A)$ such that $l(C') = l'$. Since A has zero diagonal, there are no loops in $S(A)$ and so $l' \geq 3$. Without loss of generality, we assume that C' is an odd cycle with the least length in $S(A)$.

Case 1. There exists at least one negative even cycle in $S(A)$.

Let C be a negative even cycle in $S(A)$. Without loss of generality, we assume that C is a negative even cycle with the least length in $S(A)$. Since all 2 cycles in $S(A)$ are positive, $l(C) = l \geq 4$.

Subcase 1.1. C and C' have no common vertices.

Let P be the shortest path from C to C'. Suppose P intersects C at vertex u and intersects C' at vertex v and there are k vertices on P where $k \geq 2$. Let $G_0 = C \cup P \cup C'$. Let x and y be any two (not necessarily distinct) vertices in $S(A)$. Suppose that P_1 is the shortest path from x to G_0 and intersects G_0 at vertex x' and P_2 is the shortest path from y to G_0 and intersects G_0 at vertex y'. Then $0 \leq l(P_1), l(P_2) \leq n - l - l' - k + 2$, where $l \geq 4$ and $l' \geq 3$. By the proof of Case 1 of Lemma 4.5 in [3], there exists a pair of SSSD walks from x to y of length $2n - 2$. Therefore, by Lemma 2.6, $l(A) \leq 2n - 2$.

Subcase 1.2. C and C' have at least one common vertex.

Let $G_1 = C \cup C'$. Let x and y be any two (not necessarily distinct) vertices in $S(A)$. Suppose that P_1 is the shortest path from x to G_1 and intersects G_1 at vertex x' and P_2 is the shortest path from y to G_1 and intersects G_1 at vertex y'. Denote $C \cap C'$ by R. Assume R has m vertices, where $1 \leq m \leq \min(l, l')$, then

$$0 \leq l(P_1), \qquad l(P_2) \leq n - l - l' + m. \tag{3.1}$$

If $m \geq 2$, then R is a path with vertex set $V(R) = \{v_1, v_2, \ldots, v_m\}$ and edge set $E(R) = \{(v_1, v_2), (v_2, v_3), \ldots, (v_{m-1}, v_m)\}$. In this case, if $m = \min(l, l') = l$, then there exists another odd cycle with length $l' - l + 2 < l'$, which contradicts our assumption that C' is an odd cycle with the least length in $S(A)$. Therefore, $m < l$ and if $m = \min(l, l')$, then $m = l'$.

Subcase 1.2.1. $x' = y'$ and $1 \leq m \leq \min(l, l')$.

Subcase 1.2.1.1. $x' = y' \in C \setminus R$. See Figure 2(a).

We consider two subcases: the subcase $1 \leq m < \min(l, l')$ and the subcase $m = \min(l, l') = l'$.

First, we consider the subcase $1 \leq m < \min(l, l')$. Without loss of generality, we assume that $l(Q_{C \setminus R}(x' \to v_1)) \leq l(Q_{C \setminus R}(x' \to v_m))$. Let $w = l(P_1) + l(C) + l(P_2)$. If w is even, set

$$W' = P_1 + C + P_2. \tag{3.2}$$

Otherwise, set

$$W'' = P_1 + Q_{C \setminus R}(x' \longrightarrow v_1) + Q_{C' \setminus R}(v_1 \longrightarrow v_m) + Q_{C \setminus R}(v_m \longrightarrow y') + P_2. \tag{3.3}$$

Let

$$W_1 = \begin{cases} W' + C, & w \text{ is even,} \\ W'' + C, & \text{otherwise,} \end{cases} \qquad W_2 = \begin{cases} W' + \dfrac{l}{2}C_2, & w \text{ is even,} \\ W'' + \dfrac{l}{2}C_2, & \text{otherwise,} \end{cases} \tag{3.4}$$

where C_2 is a positive 2-cycle that contains vertex x. Since C' is odd, $l(Q_{C'\backslash R}(v_1 \rightarrow v_m))$ and $l(Q_R(v_1 \rightarrow v_m))$ have different parity. Therefore, both $l(W_1)$ and $l(W_2)$ are even. Then, if w is even,

$$l(W_1) = l(W_2) \leq 2(n - l - l' + m) + 2l = 2n - 2l' + 2m. \tag{3.5}$$

Otherwise,

$$l(W_1) = l(W_2) \leq 2(n - l - l' + m) + (l - m + 1) + (l' - m + 1) + l = 2n - l' + 2. \tag{3.6}$$

Since both $l(W_1)$ and $l(W_2)$ are even, it follows that $l(W_1) = l(W_2) \leq 2n - 2$. We see that the pair W_1, W_2 is a pair of SSSD walks with even length. Therefore, there exists a pair of SSSD walks from x to y with length $2n - 2$.

Then, we consider the subcase $m = \min(l, l') = l'$. Without loss of generality, we assume that $l(Q_{C\backslash R}(x' \rightarrow v_1)) \leq l(Q_{C\backslash R}(x' \rightarrow v_m))$. Let

$$w = l(P_1) + 2l(Q_{C\backslash R}(x' \longrightarrow v_1)) + l(P_2). \tag{3.7}$$

If w is even, set

$$W' = P_1 + Q_{C\backslash R}(x' \longrightarrow v_1) + Q_{C\backslash R}(v_1 \longrightarrow x') + P_2. \tag{3.8}$$

Otherwise, set

$$W'' = P_1 + Q_{C\backslash R}(x' \longrightarrow v_1) + Q_{C'\backslash R}(v_1 \longrightarrow v_m) + Q_{C\backslash R}(v_m \longrightarrow x') + P_2. \tag{3.9}$$
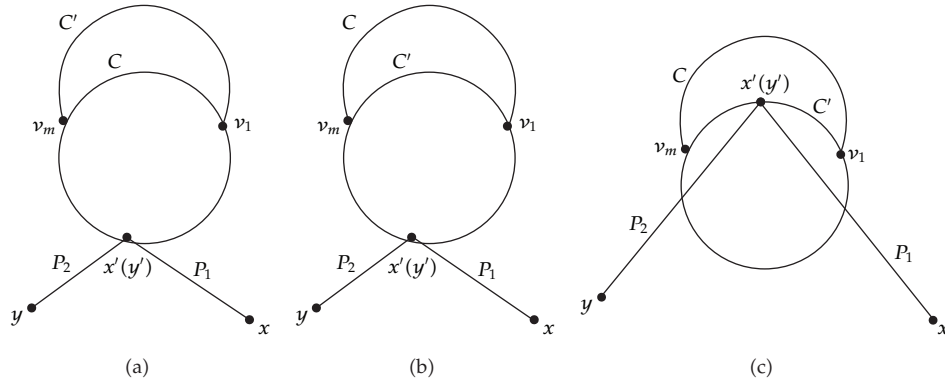
Let

$$W_1 = \begin{cases} W' + C, & w \text{ is even,} \\ W'' + C, & \text{otherwise,} \end{cases} \qquad W_2 = \begin{cases} W' + \dfrac{l}{2}C_2, & w \text{ is even,} \\ W'' + \dfrac{l}{2}C_2, & \text{otherwise,} \end{cases} \tag{3.10}$$

where C_2 is a positive 2-cycle that contains vertex x. Since $Q_{C\backslash R}(x' \rightarrow v_1) + Q_{C'\backslash R}(v_1 \rightarrow v_m) + Q_{C\backslash R}(v_m \rightarrow x')$ is an odd cycle, $l(Q_{C\backslash R}(x' \rightarrow v_1))$ and $l(Q_{C'\backslash R}(v_1 \rightarrow v_m) + Q_{C\backslash R}(v_m \rightarrow x'))$ have different parity. Therefore, both $l(W_1)$ and $l(W_2)$ are even. Then, if w is even,

$$l(W_1) = l(W_2) \leq 2(n - l) + (l - l' + 1) + l = 2n - l' + 1. \tag{3.11}$$

Otherwise,

$$l(W_1) = l(W_2) \leq 2(n - l) + (l - l' + 2) + l = 2n - l' + 2. \tag{3.12}$$

Figure 2: Illustrate for Subcase 1.2.1 of **Lemma 3.1**.

Since both $l(W_1)$ and $l(W_2)$ are even, it follows that $l(W_1) = l(W_2) \leq 2n - 2$. We see that the pair W_1, W_2 is a pair of SSSD walks with even length. Therefore, there exists a pair of SSSD walks from x to y with length $2n - 2$.

Subcase 1.2.1.2. $x' = y' \in C' \setminus R$. See Figure 2(b).
   The proof for this subcase is similar to that of the Subcase 1.2.1.1 and is omitted.

