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Research Article

Discretization of Optimal Control Problems Governed by *p*-Laplacian Elliptic Equations

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In this paper, a state-constrained optimal control problem governed by p-Laplacian elliptic equations is studied. The feasible control set or the cost functional may be nonconvex, and the purpose is to obtain the convergence of a solution of the discretized control problem to an optimal control of the relaxed continuous problem.

1. Introduction and the Optimal Control Problem

Let Ω be a bounded open convex domain of $\mathbb{R}^n$, $n = 2, 3$, with a Lipschitz continuous boundary Γ. Let U be a compact subset of $\mathbb{R}^n$, and we denote by $\mathscr{U}$ the set of measurable functions $u : \Omega \to U$. For each $u \in \mathscr{U}$, we consider the following state equation

$$\begin{cases} -\text{div}\left(|\nabla y|^{p-2}\nabla y\right) = f(x, y(x), u(x)) & \text{in } \Omega, \\ y = 0 & \text{on } \Gamma, \end{cases} \tag{1}$$

where $3 < p < +\infty$.

We first make the following assumptions on f:

(S1) The function $f(\cdot, y, u)$ is measurable in Ω, $f(x, \cdot, u)$ is in $C^1(\mathbb{R})$, $f(x, \cdot, \cdot)$, $f_y(x, \cdot, \cdot)$ are continuous in $\mathbb{R} \times U$. Moreover,

$$f_y(x, y, u) \le 0 \quad \forall (x, y, u) \in \Omega \times \mathbb{R} \times U, \tag{2}$$

and for any $R > 0$, there exists a constant $M_R > 0$ such that

$$|f(x, y, u)| + |f_y(x, y, u)| \le M_R \quad \forall (x, u) \in \Omega \times U, \ |y| \le R. \tag{3}$$

The next theorem claims the well-posedness of the state equation.

Proposition 1. *Suppose that (S1) holds. Then for any $u \in \mathscr{U}$, there exists a unique weak solution $y_u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ of (1). Moreover, there exists a constant $C > 0$, independent of $u \in \mathscr{U}$, such that*

$$\|y_u\|_{W_0^{1,p}(\Omega)\cap L^\infty(\Omega)} \le C. \tag{4}$$

The estimate of $\|\cdot\|_{W_0^{1,p}(\Omega)}$ can be obtained by the same arguments in the proof of Theorem 6.11 in [Chapter 2, 9] and the remained results of this theorem can be deduced from Lemma 3.1 in [1].

Remark 2. Since $3 < p < +\infty, W_0^{1,p}(\Omega)$ can be compactly embedded into $C(\overline{\Omega})$, which shows that there exists a constant $C > 0$, independent of $u \in \mathscr{U}$, such that

$$\|y_u\|_{C(\overline{\Omega})} \le C, \tag{5}$$

where y_u is the solution of (1) corresponding to $u \in \mathscr{U}$.

Let us consider another function that satisfies the following properties:

(S2) $L : \Omega \times (\mathbb{R} \times U) \to \mathbb{R}$ is a Carathéodory function which satisfies that for any $M > 0$, there exists a nonnegative function $\phi_M \in L^1(\Omega)$ such that

$$|L(x, y, u)| \le \phi_M(x) \ \text{ a.e. } x \in \Omega, \quad \forall |y| \le M, \ \forall u \in U. \tag{6}$$

Now our optimal control problem can be stated as follows.

$$(\mathrm{P}_\delta)\begin{cases}\text{Minimize } J(u)=\int_\Omega L(x,y_u(x),u(x))dx,\\ u\in\mathcal{U},\ \ g(x,y_u(x))\le\delta\ \ \forall x\in\overline{\Omega},\end{cases}\tag{7}$$

where $\delta\in\mathbb{R}$ and $g:\overline{\Omega}\times\mathbb{R}\to\mathbb{R}$ is a continuous function.

