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Research Article

Interval Continuous Plant Identification from Value Sets

R. Hernández,[1] J. A. García,[2] and C. Mañoso[1]

[1] *Departamento de Sistemas de Comunicación y Control, UNED, c/Juan del Rosal 16, 28040 Madrid, Spain*
[2] *Departamento de Tecnología de Computadores y Comunicaciones, Universidad de Extremadura, 06800 Madrid, Spain*

Correspondence should be addressed to R. Hernández, roberto@scc.uned.es

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This paper shows how to obtain the values of the numerator and denominator Kharitonov polynomials of an interval plant from its value set at a given frequency. Moreover, it is proven that given a value set, all the assigned polynomials of the vertices can be determined if and only if there is a complete edge or a complete arc lying on a quadrant. This algorithm is nonconservative in the sense that if the value-set boundary of an interval plant is exactly known, and particularly its vertices, then the Kharitonov rectangles are exactly those used to obtain these value sets.

1. Introduction

In reference to the identification problem, these have been widely motivated and analysed over recent years [1]. Van Overschee and De Moor in [2] explains a subspace identification algorithm. In [3] the authors present a robust identification procedure for a priori classes of models in H_∞; the authors consider casual, linear time invariant, stable, both continuous or discrete time models, and only SISO systems.

Interval plants have been widely motivated and analysed over recent years. For further engineering motivation, among the numerous papers and books, [4–9] must be pointed out and the references thereof.

The identification problem using the interval plant framework, that is, to compute an interval plant from the frequency response, has not been completely solved. Interval plant identification was investigated by Bhattacharyya et al. [5], who developed a method in which identification is carried out for interval plants so that the numerator and denominator have the same degree, starting from the variation of the coefficient values of a nominal transfer function at certain intervals. So, the identification of a nominal transfer function is carried out first, and then the intervals of variation of the coefficients are determined.

A different approach was developed by Hernández et al. [10] studying the problem from the extreme point results point of view. This was a first step for the identification of an interval plant, showing three main properties to characterize the value set lying on a quadrant. Then an algorithm for the identification of interval plants from the vertices of the value sets is obtained. However, this algorithm solves the identification problem when the value set contains at least five vertices in a quadrant.

This paper improves the results obtained in [10] and shows how to obtain the values of the numerator and denominator Kharitonov polynomials when the value sets have less than five vertices in the same quadrant. Identification with such an interval plant allows engineers predict the worst case performance and stability margins using the results on interval systems, particularly extreme point results.

2. Problem Statement

Let us consider a linear interval plant of real coefficients, of the form

$$P(s,a,b) = \frac{N_p(s,a)}{D_P(s,b)}, \tag{2.1}$$

where $N_p(s,a)$ and $D_P(s,b)$ are interval polynomials given as

$$\begin{aligned} N_p(s,a) &= a_m s^m + a_{m-1}s^{m-1} + \cdots + a_0, \quad a \in A = \{a : a_i^- \le a_i \le a_i^+,\ i = 0,\dots,m\}, \\ D_P(s,b) &= b_n s^n + b_{n-1}s^{n-1} + \cdots + b_0, \quad b \in B = \{b : b_i^- \le b_i \le b_i^+,\ i = 0,\dots,n\}, \end{aligned} \tag{2.2}$$

with $m \ge 1$, $n \ge 1$, $0 \notin D_p(s,b)$, and where vectors $a = [a_0, a_1, \dots, a_m]$, $a_m \ne 0$, and $b = [b_0, b_1, \dots, b_n]$, $b_n \ne 0$ are the uncertainty parameters that lie in the hyperrectangles A and B, respectively.

Numerator and denominator polynomial families are characterized by their respective Kharitonov polynomials, and they can be expressed in terms of their even and odd parts, at $s = j\omega$, as follows:

Family $N_p(s)$:

$$\begin{aligned} k_{n1} &= p_{e\min}(j\omega) + jp_{o\min}(j\omega), \qquad k_{n2} = p_{e\max}(j\omega) + jp_{o\min}(j\omega), \\ k_{n3} &= p_{e\max}(j\omega) + jp_{o\max}(j\omega), \qquad k_{n4} = p_{e\min}(j\omega) + jp_{o\max}(j\omega), \end{aligned} \tag{2.3}$$

where

$$\begin{aligned} p_{e\min}(j\omega) &= a_0^- - a_2^+\omega^2 + a_4^-\omega^4 - a_6^+\omega^6 + \cdots, \qquad p_{e\max}(j\omega) = a_0^+ - a_2^-\omega^2 + a_4^+\omega^4 - a_6^-\omega^6 + \cdots, \\ p_{o\min}(j\omega) &= a_1^-\omega - a_3^+\omega^3 + a_5^-\omega^5 - a_7^+\omega^7 + \cdots, \qquad p_{o\max}(j\omega) = a_1^+\omega - a_3^-\omega^3 + a_5^+\omega^5 - a_7^-\omega^7 + \cdots. \end{aligned} \tag{2.4}$$

Family $D_p(s)$:

$$k_{d1} = q_{e\min}(j\omega) + jq_{o\min}(j\omega), \qquad k_{d2} = q_{e\max}(j\omega) + jq_{o\min}(j\omega),$$
$$k_{d3} = q_{e\max}(j\omega) + jq_{o\max}(j\omega), \qquad k_{d4} = q_{e\min}(j\omega) + jq_{o\max}(j\omega), \tag{2.5}$$

where

$$q_{e\min}(j\omega) = b_0^- - b_2^+\omega^2 + b_4^-\omega^4 - b_6^+\omega^6 + \cdots, \qquad q_{e\max}(j\omega) = b_0^+ - b_2^-\omega^2 + b_4^+\omega^4 - b_6^-\omega^6 + \cdots,$$
$$q_{o\min}(j\omega) = b_1^-\omega - b_3^+\omega^3 + b_5^-\omega^5 - b_7^+\omega^7 + \cdots, \qquad q_{o\max}(j\omega) = b_1^+\omega - b_3^-\omega^3 + b_5^+\omega^5 - b_7^-\omega^7 + \cdots. \tag{2.6}$$

As is well known, the values $G(j\omega)$ of the complex plane obtained for the transfer function $G(s)$ at a given frequency are denominated as a *value set*. The identification of the system consists in determining the transfer function coefficients from the value set.

As can be observed in [10], when the values $\{k_{n1}(j\omega), k_{n2}(j\omega), k_{n3}(j\omega), k_{n4}(j\omega)\}$ and $\{k_{d1}(j\omega), k_{d2}(j\omega), k_{d3}(j\omega), k_{d4}(j\omega)\}$ are known, then the system of equations given in [10, equation 14] can be solved and therefore the interval plant is identified (see [10] for details).

As is shown [10] the vertices of the value-set boundary of an interval plant can be assigned as

$$v_i = \frac{n_j}{d_k}, \tag{2.7}$$

where n_j, $j = 1,2,3,4$ and d_k, $k = 1,2,3,4$ are the assigned polynomials numerator and denominator, respectively. When they are in the same quadrant they are a *Sorted Set of Vertices (SSV)*.

As is well known, the Kharitonov polynomials values can be obtained from

$$\begin{aligned}
k_{n1}(j\omega) &= \min[\mathrm{Re}(n_1, n_3)] + j\min[\mathrm{Im}(n_1, n_3)],\\
k_{n2}(j\omega) &= \max[\mathrm{Re}(n_1, n_3)] + j\min[\mathrm{Im}(n_1, n_3)],\\
k_{n3}(j\omega) &= \max[\mathrm{Re}(n_1, n_3)] + j\max[\mathrm{Im}(n_1, n_3)],\\
k_{n4}(j\omega) &= \min[\mathrm{Re}(n_1, n_3)] + j\max[\mathrm{Im}(n_1, n_3)],\\
k_{d1}(j\omega) &= \min[\mathrm{Re}(d_1, d_3)] + j\min[\mathrm{Im}(d_1, d_3)],\\
k_{d2}(j\omega) &= \max[\mathrm{Re}(d_1, d_3)] + j\min[\mathrm{Im}(d_1, d_3)],\\
k_{d3}(j\omega) &= \max[\mathrm{Re}(d_1, d_3)] + j\max[\mathrm{Im}(d_1, d_3)],\\
k_{d4}(j\omega) &= \min[\mathrm{Re}(d_1, d_3)] + j\max[\mathrm{Im}(d_1, d_3)].
\end{aligned} \tag{2.8}$$

It must be pointed out that the results presented in [10] must be considered as the background necessary for this work. Thus, the geometry of the value set is described in [10] and the concepts necessary for its description are defined, (such as the successor, predecessor element, etc.) and the fundamental properties on which this work is based are proven.

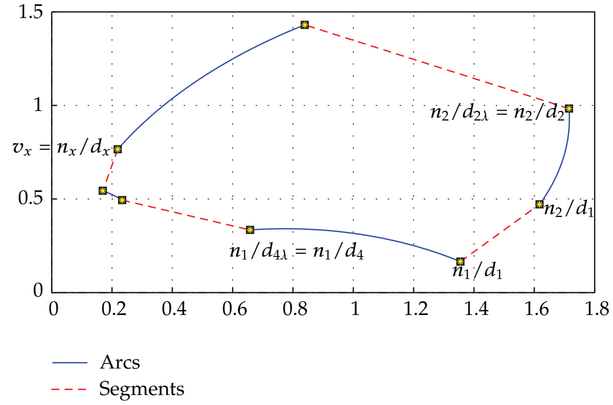


Figure 1: egment and complete arcs.

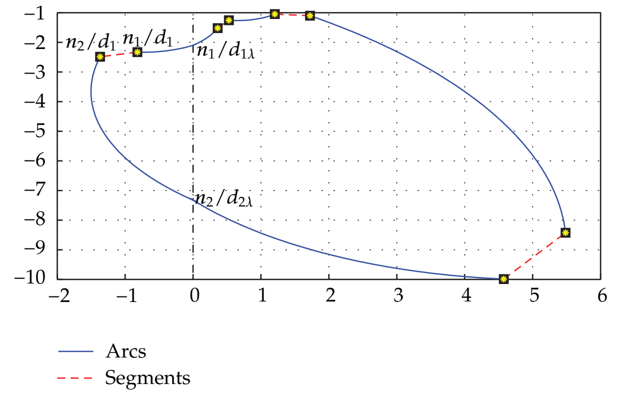


Figure 2: Segment and no complete arcs.

This paper is organized as follows. Section 3 shows how to determine the assigned polynomial with the only condition that there is a complete segment in a quadrant. Similarly Section 4 shows it when there is an arc in a quadrant. Section 5 illustrates the algorithm and examples. Finally, the conclusions are shown in Section 6.

3. Assigned Polynomial Determination When There Is a Complete Segment in a Quadrant

In order to determine the polynomials numerator and denominator associated to a vertex of the value set boundary with the minimum number of elements, the situation of a segment in a quadrant will be considered. So, let S_1 be a segment of the value-set boundary with vertices $v_1 = n_1/d_1$ and $v_2 = n_2/d_1$. Continuity segment-arc in a quadrant (see [10, Theorem 2]) implies that there will be a successor arc with vertices $v_2 = n_2/d_1$, $v_{2\,\text{succ}} = n_2/d_{2\lambda}$ counter-clockwise and a predecessor arc with vertices $v_{1\,\text{pred}} = n_1/d_{4\lambda}$ counter-clockwise. When these arcs are completed the denominators are vertices of the Kharitonov rectangle. Figures 1 and 2 show this situation.

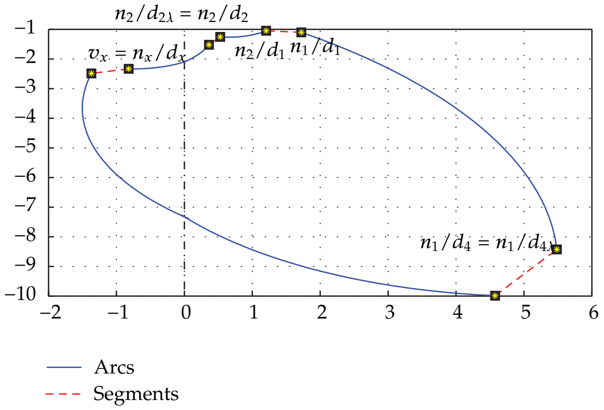


Figure 3: v_x vertex of two elements, arc-segment.

As was shown, the values of n_1, n_2, and d_1 can be calculated from the complete segment based on a normalization (see [10, Theorem 4]). The following normalization simplifies the nomenclature.

Lemma 3.1 (segment normalization). *Let S_1 be a complete segment of the value-set boundary with vertices $v_1 = n_1/d_1$ and $v_2 = n_2/d_1$ and the normalization $d_1 = \cos(\varphi(d_1)) + j\sin(\varphi(d_1))$, where $\varphi(d_1) = 360° - \arg(v_2 - v_1)$ $\arg(v_2 - v_1)$ being the argument of the segment $v_2 - v_1$. Then $n_1 = v_1 d_1$, $n_2 = v_2 d_1$, $d_{2\lambda} = n_2/v_{2\,\text{succ}}$, and $d_{4\lambda} = n_1/v_{1\,\text{pred}}$, where $v_{2\,\text{succ}}$ ($v_{1\,\text{pred}}$) is any point of the next (previous) arc of the segment S_1.*

Proof. It is trivial. This normalization is one of the infinite possible solutions [10] for a value set. This normalization implies fitting d_1 with modulus $|d_1| = 1$ and angle so that the segment of the Kharitonov polynomial numerator with vertices n_1 and n_2 will be parallel to the real axis counter-clockwise. Thus, from the information with a complete segment in a quadrant the values of d_1, n_1, n_2, $d_{2\lambda}$, and $d_{4\lambda}$ can be calculated. □

This paper deals with the general case where $n_{2R} \neq 0$, $n_{2I} \neq 0$, $n_{1R} \neq 0$, and $n_{1I} \neq 0$.

Given a vertex $v_x = n_x/d_x$ in a quadrant, the target is to determine the polynomials n_x and d_x. The vertex v_x belongs to a part of a segment and a part of an arc, due to the continuity segment-arc in a quadrant. So, v_x will be the vertex of two elements, arc-segment (Figure 3) or segment-arc (Figure 4).

The following Lemma shows the necessary conditions on the denominator d_x to be a solution of $v_x = n_x/d_x$.

Lemma 3.2 (denominator condition). *Let S_1 be a complete segment in a quadrant and let d_x be the denominator of a vertex $v_x = n_x/d_x$ in a quadrant. Then it is a necessary condition that d_x satisfies one of the following conditions:*

(1) $(d_{1R} < d_{2\lambda R}$ *and* $d_{1I} < d_{4\lambda I})$ *and* $\{(d_{xR} = d_{1R}$ *and* $d_{xI} = d_{1I})$ $[d_x = d_1]$ *or* $(d_{xR} = d_{1R}$ *and* $d_{xI} \geq d_{1I})$ $[d_x = d_4]$ *or* $(d_{xI} = d_{1I}$ *and* $d_{xR} \geq d_{1R})$ $[d_x = d_2]$ *or* $(d_{xR} > d_{1R}$ *and* $d_{xI} > d_{1I})$ $[d_x = d_3]\}$,

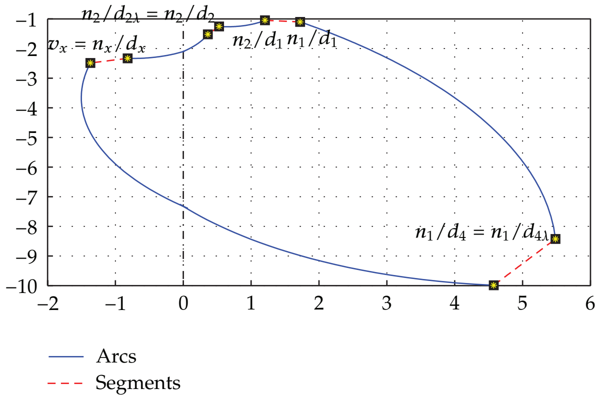


Figure 4: v_x vertex of two elements, segment-arc.