Subcase 1.2.1.3. $x' = y' \in R$. See Figure 2(c).
   Let $w = l(P_1) + l(P_2)$. If w is even, set

$$W' = P_1 + P_2. \tag{3.13}$$

Otherwise, set

$$W'' = P_1 + P_2 + C'. \tag{3.14}$$

Let

$$W_1 = \begin{cases} W' + C, & w \text{ is even,} \\ W'' + C, & \text{otherwise,} \end{cases} \qquad W_2 = \begin{cases} W' + \dfrac{l}{2}C_2, & w \text{ is even,} \\ W'' + \dfrac{l}{2}C_2, & \text{otherwise,} \end{cases} \tag{3.15}$$

where C_2 is a positive 2-cycle that contains vertex x. Since l' is odd, we see that both $l(W_1)$ and $l(W_2)$ are even. Then, if w is even,

$$l(W_1) = l(W_2) \leq 2(n - l - l' + m) + l = 2n - l - 2l' + 2m. \tag{3.16}$$

Otherwise,

$$l(W_1) = l(W_2) \leq 2(n - l - l' + m) + l' + l = 2n - l - l' + 2m. \tag{3.17}$$
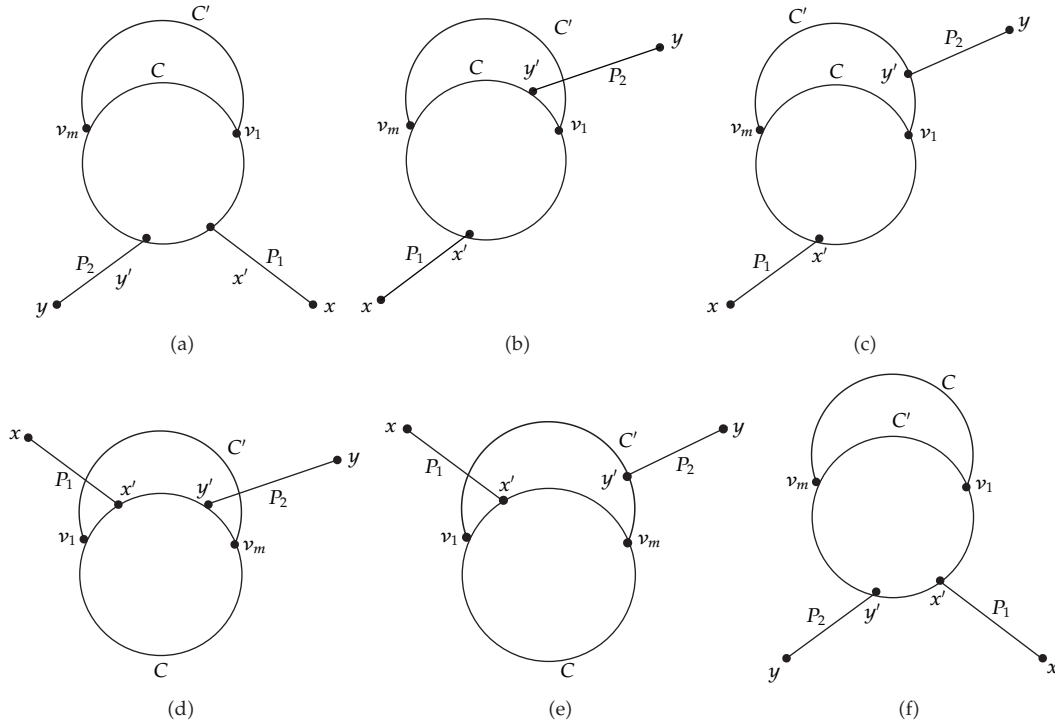
Figure 3: Illustration for Subcases 1.2.2 and 2.2 of Lemma 3.1.

Since both $l(W_1)$ and $l(W_2)$ are even, it follows that $l(W_1) = l(W_2) \leq 2n - 2$. We see that the pair W_1, W_2 is a pair of SSSD walks with even length. Therefore, there exists a pair of SSSD walks from x to y with length $2n - 2$.

Subcase 1.2.2. $x' \neq y'$ and $2 \leq m \leq \min(l, l')$.

Subcase 1.2.2.1. $x' \in C \setminus R$ and $y' \in C \setminus R$. See Figure 3(a).

Without loss of generality, we assume that $l(Q_{C \setminus R}(x' \to v_1)) \leq l(Q_{C \setminus R}(y' \to v_1))$. Let $w = l(P_1) + l(Q_{C \setminus R}(x' \to v_1)) + l(Q_R(v_1 \to v_m)) + l(Q_{C \setminus R}(v_m \to y')) + l(P_2)$. If w is even, set

$$W' = P_1 + Q_{C \setminus R}(x' \longrightarrow v_1) + Q_R(v_1 \longrightarrow v_m) + Q_{C \setminus R}(v_m \longrightarrow y') + P_2. \tag{3.18}$$

Otherwise, set

$$W'' = P_1 + Q_{C \setminus R}(x' \longrightarrow v_1) + Q_{C' \setminus R}(v_1 \longrightarrow v_m) + Q_{C \setminus R}(v_m \longrightarrow y') + P_2. \tag{3.19}$$

Let

$$W_1 = \begin{cases} W' + C, & w \text{ is even,} \\ W'' + C, & \text{otherwise,} \end{cases} \qquad W_2 = \begin{cases} W' + \dfrac{l}{2}C_2, & w \text{ is even,} \\ W'' + \dfrac{l}{2}C_2, & \text{otherwise,} \end{cases} \tag{3.20}$$

where C_2 is a positive 2-cycle that contains vertex x. Since C' is odd, $l(Q_{C' \setminus R}(v_1 \to v_m))$ and $l(Q_R(v_1 \to v_m))$ have different parity. Therefore, both $l(W_1)$ and $l(W_2)$ are even. Noting that

$W_3 = P_1 + Q_{C\backslash R}(x' \to v_1) + Q_R(v_1 \to v_m) + Q_{C\backslash R}(v_m \to y')$ and $W_4 = P_1 + Q_{C\backslash R}(x' \to v_1) + Q_{C'\backslash R}(v_1 \to v_m) + Q_{C\backslash R}(v_m \to y')$ are paths of $S(A)$, we have $l(W_3), l(W_4) \leq n-1$. Then

$$l(W_1) = l(W_2) \leq (n-1) + (n - l - l' + m) + l = 2n - l' + m - 1. \tag{3.21}$$

Since both $l(W_1)$ and $l(W_2)$ are even, it follows that $l(W_1) = l(W_2) \leq 2n - 2$. We see that the pair W_1, W_2 is a pair of SSSD walks with even length. Therefore, there exists a pair of SSSD walks from x to y with length $2n - 2$.

Subcase 1.2.2.2. $x' \in C \setminus R$ and $y' \in R$. See Figure 3(b).

Without loss of generality, we assume that $l(Q_{C\backslash R}(x' \to v_1)) \leq l(Q_{C\backslash R}(x' \to v_m))$. Then we consider two subcases: the subcase $x' \in C \setminus R$, $y' \in R$ and $y' \neq v_1$ and the subcase $x' \in C \setminus R$ and $y' = v_1$.

First, we consider the subcase $x' \in C \setminus R$, $y' \in R$ and $y' \neq v_1$. Let $w = l(P_1) + l(Q_{C\backslash R}(x' \to v_1)) + l(Q_R(v_1 \to y')) + l(P_2)$. If w is even, set

$$W' = P_1 + Q_{C\backslash R}(x' \longrightarrow v_1) + Q_R(v_1 \longrightarrow y') + P_2. \tag{3.22}$$

Otherwise, set

$$W'' = P_1 + Q_{C\backslash R}(x' \longrightarrow v_1) + Q_{C'\backslash R}(v_1 \longrightarrow v_m) + Q_R(v_m \longrightarrow y') + P_2. \tag{3.23}$$

Let

$$W_1 = \begin{cases} W' + C, & w \text{ is even,} \\ W'' + C, & \text{otherwise,} \end{cases} \qquad W_2 = \begin{cases} W' + \dfrac{l}{2}C_2, & w \text{ is even,} \\ W'' + \dfrac{l}{2}C_2, & \text{otherwise,} \end{cases} \tag{3.24}$$

where C_2 is a positive 2-cycle that contains vertex x. Since C' is odd, $l(Q_R(v_1 \to y'))$ and $l(Q_{C'\backslash R}(v_1 \to v_m)) + l(Q_R(v_m \to y'))$ have different parity. Therefore, both $l(W_1)$ and $l(W_2)$ are even. Noting that $W_3 = P_1 + Q_{C\backslash R}(x' \to v_1) + Q_R(v_1 \to y')$ and $W_4 = P_1 + Q_{C\backslash R}(x' \to v_1) + Q_{C'\backslash R}(v_1 \to v_m) + Q_R(v_m \to y')$ are two paths of $S(A)$, then we have $l(W_3), l(W_4) \leq n-1$. Then

$$l(W_1) = l(W_2) \leq (n-1) + (n - l - l' + m) + l = 2n - l' + m - 1. \tag{3.25}$$

Since both $l(W_1)$ and $l(W_2)$ are even, it follows that $l(W_1) = l(W_2) \leq 2n - 2$. We see that the pair W_1, W_2 is a pair of SSSD walks with even length. Therefore, there exists a pair of SSSD walks from x to y with length $2n - 2$.

Then, we consider the subcase $x' \in C \setminus R$ and $y' = v_1$. Let $w = l(P_1) + l(Q_{C\backslash R}(x' \to v_m)) + l(Q_R(v_m \to v_1(y'))) + l(P_2)$. If w is even, set

$$W' = P_1 + Q_{C\backslash R}(x' \longrightarrow v_m) + Q_R(v_m \longrightarrow v_1(y')) + P_2. \tag{3.26}$$

Otherwise, set

$$W'' = P_1 + Q_{C \setminus R}(x' \longrightarrow v_m) + Q_{C' \setminus R}(v_m \longrightarrow v_1(y')) + P_2. \tag{3.27}$$

Let

$$W_1 = \begin{cases} W' + C, & w \text{ is even,} \\ W'' + C, & \text{otherwise,} \end{cases} \qquad W_2 = \begin{cases} W' + \dfrac{l}{2}C_2, & w \text{ is even,} \\ W'' + \dfrac{l}{2}C_2, & \text{otherwise,} \end{cases} \tag{3.28}$$

where C_2 is a positive 2-cycle that contains vertex x. Since C' is odd, $l(Q_R(v_m \rightarrow v_1(y')))$ and $l(Q_{C' \setminus R}(v_m \rightarrow v_1(y')))$ have different parity. Therefore, both $l(W_1)$ and $l(W_2)$ are even. Noting that $W_3 = P_1 + Q_{C \setminus R}(x' \rightarrow v_m) + Q_R(v_m \rightarrow v_1(y'))$ and $W_4 = P_1 + Q_{C \setminus R}(x' \rightarrow v_m) + Q_{C' \setminus R}(v_m \rightarrow v_1(y'))$ are two paths of $S(A)$, we have $l(W_3), l(W_4) \leq n - 1$. Then

$$l(W_1) = l(W_2) \leq (n - 1) + (n - l - l' + m) + l = 2n - l' + m - 1. \tag{3.29}$$

Since both $l(W_1)$ and $l(W_2)$ are even, it follows that $l(W_1) = l(W_2) \leq 2n - 2$. We see that the pair W_1, W_2 is a pair of SSSD walks with even length. Therefore, there exists a pair of SSSD walks from x to y with length $2n - 2$.