In the case of no convexity assumption, optimal control problems do not have classical solutions generally, whereas the corresponding relaxed problems have solutions if some reasonable assumptions are made. To deal with these problems numerically, one needs to discretize them in some way, and then by applying some optimization method to the discrete problems to find some discrete optimal solution. Since the structures of the continuous problems are basically different from the discrete ones, it is necessary to know whether discrete optimality converges to continuous optimality.

Similar problems were considered by Casas [2] and Chryssoverghi and Kokkinis [3]. In the field of finite element approximations for optimal controls governed by PDEs, we refer the readers to the papers [4–10] and the references therein. This present paper is mainly motivated by the work of [2] where the author considered the following state equation

$$\begin{cases}Ay=f(x,y(x),u(x)) & \text{in } \Omega,\\ y=0 \quad \text{on } \Gamma,\end{cases}\tag{8}$$

with

$$Ay=-\sum_{i,j=1}^{n}\partial_{x_j}\left[a_{ij}(x)\partial_{x_i}y\right].\tag{9}$$

Our main goal is to generalize the results in [2] to the case of p-Laplacian. Such models arise from fluid mechanics, nonlinear diffusion and nonlinear elasticity (see [11]).

Now, we first introduce the stability concept of (P_δ) with respect to perturbations of the set of feasible states.

Definition 3 [1, Definition 1]. We will say that (P_δ) is stable to the right if

$$\lim_{\delta'\searrow\delta}\inf(\mathrm{P}_{\delta'})=\inf(\mathrm{P}_\delta).\tag{10}$$

Analogously, (P_δ) is stable to the left if

$$\lim_{\delta'\nearrow\delta}\inf(\mathrm{P}_{\delta'})=\inf(\mathrm{P}_\delta).\tag{11}$$

(P_δ) is said stable if it is stable to the left and to the right simultaneously.

The following result shows that problem (P_δ) is stable under what cases, and which can be proved by the same arguments as that in the proof of Theorem 2 [2]. However, we still present the details for readers' convenience.

Lemma 4. *Suppose that (S1) and (S2) hold. There exists $\delta_0\in\mathbb{R}$ such that (P_δ) has no feasible control for $\delta<\delta_0$. For every $\delta>\delta_0$, except at most a countable number of them, problem (P_δ) is stable.*

Proof. From (5), there exists a constant $M>0$ such that $|y_u(x)|\le M$ for all $x\in\Omega$ and $u\in\mathcal{U}$. The minimum and maximum of g over $\overline{\Omega}\times[-M,M]$ are denoted by λ_M and Λ_M, respectively. Then we can claim that for $\delta<\lambda_M$, (P_δ) admits no a feasible control, while every element of $\mathcal{U}$ is a feasible control for any $\delta\ge\Lambda_M$. Let $\delta_0=\inf\{\delta:(\mathrm{P}_\delta)\text{ has at least one feasible control}\}$, and then we have that $\lambda_M\le\delta_0\le\Lambda_M$.

Next we show that, for almost all $\delta>\delta_0$, (P_δ) is stable. We consider a function $h:(\delta_0,+\infty)\to\mathbb{R}$ defined by $h(\delta)=\inf(\mathrm{P}_\delta)$. Then except for at most a countable number of δ, we find that h is monotone, nonincreasing and continuous. Moreover, it is easy to see that the continuity of h in δ is equivalent to the stability of (P_δ). Thus the lemma is proved. □

2. The Relaxed Control Problem

In this section, we would like to apply the relaxation theory. That is the control set U can be extended to a bigger space such that the new control problem has at least one solution. For this reason, we recall the concept of relaxed controls and the relations between classical controls and relaxed controls given by Warga [12] first.