(2) $(d_{1R} > d_{4\lambda R}$ *and* $d_{1I} < d_{2\lambda I})$ *and* $\{(d_{xR} = d_{1R}$ *and* $d_{xI} = d_{1I})$ $[d_x = d_1]$ *or* $(d_{xR} = d_{1R}$ *and* $d_{xI} \geq d_{1I})$ $[d_x = d_2]$ *or* $(d_{xI} = d_{1I}$ *and* $d_{xR} \leq d_{1R})$ $[d_x = d_4]$ *or* $(d_{xR} < d_{1R}$ *and* $d_{xI} > d_{1I})$ $[d_x = d_3]\}$,

(3) $(d_{1R} > d_{2\lambda R}$ *and* $d_{1I} > d_{4\lambda I})$ *and* $\{(d_{xR} = d_{1R}$ *and* $d_{xI} = d_{1I})$ $[d_x = d_1]$ *or* $(d_{xR} = d_{1R}$ *and* $d_{xI} \leq d_{1I})$ $[d_x = d_4]$ *or* $(d_{xI} = d_{1I}$ *and* $d_{xR} \leq d_{1R})$ $[d_x = d_2]$ *or* $(d_{xR} < d_{1R}$ *and* $d_{xI} < d_{1I})$ $[d_x = d_3]\}$,

(4) $(d_{1R} < d_{4\lambda R}$ *and* $d_{1I} > d_{2\lambda I})$ *and* $\{(d_{xR} = d_{1R}$ *and* $d_{xI} = d_{1I})$ $[d_x = d_1]$ *or* $(d_{xR} = d_{1R}$ *and* $d_{xI} \leq d_{1I})$ $[d_x = d_2]$ *or* $(d_{xI} = d_{1I}$ *and* $d_{xR} \geq d_{1R})$ $[d_x = d_4]$ *or* $(d_{xR} > d_{1R}$ *and* $d_{xI} < d_{1I})$ $[d_x = d_3]\}$,

where d_{iR} *is the real part of* d_i *and* d_{iI} *is the imaginary part of* d_i*, and the corresponding assigned denominator is shown between brackets.*

Proof. The proof is obtained directly from the information of a complete segment in a quadrant and the properties of the Kharitonov rectangle. So, from the complete segment and the normalization (Lemma 3.1), the values of d_1, $d_{2\lambda}$, and $d_{4\lambda}$ are known. Then, d_1 can be established as k_{d1}, k_{d2}, k_{d3}, or k_{d4}.

(1) If $(d_{1R} < d_{2\lambda R}$ and $d_{1I} < d_{4\lambda I})$ then d_1 is k_{d1}. Given a value d_x, it will be a vertex of the Kharitonov rectangle denominator only if $d_{xR} = d_{1R}$ and $d_{xI} = d_{1I}$ (d_x is $d_1 = k_{d1}$) or $d_{xR} = d_{1R}$ and $d_{xI} > d_{1I}$ (d_x is $d_4 = k_{d4}$) or $d_{xI} = d_{1I}$ and $d_{xR} > d_{1R}$ (d_x is $d_2 = k_{d2}$) or $d_{xR} > d_{1R}$ and $d_{xI} > d_{1I}$ (d_x is $d_3 = k_{d3}$). (Figures 5(a), 5(b), 5(c), and 5(d)).

Note that if any of these conditions is not satisfied, then d_x cannot be a solution. For example, if $d_{xR} = d_{1R}$ and $d_{xI} < d_{1I}$, d_x does not belong to the rectangle with vertex d_1, $d_{2\lambda}$, and $d_{4\lambda}$ are elements of the successor and predecessor edges. Figure 6 shows these considerations.

(2) Similarly, if $(d_{1R} > d_{4\lambda R}$ and $d_{1I} < d_{2\lambda I})$ then d_1 is k_{d2}. Given a value d_x, it will be a vertex of the Kharitonov rectangle denominator only if $d_{xR} = d_{1R}$ and $d_{xI} = d_{1I}$ (d_x is $d_1 = k_{d2}$) or $d_{xR} = d_{1R}$ and $d_{xI} > d_{1I}$ (d_x is $d_2 = k_{d3}$) or $d_{xI} = d_{1I}$ and $d_{xR} < d_{1R}$ (d_x is $d_4 = k_{d1}$) or $d_{xR} < d_{1R}$ and $d_{xI} > d_{1I}$ (d_x is $d_3 = k_{d4}$).

(3) If $d_{1R} > d_{2\lambda R}$ and $d_{1I} > d_{4\lambda I}$ then d_1 is k_{d3}. Given a value d_x, it will be a vertex of the Kharitonov rectangle denominator only if $d_{xR} = d_{1R}$ and $d_{xI} = d_{1I}$ (d_x is

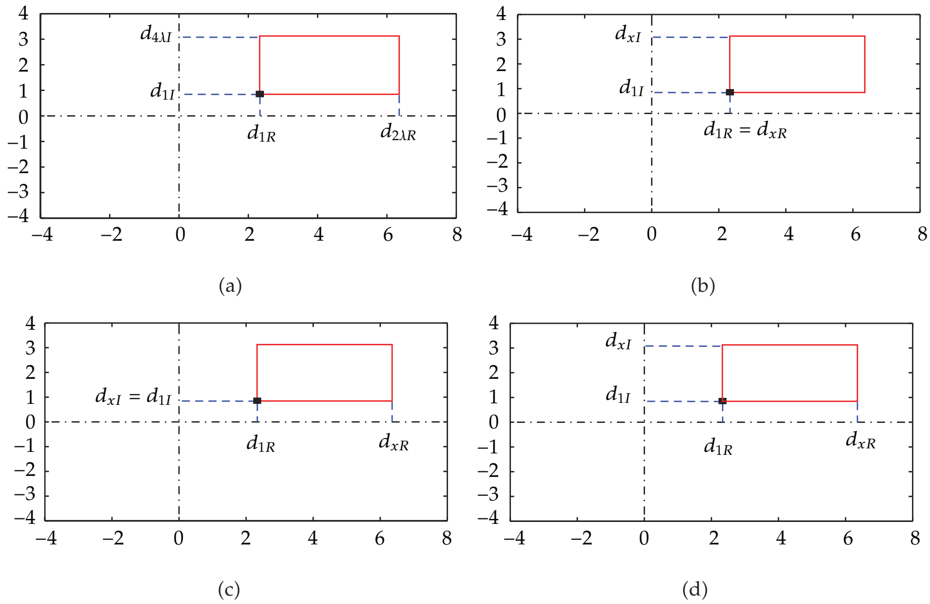


Figure 5: Cases where d_x is a vertex of the kharitonov rectangle denominator.

$d_1 = k_{d3}$) or $d_{xR} = d_{1R}$ and $d_{xI} < d_{1I}$ (d_x is $d_4 = k_{d2}$) or $d_{xI} = d_{1I}$ and $d_{xR} < d_{1R}$ (d_x is $d_2 = k_{d4}$) or $d_{xR} < d_{1R}$ and $d_{xI} < d_{1I}$ (d_x is $d_3 = k_{d1}$).

(4) Finally, if $d_{1R} < d_{4\lambda R}$ and $d_{1I} > d_{2\lambda I}$ then d_1 is k_{d4}. Given a value d_x, it will be a vertex of the Kharitonov rectangle denominator only if $d_{xR} = d_{1R}$ and $d_{xI} = d_{1I}$ (d_x is $d_1 = k_{d4}$) or $d_{xR} = d_{1R}$ and $d_{xI} < d_{1I}$ (d_x is $d_2 = k_{d1}$) or $d_{xI} = d_{1I}$ and $d_{xR} > d_{1R}$ (d_x is $d_4 = k_{d3}$) or $d_{xR} > d_{1R}$ and $d_{xI} < d_{1I}$ (d_x is $d_3 = k_{d2}$). □

On the other hand, the behaviour of a segment on the complex plane when divided by a complex number is well known. The following property shows this behaviour.

Property 1. Let $S_x = S/d_x$ be a segment on the complex plane with vertices v_{x1} and v_{x2} counter-clockwise where S is a segment with vertices n_a and n_b counter-clockwise. Let d_x be a complex number with argument $\arg(d_x)$. Let $\varphi(S_x)$ be $\varphi(S_x) \equiv \arg(v_{x2} - v_{x1})$. Then the relation between the argument of d_x and $\varphi(S_x)$, is given by

(1) $\arg(d_x) = -\varphi(S_x)$ if and only if $\arg(n_b - n_a) = 0°$,

(2) $\arg(d_x) = 90° - \varphi(S_x)$ if and only if $\arg(n_b - n_a) = 90°$,

(3) $\arg(d_x) = 180° - \varphi(S_x)$ if and only if $\arg(n_b - n_a) = 180°$,

(4) $\arg(d_x) = 270° - \varphi(S_x)$ if and only if $\arg(n_b - n_a) = 270°$.

The following Theorem shows how to characterize and calculate the polynomials n_x and d_x associated with a vertex $v_x = n_x/d_x$ from the information of the boundary with a segment S_x in a quadrant, $v_x = n_x/d_x$ belonging to a segment-arc.

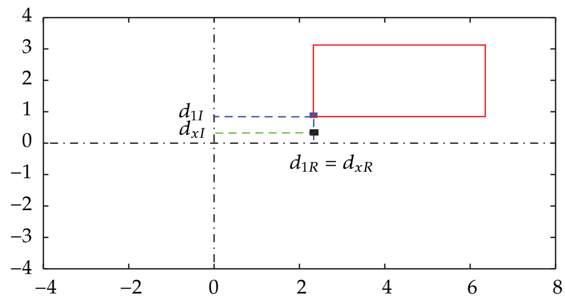


Figure 6: Cases where d_x is not a vertex of the kharitonov rectangle denominator.

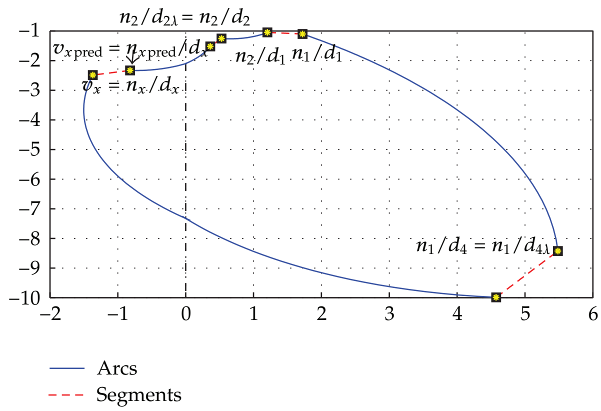


Figure 7: Vertices for the conditions of the Theorem 3.3.

Theorem 3.3 (predecessor). *Let S_1 be a complete segment of the value-set boundary with vertices $v_1 = n_1/d_1$ and $v_2 = n_2/d_1$, the successor arc with vertices $v_2 = n_2/d_1$, $v_{2\,\mathrm{succ}} = n_2/d_{2\lambda}$ counter-clockwise, and the predecessor arc with vertices $v_{1\,\mathrm{pred}} = n_1/d_{4\lambda}$, $v_1 = n_1/d_1$ counter-clockwise. Let S_x be a segment with vertices $v_{x\,\mathrm{pred}} = n_{x\,\mathrm{pred}}/d_x$ and $v_x = n_x/d_x$ counter-clockwise, where v_x belongs to the intersection of S_x and an arc of the boundary (Figure 7). Then*

(1) $\arg(v_x/v_2) = \arg(d_1) + \varphi(S_x)$ *(condition C1) and the denominator d_x of v_x defined by n_2/v_x satisfies the denominator condition (Lemma 3.2), if and only if $n_x = n_2$ and cannot be any other assigned polynomial,*

(2) *when $n_x \neq n_2$, $\arg(v_x/v_1) = \arg(d_1) + \varphi(S_x) + 90°$ (condition C2) and the denominator d_x of v_x defined by n_1/v_x satisfies the denominator condition (Lemma 3.2) if and only if $n_x = n_1$ and cannot be any other assigned polynomial,*

(3) *when $n_x \neq n_1$ and $n_x \neq n_2$, $\tan(\arg(v_x) - \varphi(S_x) + 90°)n_{2R} > n_{2I}$ (condition C3), and the denominator d_x of v_x defined by $n_{2R}[1 + j\tan(\arg(v_x) - \varphi(S_x) + 90°)]/v_x$ satisfies the denominator condition (Lemma 3.2) if and only if $n_x = n_3 = n_{2R}[1 + j\tan(\arg(v_x) - \varphi(S_x) + 90°)]$ and cannot be any other assigned polynomial,*

(4) *when $n_x \neq n_1$, $n_x \neq n_2$, and $n_x \neq n_3$, $\tan(\arg(v_x) - \varphi(S_x) + 180°)n_{1R} > n_{1I}$ (condition C4), and the denominator d_x of v_x defined by $n_{1R}[1 + j\tan(\arg(v_x) - \varphi(S_x) + 180°)]/v_x$ satisfies the denominator condition (Lemma 3.2) if and only if $n_x = n_4 = n_{1R}[1 + j\tan(\arg(v_x) - \varphi(S_x) + 180°)]$.*

Proof. From the complete segment S_1 using the normalization (Lemma 3.1) the values of d_1, $n_1 = v_1 d_1$, $n_2 = v_2 d_1$, $d_{2\lambda} = n_2/v_{2\text{succ}}$, and $d_{4\lambda} = n_1/v_{1\text{pred}}$ are known. Obviously the value v_x is known.

(1) $\Leftarrow$ If $n_x = n_2$ the value of $d_x = n_2/v_x$ can be calculated and the denominator condition (Lemma 3.2) is satisfied. On the other hand, the quotient of the vertices $v_x = n_2/d_x$ and $v_2 = n_2/d_1$ is $v_x/v_2 = d_1/d_x$, and $\arg(v_x/v_2) = \arg(d_1) - \arg(d_x)$. $S_x = S_2/d_x$, where S_2 is part of the segment with vertices n_1 and n_2, then $\arg(n_2 - n_1) = 0°$ (normalization). Thus $\arg(d_x) = -\varphi(S_x)$ (Property 1) and $\arg(v_x/v_2) = \arg(d_1) + \varphi(S_x)$; Theorem 3.3(C1) is satisfied.

$\Rightarrow$ In order to demonstrate the "only if" part, it must be proven that if Theorem 3.3(C1) and the denominator condition are satisfied then the solution $d_x = n_2/v_x$, $n_x = n_2$ is unique. It must be noted that Theorem 3.3(C1) can be satisfied when (a) $n_x = n_3$, (b) $n_x = n_4$ or (c) $n_x = n_1$ and in all the cases, the value of d_x determined, verify the denominator condition.

Let d_x be the denominator of v_x determined by n_2/v_x, verifying Theorem 3.3(C1), and denominator condition, and let $S_x = S_2/d_x$ where S_2 is part of the segment with vertices n_1 and n_2, $\arg(n_2 - n_1) = 0°$.