Subcase 1.2.2.3. $x' \in C \setminus R$ and $y' \in C' \setminus R$. See Figure 3(c).
    The proof for this subcase is similar to that of the Subcase 1.2.2.1 and is omitted.

Subcase 1.2.2.4. $x' \in R$ and $y' \in R$. See Figure 3(d).
    Let $w = l(P_1) + l(Q_{C'}(x' \rightarrow y')) + l(P_2)$. If w is even, set

$$W' = P_1 + Q_{C'}(x' \longrightarrow y') + P_2. \tag{3.30}$$

Otherwise, set

$$W'' = P_1 + C' \setminus Q_{C'}(x' \longrightarrow y') + P_2. \tag{3.31}$$

Let

$$W_1 = \begin{cases} W' + C, & w \text{ is even,} \\ W'' + C, & \text{otherwise,} \end{cases} \qquad W_2 = \begin{cases} W' + \dfrac{l}{2}C_2, & w \text{ is even,} \\ W'' + \dfrac{l}{2}C_2, & \text{otherwise,} \end{cases} \tag{3.32}$$

where C_2 is a positive 2-cycle that contains vertex x. Since C' is odd, $l(Q_{C'}(x' \to y'))$ and $l(C' \setminus Q_{C'}(x' \to y'))$ have different parity. Therefore, both $l(W_1)$ and $l(W_2)$ are even. Then if w is even,

$$l(W_1) = l(W_2) \leq 2(n - l - l' + m) + \frac{l' - 1}{2} + l = 2n + 2m - l - \frac{3l'}{2} - \frac{1}{2}. \tag{3.33}$$

Otherwise,

$$l(W_1) = l(W_2) \leq 2(n - l - l' + m) + (l' - 1) + l = 2n - l - l' + 2m - 1. \tag{3.34}$$

Since both $l(W_1)$ and $l(W_2)$ are even, it follows that $l(W_1) = l(W_2) \leq 2n - 2$. We see that the pair W_1, W_2 is a pair of SSSD walks with even length. Therefore, there exists a pair of SSSD walks from x to y with length $2n - 2$.

Subcase 1.2.2.5. $x' \in R$ and $y' \in C' \setminus R$. See Figure 3(e).
  The proof for this subcase is similar to that of Subcase 1.2.2.4, so we omit it.

Subcase 1.2.2.6. $x' \in C' \setminus R$ and $y' \in C' \setminus R$. See Figure 3(f).
  Without loss of generality, we assume that $l(Q_{C' \setminus R}(x' \to v_1)) \leq l(Q_{C' \setminus R}(y' \to v_1))$. Let $w = l(P_1) + l(Q_{C' \setminus R}(x' \to v_1)) + l(Q_R(v_1 \to v_m)) + l(Q_{C' \setminus R}(v_m \to y')) + l(P_2)$. If w is even, set

$$W' = P_1 + Q_{C' \setminus R}(x' \longrightarrow v_1) + Q_R(v_1 \longrightarrow v_m) + Q_{C' \setminus R}(v_m \longrightarrow y') + P_2. \tag{3.35}$$

Otherwise, set

$$W'' = P_1 + Q_{C' \setminus R}(x' \longrightarrow y') + Q_{C' \setminus R}(y' \longrightarrow v_m) + Q_{C' \setminus R}(v_m \longrightarrow y') + P_2. \tag{3.36}$$

Let

$$W_1 = \begin{cases} W' + C, & w \text{ is even}, \\ W'' + C, & \text{otherwise}, \end{cases} \qquad W_2 = \begin{cases} W' + \dfrac{l}{2}C_2, & w \text{ is even}, \\ W'' + \dfrac{l}{2}C_2, & \text{otherwise}, \end{cases} \tag{3.37}$$

where C_2 is a positive 2-cycle that contains vertex x. Since C' is odd, $l(Q_{C' \setminus R}(x' \to y'))$ and $l(Q_{C' \setminus R}(x' \to v_1)) + l(Q_R(v_1 \to v_m)) + l(Q_{C' \setminus R}(v_m \to y'))$ have different parity. Therefore, both $l(W_1)$ and $l(W_2)$ are even. Noting that $W_3 = P_1 + Q_{C' \setminus R}(x' \to v_1) + Q_R(v_1 \to v_m) + Q_{C' \setminus R}(v_m \to y')$ and $W_4 = P_1 + Q_{C' \setminus R}(x' \to y') + Q_{C' \setminus R}(y' \to v_m)$ are two paths of $S(A)$, then we have $l(W_3) \leq n - 1$ and $l(W_4) \leq n - m$. Then, if w is even,

$$l(W_1) = l(W_2) \leq (n - 1) + (n - l - l' + m) + l = 2n - l' + m - 1. \tag{3.38}$$

Otherwise,

$$l(W_1) = l(W_2) \leq (n - l - l' + m) + (n - m) + (l' - m - 1) + l = 2n - m - 1. \tag{3.39}$$
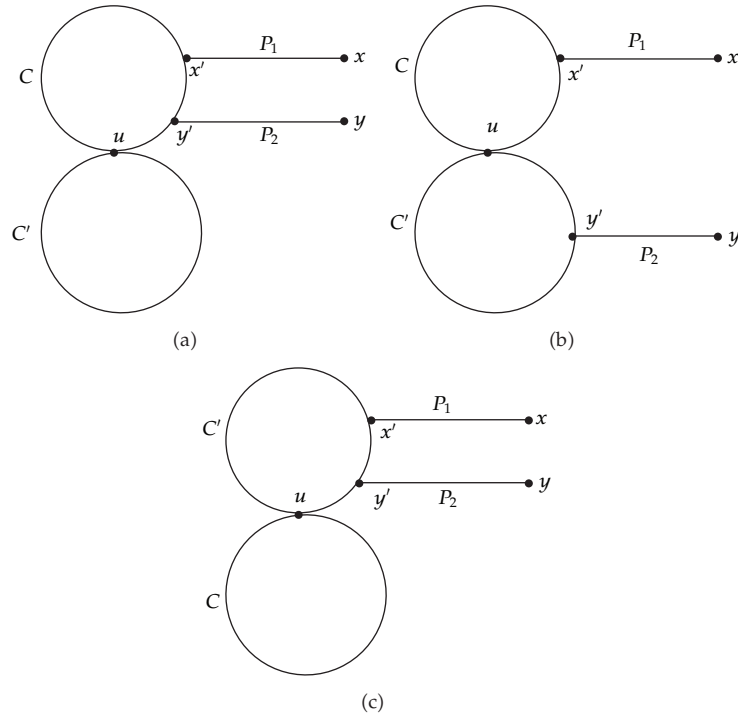
Figure 4: Illustrate for Subcase 1.2.3 of **Lemma 3.1**.

Since both $l(W_1)$ and $l(W_2)$ are even, it follows that $l(W_1) = l(W_2) \leq 2n - 2$. We see that the pair W_1, W_2 is a pair of SSSD walks with even length. Therefore, there exists a pair of SSSD walks from x to y with length $2n - 2$.

Subcase 1.2.3. $m = 1$ and $x' \neq y'$. We assume that $V(C) \cap V(C') = \{u\}$.

Subcase 1.2.3.1. $x' \in C$ and $y' \in C$. See Figure 4(a).
Let $w = l(P_1) + l(Q_C(x' \rightarrow u)) + l(Q_C(u \rightarrow y')) + l(P_2)$. If w is even, set

$$W' = P_1 + Q_C(x' \longrightarrow u) + Q_C(u \longrightarrow y') + P_2. \tag{3.40}$$

Otherwise, set

$$W'' = P_1 + Q_C(x' \longrightarrow u) + Q_C(u \longrightarrow y') + P_2 + C'. \tag{3.41}$$

Let

$$W_1 = \begin{cases} W' + C, & w \text{ is even,} \\ W'' + C, & \text{otherwise,} \end{cases} \qquad W_2 = \begin{cases} W' + \dfrac{l}{2}C_2, & w \text{ is even,} \\ W'' + \dfrac{l}{2}C_2, & \text{otherwise,} \end{cases} \tag{3.42}$$

where C_2 is a positive 2-cycle that contains vertex x. Since l' is odd, both $l(W_1)$ and $l(W_2)$ are even. Then, if w is even,

$$l(W_1) = l(W_2) \leq 2(n - l - l' + 1) + (l - 1) + l = 2n - 2l' + 1. \tag{3.43}$$

Otherwise,

$$l(W_1) = l(W_2) \leq 2(n - l - l' + 1) + (l - 1) + l' + l = 2n - l' + 1. \tag{3.44}$$

Since both $l(W_1)$ and $l(W_2)$ are even, it follows that $l(W_1) = l(W_2) \leq 2n - 2$. We see that the pair W_1, W_2 is a pair of SSSD walks with even length. Therefore, there exists a pair of SSSD walks from x to y with length $2n - 2$.