Let $C(U)$ denote the space of continuous functions endowed with the maximum norm and $\mathcal{M}(U)=C(U)^*$ is the space of Radon measures in U. Let $\mathcal{M}_+^1(U)$ be the subset of $\mathcal{M}(U)$ formed by the probability measures in U, and $\mathcal{R}$ be the subset of the Banach space $L^\infty(\Omega;\mathcal{M}(U))=L^1(\Omega;C(U))^*$ formed by all $\mathcal{M}_+^1(U)$-valued $C(U)$-weakly measurable functions in Ω. That is, $\sigma\in\mathcal{R}$ if and only if

$$\sigma(x)\in\mathcal{M}_+^1(U),\quad\forall x\in\Omega\tag{12}$$

and

$$x\mapsto\int_U h(v)\sigma(x)(dv)\text{ is measurable},\quad\forall h\in C(U).\tag{13}$$

As usual we call each member of $\mathcal{R}$ a relaxed control and an element of $\mathcal{U}$ a classical control, respectively.

It is known that $\mathcal{R}$ is convex and compact, moreover, $\mathcal{U}$ is dense in $\mathcal{R}$ with the weak star topology of $L^\infty(\Omega;\mathcal{M}(U))$ (see Warga [12, Theorem IV.2.1, p. 272 and Theorem IV.2.6, p. 275]).

We now define the relaxed control problem in the following way

$$(\mathrm{RP}_\delta)\begin{cases}\text{Minimize } J_R(\sigma)=\int_\Omega dx\int_U L(x,y_\sigma(x),v)\sigma(x)(dv),\\ \sigma\in\mathcal{R},g(x,y_\sigma(x))\le\delta\ \forall x\in\overline{\Omega},\end{cases}\tag{14}$$

where $y_\sigma\in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$ being the solution of the problem

$$\begin{cases}-\mathrm{div}\left(|\nabla y_\sigma|^{p-2}\nabla y_\sigma\right)=\int_U f(x,y_\sigma(x),v)\sigma(x)(dv) & \text{in } \Omega,\\ y_\sigma=0\quad\text{on }\Gamma.\end{cases}\tag{15}$$

Let us remark that $\mathcal{U}$ can be considered as a subset of $\mathcal{R}$ by identifying $u\in\mathcal{U}$ with Dirac measure-valued function

$\sigma = \delta_u \in \mathscr{R}$. Moreover, with this identification we have $y_\sigma = y_u$ and $J_R(\sigma) = J(u)$. On the other hand since U is dense in $\mathscr{R}$, problem (RP_δ) can be considered as an extension of (P_δ). Furthermore, we will see below that (RP_δ) has at least one solution. However we must be concerned whether $\inf(\mathrm{RP}_\delta) = \inf(\mathrm{P}_\delta)$. The following theorem gives the answer.

Theorem 5. *Suppose that (S1) and (S2) hold. Let δ_0 be as in Lemma 4. (RP_δ) has at least one solution for every $\delta > \delta_0$. Moreover $\inf(\mathrm{RP}_\delta) = \inf(\mathrm{P}_\delta)$ if and only if (P_δ) is stable to the right.*

Proof. Step 1. We would like to prove the existence of one solution of (RP_δ) for $\delta > \delta_0$. Indeed, similar to (5), there exists a constant $C > 0$, independent of $\sigma \in \mathscr{R}$, such that

$$\|y_\sigma\|_{C(\overline{\Omega})} \le C. \tag{16}$$

It follows from (16) and (S2) that

$$\inf_{\sigma\in\mathscr{R}} J_R(\sigma) > -\infty. \tag{17}$$

Therefore, there exists a minimizing sequence $\sigma_k \in \mathscr{R}$ with the property of

$$\lim_{k\to+\infty} J_R(\sigma_k) = \inf_{\sigma\in\mathscr{R}} J_R(\sigma). \tag{18}$$

Since $\mathscr{R}$ is convex and compact, there exists a $\overline{\sigma} \in \mathscr{R}$ such that (as $k \to +\infty$)

$$\sigma_k \to \overline{\sigma} \text{ weakly}^* \text{ in } L^\infty(\Omega; \mathscr{M}(U)). \tag{19}$$

Moreover, without losing generality, we can suppose that there exists a function $\overline{y} \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ such that

$$y_k \to \overline{y} \text{ weakly in } W_0^{1,p}(\Omega), \text{ strongly in } C(\overline{\Omega}) \tag{20}$$

when $k \to +\infty$, where $y_k, \overline{y}$ is the solution of (15) corresponding to $\sigma_k, \overline{\sigma}$, respectively. Finally the continuity of $J(\cdot)$ and $g(\cdot,\cdot)$ shows that $\overline{\sigma}$ is a solution of (RP_δ).