(a) Let d_x^* be the denominator of v_x determined by n_3/v_x. Then $S_x = S_3/d_x^*$ where S_3 is part of the segment with vertices n_2 and n_3, $\arg(n_3 - n_2) = 90°$ (normalization) and using Property 1 $\arg(d_x^*) = 90° - \varphi(S_x)$. As v_x is the same vertex, then $\arg(n_3/d_x^*) = \arg(n_2/d_x)$, and $\arg(n_3) = \arg(n_2) + 90°$. $n_x = n_3$ verify Theorem 3.3(C1), because

$$\arg\left(\frac{v_x}{v_2}\right) = \arg(n_2) + 90° - \arg(n_2) + \arg(d_1) - 90° + \varphi(S_x) = \arg(d_1) + \varphi(S_x). \tag{3.1}$$

Let $\alpha = \arg(n_2)$ with $\tan(\alpha) = n_{2I}/n_{2R}$. Then $\arg(n_3) = \alpha + 90°$ and $\tan(\alpha + 90°) = n_{3I}/n_{3R} = n_{3I}/n_{2R}$ (by normalization $n_{3R} = n_{2R}$). Thus $n_3 = n_{2R} + jn_{3I} = n_{2R} + j\tan(\alpha + 90°)n_{2R} = n_{2R} - j(n_{2R}^2/n_{2I})$. Moreover $\arg(d_x^*) = 90° + \arg(d_x)$, and if $d_x = d_{xR} + jd_{xI}$ then $d_x^* = \rho e^{j(\pi/2)} d_x = -\rho d_{xI} + j\rho d_{xR}$. As $v_x = n_2/d_x$ and $v_x = n_3/d_x^*$, then $n_2 d_x^* = n_3 d_x$ and they have equal real and imaginary parts.

$\text{Re}[n_2 d_x^*] = \text{Re}[n_3 d_x]$ then

$$\begin{gathered} -\rho d_{xI} n_{2R} - \rho d_{xR} n_{2I} = n_{2R} d_{xR} + \frac{n_{2R}^2}{n_{2I}} d_{xI}, \\ -\rho d_{xI} n_{2R} n_{2I} - \rho d_{xR} n_{2I}^2 = n_{2R} n_{2I} d_{xR} + n_{2R}^2 d_{xI}, \\ -(\rho n_{2I} + n_{2R}) d_{xR} n_{2I} = (n_{2R} + \rho n_{2I}) d_{xI} n_{2R}. \end{gathered} \tag{3.2}$$

Thus $d_{xI}/d_{xR} = -n_{2I}/n_{2R}$

$\text{Im}[n_2 d_x^*] = \text{Im}[n_3 d_x]$ then

$$\begin{gathered} \rho d_{xR} n_{2R} - \rho d_{xI} n_{2I} = d_{xI} n_{2R} - d_{xR} \frac{n_{2R}^2}{n_{2I}}, \\ \rho d_{xR} n_{2R} n_{2I} - \rho d_{xI} n_{2I}^2 = d_{xI} n_{2R} n_{2I} - d_{xR} n_{2R}^2, \\ (n_{2I}\rho + n_{2R}) n_{2R} d_{xR} = d_{xI} n_{2I} (n_{2R} + \rho n_{2I}). \end{gathered} \tag{3.3}$$

Thus $d_{xI}/d_{xR} = n_{2R}/n_{2I}$.

Taking into account both conditions, $n_{2R}/n_{2I} = -n_{2I}/n_{2R} \Leftrightarrow n_{2R}^2 < 0$. This relation is impossible. Therefore, if d_x is a solution then d_x^* is not, and $n_x = n_3$ is not a solution.

(b) Let d_x^* be the denominator of v_x determined by n_4/v_x. Then $S_x = S_4/d_x^*$ where S_4 is part of the segment with vertices n_3 and n_4, $\arg(n_4 - n_3) = 180°$ (normalization) and using Property 1 $\arg(d_x^*) = 180° - \varphi(S_x)$. As v_x is the same vertex, then $\arg(n_4/d_x^*) = \arg(n_2/d_x)$ and $\arg(n_4) = \arg(n_2) + 180°$. $n_x = n_4$ verify Theorem 3.3(C1), because

$$\arg\left(\frac{v_x}{v_2}\right) = \arg(n_2) + 180° - \arg(n_2) + \arg(d_1) - 180° + \varphi(S_x) \quad = \arg(d_1) + \varphi(S_x). \tag{3.4}$$

In this case the demonstration is trivial noting that $\arg(d_x^*) = 180° + \arg(d_x)$. This is not possible because the Kharitonov polynomial denominator cannot contain the zero.

(c) Let d_x^* be the denominator of v_x determined by n_1/v_x. Then $S_x = S_1/d_x^*$ where S_1 is part of the segment with vertices n_4 and n_1, $\arg(n_1 - n_4) = 270°$ (normalization) and using Property 1 $\arg(d_x^*) = 270° - \varphi(S_x)$. As v_x is the same vertex, then $\arg(n_1/d_x^*) = \arg(n_2/d_x)$, and $\arg(n_1) = \arg(n_2) + 270°$. $n_x = n_1$ verify Theorem 3.3(C1), because

$$\arg\left(\frac{v_x}{v_2}\right) = \arg(n_2) + 270° - \arg(n_2) + \arg(d_1) - 270° + \varphi(S_x) = \arg(d_1) + \varphi(S_x). \tag{3.5}$$

Let $\alpha = \arg(n_2)$ with $\tan(\alpha) = n_{2I}/n_{2R}$. Then $\arg(n_1) = \alpha + 270°$ and $\tan(\alpha + 270°) = n_{1I}/n_{1R} = n_{2I}/n_{1R}$ (by normalization $n_{3R} = n_{2R}$). Thus $n_1 = n_{1R} + jn_{2I} = (n_{2I}/\tan(\alpha + 270°)) + jn_{2I} = -(n_{2I}^2/n_{2R}) + jn_{2I}$. Moreover $\arg(d_x^*) = 270° + \arg(d_x)$, and if $d_x = d_{xR} + jd_{xI}$ then $d_x^* = \rho e^{j3(\pi/2)} d_x = \rho d_{xI} - j\rho d_{xR}$. As $v_x = n_2/d_x$ and $v_x = n_1/d_x^*$, then $n_2 d_x^* = n_1 d_x$ and they have equals real and imaginary parts.

$\mathrm{Re}[n_2 d_x^*] = \mathrm{Re}[n_1 d_x]$ then

$$\begin{gathered} +\rho d_{xI} n_{2R} + \rho d_{xR} n_{2I} = -\frac{n_{2I}^2}{n_{2R}} d_{xR} - n_{2I} d_{xI}, \\ (n_{2I} + n_{2R}\rho) d_{xI} n_{2R} = -(n_{2R}\rho + n_{2I}) d_{xR} n_{2I}. \end{gathered} \tag{3.6}$$

Thus $d_{xI}/d_{xR} = -n_{2I}/n_{2R}$.

$\mathrm{Im}[n_2 d_x^*] = \mathrm{Im}[n_1 d_x]$ then

$$\begin{gathered} -\rho d_{xR} n_{2R} + \rho d_{xI} n_{2I} = -d_{xI} \frac{n_{2I}^2}{n_{2R}} + d_{xR} n_{2I}, \\ -\rho d_{xR} n_{2R} n_{2R} + \rho d_{xI} n_{2I} n_{2R} = -d_{xI} n_{2I}^2 + d_{xR} n_{2I} n_{2R}, \\ (n_{2I} + \rho n_{2R}) d_{xI} n_{2I} = d_{xR} n_{2R} (\rho n_{2R} + n_{2I}). \end{gathered} \tag{3.7}$$

Thus $d_{xI}/d_{xR} = n_{2R}/n_{2I}$.

Taking into account both conditions, $n_{2R}/n_{2I} = -n_{2I}/n_{2R}$. This relation is impossible. Therefore, if d_x is a solution, d_x^* is not and $n_x = n_1$ cannot be a solution.

(2) $\Leftarrow$ If $n_x = n_1$ the value of $d_x = n_1/v_x$ can be calculated and the denominator condition (Lemma 3.2) is satisfied. On the other hand, the quotient of the vertices $v_x = n_1/d_x$ and

$v_2 = n_1/d_1$ is $v_x/v_1 = d_1/d_x$, and $\arg(v_x/v_1) = \arg(d_1) - \arg(d_x)$. $S_x = S_1/d_x$ where S_1 is part of the segment with vertices n_4 and n_1, then $\arg(n_1 - n_4) = 270°$ (normalization). Thus $\arg(d_x) = 270° - \varphi(S_x)$ (Property 1) and $\arg(v_x/v_1) = \arg(d_1) + \varphi(S_x) + 90°$; Theorem 3.3(C2) is satisfied.

⇒ In order to demonstrate the "only if" part, it must be proven that if Theorem 3.3(C2) and the denominator condition are satisfied then the solution $d_x = n_1/v_x$, $n_x = n_1$ is unique. It must be noted that Theorem 3.3(C2) can be satisfied when (a) $n_x = n_3$ or (b) $n_x = n_4$ and in all the cases, the value of d_x determined, verify the denominator condition.

Let d_x be the denominator of v_x determined by n_1/v_x, verifying Theorem 3.3(C2), and denominator condition, and let $S_x = S_1/d_x$ where S_1 is part of the segment with vertices n_4 and n_1, $\arg(n_2 - n_1) = 270°$.

(a) Let d_x^* be the denominator of v_x determined by n_3/v_x. Then $S_x = S_3/d_x^*$ where S_3 is part of the segment with vertices n_2 and n_3, $\arg(n_3 - n_2) = 90°$ (normalization) and using Property 1 $\arg(d_x^*) = 90° - \varphi(S_x)$. As v_x is the same vertex, then $\arg(n_3/d_x^*) = \arg(n_1/d_x)$ and $\arg(n_3) = \arg(n_1) + 180°$. $n_x = n_3$ verify Theorem 3.3(C2), because

$$\arg\left(\frac{v_x}{v_1}\right) = \arg(n_1) + 180° - \arg(n_1) + \arg(d_1) - 90° + \varphi(S_x) = \arg(d_1) + \varphi(S_x) + 90°. \quad (3.8)$$

In this case the demonstration is trivial noting that $\arg(d_x^*) = -180° + \arg(d_x)$. This is not possible because the Kharitonov polynomial denominator cannot contain the zero.

(b) Let d_x^* be the denominator determined by n_4/v_x. Then $S_x = S_4/d_x^*$ where S_4 is part of the segment with vertices n_3 and n_4, $\arg(n_4 - n_3) = 180°$ (normalization) and using Property 1 $\arg(d_x^*) = 180° - \varphi(S_x)$. As v_x is the same vertex, then $\arg(n_4/d_x^*) = \arg(n_1/d_x)$, and $\arg(n_4) = \arg(n_1) + 270°$. $n_x = n_4$ verify Theorem 3.3(C2), because

$$\arg\left(\frac{v_x}{v_1}\right) = \arg(n_1) + 270° - \arg(n_1) + \arg(d_1) - 180° + \varphi(S_x) = \arg(d_1) + \varphi(S_x) + 90°. \quad (3.9)$$

Let $\alpha = \arg(n_1)$ with $\tan(\alpha) = n_{1I}/n_{1R}$. Then $\arg(n_1) = \alpha + 270°$ and $\tan(\alpha + 270°) = n_{4I}/n_{4R} = -n_{1R}/n_{1I}$ (by normalization $n_{1R} = n_{4R}$). Thus $n_4 = n_{4R} + jn_{4I} = n_{1R} + jn_{2R}\tan(\alpha + 270°) = n_{1R} - j(n_{1R}^2/n_{1I})$. Moreover $\arg(d_x^*) = -90° + \arg(d_x)$, and if $d_x = d_{xR} + jd_{xI}$ then $d_x^* = \rho e^{j3(\pi/2)} d_x = \rho d_{xI} - j\rho d_{xR}$. How $v_x = n_1/d_x$ and $v_x = n_4/d_x^*$, then $n_1 d_x^* = n_4 d_x$ and they have equals real and imaginary parts.

$\mathrm{Re}[n_1 d_x^*] = \mathrm{Re}[n_4 d_x]$

$$\begin{aligned}
\rho d_{xI} n_{1R} + \rho d_{xR} n_{1I} &= +\frac{n_{1R}^2}{n_{1I}} d_{xI} + n_{1R} d_{xR}, \\
\rho d_{xI} n_{1R} n_{1I} + \rho d_{xR} n_{1I} n_{1I} &= +n_{1R}^2 d_{xI} + n_{1R} d_{xR} n_{1I}, \\
(\rho n_{1I} - n_{1R}) d_{xI} n_{1R} &= (n_{1R} - \rho n_{1I}) d_{xR} n_{1I}.
\end{aligned} \quad (3.10)$$

Thus $d_{xI}/d_{xR} = -n_{1I}/n_{1R}$.

$\text{Im}[n_1 d_x^*] = \text{Im}[n_4 d_x]$

$$-\rho d_{xR} n_{1R} + \rho d_{xI} n_{1I} = -d_{xR}\frac{n_{1R}^2}{n_{1I}} + d_{xI} n_{1R},$$
$$-\rho d_{xR} n_{1R} n_{1I} + \rho d_{xI} n_{1I} n_{1I} = -d_{xR} n_{1R}^2 + d_{xI} n_{1R} n_{1I},$$
$$(-\rho n_{1I} + n_{1R}) d_{xR} n_{1R} = (n_{1R} - \rho n_{1I}) d_{xI} n_{1I}. \tag{3.11}$$

and finally $d_{xI}/d_{xR} = n_{1R}/n_{1I}$.

Taking into account both conditions, $-n_{1I}/n_{1R} = n_{1R}/n_{1I}$. This relation is impossible. Therefore, if d_x is a solution, d_x^* is not and $n_x = n_4$ is not a solution.

(3) $\Leftarrow$ If $n_x = n_3$ then $d_x = n_3/v_x$ cannot be directly calculated because n_3 is not known. First, Theorem 3.3(C3) is developed. If $n_x = n_3$ then $S_x = S_3/d_x$ where S_3 is part of the segment with vertices n_2 and n_3 and $\arg(n_3 - n_2) = 90°$. Thus $\arg(d_x) = 90° - \varphi(S_x)$ (Property 1) and $\arg(n_3) = \arg(v_x) + \arg(d_x) = \arg(v_x) + 90° - \varphi(S_x)$.

As $n_{2R} = n_{3R}$, then $n_3 = n_{3R} + jn_{3I} = n_{2R} + jn_{2R}\tan(\arg(v_x) + 90° - \varphi(S_x))$. On the other hand, n_{3I} is greater than n_{2I} because it is counter-clockwise. Therefore $\tan(\arg(v_x) - \varphi(S_x) + 90°) n_{2R} > n_{2I}$ (Theorem 3.3(C3)) is satisfied and d_x can be calculated by the expression $d_x = n_3/v_x = n_{2R}[1 + j\tan(\arg(v_x) - \varphi(S_x) + 90°)]/v_x$.

$\Rightarrow$ In order to demonstrate the "only if" part, it must be proven that if Theorem 3.3(C3) and the denominator condition are satisfied then the solution $d_x = n_3/v_x$, $n_x = n_3$ is unique. If $n_x \neq n_2$ and $n_x \neq n_1$, it must be noted that Theorem 3.3(C3) can be satisfied when $n_x = n_4$.

Let d_x be the denominator of v_x determined by n_3/v_x verifying Theorem 3.3(C3) and denominator condition. $S_x = S_3/d_x^*$ where S_3 is part of the segment with vertices n_2 and n_3, $\arg(n_3 - n_2) = 90°$.

Let d_x^* be the denominator of v_x determined by n_4/v_x. Then $S_x = S_4/d_x^*$ where S_4 is part of the segment with vertices n_3 and n_4, $\arg(n_4 - n_3) = 180°$ (normalization) and using Property 1 $\arg(d_x^*) = 180° - \varphi(S_x) = \arg(d_x) + 90°$. Thus $d_x^* = \rho e^{j(\pi/2)} d_x = -\rho d_{xI} + j\rho d_{xR}$.