Subcase 1.2.3.2. $x' \in C$, $y' \in C'$, and $x' \neq y' \neq u$. See Figure 4(b).
     Let $w = l(P_1) + l(Q_C(x' \to u)) + l(Q_{C'}(u \to y')) + l(P_2)$. If w is even, set

$$W' = P_1 + Q_C(x' \longrightarrow u) + Q_{C'}(u \longrightarrow y') + P_2. \tag{3.45}$$

Otherwise, set

$$W'' = P_1 + Q_C(x' \longrightarrow u) + C' \setminus Q_{C'}(u \longrightarrow y') + P_2. \tag{3.46}$$

Let

$$W_1 = \begin{cases} W' + C, & w \text{ is even,} \\ W'' + C, & \text{otherwise,} \end{cases} \qquad W_2 = \begin{cases} W' + \dfrac{l}{2}C_2, & w \text{ is even,} \\ W'' + \dfrac{l}{2}C_2, & \text{otherwise,} \end{cases} \tag{3.47}$$

where C_2 is a positive 2-cycle that contains vertex x. Since C' is odd, $l(C' \setminus Q_{C'}(u \to y'))$ and $l(Q_{C'}(u \to y'))$ have different parity. Therefore, both $l(W_1)$ and $l(W_2)$ are even. Noting that $W_3 = P_1 + Q_C(x' \to u) + Q_{C'}(u \to y')$ and $W_4 = P_1 + Q_C(x' \to u) + C' \setminus Q_{C'}(u \to y')$ are two paths of $S(A)$, we have $l(W_3)$, $l(W_4) \leq n - 1$. Then,

$$l(W_1) = l(W_2) \leq (n - 1) + (n - l - l' + 1) + l = 2n - l'. \tag{3.48}$$

Since both $l(W_1)$ and $l(W_2)$ are even, it follows that $l(W_1) = l(W_2) \leq 2n - 2$. We see that the pair W_1, W_2 is a pair of SSSD walks with even length. Therefore, there exists a pair of SSSD walks from x to y with length $2n - 2$.

Subcase 1.2.3.3. $x' \in C'$ and $y' \in C'$. See Figure 4(c).
     The proof for this subcase is similar to that of the Subcase 1.2.3.1 and is omitted.
     Thus in each of the above subcases, there exists a pair of SSSD walks from x to y with length $2n - 2$. Therefore, we get $l(A) \leq 2n - 2$ by Lemma 2.6.

Case 2. There exists no negative even cycle in $S(A)$.

Since A is primitive, nonpowerful and there exist no negative even cycles in $S(A)$, it follows from Lemma 2.5 that there exist two odd cycles C and C' with different signs in $S(A)$. We assume that $l(C) = l$ and $l(C') = l'$. Since the trace of A is zero, there are no loops in $S(A)$ and so $l, l' \geq 3$. Without loss of generality, we assume $l \geq l' \geq 3$.

Subcase 2.1. C and C' have no common vertices.

Let P be the shortest path from C to C'. Suppose P intersects C at vertex u and intersects C' at vertex v and there are k vertices on P where $k \geq 2$. Let $G_2 = C \cup P \cup C'$. Let x and y be two arbitrary (not necessarily distinct) vertices in $S(A)$. Suppose that P_1 is the shortest path from x to G_2 and intersects G_2 at vertex x' and P_2 is the shortest path from y to G_2 and intersects G_2 at vertex y'. Then

$$0 \leq l(P_1), \qquad l(P_2) \leq n - l - l' - k + 2. \tag{3.49}$$

Since C and C' have diffident signs, if there exists an even walk W with length $l(W) \leq 2n-2-l$, then $W_1 = W + C$ and $W_2 = W + ((l-l')/2)C_2 + C'$ have the same length $l(W_1) = l(W_2) \leq 2n-2$ and different signs. As the proof of Subcase 1.1, we can construct an even walk W with length $l(W) \leq 2n - 2 - l$. So there exists a pair of SSSD walks from x to y of length $2n - 2$. It remains to consider the following subcase.

Subcase 2.2. C and C' have at least one common vertex.

Let $G_3 = C \cup C'$. Let x and y be two arbitrary (not necessarily distinct) vertices in $S(A)$. Suppose that P_1 is the shortest path from x to G_3 and intersects G_3 at vertex x' and P_2 is the shortest path from y to G_3 and intersects G_3 at vertex y'. Denote $C \cap C'$ by R. Assume R has m vertices, where $1 \leq m \leq \min(l_1, l_2)$, then

$$0 \leq l(P_1), \qquad l(P_2) \leq n - l_1 - l_2 + m. \tag{3.50}$$

If $m > 1$, then R is a path with vertex set $V(R) = \{v_1, v_2, \ldots, v_m\}$ and edge set $E(R) = \{(v_1, v_2), (v_2, v_3), \ldots, (v_{m-1}, v_m)\}$. If $m = l'$, since there exists no negative even cycles in $S(A)$, then C and C' have the same sign, a contradiction. Therefore, $1 \leq m < l'$.

Subcase 2.2.1. $x' \in C \setminus R$ and $y' \in C \setminus R$. See Figure 3(a).

Without loss of generality, we assume that $l(Q_{C \setminus R}(x' \rightarrow v_1)) \leq l(Q_{C \setminus R}(y' \rightarrow v_1))$. Let $w = l(P_1) + l(Q_{C \setminus R}(x' \rightarrow v_1)) + l(Q_R(v_1 \rightarrow v_m)) + l(Q_{C \setminus R}(v_m \rightarrow y')) + l(P_2)$. If w is odd, set

$$W' = P_1 + Q_{C \setminus R}(x' \longrightarrow v_1) + Q_R(v_1 \longrightarrow v_m) + Q_{C \setminus R}(v_m \longrightarrow y') + P_2. \tag{3.51}$$

Otherwise, set

$$W'' = P_1 + Q_{C \setminus R}(x' \longrightarrow v_1) + Q_{C' \setminus R}(v_1 \longrightarrow v_m) + Q_{C \setminus R}(v_m \longrightarrow y') + P_2. \tag{3.52}$$

Let

$$W_1 = \begin{cases} W' + C, & w \text{ is odd}, \\ W'' + C, & \text{otherwise}, \end{cases} \qquad W_2 = \begin{cases} W' + C' + \dfrac{l - l'}{2} C_2, & w \text{ is odd}, \\ W'' + C' + \dfrac{l - l'}{2} C_2, & \text{otherwise}, \end{cases} \tag{3.53}$$

where C_2 is a positive 2-cycle that contains x. Since C' is odd, $l(Q_R(v_1 \rightarrow v_m))$ and $l(Q_{C' \backslash R}(v_1 \rightarrow v_m))$ have different parity. Therefore, both $l(W_1)$ and $l(W_2)$ are even. Then, if w is odd,

$$l(W_1) = l(W_2) \le 2(n - l - l' + m) + l + l = 2n - 2l' + 2m. \tag{3.54}$$

Otherwise,

$$l(W_1) = l(W_2) \le 2(n - l - l' + m) + (l - m + 1) + (l' - m + 1) + l = 2n - l' + 2. \tag{3.55}$$

Since both $l(W_1)$ and $l(W_2)$ are even, it follows that $l(W_1) = l(W_2) \le 2n - 2$. We see that the pair W_1, W_2 is a pair of SSSD walks with even length. Therefore, there exists a pair of SSSD walks from x to y with length $2n - 2$.

Subcase 2.2.2. $x' \in C \setminus R$ and $y' \in R$. See Figure 3(b).
     Let $w = l(P_1) + l(Q_C(x' \rightarrow y')) + l(P_2)$. If w is odd, set

$$W' = P_1 + Q_C(x' \longrightarrow y') + P_2. \tag{3.56}$$

Otherwise, set

$$W'' = P_1 + C \setminus Q_C(x' \longrightarrow y') + P_2. \tag{3.57}$$

Let

$$W_1 = \begin{cases} W' + C, & w \text{ is odd}, \\ W'' + C, & \text{otherwise}, \end{cases} \qquad W_2 = \begin{cases} W' + C' + \dfrac{l - l'}{2} C_2, & w \text{ is odd}, \\ W'' + C' + \dfrac{l - l'}{2} C_2, & \text{otherwise}, \end{cases} \tag{3.58}$$

where C_2 is a positive 2-cycle that contains x. Since C is odd, $C \setminus Q_C(x' \rightarrow y')$ and $Q_C(x' \rightarrow y')$ have different parity, Therefore, both $l(W_1)$ and $l(W_2)$ are even. Then, if w is odd,

$$l(W_1) = l(W_2) \le 2(n - l - l' + m) + \frac{l - 1}{2} + l = 2n - 2l' + 2m - \frac{l + 1}{2}. \tag{3.59}$$

Otherwise,

$$l(W_1) = l(W_2) \le 2(n - l - l' + m) + (l - 1) + l = 2n - 2l' + 2m - 1. \tag{3.60}$$

Since both $l(W_1)$ and $l(W_2)$ are even, it follows that $l(W_1) = l(W_2) \leq 2n - 2$. We see that the pair W_1, W_2 is a pair of SSSD walks with even length. Therefore, there exists a pair of SSSD walks from x to y with length $2n - 2$.