Step 2. We deal with the remainder part of the theorem. To do this, we first sate the following inequalities

$$\inf(\mathrm{RP}_{\delta'}) \le \inf(\mathrm{P}_{\delta'}) \le \inf(\mathrm{RP}_\delta) \le \inf(\mathrm{P}_\delta) \quad \text{for every } \delta' > \delta. \tag{21}$$

The first and the last inequalities can be deduced from the identification of every feasible control for (P_δ) (resp. $(\mathrm{P}_{\delta'})$) with a feasible control for (RP_δ) (resp. $(\mathrm{RP}_{\delta'})$). We only need prove the second inequality. Since $\mathscr{U}$ is dence in $\mathscr{R}$, if $\sigma \in \mathscr{R}$ is a feasible control for (RP_δ), then there exists $\{u_k\}_{k=1}^\infty \subset \mathscr{U}$ such that (as $k \to +\infty$)

$$\delta_{u_k} \to \sigma \quad \text{weakly}^* \text{ in } L^\infty(\Omega; \mathscr{M}(U)). \tag{22}$$

That is,

$$\int_\Omega dx \int_U z(x,v)\delta_{u_k(x)}(dv) \to \int_\Omega dx \int_U z(x,v)\sigma(x)(dv), \quad \forall z \in L^1(\Omega; C(U)). \tag{23}$$

Let y_σ be the solution of (15) corresponding to $\sigma \in \mathscr{R}$. From (23), we have that

$$\lim_{k\to+\infty} \int_\Omega f(x, y_\sigma, u_k)dx = \lim_{k\to+\infty} \int_\Omega dx \int_U f(x, y_\sigma, v)\delta_{u_k(x)}(dv) = \int_\Omega dx \int_U f(x, y_\sigma, v)\sigma(x)(dv), \tag{24}$$

which means that $y_{u_k} \to y_\sigma$ uniformly in $\overline{\Omega}$, then

$$\delta \ge g(x, y_\sigma(x)) = \lim_{k\to+\infty} g(x, y_{u_k}(x)), \tag{25}$$

therefore $g(x, y_{u_k}(x)) \le \delta'$ for any $x \in \overline{\Omega}$ and k bigger than some k_0, only depending on δ'. Thus $\{u_k\}_{k\ge k_0}$ are feasible solutions for problem $(\mathrm{P}_{\delta'})$ and

$$J_R(\sigma) = \lim_{k\to+\infty} J(u_k) \ge \inf(\mathrm{P}_{\delta'}), \tag{26}$$

the desired inequality is obtained.

Next, we only need to prove that

$$\lim_{\delta'\searrow\delta} \inf(\mathrm{RP}_{\delta'}) = \inf(\mathrm{RP}_\delta). \tag{27}$$

Let $\sigma_{\delta'}$ be a solution of $(\mathrm{RP}_{\delta'})$ for every $\delta' > \delta$. Since $\mathscr{R}$ is compact, one can take a sequence $\{\sigma_{\delta_j}\}_{j=1}^\infty$, with $\delta_j \searrow \delta$, such that $\sigma_{\delta_j} \to \sigma$ weakly* for some $\sigma \in \mathscr{R}$. By the uniform convergence $y_{\sigma_{\delta_j}} \to y_\sigma$ in $\overline{\Omega}$, for every $x \in \overline{\Omega}$, we have that

$$g(x, y_\sigma(x)) = \lim_{j\to+\infty} g\left(x, y_{\sigma_{\delta_j}}(x)\right) \le \lim_{j\to+\infty} \delta_j = \delta, \tag{28}$$

this shows that σ is a feasible control for (RP_δ). Hence we obtain that

$$\inf(\mathrm{RP}_\delta) \le J_R(\sigma) = \lim_{j\to+\infty} J_R\left(\sigma_{\delta_j}\right) = \lim_{\delta'\searrow\delta} \inf(\mathrm{RP}_{\delta'}) \le \inf(\mathrm{RP}_\delta). \tag{29}$$