As v_x is the same vertex, $\arg(n_4/d_x^*) = \arg(n_3/d_x)$, and then $\arg(n_4) = \arg(n_3) + 90°$. Let $\alpha = \arg(n_3)$, then $\alpha + 90° = \arg(n_4) = \arg(v_x) + \arg(d_x^*) = \arg(v_x =) + 180° - \varphi(S_x)$, and because $\arg(n_3)$ verifies $n_3 = n_{2R}\tan(\alpha) > n_{2I}$ (by normalization), Theorem 3.3(C3) is satisfied.

$n_3 = n_{2R} + j\tan(\alpha) n_{2R} = n_{2R} + jn_{2R}(n_{3R}/n_{3I})$. If $n_x = n_4$ then $n_4 = n_{1R} + j\tan(\alpha + 90°) n_{1R} = n_{1R} - jn_{1R}(n_{3R}/n_{3I})$. As $v_x = n_3/d_x$ and $v_x = n_4/d_x^*$, then $n_2 d_x^* = n_3 d_x$ and they have equal real and imaginary parts.

$\text{Re}[n_3 d_x^*] = \text{Re}[n_4 d_x]$

$$-n_{2R}\rho d_{xI} - n_{3I}\rho d_{xR} = n_{1R} d_{xR} + d_{xI} n_{1R}\frac{n_{2R}}{n_{3I}},$$
$$-n_{2R} n_{3I}\rho d_{xI} - n_{3I} n_{3I}\rho d_{xR} = n_{1R} n_{3I} d_{xR} + d_{xI} n_{1R} n_{3R},$$
$$-(n_{3I}\rho + n_{1R}) n_{2R} d_{xI} = (n_{1R} + n_{3I}\rho) n_{3I} d_{xR}, \tag{3.12}$$

and finally $d_{xI}/d_{xR} = -n_{3I}/n_{3R}$.

$\mathrm{Im}[n_3 d_x^*] = \mathrm{Im}[n_4 d_x]$

$$
\begin{aligned}
-n_{3I}\rho d_{xI} + n_{2R}\rho d_{xR} &= d_{xI} n_{1R} - d_{xR} n_{1R} \frac{n_{2R}}{n_{3I}}, \\
-n_{3I} n_{3I}\rho d_{xI} + n_{3I} n_{2R}\rho d_{xR} &= d_{xI} n_{1R} n_{3I} - d_{xR} n_{1R} n_{2R}, \\
-(n_{3I}\rho + n_{1R}) d_{xI} n_{3I} &= -(n_{3I}\rho + n_{1R}) d_{xR} n_{2R}
\end{aligned}
\tag{3.13}
$$

and finally $d_{xI}/d_{xR} = n_{3R}/n_{3I}$.

Taking into account both conditions, $-n_{3I}/n_{3R} = n_{3R}/n_{3I}$. This relation is impossible. Therefore, if d_x is a solution, d_x^* is not, and $n_x = n_3$ is not a solution.

(4) $\Leftarrow$ If $n_x = n_4$ then $d_x = n_4/v_x$ cannot be directly calculated because n_4 is not known. First, Theorem 3.3(C4) is developed.

If $n_x = n_4$ then $S_x = S_4/d_x$ where S_4 is part of the segment with vertices n_3 and n_4 verifying that $\arg(n_4 - n_3) = 180°$. Thus $\arg(d_x) = 180° - \varphi(S_x)$ (Property 1) and $\arg(n_4) = \arg(v_x) + \arg(d_x) = \arg(v_x) + 180° - \varphi(S_x)$. Moreover, $n_{1R} = n_{4R}$. Then $n_4 = n_{4R} + jn_{4I} = n_{1R} + jn_{1R}\tan(\arg(v_x) + 180° - \varphi(S_x))$. On the other hand, n_{4I} is greater than n_{1I} because it is counter-clockwise.

Therefore the condition $\tan(\arg(v_x) - \varphi(S_x) + 180°)n_{1R} > n_{1I}$ Theorem 3.3(C4) is satisfied and d_x can be calculated using the expression $d_x = n_4/v_x = n_{1R}[1 + j\tan(\arg(v_x) - \varphi(S_x) + 180°)]/v_x$.

$\Rightarrow$ If $n_x \neq n_2$, $n_x \neq n_1$ and $n_x \neq n_3$ it is $n_x = n_4$. □

Remark 3.4. This theorem is used in the example of Section 5, for the value set III (frequency $w = 1.2$) in order to assign the second and fifth vertices.

The following Theorem is analogous to Theorem 3.3 when S_x is a segment with vertices $v_x = n_x/d_x$ and $v_{x\mathrm{succ}} = n_{x\mathrm{succ}}/d_x$ counter-clockwise, and belonging to an arc-segment.

Theorem 3.5 (successor). *Let S_1 be a complete segment of the value-set boundary with vertices $v_1 = n_1/d_1$ and $v_2 = n_2/d_1$, the successor arc to S_1, with vertices $v_2 = n_2/d_1$, $v_{2\,\mathrm{succ}} = n_2/d_{2\lambda}$ counter-clockwise, and the predecessor arc to S_1 with vertices $v_{1\,\mathrm{pred}} = n_1/d_{4\lambda}$, $v_1 = n_1/d_1$ counter-clockwise. Let S_x be a boundary segment with vertices $v_x = n_x/d_x$ and $v_{x\,\mathrm{succ}} = n_{x\,\mathrm{succ}}/d_x$ counter-clockwise, where v_x belongs to the intersection of an arc of the boundary and S_x. Then*

(1) $\arg(v_x/v_2) = \arg(d_1) + \varphi(S_x) - 90°$ *(condition C1) and the denominator d_x of v_x defined by n_2/v_x satisfies the denominator condition (Lemma 3.2), if and only if $n_x = n_2$ and cannot be any other assigned polynomial,*

(2) *when $n_x \neq n_2$,* $\arg(v_x/v_1) = \arg(d_1) + \varphi(S_x)$ *(condition C2) and the denominator d_x of v_x of defined by n_1/v_x satisfies the denominator condition (Lemma 3.2) if and only if $n_x = n_1$ and cannot be any other assigned polynomial,*

(3) *when $n_x \neq n_1$ and $n_x \neq n_2$,* $\tan(\arg(v_x) - \varphi(S_x) + 180°)n_{2R} > n_{2I}$ *(condition C3), and the denominator d_x of v_x defined by* $n_{2R}[1 + j\tan(\arg(v_x) - \varphi(S_x) + 180°)]/v_x$ *satisfies the denominator condition (Lemma 3.2) if and only if* $n_x = n_3 = n_{2R}[1 + j\tan(\arg(v_x) - \varphi(S_x) + 180°)]$ *and cannot be any other assigned polynomial,*

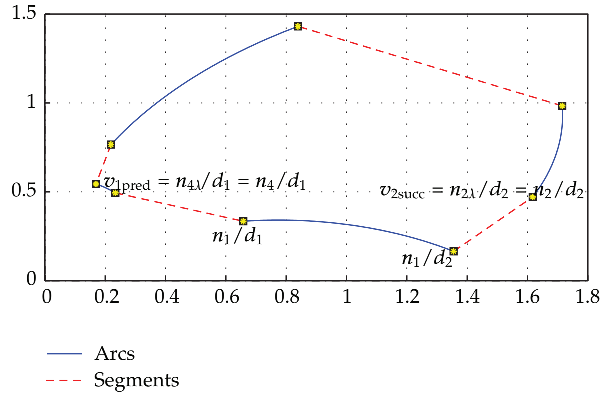


Figure 8: Arc and two complete segments.

(4) *when* $n_x \neq n_1$, $n_x \neq n_2$, *and* $n_x \neq n_3$, $\tan(\arg(v_x) - \varphi(S_x) + 270°)n_{1R} > n_{1I}$ *(condition C4), and the denominator* d_x *of* v_x *defined by* $n_{1R}[1 + j\tan(\arg(v_x) - \varphi(S_x) + 270°]/v_x$ *satisfies the denominator condition (Lemma 3.2) if and only if* $n_x = n_4 = n_{1R}[1 + j\tan(\arg(v_x) - \varphi(S_x) + 270°)]$.

Proof. Analogous to Theorem 3.3. □

Remark 3.6. This theorem is used in the example of Section 5, for the value set III (frequency $w = 1.2$) in order to assign the third, fifth, and sixth vertices.

4. Assigned Polynomial Determination When There Is a Complete Arc in a Quadrant

In order to determine the polynomials numerator and denominator associated to a vertex of the value set boundary with the minimum number of elements, the situation of an arc in a quadrant will be considered. So, let A_1 be an arc of the value-set boundary with vertices $v_1 = n_1/d_1$ and $v_2 = n_1/d_2$. A continuity arc-segment in a quadrant (see [10, Theorem 2]) implies that there will be a successor segment with vertices $v_2 = n_1/d_2$, $v_{2\text{succ}} = n_{2\lambda}/d_2$ counter-clockwise and a predecessor segment with vertices $v_1 = n_1/d_1$ and $v_{1\text{pred}} = n_{4\lambda}/d_1$ counter-clockwise.

When these segments are completed the denominators are vertices of the Kharitonov rectangle. Figure 8 shows this situation.

As was shown, the values of d_1, d_2, and n_1 can be calculated from the complete arc based on a normalization (see [10, Theorem 5]). The following normalization simplifies the nomenclature.

Lemma 4.1 (arc normalization). *Let* A_1 *be a complete arc of the value-set boundary with vertices* $v_1 = n_1/d_1$ *and* $v_2 = n_1/d_2$, *the normalization* $n_1 = \cos(\varphi(n_1)) + j\sin(\varphi(n_1))$, *where* $\varphi(n_1) = 360° - \arg(1/v_2 - 1/v_1)$, $\arg(1/v_2 - 1/v_1)$ *being the argument of the segment* $1/v_2 - 1/v_1$. *Then* $d_1 = n_1/v_1$, $d_2 = n_1/v_2$, $n_{4\lambda} = d_1 v_{1\text{ pred}}$, *and* $n_{2\lambda} = d_2 v_{2\text{ succ}}$, *where* $v_{2\text{ succ}}$ $(v_{1\text{ pred}})$ *is any point of the next (previous) segment of the arc* A_1.

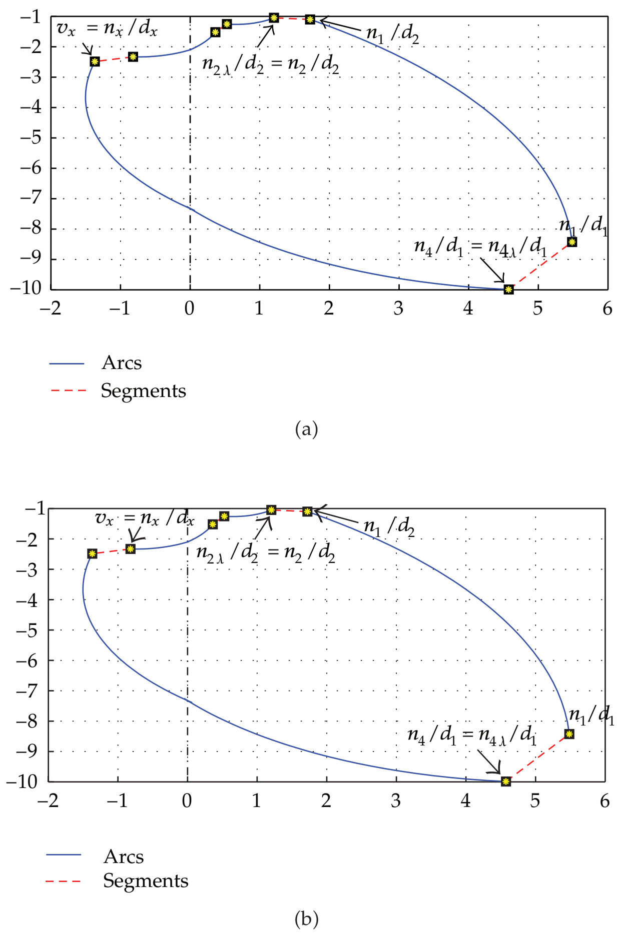


Figure 9: (a) v_x vertex of two elements, segment-arc. (b) v_x vertex of two elements, arc-segment.

Proof. It is trivial. This normalization is one of the infinite possible solutions for a value set. This normalization implies fitting n_1 with modulus $|n_1| = 1$ and angle so that the segment of the Kharitonov polynomial denominator with vertices d_1 and d_2 will be parallel to the real axis counter-clockwise. Thus, from the information with a complete arc in a quadrant the values of d_1, d_2, n_1, $n_{2\lambda}$, and $n_{4\lambda}$ can be calculated. □

This paper deals with the general case where $d_{2R} \neq 0$, $d_{2I} \neq 0$, $d_{1R} \neq 0$, and $d_{1I} \neq 0$.

Given a vertex $v_x = n_x / d_x$ in a quadrant, the target is to determine the polynomials n_x and d_x. The vertex v_x belongs to a part of an arc and a part of a segment, due to the continuity arc-segment in a quadrant. So, v_x will be the vertex of two elements, segment-arc (Figure 9(a)) or arc-segment (Figure 9(b)).

The following Lemma shows the necessary conditions on the denominator d_x to be a solution of $v_x = n_x / d_x$.

Lemma 4.2 (numerator condition). *Let A_1 be a complete arc in a quadrant and let n_x be the numerator of a vertex $v_x = n_x/d_x$ in a quadrant. Then it is a necessary condition that n_x satisfies one of the following conditions:*

(1) $(n_{1R} < n_{2\lambda R}$ and $n_{1I} < n_{4\lambda I})$ *and* $\{(n_{xR} = n_{1R}$ and $n_{xI} = n_{1I})\,[n_x = n_1]$ *or* $(n_{xR} = n_{1R}$ and $n_{xI} \geq n_{1I})\,[n_x = n_4]$ *or* $(n_{xI} = n_{1I}$ and $n_{xR} \geq n_{1R})\,[n_x = n_2]$ *or* $(n_{xR} > n_{1R}$ and $n_{xI} > n_{1I})\,[n_x = n_3]\}$,

(2) $(n_{1R} > n_{4\lambda R}$ and $n_{1I} < n_{2\lambda I})$ *and* $\{(n_{xR} = n_{1R}$ and $n_{xI} = n_{1I})\,[n_x = n_1]$ *or* $(n_{xR} = n_{1R}$ and $n_{xI} \geq n_{1I})\,[n_x = n_2]$ *or* $(n_{xI} = n_{1I}$ and $n_{xR} \leq n_{1R})\,[n_x = n_4]$ *or* $(n_{xR} < n_{1R}$ and $n_{xI} > n_{1I})\,[n_x = n_3]\}$,

(3) $(n_{1R} > n_{2\lambda R}$ and $n_{1I} > n_{4\lambda I})$ *and* $\{(n_{xR} = n_{1R}$ and $n_{xI} = n_{1I})\,[n_x = n_1]$ *or* $(n_{xR} = n_{1R}$ and $n_{xI} \leq n_{1I})\,[n_x = n_4]$ *or* $(n_{xI} = n_{1I}$ and $n_{xR} \leq n_{1R})\,[n_x = n_2]$ *or* $(n_{xR} < n_{1R}$ and $n_{xI} < n_{1I})\,[n_x = n_3]\}$,

(4) $(n_{1R} < n_{4\lambda R}$ and $n_{1I} > n_{2\lambda I})$ *and* $\{(n_{xR} = n_{1R}$ and $n_{xI} = n_{1I})\,[n_x = n_1]$ *or* $(n_{xR} = n_{1R}$ and $n_{xI} \leq n_{1I})\,[n_x = n_2]$ *or* $(n_{xI} = n_{1I}$ and $n_{xR} \geq n_{1R})\,[n_x = n_4]$ *or* $(n_{xR} > n_{1R}$ and $n_{xI} < n_{1I})\,[n_x = n_3]\}$,

where n_{iR} is the real part of n_i and n_{iI} is the imaginary part of n_i, and the corresponding assigned numerator is shown between brackets.