Subcase 2.2.3. $x' \in C \setminus R$ and $y' \in C' \setminus R$. See Figure 3(c).

Without loss of generality, we assume that $l(Q_{C \setminus R}(x' \to v_1)) \leq l(Q_{C \setminus R}(x' \to v_m))$. Let $w = l(P_1) + l(Q_{C \setminus R}(x' \to v_1)) + l(Q_{C' \setminus R}(v_1 \to y')) + l(P_2)$. If w is odd, set

$$W' = P_1 + Q_{C \setminus R}(x' \longrightarrow v_1) + Q_{C' \setminus R}(v_1 \longrightarrow y') + P_2. \tag{3.61}$$

Otherwise, set

$$W'' = P_1 + Q_{C \setminus R}(x' \longrightarrow v_1) + Q_R(v_1 \longrightarrow v_m) + Q_{C' \setminus R}(v_m \longrightarrow y') + P_2. \tag{3.62}$$

Let

$$W_1 = \begin{cases} W' + C, & w \text{ is odd,} \\ W'' + C, & \text{otherwise,} \end{cases} \qquad W_2 = \begin{cases} W' + C' + \dfrac{l - l'}{2} C_2, & w \text{ is odd,} \\ W'' + C' + \dfrac{l - l'}{2} C_2, & \text{otherwise,} \end{cases} \tag{3.63}$$

where C_2 is a positive 2-cycle that contains x. Since C' is odd, $l(Q_{C' \setminus R}(v_1 \to y'))$ and $l(Q_R(v_1 \to v_m)) + l(Q_{C' \setminus R}(v_m \to y'))$ have different parity. Therefore, both $l(W_1)$ and $l(W_2)$ are even. Noting that $W_3 = P_1 + Q_{C \setminus R}(x' \to v_1) + Q_{C' \setminus R}(v_1 \to y')$ and $W_4 = P_1 + Q_{C \setminus R}(x' \to v_1) + Q_R(v_1 \to v_m) + Q_{C' \setminus R}(v_m \to y')$ are paths of $S(A)$, then we have $l(W_3)$, $l(W_4) \leq n - 1$. Therefore,

$$l(W_1) = l(W_2) \leq (n - 1) + (n - l - l' + m) + l = 2n - l' + m - 1. \tag{3.64}$$

Since both $l(W_1)$ and $l(W_2)$ are even, it follows that $l(W_1) = l(W_2) \leq 2n - 2$. We see that the pair W_1, W_2 is a pair of SSSD walks with even length. Therefore, there exists a pair of SSSD walks from x to y with length $2n - 2$.

Subcase 2.2.4. $x' \in R$ and $y' \in R$. See Figure 3(d).

The proof for this subcase is similar to that of Subcase 2.2.2, so we omit it.

Subcase 2.2.5. $x' \in R$ and $y' \in C' \setminus R$. See Figure 3(e).

The proof for this subcase is similar to that of Subcase 2.2.2, so we omit it.

Subcase 2.2.6. $x' \in C' \setminus R$ and $y' \in C' \setminus R$. See Figure 3(f).

The proof for this subcase is similar to that of Subcase 2.2.1, so we omit it.

From all the above subcases, there exists a pair of SSSD walks from x to y with length $2n - 2$. Therefore, we get $l(A) \leq 2n - 2$ by Lemma 2.6. □

Lemma 3.2. *Let A be an $n \times n$ primitive nonpowerful ZS sign pattern matrix with zero diagonal. If there exist a negative 2-cycle and a positive 2-cycle in $S(A)$, then $l(A) \leq 2n - 2$.*
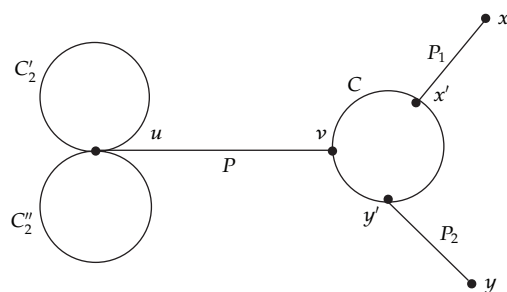
Figure 5: Illustration for Case 3 of Lemma 3.2.

Proof. Since A is primitive, it follows from Lemma 2.4 that $S(A)$ is strongly connected and there is an odd cycle C in $S(A)$ with length $l(C) = l$. Since A has zero diagonal, there are no loops in $S(A)$ and so $l \geq 3$. Without loss of generality, we assume that C is an odd cycle with the least length in $S(A)$. Since A is ZS and $S(A)$ contain a positive 2-cycle and a negative 2-cycle, there exists a positive 2-cycle C_2' and a negative 2-cycle C_2'' such that $V(C_2') \cap V(C_2'') \neq \emptyset$. Let $u \in V(C_2') \cap V(C_2'')$.

Let P be the shortest path from u to C. Let x and y be two arbitrary (not necessarily distinct) vertices in $S(A)$. Suppose there are k vertices on P and P intersects C at v, where $k \geq 1$. Suppose P_1 is the shortest path from x to $P \cup C$ and P_1 intersects $P \cup C$ at x' and P_2 is the shortest path from y to $P \cup C$ and P_2 intersects $P \cup C$ at y' where $0 \leq l(P_1), l(P_2) \leq n - l - k + 1$. We consider the following three cases.

Case 1. $x' \in P$ and $y' \in P$.

By the Subcase 2.1 of Lemma 4.2 in [3], there exists a pair of SSSD walks from x to y of length $2n - 2$.

Case 2. Only one of x' and y' belongs to P.

By the Subcase 2.2 of Lemma 4.2 in [3], there exists a pair of SSSD walks from x to y of length $2n - 2$.

It remains to consider the following case.

Case 3. $x' \notin P$ and $y' \notin P$. See Figure 5.

Subcase 3.1. $x' = y'$.

Let $w = l(P_1) + l(Q_C(x' \rightarrow v)) + 2l(P) + l(Q_C(v \rightarrow y')) + l(P_2)$. If w is even, set

$$W' = P_1 + Q_C(x' \longrightarrow v) + 2P + Q_C(v \longrightarrow y') + P_2. \tag{3.65}$$

Otherwise, set

$$W'' = P_1 + Q_C(x' \longrightarrow v) + 2P + C \setminus Q_C(v \longrightarrow y') + P_2. \tag{3.66}$$

Let

$$W_1 = \begin{cases} W' + C_2', & w \text{ is even,} \\ W'' + C_2', & \text{otherwise,} \end{cases} \qquad W_2 = \begin{cases} W' + C_2'', & w \text{ is even,} \\ W'' + C_2'', & \text{otherwise.} \end{cases} \tag{3.67}$$

Since C is odd, $l(Q_C(v \rightarrow x'))$ and $l(C \setminus Q_C(v \rightarrow x'))$ have different parity. Therefore, both $l(W_1)$ and $l(W_2)$ are even. Then, if w is odd,

$$l(W_1) = l(W_2) \le 2(n - l - k + 1) + 2(k - 1) + (l - 1) + 2 = 2n - l + 1. \tag{3.68}$$

Otherwise,

$$l(W_1) = l(W_2) \le 2(n - l - k + 1) + 2(k - 1) + l + 2 = 2n - l + 2. \tag{3.69}$$

Since both $l(W_1)$ and $l(W_2)$ are even, it follows that $l(W_1) = l(W_2) \le 2n - 2$. We see that the pair W_1, W_2 is a pair of SSSD walks with even length. Therefore, there exists a pair of SSSD walks from x to y with length $2n - 2$.

Subcase 3.2. $x' \neq y'$.
    Without loss of generality, we assume that $l(Q_C(x' \rightarrow v)) \le l(Q_C(y' \rightarrow v))$. Let $P_3 = Q_C(x' \rightarrow v)$ and $P_4 = Q_{C \setminus P_3}(v \rightarrow y')$. Let $w = l(P_1) + l(P_3) + l(P_4) + 2l(P) + l(P_2)$, If w is odd, set

$$W' = P_1 + Q_C(x' \longrightarrow v) + 2P + P_4 + P_2. \tag{3.70}$$

Otherwise, set

$$W'' = P_1 + Q_C(x' \longrightarrow v) + 2P + Q_C(v \longrightarrow x') + Q_{C \setminus \{P_3 + P_4\}}(x' \longrightarrow y') + P_2. \tag{3.71}$$

Let

$$W_1 = \begin{cases} W' + C_2', & w \text{ is even,} \\ W'' + C_2', & \text{otherwise,} \end{cases} \qquad W_2 = \begin{cases} W' + C_2'', & w \text{ is even,} \\ W'' + C_2'', & \text{otherwise.} \end{cases} \tag{3.72}$$

Since C is odd, $l(P_3) + l(P_4)$ and $l(Q_{C \setminus \{P_3 + P_4\}}(x' \rightarrow y'))$ have different parity. Therefore, both $l(W_1)$ and $l(W_2)$ are even. Then, if w is odd,

$$l(W_1) = l(W_2) \le 2(n - l - k + 1) + 2(k - 1) + (l - 1) + 2 = 2n - l + 1. \tag{3.73}$$

Otherwise,

$$l(W_1) = l(W_2) \le 2(n - l - k + 1) + 2(k - 1) + (l - 1) + (l - 2) + 2 = 2n - 1. \tag{3.74}$$

Since both $l(W_1)$ and $l(W_2)$ are even, it follows that $l(W_1) = l(W_2) \leq 2n - 2$. We see that the pair W_1, W_2 is a pair of SSSD walks with even length. Therefore, there exists a pair of SSSD walks from x to y with length $2n - 2$.