Finally, it follows from (22) and (27) the proof can be deduced. In fact, if $\inf(\mathrm{RP}_\delta) = \inf(\mathrm{P}_\delta)$, then we have that

$$\lim_{\delta'\searrow\delta} \inf(\mathrm{P}_{\delta'}) \le \inf(\mathrm{P}_\delta) = \inf(\mathrm{RP}_\delta) = \lim_{\delta'\searrow\delta} \inf(\mathrm{RP}_{\delta'}) \le \lim_{\delta'\searrow\delta} \inf(\mathrm{P}_{\delta'}), \tag{30}$$

that is (P_δ) is stable to the right. On the other hand, if $\lim_{\delta'\searrow\delta} \inf(\mathrm{P}_{\delta'}) = \inf(\mathrm{P}_\delta)$, then

$$\inf(\mathrm{RP}_\delta) = \lim_{\delta'\searrow\delta} \inf(\mathrm{RP}_{\delta'}) \le \inf(\mathrm{P}_\delta) = \lim_{\delta'\searrow\delta} \inf(\mathrm{P}_{\delta'}) \le \inf(\mathrm{RP}_\delta), \tag{31}$$

and the proof is completed. □

Corollary 6. *Suppose that (S1) and (S2) hold. If problem (P_δ) is stable to the right and it has a solution $\overline{u}$, then $\overline{\sigma} = \delta_{\overline{u}}$ is also a solution of (RP_δ).*

3. Numerical Approximation of the Control Problem

In this section the numerical discretization of problem (P_δ) will be considered, and the convergence of optimal discrete controls to optimal relaxed controls in some topology will be proved.

We first give some standard notations to use the finite element method (see Ciarlet [13] or Casas [2]). Let $\{\mathcal{T}_h\}_{h>0}$ be a regular family of triangulations in $\overline{\Omega}$ satisfying the inverse assumption. Let us take $\overline{\Omega}_h = \bigcup_{T\in\mathcal{T}_h} T$ with the interior Ω_h and the boundary Γ_h. Then we assume that $\overline{\Omega}_h$ is convex and the vertices of $\mathcal{T}_h$ placed on the boundary Γ_h are points of Γ. To any boundary triangle T of $\mathcal{T}_h$ we associate another triangle $\tilde{T} \subset \overline{\Omega}$ with two interior sides to Ω coincident with two sides of T and the third side is the curvilinear arc of Γ limited by the other two sides. Denote by $\widetilde{\mathcal{T}_h}$ the family formed by these boundary triangles with a curvilinear side and the interior triangles to Ω of $\mathcal{T}_h$, and thus $\overline{\Omega} = \bigcup_{T\in\widetilde{\mathcal{T}_h}} T$. We now consider the spaces

$$\mathcal{U}_h = \left\{u_h \in \mathcal{U} : u_h\big|_T \text{ is constant } \forall T \in \widetilde{\mathcal{T}_h}\right\}, \tag{32}$$

$$V_h = \left\{y_h \in C(\overline{\Omega}) : y_h\big|_T \in \mathcal{P}_1 \ \forall T \in \mathcal{T}_h \text{ and } y_h(x) = 0 \ \forall x \in \overline{\Omega}\backslash\Omega_h\right\}, \tag{33}$$

where $\mathcal{P}_1$ denotes the space of the polynomials of degree less than or equal to 1. It is noticed that since we assume the set $\overline{\Omega}$ is convex, the inclusion $V_h \subset W_0^{1,p}(\Omega)$ holds. For any $u_h \in \mathcal{U}_h$ we denote by $y_h(u_h)$ the unique element of V_h that satisfies (for any $v_h \in V_h$):

$$\int_\Omega |\nabla y_h(u_h)|^{p-2}\nabla y_h(u_h)\cdot\nabla v_h dx = \int_\Omega f(x, y_h(u_h)(x), u_h(x))v_h(x)dx. \tag{34}$$