Proof. The proof is obtained directly from the information of a complete arc in a quadrant and the properties of the Kharitonov rectangle. So, from the complete arc and the normalization (Lemma 3.2), the values of n_1, $n_{2\lambda}$, and $n_{4\lambda}$ are known. Then, n_1 can be established as k_{n1}, k_{n2}, k_{n3}, or k_{d4}.

(1) If $(n_{1R} < n_{2\lambda R}$ and $n_{1I} < n_{4\lambda I})$ then n_1 is k_{n1}. Given a value n_x, it will be a vertex of the Kharitonov rectangle numerator only if $n_{xR} = n_{1R}$ and $n_{xI} = n_{1I}$ (n_x is $n_1 = k_{n1}$) or $n_{xR} = n_{1R}$ and $n_{xI} > n_{1I}$ (n_x is $n_4 = k_{n4}$) or $n_{xI} = n_{1I}$ and $n_{xR} > n_{1R}$ (n_x is $n_2 = k_{n2}$) or $n_{xR} > n_{1R}$ and $n_{xI} > n_{1I}$ (n_x is $n_3 = k_{n3}$). Note that if any of these conditions is not satisfied, then n_x cannot be a solution. For example, if $n_{xR} = n_{1R}$ and $n_{xI} < n_{1I}$, n_x does not belong to the rectangle with vertex n_1, $n_{2\lambda}$, and $n_{4\lambda}$ are elements of the successor and predecessor edge.

(2) Similarly, if $(n_{1R} > n_{4\lambda R}$ and $n_{1I} < n_{2\lambda I})$ then n_1 is k_{n2}. Given a value n_x, it will be a vertex of the Kharitonov rectangle numerator only if $n_{xR} = n_{1R}$ and $n_{xI} = n_{1I}$ (n_x is $n_1 = k_{n2}$) or $n_{xR} = n_{1R}$ and $n_{xI} > n_{1I}$ (n_x is $n_2 = k_{n3}$) or $n_{xI} = n_{1I}$ and $n_{xR} < n_{1R}$ (n_x is $n_4 = k_{n1}$) or $n_{xR} < n_{1R}$ and $n_{xI} > n_{1I}$ (n_x is $n_3 = k_{n4}$).

(3) If $n_{1R} > n_{2\lambda R}$ and $n_{1I} > n_{4\lambda I}$ then n_1 is k_{n3}. Given a value n_x, it will be a vertex of the Kharitonov rectangle numerator only if $n_{xR} = n_{1R}$ and $n_{xI} = n_{1I}$ (n_x is $n_1 = k_{n3}$) or $n_{xR} = n_{1R}$ and $n_{xI} < n_{1I}$ (n_x is $n_4 = k_{n2}$) or $n_{xI} = n_{1I}$ and $n_{xR} < n_{1R}$ (n_x is $n_2 = k_{n4}$) or $n_{xR} < n_{1R}$ and $n_{xI} < n_{1I}$ (n_x is $n_3 = k_{n1}$).

(4) Finally, if $n_{1R} < n_{4\lambda R}$ and $n_{1I} > n_{2\lambda I}$ then n_1 is k_{n4}. Given a value n_x, it will be a vertex of the Kharitonov rectangle numerator only if $n_{xR} = n_{1R}$ and $n_{xI} = n_{1I}$ (n_x is $n_1 = k_{n4}$) or $n_{xR} = n_{1R}$ and $n_{xI} < n_{1I}$ (n_x is $n_2 = k_{n1}$) or $n_{xI} = n_{1I}$ and $n_{xR} > n_{1R}$ (n_x is $n_4 = k_{n3}$) or $n_{xR} > n_{1R}$ and $n_{xI} < n_{1I}$ (n_x is $n_3 = k_{n2}$). □

On the other hand, the behaviour of an arc on the complex plane when it is divided by a complex number is well known. The following property shows this behaviour.

Property 2. Let $A_x = n_x/S$ be an arc on the complex plane with vertices v_{x1} and v_{x2} counter-clockwise where S is a segment with vertices d_a and d_b counter-clockwise. Let n_x be a complex number with argument $\arg(n_x)$. Let $\varphi(A_x)$ be $\varphi(A_x) \equiv \arg(1/v_{x2} - 1/v_{x1})$. Then the relation between the argument of n_x and $\varphi(A_x)$, is given by

(1) $\arg(n_x) = -\varphi(A_x)$ if and only if $\arg(d_b - d_a) = 0°$,

(2) $\arg(n_x) = 90° - \varphi(A_x)$ if and only if $\arg(d_b - d_a) = 90°$,

(3) $\arg(n_x) = 180° - \varphi(A_x)$ if and only if $\arg(d_b - d_a) = 180°$,

(4) $\arg(n_x) = 270° - \varphi(A_x)$ if and only if $\arg(d_b - d_a) = 270°$.

The following Theorem shows how to characterize and calculate the polynomials n_x and d_x associated with a vertex $v_x = n_x/d_x$ from the information of the boundary with an arc A_x in a quadrant, belonging to an arc-segment.

Theorem 4.3 (predecessor). *Let A_1 be an arc of the value-set boundary with vertices $v_1 = n_1/d_1$ and $v_2 = n_1/d_2$, the successor segment with vertices $v_2 = n_1/d_2$, $v_{2\,\text{succ}} = n_{2\lambda}/d_2$ counter-clockwise, and the predecessor segment with vertices $v_{1\,\text{pred}} = n_{4\lambda}/d_1$, $v_1 = n_1/d_1$ counter-clockwise Let A_x be an arc with vertices $v_{x\,\text{pred}} = n_x/d_{x\,\text{pred}}$ and $v_x = n_x/d_x$ counter-clockwise. Then*

(1) $\arg(v_2/v_x) = \arg(n_1) + \varphi(A_x)$ *(condition C1) and n_x satisfies the numerator condition, where $n_x = d_2 v_x$, if and only if $d_x = d_2$ and cannot be any other assigned polynomial,*

(2) *when $d_x \neq d_2$,* $\arg(v_1/v_x) = \arg(n_1) + \varphi(A_x) + 90°$ *(condition C2) and $n_x = d_1 v_x$ satisfies the numerator condition if and only if $d_x = d_1$ and cannot be any other assigned polynomial,*

(3) *when $d_x \neq d_1$ and $d_x \neq d_2$,* $\tan(\arg(1/v_x) - \varphi(A_x) + 90°)d_{2R} > d_{2I}$ *(condition C3), and $n_x = d_{2R}[1 + j\tan(\arg(1/v_x) - \varphi(A_x) + 90°)]v_x$ satisfies the numerator condition if and only if $d_x = d_3 = d_{2R}(1 + j\tan(\arg(1/v_x) - \varphi(A_x) + 90°))$ and cannot be any other assigned polynomial,*

(4) *when $d_x \neq d_1$, $d_x \neq d_2$, and $d_x \neq d_3$,* $\tan(\arg(1/v_x) - \varphi(A_x) + 180°)d_{1R} > d_{1I}$ *(condition C4), and $n_x = d_{1R}[1+j\tan(\arg(1/v_x)-\varphi(A_x)+180°)]v_x$ satisfies the numerator condition if and only if $d_x = d_4 = d_{1R}(1 + j\tan(\arg(1/v_x) - \varphi(A_x) + 180°))$.*

Proof. Analogous to Theorem 3.3. □

Remark 4.4. This theorem is used in the example of Section 5, for the value set I (frequency $w = 1.0$) in order to assign the fifth and seventh vertices, and for the value set II (frequency $w = 1.1$) to assign the third, fifth, and seventh vertices.

The following theorem is analogous to Theorem 4.3 when A_x is an arc with vertices $v_x = n_x/d_x$ and $v_{x\text{succ}} = n_x/d_{x\text{succ}}$ counter-clockwise, and belonging to a segment-arc.

Theorem 4.5 (successor). *Let A_1 be a complete arc of the value-set boundary with vertices $v_1 = n_1/d_1$ and $v_2 = n_1/d_2$, the successor segment with vertices $v_2 = n_1/d_2$, $v_{2\,\text{succ}} = n_{2\lambda}/d_2$ counter-clockwise and the predecessor segment with vertices $v_{1\,\text{pred}} = n_{4\lambda}/d_1$, $v_1 = n_1/d_1$ counter-clockwise. Let A_x be an arc with vertices $v_{x\,\text{succ}} = n_x/d_{x\,\text{succ}}$ and $v_x = n_x/d_x$ counter-clockwise*

Then

(1) $\arg(v_2/v_x) = \varphi(A_x) + \arg(n_1) - 90°$ *(condition C1) and n_x satisfies the numerator condition, where $n_x = d_2 v_x$, if and only if $d_x = d_2$ and cannot be any other assigned polynomial,*

(2) *when* $d_x \neq d_2$, $\arg(v_1/v_x) = \varphi(A_x) + \arg(n_1)$ *(condition C2) and* $n_x = d_1 v_x$ *satisfies the numerator condition if and only if* $d_x = d_1$ *and cannot be any other assigned polynomial,*

(3) *when* $d_x \neq d_1$ *and* $d_x \neq d_2$, $\tan(\arg(1/v_x) - \varphi(A_x) + 180°)d_{2R} > d_{2I}$ *(condition C3), and* $n_x = d_{2R}[1 + j\tan(\arg(1/v_x) - \varphi(A_x) + 180°)]v_x$ *satisfies the numerator condition if and only if* $d_x = d_3 = d_{2R}(1 + j\tan(\arg(1/v_x) - \varphi(A_x) + 180°))$ *and cannot be any other assigned polynomial,*

(4) *when* $d_x \neq d_1$, $d_x \neq d_2$, *and* $d_x \neq d_3$, $\tan(\arg(1/v_x) - \varphi(A_x) + 270°)d_{1R} > d_{1I}$ *(condition C4), and* $n_x = d_{1R}[1+j\tan(\arg(1/v_x)-\varphi(A_x)+270°)]v_x$ *satisfies the numerator condition if and only if* $d_x = d_4 = d_{1R}(1 + j\tan(\arg(1/v_x) - \varphi(A_x) + 270°))$.

Proof. Analogous to Theorem 3.3. □

Remark 4.6. This theorem is used in the example of Section 5, for the value set I (frequency $w = 1.0$) in order to assign the third, fourth, and sixth vertices, and for the value set II (frequency $w = 1.1$) to assign the fourth and sixth vertices.

Finally, the following theorem points out the necessary and sufficient condition.

Theorem 4.7. *Given a value set, all the assigned polynomials of the vertices can be determined if and only if there is a complete edge or a complete arc lying on a quadrant when the normalized edge satisfies* $n_{2R} \neq 0$, $n_{2I} \neq 0$, $n_{1R} \neq 0$, *and* $n_{1I} \neq 0$ *or the normalized arc satisfies* $d_{2R} \neq 0$, $d_{2I} \neq 0$, $d_{1R} \neq 0$, *and* $d_{1I} \neq 0$.

Proof. It is obvious from Theorems 3.3–4.5. □

5. Algorithm and Examples

Algorithm 5.1. Given a value set with a complete segment or a complete arc in a quadrant, to obtain the Kharitonov polynomials the following.

(1) If there is a complete segment in a quadrant, S_1, with vertices $v_1 = n_1/d_1$ and $v_2 = n_2/d_1$, the successor arc with vertices $v_2 = n_2/d_1$, $v_{2\text{succ}} = n_2/d_{2\lambda}$ counter-clockwise and the predecessor arc with vertices $v_{1\text{pred}} = n_1/d_{4\lambda}$, $v_1 = n_1/d_1$ counter-clockwise then for all vertex $v_x = n_x/d_x$:

   (a) if $v_x = n_x/d_x$ is a vertex intersection of a segment and an arc counter-clockwise, then the assigned polynomials numerator and denominator, n_x and d_x, determine applying Theorem 3.3,

   (b) if $v_x = n_x/d_x$ is a vertex intersection of an arc and a segment counter-clockwise, then the assigned polynomials numerator and denominator, n_x and d_x, determine applying Theorem 3.5.

(2) If there is a complete arc in a quadrant, A_1, with vertices $v_1 = n_1/d_1$ and $v_2 = n_1/d_2$, the successor segment with vertices $v_2 = n_1/d_2$, $v_{2\text{succ}} = n_{2\lambda}/d_2$ counter-clockwise and the predecessor segment with vertices $v_{1\text{pred}} = n_{4\lambda}/d_1$, $v_1 = n_1/d_1$ counter-clockwise, then given a vertex $v_x = n_x/d_x$:

   (a) if $v_x = n_x/d_x$ is a vertex intersection of an arc and a segment counter-clockwise, then the assigned polynomials numerator and denominator, n_x and d_x, determine applying Theorem 4.3,

Table 1: Value set boundary information.

	$\omega = 1.0$			$\omega = 1.1$			$\omega = 1.2$	
(a)	(b)	(c)	(a)	(b)	(c)	(a)	(b)	(c)
v_1	$1.5676 + 2.5946j$	0	v_1	$-2.8422 + 2.9830j$	0	v_1	$6.1015 + 5.2779j$	1
v_2	$2.0000 + 8.0000j$	1	v_2	$-0.9808 + 2.4599j$	1	v_2	$6.5135 + 6.8573j$	0
v_3	$0.8000 + 10.4000j$	0		$0 + 3.0420j$	0		$0 + 8.5560j$	0
	$0 + 10.0000j$	1	v_3	$0.4996 + 3.0386j$	1	v_3	$-3.0339 + 6.1294j$	1
v_4	$-4.8000 + 7.6000j$	0	v_4	$2.3317 + 3.0261j$	0	v_4	$-2.2110 + 5.1007j$	0
v_5	$-3.5862 + 1.0345j$	1	v_5	$5.1859 + 6.6181j$	1	v_5	$-0.4710 + 3.4462j$	1
v_6	$2.5517 + 0.6207j$	0	v_6	$5.2164 + 8.6623j$	0		$0 + 3.6463j$	0
v_7	$-1.3443 + 1.2131j$	1		$0 + 8.7404j$	0	v_6	$1.4690 + 3.4428j$	1
	$0 + 2.3336j$	1	v_7	$-3.8291 + 3.7385j$	1	v_7	$2.9559 + 3.2369j$	0

(a): Vertex (v_i) or cut point (blank) with an axis. (b): Value of the vertex or cut point.
(c): Edge (2.1) or arc (0) between this element and the next element. If the element is the last, the next element is the first.

(b) if $v_x = n_x/d_x$ is a vertex intersection of a segment and an arc counterclockwise, then the assigned polynomials numerator and denominator, n_x and d_x, determine applying Theorem 4.5.

(3) Calculate the values of the assigned polynomials n_j, d_k, solving the equation system (2.7):

$$v_i = \frac{n_j}{d_k}. \tag{5.1}$$

(4) Calculate the numerator and denominator rectangles with Kharitonov polynomial values $N = (k_{n1}(j\omega), k_{n2}(j\omega), k_{n3}(j\omega), k_{n4}(j\omega))$, $D = (k_{d1}(j\omega), k_{d2}(j\omega), k_{d3}(j\omega), k_{d4}(j\omega))$ applying (2.8).

