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Research Article

Nonlinear Analysis of the Dynamics of Criminality and Victimisation: A Mathematical Model with Case Generation and Forwarding

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In this paper, a system of deterministic model is presented for the dynamical analysis of the interactional consequence of criminals and criminality on victimisation under two distinguishable forms of rehabilitation—the behavioural reformation of criminals and the emotional psychotherapy of victims. A threshold value, $\mathscr{R}_0 = \max\{\mathscr{R}_K, \mathscr{R}_V\}$, responsible for the persistence of crime/criminality and victimisation, is obtained and, using it, stability analyses on the model performed. The impact of an effective implementation of the two forms of rehabilitation was found to be substantial on crime and criminality, while an ineffective implementation of same was observed to have a detrimental consequence. The prevention of repeat victimisation was seen to present a more viable option for containing crime than the noncriminalisation of victims. Further, the removal of criminals, either through quitting or death, among others, was also found to have a huge positive impact. Numerical simulations were performed for a variety of mixing criminal scenarios to verify the analytical results obtained.

1. Introduction

Crime is a complex dynamical phenomenon [1–8] that no one global community is free of [5, 7, 9–12]. While crime is ubiquitous, it does neither, however, appear to be uniformly distributed [7, 10–15] nor is its perception, severity, categorisation, and punishment across regions the same [12, 15]. It is thus quite challenging to provide a consistent and comprehensive definition of crime [4, 7, 16]. The common denominator for what constitutes a 'crime', however, consists of an unlawful act or the deviant behaviour of a perpetrator, its appropriate punishment, as prescribed by a criminal legislating institution [5, 16] and the victim of such acts [1, 16, 17]. Criminality is basically influenced by the convergence of three factors—a motivated offender, a suitable target, and the absence of a capable guardian [14, 18–20]. The rise in global crime and criminality, together with the attendant effect on victims is, in recent times, very worrisome [7, 10, 12, 17, 21–25]. Though most disturbing and absolutely unjustifiable, the assumption elsewhere is that offenders have needs that directly cause their criminal behaviour [23]. Other influencing risks factors [12, 23, 25–27] are equally very disheartening. It is expedient, therefore, to correct this dysfunctional mind-set through a systematic capacity building initiative which primarily focuses on assisting criminals to reconstruct personally meaningful and socially acceptable identities [28]. A comprehensive listing and categorisation of crime can be looked up in [8, 23]. The economic implication of crime is enormous [1, 15] and the ensuing psychological and/or physical aftermath of victimisation, to say the least, is colossal [1, 23, 26, 29–31]. Criminal activities are widely recognised to concentrate among a relatively few victims [32]. Further, empirical evidence has associated victimisation with a temporal future risk elevation of repeat and near repeat [1, 12, 23, 30–32]. Heterogenic association, either through direct contact with crime victims or indirectly through media outlets, that regularly publicise victimisation, could aggravate the fear of crime [12, 29, 31–33] or even the crime itself. Victimisation estimates should sufficiently index all occurrences of crime. However, there is a wide gap between occurrences and their records [6, 7] especially in developing countries. Victims are often reluctant or unwilling to report their ordeals for factors deserving institutional attention [7, 18, 34]. The challenges militating against easy and willful flow of crime information, as at when due, have continued to create avoidable bottlenecks in the dispensation of criminal justice [6, 7]. The social media has been explored in this regard in [35]. The short message service (SMS), like the Twitter-handle [35], has proven worthwhile in facilitating reportage [36, 37] and guaranteeing secrecy and confidentiality [9, 18, 36, 38]. A sufficient volume of research has proffered profound mathematical insight into the extent of and remedies to the menace of crime and criminality [2, 9, 13, 15, 27, 34, 39, 40–44]. The present study is motivated by the phenomenal successes in the modelling of systemic interactional dynamics [45–50]. The remainder of the paper is organised as follows. The model is formulated in Section 2. The analyses of the model (for the stability of the associated crime-free and crime persistence equilibria) are done in Section 3 and numerical simulations and discussion are carried out in Section 4. Finally, in Section 5 the conclusion is drawn.