Now we state the finite dimensional optimal control problem as follows:

$$(\mathrm{P}_{\delta h})\begin{cases}\text{Minimize } J_h(u_h) = \int_{\Omega_h} L(x, y_h(u_h)(x), u_h(x))dx \\ \text{subject to } u_h \in \mathcal{U}_h \text{ and } g\left(x_j, y_h(u_h)\left(x_j\right)\right) \le \delta \ \ 1 \le j \le n(h),\end{cases} \tag{35}$$

where $\{x_j\}_{j=1}^{n(h)}$ is the set of vertices of $\mathcal{T}_h$.

From Theorem 5.3.2 in [13], we can prove the discrete solution converges to the solution y_u of (1), as we now show.

Lemma 7. *Suppose that (S1) holds. Let there be given a family of finite element spaces as previously described. If y_u, $y_h(u_h)$ are the solutions of (1), (34), respectively. Then*

$$\lim_{h\to 0}\|y_u - y_h(u_h)\|_{W_0^{1,p}(\Omega)} = 0. \tag{36}$$

The following result shows that problem $(\mathrm{P}_{\delta h})$ has at least one solution.

Lemma 8. *Suppose that (S1) and (S2) hold. For every $\delta > \delta_0$ there exists $h_\delta > 0$ such that $(\mathrm{P}_{\delta h})$ has at least one solution $\overline{u}_h$ for all $h < h_\delta$.*

Proof. By the compactness of $\mathcal{U}_h$ and the continuity of J_h, we can claim that $(\mathrm{P}_{\delta h})$ has one solution if one can show the set of feasible controls is nonempty. In fact, let $u_0 \in U$ be a feasible control for $\left(\mathrm{P}_{\delta_0}\right)$ and we take $u_{0h} \in \mathcal{U}_h$ such that $u_{0h}(x) \to u_0(x)$ for almost all $x \in \Omega$ as $h \to 0$. Then it follows from (36) that $y_h(u_{0h}) \to y_{u_0}$ uniformly in $\overline{\Omega}$. Using this uniform convergence, we have that $\lim_{h\to 0} g(x, y_h(u_{0h})(x)) = g\left(x, y_{u_0}(x)\right) \le \delta_0$ for any $x \in \overline{\Omega}$. Thus for $\delta > \delta_0$, there exists a constant $h_\delta > 0$ such that $g(x, y_h(u_{0h})(x)) \le \delta$ holds for all $x \in \overline{\Omega}$ and each $h \le h_\delta$. That is to say that u_{0h} is feasible for $(\mathrm{P}_{\delta h})$ and thus the proof is over. □

Finally, we will prove the main result in this paper.

Theorem 9. *Suppose that (S1) and (S2) hold. Let us assume that (P_δ) is stable and let $h_\delta > 0$ be as in Lemma 8. Given a family of controls $\{\overline{u}_h\}_{h<h_\delta}$, $\overline{u}_h$ being a solution of $(\mathrm{P}_{\delta h})$, there exist subsequences $\{\overline{u}_{h_k}\}_{k\in\mathbb{N}}$, with $h_k \to 0$ as $k \to +\infty$, and elements $\overline{\sigma} \in \mathcal{R}$ such that $\overline{\sigma}_{h_k} = \delta_{\overline{u}_{h_k}} \to \overline{\sigma}$ in the weakly* topology of $L^\infty(\Omega; \mathcal{M}(U))$. Each one of these limit points is a solution of (RP_δ). Moreover we have that*

$$\lim_{h\to 0} J_h(\overline{u}_h) = \inf(\mathrm{RP}_\delta) = \inf(\mathrm{P}_\delta). \tag{37}$$