Example 5.2. Figure 10 shows three value sets of an interval plant. The necessary information (Table 1) is

(i) the vertices,

(ii) the intersections with the axis,

(iii) the shape of the boundary's elements: arc or segment.

This example illustrates how to obtain the assigned polynomials and the numerator and denominator rectangles for each value set, and remarks the theorem used in each step.

5.1. Value Set at Frequency $\omega = 1.0$

The complete arc with vertices $v_1 = n_1/d_1 = 1.5676+2.5946j$ and $v_2 = n_1/d_2 = 2.0000+8.0000j$ is taken as initial element. Then Theorems 4.3 and 4.5 will be applied. So

$$v_{2\text{succ}} = \frac{n_{2\lambda}}{d_2} = 0.8000 + 10.4000j, \qquad v_{1\text{pred}} = \frac{n_{4\lambda}}{d_1} = 2.3336j. \tag{5.2}$$

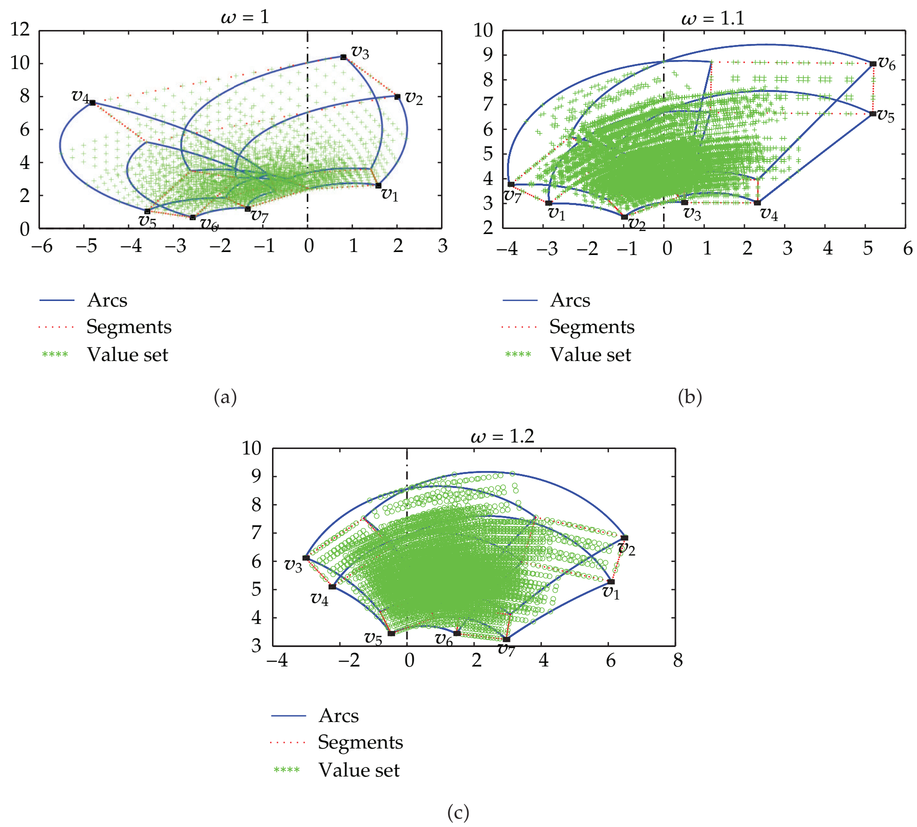


Figure 10: Three value sets of an interval plant.

Applying the arc normalization (Lemma 4.1) the following data are obtained

$$\varphi(n_1) = 229.40, \quad n_1 = -0.6508 - 0.7592j, \quad d_1 = -0.3254 + 0.0542j, \quad d_2 = -0.1085 + 0.0542j,$$
$$n_{4\lambda} = -0.1266 - 0.7594j, \qquad n_{2\lambda} = -0.6508 - 1.0846j. \tag{5.3}$$

Then, all the other vertices are assigned as follows.

(1) Vertex $v_3 = v_x = n_x/d_x = 0.8000 + 10.4000j$. Then $v_{x\text{pred}} = n_x/d_{x\text{pred}} = 2.0000 + 8.0000j$. These are the vertices of an edge, and Theorem 4.5 is applied, $v_{x\text{succ}} = 10.0000j$, $\varphi(A_x) = 210.97$.

Case 1. Theorem 4.5(C1) is satisfied: $\arg(v_2/v_x) = \varphi(A_x) + \arg(n_1) - 90 = 350.36$ and $n_x = d_2 v_x = -0.6508 - 1.0846j$ satisfies the Numerator Condition (Lemma 4.2(4), $n_x = n_2$):

$$\left(n_{1R} = -0.6508 < n_{4\lambda R} = -0.1266,\ n_{1I} = -0.7592 > n_{2\lambda I} = -1.0846\right),$$
$$\left(n_{xR} = n_{1R} = -0.6508,\ n_{xI} = -1.0846 \le n_{1I} = -0.7592\right). \tag{5.4}$$

Then $d_x = d_2 = -0.1085 + 0.0542j$. Therefore $v_3 = v_x = n_2/d_2$.

(2) Vertex $v_4 = v_x = n_x/d_x = -4.8000+7.6000j$. Then $v_{x\text{pred}} = 10j$. These are the vertices of an edge, and Theorem 4.5 is applied: $v_{x\text{succ}} = -3.5862 + 1.0345j$, $\varphi(A_x) = 174.29$.

Case 1. Theorem 4.5(C1) is satisfied: $\arg(v_2/v_x) = \varphi(A_x) + \arg(n_1) - 90 = 313.69$ and $n_x = d_2 v_x = 0.1084 - 1.0847j$ satisfies the Numerator Condition (Lemma 4.2(4), $n_x = n_3$). Then $d_x = d_2 = -0.1085 + 0.0542j$. Therefore $v_4 = v_x = n_3/d_2$.

(3) Vertex $v_5 = v_x = n_x/d_x = -3.5862 + 1.0345j$. Then $v_{x\text{pred}} = n_x/d_{x\text{pred}} = -4.8000 + 7.6000j$. These are the vertices of an arc, and Theorem 4.3 is applied: $\varphi(A_x) = 174.29$.

Case 1. Theorem 4.3(C1) is not satisfied: $\arg(v_2/v_x) = 272.06 \neq \arg(n_1) + \varphi(A_x) = 43.69$.

Case 2. Theorem 4.3(C2) is not satisfied:$\arg(v_1/v_x) = 254.95 \neq \arg(n_1) + \varphi(A_x) + 90 = 133.69$.

Case 3. Theorem 4.3(C3) is satisfied: $\tan(\arg(1/v_x) - \varphi(A_x) + 90)d_{2R} = 0.2712 > d_{2I} = 0.0542$ and $n_x = 0.1085 - 1.0846j$ satisfies the Numerator Condition (Lemma 4.2(4)) $n_x = n_3$. Then $d_x = d_3 = -0.1085 + 0.2712j$ $v_5 = v_x = n_3/d_3$.

(4) Vertex $v_6 = v_x = n_x/d_x = -2.5517 + 0.6207j$. Then $v_{x\text{pred}} = n_x/d_{x\text{pred}} = -3.5862 + 1.0345j$. These are the vertices of an edge, and Theorem 4.5 is applied: $v_{x\text{succ}} - 1.3443 + 1.2131j$, $\varphi(A_x) = 261.87$.

Case 1. Theorem 4.5(C1) is not satisfied: $\arg(v_2/v_x) = 269.64 \neq \varphi(A_x) + \arg(n_1) - 90 = 41.27$.

Case 2. Theorem 4.5(C2) is not satisfied: $\arg(v_1/v_x) = 252.53 \neq \varphi(A_x) + \arg(n_1) = 131.27$.

Case 3. Theorem 4.5(C3) is satisfied: $\tan(\arg(1/v_x) - \varphi(A_x) + 180)d_{2R} = 0.2712 > d_{2I} = 0.0542$ and $n_x = 0.1085 - 0.7592j$ satisfies the Numerator Condition (Lemma 4.2(3)) $n_x = n_4$: then $d_x = d_3 = -0.1085 + 0.2712j$ and $v_6 = v_x = n_4/d_3$.

(5) Vertex $v_7 = v_x = n_x/d_x = -1.3443 + 1.2131j$. Then $v_{x\text{pred}} = n_x/d_{x\text{pred}} = -2.5517 + 0.6207j$. These are the vertices of an arc, and Theorem 4.3 is applied: $\varphi(A_x) = 261.87$.

Case 1. Theorem 4.3(C1) is not satisfied: $\arg(v_2/v_x) = 298.03 \neq \arg(n_1) + \varphi(A_x) = 131.27$.

Case 2. Theorem 4.3(C2) is not satisfied: $\arg(v_1/v_x) = 280.93 \neq \arg(n_1) + \varphi(A_x) + 90 = 221.27$.

Case 3. Theorem 4.3(C3) is not satisfied: $\tan(\arg(1/v_x) - \varphi(A_x) + 90)d_{2R} = -0.1302 < d_{2I} = 0.0542$.

Case 4. Theorem 4.3(C4) is satisfied: $\tan(\arg(1/v_x) - \varphi(A_x) + 180)d_{1R} = 0.2712 > d_{1I} = 0.0542$ and $n_x = 0.1085 - 0.7592j$ satisfies the Numerator Condition (Lemma 4.2(4)) $n_x = n_4$. Then $d_x = d_4 = -0.3254 + 0.2712j$; $v_7 = v_x = n_4/d_4$.

In summary, the assigned polynomials are

$$v_1 = \frac{n_1}{d_1}, \quad v_2 = \frac{n_1}{d_2}, \quad v_3 = \frac{n_2}{d_2}, \quad v_4 = \frac{n_3}{d_2}, \quad v_5 = \frac{n_3}{d_3}, \quad v_6 = \frac{n_4}{d_3}, \quad v_7 = \frac{n_4}{d_4}, \tag{5.5}$$

and the values can be calculated: from normalization,

$$n_1 = -0.6508 - 0.7592j, \quad d_1 = -0.3254 + 0.0542j, \quad d_2 = -0.1085 + 0.0542j, \tag{5.6}$$

and from the vertices,

$$\begin{aligned}
&v_3: n_2 = -0.6508 - 1.0846j, \qquad d_2 = -0.1085 + 0.0542j,\\
&v_4: n_3 = 0.1084 - 1.0847j, \qquad d_2 = -0.1085 + 0.0542j,\\
&v_5: n_3 = 0.1085 - 1.0846j, \qquad d_3 = -0.8464 + 2.0152j,\\
&v_6: n_4 = 0.1085 - 0.7592j, \qquad d_3 = -0.1085 + 0.2712j,\\
&v_7: n_4 = 0.1085 - 0.7593j, \qquad d_4 = -0.3254 + 0.2712j.
\end{aligned} \tag{5.7}$$

Then

$$\begin{aligned}
k_{n1}(j\omega) &= -0.6508 - 1.0847j, \qquad k_{n2}(j\omega) = 0.1085 - 1.0847j,\\
k_{n3}(j\omega) &= 0.1085 - 0.7592j, \qquad k_{n4}(j\omega) = -0.6508 - 0.7592j,\\
k_{d1}(j\omega) &= -0.3254 + 0.0542j, \qquad k_{d2}(j\omega) = -0.1085 + 0.0542j,\\
k_{d3}(j\omega) &= -0.1085 + 0.2712j, \qquad k_{d4}(j\omega) = -0.3254 + 0.2712j.
\end{aligned} \tag{5.8}$$

Table 2 shows the results of the algorithm for the value set at frequency $\omega = 1.0$.

From these Kharitonov rectangles the value set given in Figure 11(a) is directly obtained.

5.2. Value Set at Frequency $\omega = 1.1$

The complete arc with vertices $v_1 = n_1/d_1 = -2.8422 + 2.9830j$ and $v_2 = n_1/d_2 = -0.9808 + 2.4599j$ is taken as initial element. Then Theorems 4.3 and 4.5 will be applied. So

$$v_{2\text{succ}} = \frac{n_{2\lambda}}{d_2} = 3.0420j, \qquad v_{1\text{pred}} = \frac{n_{4\lambda}}{d_1} = -3.8291 + 3.7385j. \tag{5.9}$$

Applying the arc normalization (Lemma 4.1) the following data are obtained:

$$\begin{gathered}
\varphi(n_1) = 360 - \arg\left(\frac{1}{v_2} - \frac{1}{v_1}\right) = 81.05, \qquad n_1 = 0.1556 + 0.9878j,\\
d_1 = \frac{n_1}{v_1} = 0.1475 - 0.1927j;
\end{gathered} \tag{5.10}$$

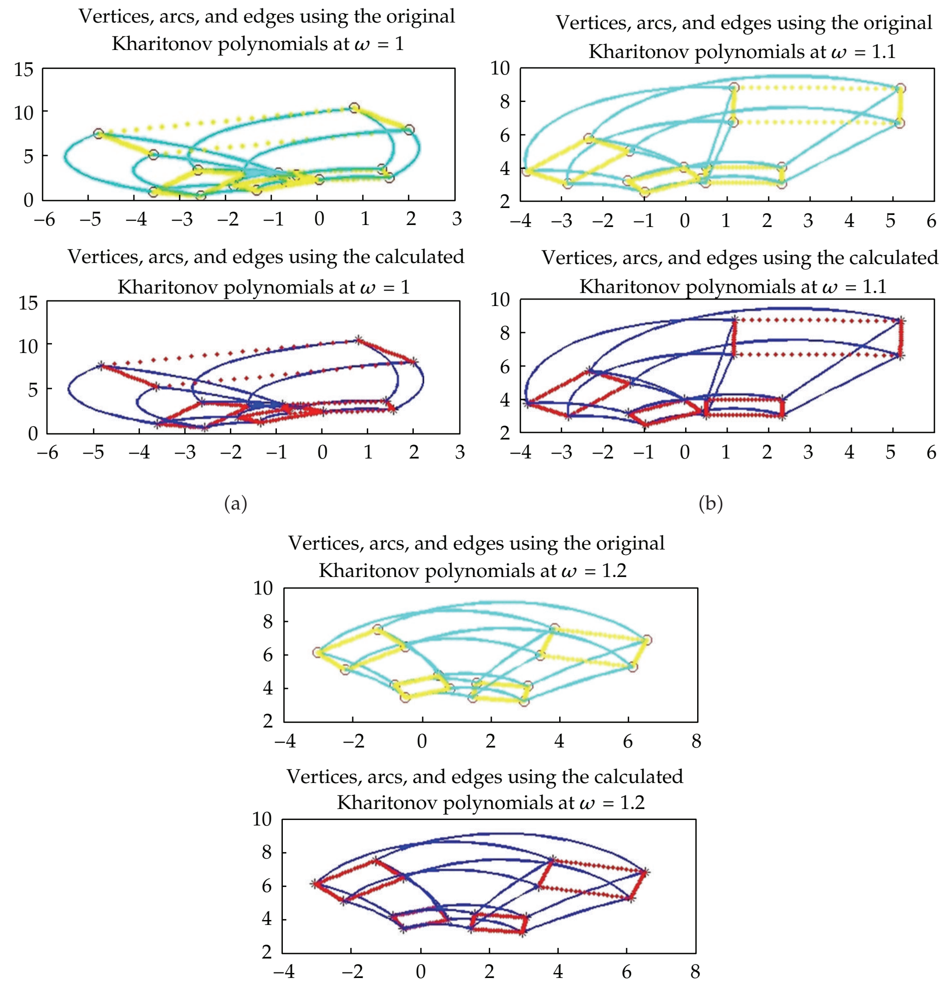
$$\begin{gathered}
d_2 = \frac{n_1}{v_2} = 0.3247 - 0.1927j, \qquad n_{4\lambda} = d_1 v_{1\text{pred}} = 0.1556 + 1.2895j,\\
n_{2\lambda} = d_2 v_{2\text{succ}} = 0.5862 + 0.9878j.
\end{gathered} \tag{5.11}$$

Then, all the other vertices are assigned as follows.