Therefore, $l(A) \leq 2n - 2$ by Lemma 2.6. $\qquad\square$

Lemma 3.3. *Let A be an $n \times n$ primitive nonpowerful ZS sign pattern matrix with zero diagonal. If $l(A) = 2n - 1$, then all 2-cycles in $S(A)$ are negative.*

Proof. Assume that $l(A) = 2n - 1$. By Lemma 3.1, $S(A)$ has at least one negative 2-cycle. It then follows from Lemma 3.2 that all 2-cycles in $S(A)$ are negative. $\qquad\square$

Lemma 3.4. *Let A be an $n \times n$ primitive nonpowerful ZS sign pattern matrix with zero diagonal. Suppose that $l(A) = 2n - 1$. Then there exists an odd cycle C in $S(A)$ and C is the only cycle of length at least 3 in $S(A)$.*

Proof. Since A is primitive, it follows from Lemma 2.4 that $S(A)$ is strongly connected and there is an odd cycle $C = (v_1, v_2, \ldots, v_l)$ in $S(A)$ with length $l(C) = l$. Since A has zero diagonal, there is no loop in $S(A)$ and so $l \geq 3$. Consider the following two directed cycles:

$$C^1 = (v_1, v_2, \ldots, v_l, v_1), \qquad C^2 = (v_1, v_l, \ldots, v_2, v_1). \tag{3.75}$$

Since there exists no positive 2-cycle in $S(A)$ by Lemma 3.3, the arcs (v_i, v_{i+1}) and (v_{i+1}, v_i) have different signs. Thus $\operatorname{sgn}(C^1) = -\operatorname{sgn}(C^2)$ by the fact that l is odd. Without loss of generality, we assume that C is an odd cycle with the least length in $S(A)$.

If $l(A) = 2n - 1$, suppose there exists another cycle C' with length $l(C') = l' \geq 3$, we consider the following two cases.

Case 1. C and C' have no common vertices.

Let P be the shortest path from C to C'. Suppose P intersects C at vertex u and intersects C' at vertex v and there are k vertices on P where $k \geq 2$. Let $G_4 = C \cup P \cup C'$. Let x and y be two arbitrary (not necessarily distinct) vertices in $S(A)$. Suppose that P_1 is the shortest path from x to G_4 and intersects G_4 at vertex x' and P_2 is the shortest path from y to G_4 and intersects G_4 at vertex y'. Then

$$0 \leq l(P_1), \qquad l(P_2) \leq n - l - l' - k + 2. \tag{3.76}$$

We consider the following six subcases.

Subcase 1.1. $x' \in C$ and $y' \in C$. See Figure 6(a).

Let $w = l(P_1) + l(Q_C(x' \rightarrow y')) + l(P_2)$. If w is odd, set

$$W' = P_1 + Q_C(x' \longrightarrow y') + P_2. \tag{3.77}$$

Otherwise, set

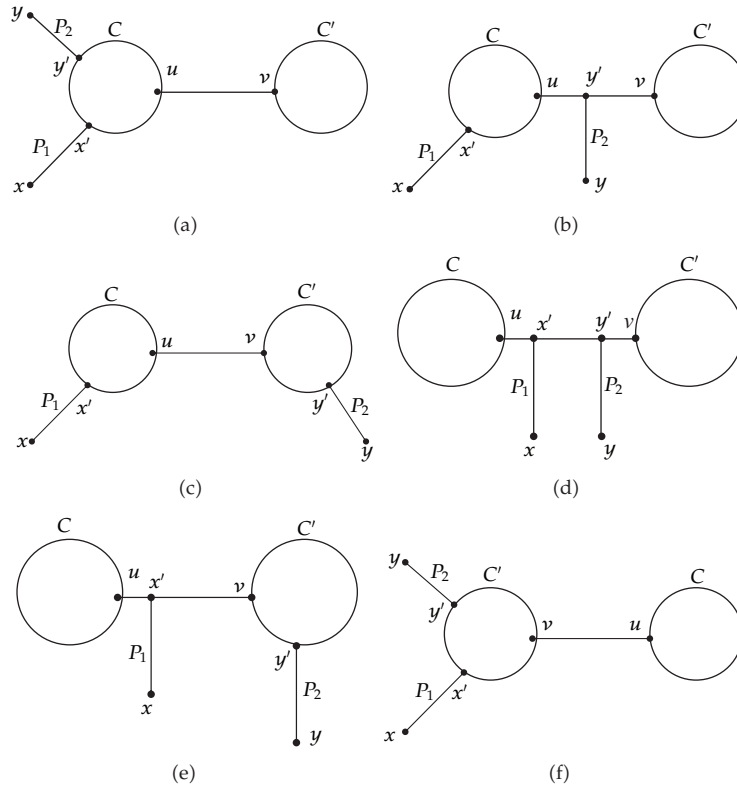$$W'' = P_1 + C \setminus Q_C(x' \longrightarrow y') + P_2. \tag{3.78}$$

Figure 6: Illustration for Lemma 3.4.

Let

$$W_1 = \begin{cases} W' + C^1, & w \text{ is odd,} \\ W'' + C^1, & \text{otherwise,} \end{cases} \qquad W_2 = \begin{cases} W' + C^2, & w \text{ is odd,} \\ W'' + C^2, & \text{otherwise.} \end{cases} \tag{3.79}$$

Since C is odd, $l(Q_C(x' \to y'))$ and $l(C \setminus Q_C(x' \to y'))$ have different parity. Therefore, both $l(W_1)$ and $l(W_2)$ are even. Then, if w is odd,

$$l(W_1) = l(W_2) \leq 2(n - l - l' - k + 2) + \frac{l-1}{2} + l = 2n - 2l' - 2k - \frac{l}{2} + \frac{7}{2}. \tag{3.80}$$

Otherwise,

$$l(W_1) = l(W_2) \leq 2(n - l - l' - k + 2) + l + l = 2n - 2l' - 2k + 4. \tag{3.81}$$

Since both $l(W_1)$ and $l(W_2)$ are even, it follows that $l(W_1) = l(W_2) \leq 2n - 2$. We see that the pair W_1, W_2 is a pair of SSSD walks with even length. Therefore, there exists a pair of SSSD walks from x to y with length $2n - 2$.

Subcase 1.2. $x' \in C$ and $y' \in P$. See Figure 6(b).

Let $W = l(P_1) + l(Q_C(x' \rightarrow u)) + l(Q_P(u \rightarrow y')) + l(P_2)$. If w is odd, set

$$W' = P_1 + Q_C(x' \longrightarrow u) + Q_P(u \longrightarrow y') + P_2. \tag{3.82}$$

Otherwise, set

$$W' = P_1 + C \setminus Q_C(x' \longrightarrow u) + Q_P(u \longrightarrow y') + P_2. \tag{3.83}$$

Let

$$W_1 = \begin{cases} W' + C^1, & w \text{ is odd}, \\ W'' + C^1, & \text{otherwise}, \end{cases} \qquad W_2 = \begin{cases} W' + C^2, & w \text{ is odd}, \\ W'' + C^2, & \text{otherwise}. \end{cases} \tag{3.84}$$

Since C is odd, $l(Q_C(x' \rightarrow u))$ and $l(C \setminus Q_C(x' \rightarrow u))$ have different parity. Therefore, both $l(W_1)$ and $l(W_2)$ are even. Then, if w is odd,

$$l(W_1) = l(W_2) \leq 2(n - l - l' - k + 2) + \frac{l - 1}{2} + (k - 1) + l = 2n - 2l' - k - \frac{l}{2} + \frac{5}{2}. \tag{3.85}$$

Otherwise,

$$l(W_1) = l(W_2) \leq 2(n - l - l' - k + 2) + (k - 1) + l + l = 2n - 2l' - k + 3. \tag{3.86}$$

Since both $l(W_1)$ and $l(W_2)$ are even, it follows that $l(W_1) = l(W_2) \leq 2n - 2$. We see that the pair W_1, W_2 is a pair of SSSD walks with even length. Therefore, there exists a pair of SSSD walks from x to y with length $2n - 2$.

Subcase 1.3. $x' \in C$ and $y' \in C'$. See Figure 6(c).
Let $w = l(P_1) + l(Q_C(x' \rightarrow u)) + l(P) + l(Q_{C'}(v \rightarrow y')) + l(P_2)$. If w is odd, set

$$W' = P_1 + Q_C(x' \longrightarrow u) + P + Q_{C'}(v \longrightarrow y') + P_2. \tag{3.87}$$

Otherwise, set

$$W'' = P_1 + C \setminus Q_C(x' \longrightarrow u) + P + Q_{C'}(v \longrightarrow y') + P_2. \tag{3.88}$$

Let

$$W_1 = \begin{cases} W' + C^1, & w \text{ is odd}, \\ W'' + C^1, & \text{otherwise}, \end{cases} \qquad W_2 = \begin{cases} W' + C^2, & w \text{ is odd}, \\ W'' + C^2, & \text{otherwise}. \end{cases} \tag{3.89}$$

Since C is odd, $l(Q_C(x' \to u))$ and $l(C \setminus Q_C(x' \to u))$ have different parity. Therefore, both $l(W_1)$ and $l(W_2)$ are even. Then, if w is odd,

$$l(W_1) = l(W_2) \le 2(n - l - l' - k + 2) + \frac{l-1}{2} + (k-1) + \frac{l'}{2} + l = 2n - k - \frac{l}{2} - \frac{3l'}{2} + \frac{5}{2}. \quad (3.90)$$

Otherwise,

$$l(W_1) = l(W_2) \le 2(n - l - l' - k + 2) + l + (k-1) + \frac{l'}{2} + l = 2n - k - \frac{3l'}{2} + 3. \quad (3.91)$$

Since both $l(W_1)$ and $l(W_2)$ are even, it follows that $l(W_1) = l(W_2) \le 2n - 2$. We see that the pair W_1, W_2 is a pair of SSSD walks with even length. Therefore, there exists a pair of SSSD walks from x to y with length $2n - 2$.