Proof. Let $\overline{y}_h$ be the state corresponding to $\overline{u}_h$ and we set $\overline{\sigma}_h(x) = \delta_{[\overline{u}_h(x)]}$. Since $\mathcal{R}$ is a weakly* compact subset of the space $L^\infty(\Omega; \mathcal{M}(U))$ and $\{\tilde{\sigma}_h\}_{h\le h_\delta} \subset \mathcal{R}$, there exists a subsequence $\overline{\sigma}_{h_k}$ such that $h_k \to 0$ and $\overline{\sigma}_{h_k} = \delta_{\overline{u}_{h_k}} \to \overline{\sigma}$ weakly* in $L^\infty(\Omega; \mathcal{M}(U))$ for some $\overline{\sigma} \in \mathcal{R}$. Now we show that $\overline{\sigma}$ is a solution of (RP_δ). Let $\overline{y}$ be the state associated to $\overline{\sigma}$. Similar to the proof of Lemma 8, since $\overline{y}_{h_k}$ converges to $\overline{y}$ uniformly in $\overline{\Omega}$ and $g\left(x_j, \overline{y}_{h_k}\left(x_j\right)\right) \le \delta$ for any $1 \le j \le n(h)$, therefore $g(x, \overline{y}(x)) \le \delta$, which shows that $\overline{\sigma}$ is feasible for the problem (RP_δ).

For $\delta' \in [\delta_0 + \varepsilon, \delta\ )$, with $0 < \varepsilon < \delta - \delta_0$ fixed, and we let $\sigma_{\delta'}$ be a solution of $(\mathrm{RP}_{\delta'})$. Since $\mathcal{U}$ is dense in $\mathcal{R}$, there exists sequence $\{u^j\}_{j=1}^\infty \subset \mathcal{U}$ such that $u^j \to \sigma_{\delta'}$ weakly* in $L^\infty(\Omega; \mathcal{M}(U))$. By the uniform convergence $y_{u^j} \to y_{\sigma_{\delta'}}$, one can claim that there exists $j_{\delta'} \in \mathbb{N}$ such that $g(x, y_{u^j}(x)) \le \delta' + \varepsilon/2$ for every $x \in \overline{\Omega}$ and $j \ge j_{\delta'}$. For any j fixed we can take a sequence $\{u_h\}_{h>0} \subset \mathcal{U}_h$ such that $u_h(x) \to u^j(x)$ for almost all $x \in \Omega$. It follows from the uniform convergence $y_h(u_h) \to y_{u^j}$ and $g(x, y_h(u_h)(x)) \le \delta$ (for each $x \in \overline{\Omega}, h \le h^j$) that u_h is a feasible control for $(\mathrm{P}_\delta h)$. Hence, we have that $J_{h_k}\left(\overline{\sigma}_{h_k}\right) = J_{h_k}\left(\overline{u}_{h_k}\right) \le J_{h_k}\left(u_{h_k}\right)$ whenever $h_k \le h^j$. Therefore, we get

$$J_R(\overline{\sigma}) = \lim_{k\to+\infty} J_{h_k}\left(\overline{\sigma}_{h_k}\right) \le \lim_{k\to+\infty} J_{h_k}\left(u_{h_k}\right) = J\left(u^j\right). \tag{38}$$

Now passing to the limit as $j \to +\infty$, we gain that $J_R(\overline{\sigma}) \le J_R(\sigma_{\delta'})$. Finally, from the feasibility of $\overline{\sigma}$ for (RP_δ) and the stability condition (Definition 3), we conclude that

$$\begin{aligned}\inf(\mathrm{RP}_\delta) \le J_R(\overline{\sigma}) &\le \lim_{\delta'\nearrow\delta} J_R(\sigma_{\delta'}) = \lim_{\delta'\nearrow\delta}\inf(\mathrm{RP}_{\delta'}) \\ &\le \lim_{\delta'\nearrow\delta}\inf(\mathrm{P}_{\delta'}) = \inf(\mathrm{P}_\delta) = \inf(\mathrm{RP}_\delta),\end{aligned} \tag{39}$$

which shows that $\overline{\sigma}$ is a solution of (RP_δ). The rest of this theorem is obvious. □

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares that they have no conflicts of interest.

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