(1) Vertex $v_3 = v_x = n_x/d_x = 0.4996 + 3.0386j$. Then $v_{x\text{pred}} = n_x/d_{x\text{pred}} = 3.0420j$. These are the vertices of an arc, and Theorem 4.3 is applied: $\varphi(A_x) = 8.95$.

Table 2: Results of the algorithm for the value set at frequency $\omega = 1.0$.

	$v_1 - v_2$ arc		v_3 $0.8000 + 10.4000j$	v_4 $-4.8000 + 7.6000j$	v_5 $-3.5862 + 1.0345j$	v_6 $-2.5517 + 0.6207j$	v_7 $-1.3443 + 1.2131j$	Kharitonov rectangles calculated
v_1	$1.5676 + 2.5946j$	Theorem applied	Theorem 4.5	Theorem 4.5	Theorem 4.3	Theorem 4.5	Theorem 4.3	$k_{n1}(j\omega) = -0.6508 - 1.0847j$
v_2	$2.0000 + 8.0000j$	$v_{x\,\text{pred}}$	$2.0000 + 8.0000j$	$10.0000j$	$-4.8000 + 7.6000j$	$-3.5862 + 1.0345j$	$-2.5517 + 0.6207j$	$k_{n2}(j\omega) = 0.1085 - 1.0847j$
$v_{2\,\text{succ}}$	$0.8000 + 10.4000j$	$v_{x\,\text{succ}}$	$10.0000j$	$-3.5862 + 1.0345j$	$-2.5517 + 0.6207j$	$-1.3443 + 1.2131j$	$2.3336j$	$k_{n3}(j\omega) = 0.1085 - 0.7592j$
$v_{1\,\text{pred}}$	$0 + 2.3336j$	$\varphi(A_x)$	210.97	174.29	174.29	261.87	261.87	$k_{n4}(j\omega) = -0.6508 - 0.7592j$
$\varphi(n_1)$	229.40	Condition verified	Theorem 4.5(C1)	Theorem 4.5(C1)	Theorem 4.3(C3)	Theorem 4.5(C3)	Theorem 4.3(C4)	$k_{d1}(j\omega) = -0.3254 + 0.0542j$
n_1	$-0.6508 - 0.7592j$	n_x	$-0.6508 - 1.0846j$	$0.1084 - 1.0847j$	$0.1085 - 1.0846j$	$0.1085 - 0.7592j$	$0.1085 - 0.7592j$	$k_{d2}(j\omega) = -0.1085 + 0.0542j$
d_1	$-0.3254 + 0.0542j$	d_x	$-0.1085 + 0.0542j$	$-0.1085 + 0.0542j$	$-0.1085 + 0.2712j$	$-0.1085 + 0.2712j$	$-0.3254 + 0.2712j$	$k_{d3}(j\omega) = -0.1085 + 0.2712j$
d_2	$-0.1085 + 0.0542j$	v_x	n_2/d_2	n_3/d_2	n_3/d_3	n_4/d_3	n_4/d_4	$k_{d4}(j\omega) = -0.3254 + 0.2712j$
$n_{4\lambda}$	$-0.1266 - 0.7594j$							
$n_{2\lambda}$	$-0.6508 - 1.0846j$							



(a) (b) (c)

Figure 11

Cases 1 and 2. Theorem 4.3(C1) and (C2) are not satisfied.

Case 3. Theorem 4.3(C3) is satisfied: $\tan(\arg(1/v_x) - \varphi(A_x) + 90)d_{2R} = 0.0022 > d_{2I} = -0.1927$ and $n_x = 0.1555 + 0.9878j$ satisfies the Numerator Condition (Lemma 4.2(1)) $n_x = n_1$. Then $d_x = d_3 = 0.3247 + 0.0022j$ $v_3 = v_x = n_1/d_3$.

(2) Vertex $v_4 = v_x = n_x/d_x = 2.3317 + 3.0261j$. Then $v_{x\text{pred}} = n_x/d_{x\text{pred}} = 0.4996 + 3.0386j$. These are the vertices of an edge, and Theorem 4.5 is applied: $v_{x\text{succ}} = 5.1859 + 6.6181j$ and $\varphi(A_x) = 127.23$.

Cases 1 and 2. Theorem 4.5(C1) and (C2) are not satisfied.

Case 3. Theorem 4.5(C3) is satisfied: $\tan(\arg(1/v_x) - \varphi(A_x) + 180)d_{2R} = 0.0022 > d_{2I} = -0.1927$ and $n_x = 0.7505 + 0.9878j$ satisfies the Numerator Condition (Lemma 4.2(1)) $n_x = n_2$. Then $d_x = d_3 = 0.3247 + 0.0022j$. $v_4 = v_x = n_2/d_3$.

(3) Vertex $v_5 = v_x = n_x/d_x = 5.1859 + 6.6181j$. Then $v_{x\text{pred}} = n_x/d_{x\text{pred}} = 2.3317 + 3.0261j$. These are the vertices of an arc, and Theorem 4.3 is applied: $\varphi(A_x) = 127.23$.

Cases 1, 2, and 3. Theorem 4.3(C1), (C2), and (C3) are not satisfied.

Case 4. Theorem 4.3(C4) is satisfied: $\tan(\arg(1/v_x) - \varphi(A_x) + 180)d_{1R} = 0.0022 > d_{1I} = -0.1927$ and $n_x = 0.7505 + 0.9878j$ satisfies the Numerator Condition (Lemma 4.2(1)) $n_x = n_2$. Then $d_x = d_4 = 0.1475 + 0.0022j$ $v_5 = v_x = n_2/d_4$.

(4) Vertex $v_6 = v_x = n_x/d_x = 5.2164 + 8.6623j$. Then $v_{x\text{pred}} = n_x/d_{x\text{pred}} = 5.1859 + 6.6181j$. These are the vertices of an edge, and Theorem 4.5 is applied: $v_{x\text{succ}} = 8.7404j$, $\varphi(A_x) = 210.20$.

Cases 1, 2, and 3. Theorem 4.5(C1), (C2), and (C3) are not satisfied.

Case 4. Theorem 4.5(C4) is satisfied: $\tan(\arg(1/v_x) - \varphi(A_x) + 270)d_{1R} = 0.0022 > d_{1I} = -0.1927$ and $n_x = 0.7505 + 1.2895j$ satisfies the Numerator Condition (Lemma 4.2(1)) $n_x = n_3$: then $d_x = d_4 = 0.1475 + 0.0022j$, $v_6 = v_x = n_3/d_4$.

(5) Vertex $v_7 = v_x = n_x/d_x = -3.8291 + 3.7385j$. Then $v_{x\text{pred}} = n_x/d_{x\text{pred}} = 8.7404j$. These are the vertices of an arc, and Theorem 4.3 is applied: $\varphi(A_x) = 186.88$.

Case 1. Theorem 4.3(C1) is not satisfied.

Case 2. Theorem 4.3(C2) is satisfied: $\arg(v_1/v_x) = \arg(n_1) + \varphi(A_x) + 90 = 357.93$ and $n_x = d_1 v_x = 0.1556 + 1.2895j$ satisfies the Numerator Condition (Lemma 4.2(1)) $n_x = n_4$. Then $d_x = d_1 = 0.1475 - 0.1927j$, $v_7 = v_x = n_4/d_1$.

In summary, the assigned polynomials are

$$v_1 = \frac{n_1}{d_1}, \qquad v_2 = \frac{n_1}{d_2}, \qquad v_3 = \frac{n_1}{d_3}, \qquad v_4 = \frac{n_2}{d_3}, \qquad v_5 = \frac{n_2}{d_4}, \qquad v_6 = \frac{n_3}{d_4}, \qquad v_7 = \frac{n_4}{d_1} \tag{5.12}$$

and the values can be calculated: from normalization,

$$n_1 = 0.1556 + 0.9878j, \qquad d_1 = 0.1475 - 0.1927j, \qquad d_2 = 0.3247 - 0.1927j, \tag{5.13}$$

and from the vertices,

$$\begin{aligned} v_3 &: n_1 = 0.1555 + 0.9878j, & d_3 &= 0.3247 + 0.0022j, \\ v_4 &: n_2 = 0.7505 + 0.9878j, & d_3 &= 0.3247 + 0.0022j, \\ v_5 &: n_2 = 0.7505 + 0.9878j, & d_4 &= 0.1475 + 0.0022j, \\ v_6 &: n_3 = 0.7505 + 1.2895j, & d_4 &= 0.1475 + 0.0022j, \\ v_7 &: n_4 = 0.1556 + 1.2895j, & d_1 &= 0.1475 - 0.1927j. \end{aligned} \tag{5.14}$$

Then

$$
\begin{aligned}
k_{n1}(j\omega) &= 0.1555 + 0.9878j, & k_{n2}(j\omega) &= 0.7505 + 0.9878j,\\
k_{n3}(j\omega) &= 0.7505 + 1.2895j, & k_{n4}(j\omega) &= 0.1556 + 1.2895j,\\
k_{d1}(j\omega) &= 0.1475 - 0.1927j, & k_{d2}(j\omega) &= 0.3247 - 0.1927j,\\
k_{d3}(j\omega) &= 0.3247 + 0.0022j, & k_{d4}(j\omega) &= 0.1475 + 0.0022j.
\end{aligned}
\tag{5.15}
$$

Table 3 shows the results of the algorithm for the value set at frequency $\omega = 1.1$.

From these Kharitonov rectangles the value set given in Figure 11(b) is directly obtained.

5.3. Value Set at Frequency $\omega = 1.2$

The complete edge with vertices $v_1 = n_1/d_1 = 6.1015 + 5.2779j$ and $v_2 = n_2/d_1 = 6.5135 + 6.8573j$ is taken as initial element. Then Theorems 3.3 and 3.5 will be applied. So

$$
v_{2\text{succ}} = \frac{n_2}{d_{2\lambda}} = 8.5560j, \qquad v_{1\text{pred}} = \frac{n_1}{d_{4\lambda}} = 2.9559 + 3.2369j. \tag{5.16}
$$

Applying the edge normalization (Lemma 3.1) the following data are obtained:

$$
\begin{gathered}
\phi(d_1) = 360 - \arg(v_2 - v_1) = 284.62, \quad d_1 = \cos(\varphi(d_1)) + j\sin(\varphi(d_1)) = 0.2524 - 0.9676j,\\
n_1 = v_1 d_1 = 6.6471 - 4.5717j, \qquad n_2 = v_2 d_1 = 8.2793 - 4.5717j,\\
d_{2\lambda} = \frac{n_2}{v_{2\text{succ}}} = -0.5343 - 0.9677j, \qquad d_{4\lambda} = \frac{n_1}{v_{1\text{pred}}} = 0.2524 - 1.8230j.
\end{gathered}
\tag{5.17}
$$

Then, all the other vertices are assigned as follows.

(1) Vertex $v_3 = v_x = n_x/d_x = -3.0339 + 6.1294j$. Then $v_{x\text{pred}} = n_x/d_{x\text{pred}} = 8.5560j$. These are the vertices of an arc, and Theorem 3.5 is applied: $v_{x\text{succ}} = -2.2110 + 5.1007j$ and $\varphi(S_x) = \arg(v_{x\text{succ}} - v_x) = 308.66$.

Cases 1 and 2. Theorem 3.5(C1) and (C2) are not satisfied.

Case 3. Theorem 3.5(C3) is satisfied: $\tan(\arg(v_x) - \varphi(S_x) + 180) n_{2R} = -1.8087 > n_{2I} = -4.571$ and $d_x = n_{2R}[1 + j\tan(\arg(v_x) - \varphi(S_x) + 180)]/v_x = -0.7740 - 0.9676j$ satisfies the Denominator Condition (Lemma 3.2(3)) $d_x = d_2$: then $n_x = n_3 = n_{2R}[1 + j\tan(\arg(v_x) - \varphi(S_x) + 180)] = 8.2793 - 1.8087j$; $v_3 = v_x = n_3/d_2$.

(2) Vertex $v_4 = v_x = n_x/d_x = -2.211 + 5.1007j$. Then $v_{x\text{pred}} = n_x/d_{x\text{pred}} = -3.0339 + 6.1294j$. These are the vertices of an edge, and Theorem 3.3 is applied: $\varphi(S_x) = \arg(v_x - v_{x\text{pred}}) = 308.66$.

Cases 1 and 2. Theorem 3.3(C1) and (C2) are not satisfied.

Case 3. Theorem 3.3(C3) is satisfied: $\tan(\arg(v_x) - \varphi(S_x) + 90) n_{2R} = 30.4258 > n_{2I} = -4.5717$ but $d_x = n_{2R}[1 + j\tan(\arg(v_x) - \varphi(S_x) + 90)]/v_x = 4.4292 - 3.5431j$ does not satisfy the Denominator

Table 3: Results of the algorithm for the value set at frequency $\omega = 1.1$.

	$v_1 - v_2$ arc		v_3 $0.4996 + 3.0386j$	v_4 $2.3317 + 3.0261j$	v_5 $5.1859 + 6.6181j$	v_6 $5.2164 + 8.6623j$	v_7 $-3.8291 + 3.7385j$	Kharitonov rectangles calculated
v_1	$-2.8422 + 2.9830j$	Theorem applied	Theorem 4.3	Theorem 4.5	Theorem 4.3	Theorem 4.5	Theorem 4.3	$k_{n1}(j\omega) = 0.1555 + 0.9878j$
v_2	$-0.9808 + 2.4599j$	$v_{x\,\mathrm{pred}}$	$3.0420j$	$0.4996 + 3.0386j$	$2.3317 + 3.0261j$	$5.1859 + 6.6181j$	$8.7404j$	$k_{n2}(j\omega) = 0.7505 + 0.9878j$
$v_{2\,\mathrm{scuss}}$	$0 + 3.0420j$	$v_{x\,\mathrm{succ}}$	$2.3317 + 3.0261j$	$5.1859 + 6.6181j$	$5.2164 + 8.6623j$	$8.7404j$	$-2.8422 + 2.9830j$	$k_{n3}(j\omega) = 0.7505 + 1.2895j$
$v_{1\,\mathrm{pred}}$	$-3.8291 + 3.7385j$	$\varphi(A_x)$	8.95	127.23	127.23	210.20	186.88	$k_{n4}(j\omega) = 0.1556 + 1.2895j$
$\varphi(n_1)$	81.05	Condition verified	Theorem 4.3(C3)	Theorem 4.5(C3)	Theorem 4.3(C4)	Theorem 4.5(C4)	Theorem 4.3(C2)	$k_{d1}(j\omega) = 0.1475 - 0.1927j$
n_1	$0.1556 + 0.9878j$	n_x	$-0.1555 + 0.9878j$	$0.7505 + 0.9878j$	$0.7505 + 0.9878j$	$0.7505 + 1.2895j$	$0.1556 + 1.2895j$	$k_{d2}(j\omega) = 0.3247 - 0.1927j$
d_1	$0.1475 - 0.1927j$	d_x	$0.3247 + 0.0022j$	$0.3247 + 0.0022j$	$0.1475 + 0.0022j$	$0.1475 + 0.0022j$	$0.1475 - 0.1927j$	$k_{d3}(j\omega) = 0.3247 + 0.0022j$
d_2	$0.3247 - 0.1927j$	v_x	n_1/d_3	n_2/d_3	n_2/d_4	n_3/d_4	n_4/d_1	$k_{d4}(j\omega) = 0.1475 + 0.0022j$
$n_{4\lambda}$	$0.1556 + 1.2895j$							
$n_{2\lambda}$	$0.5862 + 0.9878j$							

Condition: ($d_{1R} = 0.2524 > d_{2\lambda R} = -0.5343$ and $d_{1I} = -0.9676 > d_{4\lambda I} = -1.8230$) (Case 3) but ($d_{xR} = 4.4292 \neq d_{1R} = 0.2524$) then $d_x \neq d_1$ and $d_x \neq d_4$ ($d_{xI} = -3.5431 \neq d_{1I} - 0.9676$) then $d_x \neq d_2$($d_{xR} = 4.4292 > d_{1R} = 0.2524$ and $d_{xI} = -3.5431 < d_{1I} = -0.9676$) then $d_x \neq d_3$.