Subcase 1.4. $x' \in P$ and $y' \in P$. See Figure 6(d).
    Without loss of generality, we assume that $l(Q_P(x' \to u)) \le l(Q_P(y' \to u))$. Let $w = l(P_1) + l(Q_P(x' \to u)) + l(P) + l(Q_P(v \to y')) + l(P_2)$. If w is odd, set

$$W' = P_1 + Q_P(x' \longrightarrow u) + P + Q_P(v \longrightarrow y') + P_2. \quad (3.92)$$

Otherwise, set

$$W'' = P_1 + Q_P(x' \longrightarrow u) + P + Q_P(v \longrightarrow y') + P_2 + C. \quad (3.93)$$

Let

$$W_1 = \begin{cases} W' + C^1, & w \text{ is odd,} \\ W'' + C^1, & \text{otherwise,} \end{cases} \qquad W_2 = \begin{cases} W' + C^2, & w \text{ is odd,} \\ W'' + C^2, & \text{otherwise.} \end{cases} \quad (3.94)$$

Therefore, both $l(W_1)$ and $l(W_2)$ are even. Then, if w is odd,

$$l(W_1) = l(W_2) \le 2(n - l - l' - k + 2) + 2(k-1) + l = 2n - l - 2l' + 2. \quad (3.95)$$

Otherwise,

$$l(W_1) = l(W_2) \le 2(n - l - l' - k + 2) + 2(k-1) + l + l = 2n - 2l' + 2. \quad (3.96)$$

Since both $l(W_1)$ and $l(W_2)$ are even, it follows that $l(W_1) = l(W_2) \le 2n - 2$. We see that the pair W_1, W_2 is a pair of SSSD walks with even length. Therefore, there exists a pair of SSSD walks from x to y with length $2n - 2$.

Subcase 1.5. $x' \in P$ and $y' \in C'$. See Figure 6(e).
    Let $w = l(P_1) + l(Q_P(x' \to u)) + l(P) + l(Q_{C'}(v \to y')) + l(P_2)$, if w is odd, set

$$W' = P_1 + Q_P(x' \longrightarrow u) + P + Q_{C'}(v \longrightarrow y') + P_2. \quad (3.97)$$

Otherwise, set

$$W'' = P_1 + Q_P(x' \longrightarrow u) + P + Q_{C'}(v \longrightarrow y') + P_2 + C. \tag{3.98}$$

Let

$$W_1 = \begin{cases} W' + C^1, & w \text{ is odd,} \\ W'' + C^1, & \text{otherwise,} \end{cases} \qquad W_2 = \begin{cases} W' + C^2, & w \text{ is odd,} \\ W'' + C^2, & \text{otherwise.} \end{cases} \tag{3.99}$$

Since C is odd, both $l(W_1)$ and $l(W_2)$ are even. Then, if w is odd,

$$l(W_1) = l(W_2) \le 2(n - l - l' - k + 2) + 2(k - 1) + \frac{l'}{2} + l = 2n - l - \frac{3l'}{2} + 2. \tag{3.100}$$

Otherwise,

$$l(W_1) = l(W_2) \le 2(n - l - l' - k + 2) + 2(k - 1) + \frac{l'}{2} + l + l = 2n - \frac{3l'}{2} + 2. \tag{3.101}$$

Since both $l(W_1)$ and $l(W_2)$ are even, it follows that $l(W_1) = l(W_2) \le 2n - 2$. We see that the pair W_1, W_2 is a pair of SSSD walks with even length. Therefore, there exists a pair of SSSD walks from x to y with length $2n - 2$.

Subcase 1.6. $x' \in C'$ and $y' \in C'$. See Figure 6(f).
 Let $w = l(P_1) + l(Q_{C'}(x' \to v)) + l(Q_{C'}(v \to y')) + 2l(P) + l(P_2)$. If w is odd, set

$$W' = P_1 + Q_{C'}(x' \longrightarrow v) + 2P + Q_{C'}(v \longrightarrow y') + P_2. \tag{3.102}$$

Otherwise, set

$$W'' = P_1 + Q_{C'}(x' \longrightarrow v) + 2P + Q_{C'}(v \longrightarrow y') + P_2 + C. \tag{3.103}$$

Let

$$W_1 = \begin{cases} W' + C^1, & w \text{ is odd,} \\ W'' + C^1, & \text{otherwise,} \end{cases} \qquad W_2 = \begin{cases} W' + C^2, & w \text{ is odd,} \\ W'' + C^2, & \text{otherwise.} \end{cases} \tag{3.104}$$

Since C is odd, both $l(W_1)$ and $l(W_2)$ are even. Then if w is odd,

$$l(W_1) = l(W_2) \le 2(n - l - l' - k + 2) + 2(k - 1) + l' + l = 2n - l - l' + 2. \tag{3.105}$$

Otherwise,

$$l(W_1) = l(W_2) \le 2(n - l - l' - k + 2) + 2(k - 1) + l' + l + l = 2n - l' + 2. \tag{3.106}$$

Since both $l(W_1)$ and $l(W_2)$ are even, it follows that $l(W_1) = l(W_2) \leq 2n - 2$. We see that the pair W_1, W_2 is a pair of SSSD walks with even length. Therefore, there exists a pair of SSSD walks from x to y with length $2n - 2$.

Therefore, $l(A) \leq 2n - 2$ by Lemma 2.6, a contradiction.

Case 2. C and C' have at least one common vertex.

Let $G_5 = C \cup C'$. Let x and y be two arbitrary (not necessarily distinct) vertices in $S(A)$. Suppose that P_1 is the shortest path from x to G_5 and intersects G_5 at vertex x' and P_2 is the shortest path from y to G_5 and intersects G_5 at vertex y'. Denote $C \cap C'$ by R. Assume R has m vertices, where $1 \leq m \leq \min(l_1, l_2)$, then

$$0 \leq l(P_1), \qquad l(P_2) \leq n - l_1 - l_2 + m. \tag{3.107}$$

If $m > 1$, then R is a path with vertex set $V(R) = \{v_1, v_2, \ldots, v_m\}$ and edge set $E(R) = \{(v_1, v_2), (v_2, v_3), \ldots, (v_{m-1}, v_m)\}$. If l' is odd, $m = \min(l, l') = l$ by the minimality of C. If l' is even, suppose $\min(l, l') = l'$, then there exists an odd cycle C' with $l(C') < l$ which contradicts our assumption that C is an odd cycle with the least length in $S(A)$. Therefore, if $m = \min(l, l')$, then $m = l < l'$. We consider the following three subcases.

Subcase 2.1. $x' \in C$ and $y' \in C$.

Let $w = l(P_1) + l(Q_C(x' \to y')) + l(P_2)$. If w is odd, set

$$W' = P_1 + Q_C(x' \longrightarrow y') + P_2. \tag{3.108}$$

Otherwise, set

$$W'' = P_1 + C \setminus Q_C(x' \longrightarrow y') + P_2. \tag{3.109}$$

Let

$$W_1 = \begin{cases} W' + C^1, & w \text{ is odd,} \\ W'' + C^1, & \text{otherwise,} \end{cases} \qquad W_2 = \begin{cases} W' + C^2, & w \text{ is odd,} \\ W'' + C^2, & \text{otherwise.} \end{cases} \tag{3.110}$$

Since C is odd, $l(Q_C(x' \to y'))$ and $l(C \setminus Q_C(x' \to y'))$ have different parity. Therefore, both $l(W_1)$ and $l(W_2)$ are even. Then, if w is odd,

$$l(W_1) = l(W_2) \leq 2(n - l - l' + m) + \frac{l - 1}{2} + l = 2n - 2l' + 2m - \frac{l}{2} - \frac{1}{2}. \tag{3.111}$$

Otherwise,

$$l(W_1) = l(W_2) \leq 2(n - l - l' + m) + l + l = 2n - 2l' + 2m. \tag{3.112}$$

Since both $l(W_1)$ and $l(W_2)$ are even, it follows that $l(W_1) = l(W_2) \leq 2n - 2$. We see that the pair W_1, W_2 is a pair of SSSD walks with even length. Therefore, there exists a pair of SSSD walks from x to y with length $2n - 2$.

Subcase 2.2. $x' \in C \setminus R$ and $y' \in C' \setminus R$.

Without loss of generality, we assume that $l(Q_{C \setminus R}(x' \rightarrow v_1)) \leq l(Q_{C \setminus R}(x' \rightarrow v_m))$. Let $w = l(P_1) + l(Q_{C \setminus R}(x' \rightarrow v_1)) + l(Q_{C' \setminus R}(v_1 \rightarrow y')) + l(P_2)$. If w odd, set

$$W' = P_1 + Q_{C \setminus R}(x' \longrightarrow v_1) + Q_{C' \setminus R}(v_1 \longrightarrow y') + P_2. \tag{3.113}$$

Otherwise, set

$$W'' = P_1 + Q_{C \setminus R}(x' \longrightarrow v_m) + Q_R(v_m \longrightarrow v_1) + Q_{C' \setminus R}(v_1 \longrightarrow y') + P_2. \tag{3.114}$$

Let

$$W_1 = \begin{cases} W' + C^1, & w \text{ is odd,} \\ W'' + C^1, & \text{otherwise,} \end{cases} \qquad W_2 = \begin{cases} W' + C^2, & w \text{ is odd,} \\ W'' + C^2, & \text{otherwise.} \end{cases} \tag{3.115}$$

Since C is odd, $l(Q_{C \setminus R}(x' \rightarrow v_1))$ and $l(Q_{C \setminus R}(x' \rightarrow v_m)) + l(Q_R(v_m \rightarrow v_1))$ have different parity. Therefore, both $l(W_1)$ and $l(W_2)$ are even.