Case 4. Theorem 3.3(C4) is satisfied: $\tan(\arg(v_x) - \varphi(S_x) + 180)n_{1R} = -1.8088 > n_{1I} = -4.5717$ and $d_x = n_{1R}[1 + j\tan(\arg(v_x) - \varphi(S_x) + 180)]/v_x = -0.7741 - 0.9676j$ satisfies the Denominator Condition (Lemma 3.2(3)) $d_x = d_2$:

$$\begin{gathered}(d_{1R} = 0.2524 > d_{2\lambda R} = -0.5343,\ d_{1I} = -0.9676 > d_{4\lambda I} = -1.8230),\\ (d_{xI} = d_{1I} = -0.9676,\ d_{xR} = -0.7740 \leq d_{1R} = 0.2524).\end{gathered} \tag{5.18}$$

Then $n_x = n_4 = n_{1R}[1 + j\tan(\arg(v_x) - \varphi(S_x) + 180)] = 6.6471 - 1.8088j$, $v_4 = v_x = n_4/d_2$.

(3) Vertex $v_5 = v_x = n_x/d_x = -0.47099 + 3.4462j$. Then $v_{x\text{pred}} = n_x/d_{x\text{pred}} = -2.2110 + 5.1007j$. These are the vertices of an arc, and Theorem 3.5 is applied: $v_{x\text{succ}} = 3.6463j$ and $\varphi(S_x) = \arg(v_{x\text{succ}} - v_x) = 23.01$.

Cases 1, 2, and 3. Theorems 3.5(C1) and (C2) are not satisfied. Theorem 3.5(C3) is satisfied but $d_x = 8.3397 - 3.5422j$ does not satisfy the Denominator Condition (Lemma 3.2(3)).

Case 4. Theorem 3.5(C4) is satisfied: $\tan(\arg(v_x) - \varphi(S_x) + 270)n_{1R} = -1.8098 > n_{1I} = -4.5717$ and $d_x = n_{1R}[1 + j\tan(\arg(v_x) - \varphi(S_x) + 270)]/v_x = -0.7743 - 1.8230j$ satisfies the Denominator Condition (Lemma 3.2(3)).

Then $n_x = n_4 = n_{1R}[1 + j\tan(\arg(v_x) - \varphi(S_x) + 270)] = 6.6471 - 1.8098j$, $v_5 = v_x = n_4/d_3$.

(4) Vertex $v_6 = v_x = n_x/d_x = 1.469 + 3.4428j$. Then $v_{x\text{pred}} = n_x/d_{x\text{pred}} = 3.6463j$. These are the vertices of an arc, and Theorem 3.5 is applied: $v_{x\text{succ}} = 2.9559 + 3.2369j$ and $\varphi(S_x) = \arg(v_{x\text{succ}} - v_x) = 352.12$.

Cases 1, 2, and 3. Theorem 3.5(C1) and (C2) are not satisfied. Theorem 3.5(C3) is satisfied but $d_x = 8.3440 + 1.1554j$ does not satisfy the Denominator Condition (Lemma 3.2(3)).

Case 4. Theorem 3.5(C4) is satisfied: $\tan(\arg(v_x) - \varphi(S_x) + 270)n_{1R} = -1.8089 > n_{1I} = -4.5717$ and $d_x = 0.2524 - 1.8230j$ satisfies the Denominator Condition (Lemma 3.2(3)) $d_x = d_4$.

Then $n_x = n_4 = n_{1R}[1 + j\tan(\arg(v_x) - \varphi(S_x) + 270)] = 6.6471 - 1.8089j$, $v_6 = v_x = n_4/d_4$.

(5) Vertex $v_7 = v_x = n_x/d_x = 2.9559 + 3.2369j$. Then $v_{x\text{pred}} = n_x/d_{x\text{pred}} = 1.4690 + 3.4428j$. These are the vertices of an edge, and Theorem 3.3 is applied: $\varphi(S_x) = \arg(v_x - v_{x\text{pred}}) = 352.12$.

Case 1. Theorem 3.3(C1) is not satisfied.

Case 2. Theorem 3.3(C2) is satisfied: $\arg(v_x/v_1) = \arg(d_1) + \varphi(S_x) + 90 = 6.74$ and $d_x = n_1/v_x = 0.2524 - 1.8230j$ satisfies the Denominator Condition (Lemma 3.2(3)) $d_x = d_4$. Then $n_x = n_1 = 6.6471 - 4.5717j$, $v_7 = v_x = n_1/d_4$.

In summary, the assigned polynomials are

$$v_1 = \frac{n_1}{d_1}, \qquad v_2 = \frac{n_2}{d_1}, \qquad v_3 = \frac{n_3}{d_2}, \qquad v_4 = \frac{n_4}{d_2}, \qquad v_5 = \frac{n_4}{d_3}, \qquad v_6 = \frac{n_4}{d_4}, \qquad v_7 = \frac{n_1}{d_4}, \tag{5.19}$$

Table 4: Results of the algorithm for the value set at frequency $\omega = 1.2$.

	$v_1 - v_2$ edge		v_3 $-3.0339 + 6.1294j$	v_4 $-2.211 + 5.1007j$	v_5 $-0.4710 + 3.4462j$	v_6 $-1.469 + 3.4428j$	v_7 $2.9559 + 3.2369j$	Kharitonov rectangles calculated
v_1	$6.1015 + 5.2779j$	Theorem applied	Theorem 3.5	Theorem 3.3	Theorem 3.5	Theorem 3.5	Theorem 3.3	$k_{n1}(j\omega) = -6.6471 - 4.5717j$
v_2	$6.5135 + 6.8573j$	$v_{x\,\mathrm{pred}}$	$8.5560j$	$-3.0339 + 6.1294j$	$-2.211 + 5.1007j$	$-3.6463j$	$1.4690 + 3.4428j$	$k_{n2}(j\omega) = 8.2793 - 4.5717j$
$v_{2\,\mathrm{succ}}$	$8.5560j$	$v_{x\,\mathrm{succ}}$	$-2.211 + 5.1007j$	$-0.4710 + 3.4462j$	$3.6463j$	$2.9559 + 3.2369j$	$6.1015 + 5.2779j$	$k_{n3}(j\omega) = 8.2793 - 1.8087j$
$v_{1\,\mathrm{pred}}$	$2.9559 + 3.2369j$	$\varphi(S_x)$	308.66	308.66	23.01	352.12	352.12	$k_{n4}(j\omega) = 6.6471 - 1.8087j$
$\varphi(d_1)$	284.62	Condition verified	Theorem 3.5(C3)	Theorem 3.3(C4)	Theorem 3.5(C4)	Theorem 3.5(C4)	Theorem 3.3(C2)	$k_{d1}(j\omega) = -0.7743 - 1.8230j$
d_1	$0.2524 - 0.9676j$	d_x	$-0.7741 - 0.9676j$	$-0.7741 - 0.9676j$	$-0.7743 - 1.8230j$	$0.2524 - 1.8230j$	$0.2524 - 1.8230j$	$k_{d2}(j\omega) = 0.2524 - 1.8230j$
n_1	$6.6471 - 4.5717j$	n_x	$8.2793 - 1.8088j$	$6.6471 - 1.8088j$	$6.6471 - 1.8098j$	$6.6471 - 1.8089j$	$6.6471 - 4.5717j$	$k_{d3}(j\omega) = 0.2524 - 0.9676j$
n_2	$8.2793 - 4.5717j$	v_x	n_3/d_2	n_4/d_2	n_4/d_3	n_4/d_4	n_1/d_4	$k_{d4}(j\omega) = -0.7743 - 0.9676j$
$d_{4\lambda}$	$0.2524 - 1.8230j$							
$d_{2\lambda}$	$-0.5343 - 0.9677j$							

and the values can be calculated: from normalization,

$$d_1 = 0.2524 - 0.9676, \qquad n_1 = 6.6471 - 4.5717j, \qquad n_2 = 8.2793 - 4.5717j, \tag{5.20}$$

and from the vertices,

$$\begin{aligned} v_3 &: n_3 = 8.2793 - 1.8087j, & d_2 &= -0.7741 - 0.9676j, \\ v_4 &: n_4 = 6.6471 - 1.8087j, & d_2 &= -0.7741 - 0.9676j, \\ v_5 &: n_4 = 6.6471 - 1.8087j, & d_3 &= -0.7743 - 1.8230j, \\ v_6 &: n_4 = 6.6471 - 1.8087j, & d_4 &= 0.2524 - 1.8230j, \\ v_7 &: n_1 = 6.6471 - 4.5717j, & d_4 &= 0.2524 - 1.8230j. \end{aligned} \tag{5.21}$$

Then

$$\begin{aligned} k_{n1}(j\omega) &= 6.6471 - 4.5717j, & k_{n2}(j\omega) &= 8.2793 - 4.5717j, \\ k_{n3}(j\omega) &= 8.2793 - 1.8087j, & k_{n4}(j\omega) &= 6.6471 - 1.8087j, \\ k_{d1}(j\omega) &= -0.7743 - 1.8230j, & k_{d2}(j\omega) &= 0.2524 - 1.8230j, \\ k_{d3}(j\omega) &= 0.2524 - 0.9676j, & k_{d4}(j\omega) &= -0.7743 - 0.9676j. \end{aligned} \tag{5.22}$$

Table 4 shows the results of the algorithm for the value set at frequeny $w = 1.2$.

From these kharitonov rectangles the value set given in Figure 11(c) is directly obtained.

Finally, solving the equation system [10, equation (16)], the interval plant is obtained:

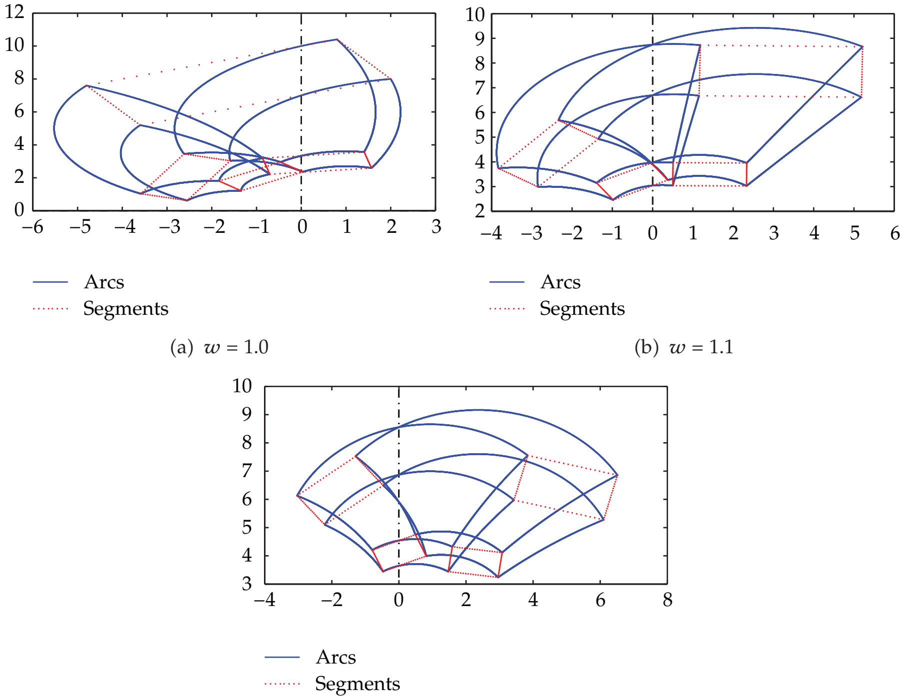
$$G_p(s) = \frac{[10 \;\; 11]s^3 + [7 \;\; 8]s^2 + [6 \;\; 6.5]s + [5 \;\; 7.5]}{[0.75 \;\; 1.25]s^3 + [2 \;\; 2.5]s^2 + [1.5 \;\; 2]s + [1 \;\; 1.5]}. \tag{5.23}$$

Applying $G_p(s = j\omega)$ at $\omega = 1.0$, $\omega = 1.1$ and $\omega = 1.2$ the value sets given in Figure 12 are obtained.

6. Conclusions

This paper shows how to obtain the values of the numerator and denominator Kharitonov polynomials of an interval plant from its value set at a given frequency. Moreover, it is proven that given a value set, all the assigned polynomials of the vertices can be determined if and only if there is a complete edge or a complete arc lying on a quadrant, that is, if there are two vertices in a quadrant. This necessary and sufficient condition is not restrictive and practically all the value sets satisfy it. Finally, the interval plant can be identified solving the equation system between the Kharitonov rectangles and the parameters of the plant.

The algorithm has been formulated using the frequency domain properties of linear interval systems. The identification procedure of multilinear (affine, polynomial) systems will be studied using the results in [11].



(a) $w = 1.0$

(b) $w = 1.1$

(c) $w = 1.2$

Figure 12: Value sets obtained at $w = 1.0$, $w = 1.1$, and $w = 1.2$.

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References

[1] Z. Gao, X. Dai, T. Breikin, and H. Wang, "Novel parameter identification by using a high-gain observer with application to a gas turbine engine," *IEEE Transactions on Industrial Informatics*, vol. 4, no. 4, pp. 271–279, 2008.

[2] P. Van Overschee and B. De Moor, *Subspace Identification for Linear Systems*, Kluwer Academic, Boston, Mass, USA, 1996.

[3] R. S. Sánchez-Peña and M. Sznaier, *Robust Systems Theory and Applications*, John Wiley & Sons, 1998.

[4] B. R. Barmish, *New Tools for Robustness of Linear Systems*, MacMillan, New York, NY, USA, 1993.

[5] S. P. Bhattacharyya, H. Chapellat, and L. H. Keel, *Robust Control: The Parametric Approach*, Prentice-Hall, 1995.

[6] A. C. Bartlett, C. V. Hollot, and H. Lin, "Root locations of an entire polytope of polynomials: it suffices to check the edges," *Mathematics of Control, Signals, and Systems*, vol. 1, no. 1, pp. 61–71, 1988.

[7] M. Fu and B. R. Barmish, "Polytopes of polynomials with zeros in a prescribed set," *IEEE Transactions on Automatic Control*, vol. 34, no. 5, pp. 544–546, 1989.

[8] B. R. Barmish, "A generalization of Kharitonov's four-polynomial concept for robust stability problems with linearly dependent coefficient perturbations," *IEEE Transactions on Automatic Control*, vol. 34, no. 2, pp. 157–165, 1989.

[9] L. Ljung, *System Identification: Theory for the User*, PTR Prentice Hall, 1999.

[10] R. Hernández, J. A. García, and A. P. de Madrid, "Interval plant identification from value sets with five vertices in a quadrant," *International Journal of Robust and Nonlinear Control*, vol. 21, no. 1, pp. 21–43, 2011.

[11] N. Tan, "Computation of the frequency response of multilinear affine systems," *IEEE Transactions on Automatic Control*, vol. 47, no. 10, pp. 1691–1696, 2002.


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