Then, if w is odd,

$$l(W_1) = l(W_2) \leq 2(n - l - l' + m) + (l - m + 1) + (l' - m + 1) + l = 2n - l' + 2. \tag{3.116}$$

Otherwise,

$$l(W_1) = l(W_2) \leq 2(n - l - l' + m) + (l - m) + (l' - 1) + l = 2n - l' + m - 1. \tag{3.117}$$

Since both $l(W_1)$ and $l(W_2)$ are even, it follows that $l(W_1) = l(W_2) \leq 2n - 2$. We see that the pair W_1, W_2 is a pair of SSSD walks with even length. Therefore, there exists a pair of SSSD walks from x to y with length $2n - 2$.

Subcase 2.3. $x' \in C' \setminus R$ and $y' \in C' \setminus R$.

The verification for this subcase is similar to that of the Subcase 2.2 and is omitted.

Therefore, $l(A) \leq 2n - 2$ by Lemma 2.6, a contradiction. □

Lemma 3.5. *Let A be an $n \times n$ primitive nonpowerful ZS sign pattern matrix with zero diagonal. If $l(A) = 2n - 1$, then $S(A)$ is isomorphic to G (see Figure 1).*

Proof. Let A be an $n \times n$ primitive nonpowerful ZS sign pattern matrix with zero diagonal. Since A is primitive, it follows from Lemma 2.4 that $S(A)$ is strongly connected and there is an odd cycle $C = (v_1, v_2, \ldots, v_l)$ in $S(A)$ with length $l(C) = l$. Since A has zero diagonal, there is no loop in $S(A)$ and so $3 \leq l \leq n$. Without loss of generality, we assume that C is an odd cycle with the least length in $S(A)$. If $l(A) = 2n-1$, then by Lemma 3.3, there exists no positive 2-cycle in $S(A)$. Then $S(A)$ is isomorphic to G by Lemma 2.7. We will give anther proof of the theorem.

Denote the vertex set of $S(A)$ by V and the vertex set of C by V'. By Lemmas 3.4 and 3.3, the cycle C is the only cycle of length at least 3 in $S(A)$ and there exists no positive 2-cycle

in $S(A)$. Consider the two directed cycles $C^1 = (v_1, v_2, \ldots, v_l, v_1)$ and $C^2 = (v_1, v_l, \ldots, v_2, v_1)$. Since there exists no positive 2-cycle in $S(A)$, the arcs (v_i, v_{i+1}) and (v_{i+1}, v_i) have different signs. Thus $\operatorname{sgn}(C^1) = -\operatorname{sgn}(C^2)$ by the fact that l is odd. Let x and y be two arbitrary (not necessarily distinct) vertices in $S(A)$. Suppose that P_1 is the shortest path from x to C and intersects C at vertex x' and P_2 is the shortest path from y to C and intersects C at vertex y'. If $l = n$, $S(A)$ is isomorphic to an odd cycle of length n. If $l < n$, it is enough to show that there exists a vertex $u \notin V'$ in $S(A)$ such that $l(P) = n - l$ where P is the shortest path from u to C and P intersects C at v. Suppose not, then

$$l(P_1), l(P_2) \leq n - l - 1. \tag{3.118}$$

Now we consider the following three cases.

Case 1. $x \in V \setminus V'$ and $y \in V \setminus V'$.
        Let $w = l(P_1) + l(Q_C(x' \rightarrow y')) + l(P_2))$. If w is odd, set

$$W'' = P_1 + Q_C(x' \longrightarrow y') + P_2. \tag{3.119}$$

Otherwise, set

$$W' = P_1 + C \setminus Q_C(x' \longrightarrow y') + P_2. \tag{3.120}$$

Let

$$W_1 = \begin{cases} W' + C^1, & w \text{ is odd,} \\ W'' + C^1, & \text{otherwise,} \end{cases} \qquad W_2 = \begin{cases} W' + C^2, & w \text{ is odd,} \\ W'' + C^2, & \text{otherwise.} \end{cases} \tag{3.121}$$

Since C is odd, $Q_C(x' \rightarrow y')$ and $C \setminus Q_C(x' \rightarrow y')$ have different parity. Therefore, both $l(W_1)$ and $l(W_2)$ are even. Then, if w is odd,

$$l(W_1) = l(W_2) \leq 2(n - l - 1) + \frac{l - 1}{2} + l = 2n - \frac{l}{2} - \frac{5}{2}. \tag{3.122}$$

Otherwise,

$$l(W_1) = l(W_2) \leq 2(n - l - 1) + l + l = 2n - 2. \tag{3.123}$$

        Since both $l(W_1)$ and $l(W_2)$ are even, it follows that $l(W_1) = l(W_2) \leq 2n - 2$. We see that the pair W_1, W_2 is a pair of SSSD walks with even length. Therefore, there exists a pair of SSSD walks from x to y with length $2n - 2$.

Case 2. $x \in V \setminus V'$ and $y \in V$.
        In this case, $y' = y$. Let $w = l(P_1) + l(Q_C(x' \rightarrow y))$. If w is odd, set

$$W'' = P_1 + Q_C(x' \longrightarrow y). \tag{3.124}$$

Otherwise, set

$$W' = P_1 + C \setminus Q_C(x' \longrightarrow y). \tag{3.125}$$

Let

$$W_1 = \begin{cases} W' + C^1, & w \text{ is odd,} \\ W'' + C^1, & \text{otherwise,} \end{cases} \qquad W_2 = \begin{cases} W' + C^2, & w \text{ is odd,} \\ W'' + C^2, & \text{otherwise.} \end{cases} \tag{3.126}$$

Since C is odd, $Q_C(x' \to y)$ and $C \setminus Q_C(x' \to y)$ have different parity. Therefore, both $l(W_1)$ and $l(W_2)$ are even. Then, if w is odd,

$$l(W_1) = l(W_2) \le (n - l - 1) + \frac{l-1}{2} + l = n + \frac{l}{2} - \frac{3}{2}. \tag{3.127}$$

Otherwise,

$$l(W_1) = l(W_2) \le (n - l - 1) + l + l = n + l - 1. \tag{3.128}$$

Since both $l(W_1)$ and $l(W_2)$ are even, it follows that $l(W_1) = l(W_2) \le 2n - 2$. We see that the pair W_1, W_2 is a pair of SSSD walks with even length. Therefore, there exists a pair of SSSD walks from x to y with length $2n - 2$.

Case 3. $x \in V$ and $y \in V$.
    In this case, $x' = x$ and $y' = y$. Let $w = Q_C(x \to y)$. If w is odd, set

$$W'' = Q_C(x \longrightarrow y). \tag{3.129}$$

Otherwise, set

$$W' = C \setminus Q_C(x \longrightarrow y). \tag{3.130}$$

Let

$$W_1 = \begin{cases} W' + C^1, & w \text{ is odd,} \\ W'' + C^1, & \text{otherwise,} \end{cases} \qquad W_2 = \begin{cases} W' + C^2, & w \text{ is odd,} \\ W'' + C^2, & \text{otherwise.} \end{cases} \tag{3.131}$$

Since C is odd, $Q_C(x \to y)$ and $C \setminus Q_C(x \to y)$ have different parity. Therefore, both $l(W_1)$ and $l(W_2)$ are even. Then, if w is odd,

$$l(W_1) = l(W_2) \le \frac{l-1}{2} + l = \frac{3l}{2} - \frac{1}{2}. \tag{3.132}$$

Otherwise,

$$l(W_1) = l(W_2) \le l + l = 2l. \tag{3.133}$$

Since both $l(W_1)$ and $l(W_2)$ are even, it follows that $l(W_1) = l(W_2) \leq 2n - 2$. We see that the pair W_1, W_2 is a pair of SSSD walks with even length. Therefore, there exists a pair of SSSD walks from x to y with length $2n - 2$.

From all the above cases, there exists a pair of SSSD walks from x to y with length $2n - 2$, therefore, $l(A) \leq 2n - 2$ by Lemma 2.6, a contradiction.   □

Proof of Theorem 1.6. We get $l(A) \leq 2n - 1$ from Theorem 2.2. Suppose that A is nonpowerful and skew symmetric and $S(A)$ is isomorphic to G. Then $l(A) = 2n - 1$ by Lemma 2.7.

Conversely, suppose that $l(A) = 2n - 1$. We need to prove that A is nonpowerful and skew symmetric and $S(A)$ is isomorphic to G. In [1], Li et al. showed that if an irreducible sign pattern matrix A is powerful, then $l(A) = l(|A|)$. Therefore, if A is powerful, then $l(A) \leq 2n - 4$ by Theorem 2.1, which contradicts $l(A) = 2n - 1$. Hence A is nonpowerful. Consequently, A is skew symmetric by Lemma 3.3 and $S(A)$ is isomorphic to G by Lemma 3.5.   □

Acknowledgments

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