2. Model Formulation and Basic Properties

We assume that the population being studied is a small, highly-criminally-prone subset of a larger population. The larger embedding population is relatively free of crime/criminality and provides a constant source for noncriminal individuals' entry into the highly-criminally-prone population. Further, the understudied population is broadly considered to be categorised into human and the density of criminal cases (forwarded through SMS). The human population at time t, given by $P(t)$, is basically divided into three mutually exclusive compartments of individuals who have not come in contact with criminals but are however at-risk of either being initiated into crime and criminality or victimisation, denoted by $S(t)$, criminals, denoted by $P_C(t)$ and victims of crime, denoted by $P_V(t)$. In addition, both the criminal and victims of crime subpopulations are further subdivided into three subclasses. The criminal subpopulation comprises of: potentially criminally minded individuals whose moral integrity has been compromised due to a sustained criminogenic contact with individuals whose willful internalisation of criminal thoughts and ideations is influencing their behaviours, denoted by $S_E(t)$ (these individuals only harbour criminal ideologies but are yet to indulge in criminality), core criminals, denoted by $C(t)$ and individuals whose degrees of blameworthiness in the commission of certain crimes have warranted their confinement in reformation/correction facilities, denoted by $C_R(t)$. The victims of crime subpopulation on the other hand is comprised of individuals who have survived a carefully planned and meticulously carried-out criminal intention and have incurred neither loss of property nor injury to life, denoted by $S_V(t)$, individuals on which such contacts were successful and have, therefore, suffered specified levels of losses, denoted by $V(t)$ and victimised individuals whose traumatic experiences are being psychotherapeutically managed in some competent rehabilitation centres, denoted by $S_R(t)$. Thus, the human population is

$$P(t) = S(t) + S_E(t) + C(t) + C_R(t) + S_V(t) + V(t) + S_R(t). \tag{1}$$

The at-risk population is assumed, at any time t, to have either never had any criminally defrauding contacts or must have substantially recovered from the effect of criminality and have reassumed a susceptibility status thus becoming at-risk of criminality again. That is, they are liable to either becoming criminals or victims of crime in the event that they condone criminally influencing risky behaviours over time. The at-risk population is assumed, after a sufficient requisite effective criminogenic interactions with criminals, to either become criminalised at a rate λ_K or become victims of crime at a rate λ_V or λ_{V1}. Thus the extent of criminogenic tendencies on criminalisation is given by

$$\lambda_K = \frac{\beta_1(C + \eta_K C_R)}{P}, \tag{2}$$

where β_1 is the effective criminalisation contact rate of criminality (criminogenic interactivity capable of recruiting new criminals). The modification parameter $0 < \eta_K < 1$ accounts for the relative reduced risk of criminalisation of reforming criminals (since it is assumed that the correction programme has the capacity to reform criminals) in comparison to that of core criminals. It is assumed that criminogenic interactivity has the capability of inflicting varying degrees of psychological injuries on a prospective victim. Thus at-risk individuals are prone to victimisation at rates given by

$$\lambda_V = \frac{\beta_2[C + \eta_{V1} C_R + \eta_V(V + \sigma S_R)]}{P} \text{ and}$$
$$\lambda_{V1} = \frac{\beta_2(C + \eta_{V1} C_R + \eta_V V)}{P}, \tag{3}$$

where β_2 is the effective victimisation contact rate (criminogenic contact capable of resulting in the loss of property or threat to life, but not leading to death). We propose the parameter $0 < \eta_V < 1$ to monitor the relative victimisation possibility (possible emotional injury resulting from fear) on individuals due to their interaction with victims of a criminal activity as compared to the victimisation causation of both career and quitting criminals. We anticipate that the rehabilitation programme for the victims of crime has the therapeutic impact of facilitating beneficiaries with the skill of relating their ordeals more objectively and so are assumed to inflict far lesser psychological injury (in the form of fear), if any, (as compared to nonrehabilitated victims) with the modification parameter $0 < \sigma < 1$ accounting for the relative reduction in victimisation causation. Further, we consider the modification parameter, $0 < \eta_{V1} < 1$ to model the reducing victimisation potentiality of quitting criminals (in comparison to career criminals).

2.1. Derivation of Model Equations

2.1.1. System Description. The at-risk population is generated by the recruitment (through birth or immigration) of

Table 1: Variables for the crime model and their descriptions.

Variable	Description
$S(t)$	At risk individuals (individuals susceptible to criminality)
$S_E(t)$	Individuals that are exposed to violent thoughts and behaviours and are fantasising criminality
$C(t)$	Active or career offenders (individuals in a criminal career with nonzero offending frequency)
$C_R(t)$	Criminal individuals in correctional (reformation) facilities
$S_V(t)$	Individuals on whom criminal assault was foiled or persons with criminally attractive properties that are close to a previous criminal event victims of criminal event
$V(t)$	Victims of criminal event
$S_R(t)$	Victims of crime being rehabilitated of trauma
$T(t)$	Density of criminal cases forwarded through text messages (SMS)

individuals (assumed susceptible) into the population at a rate Δ, or from the successful correction of the two adverse influences of criminality through reformation and rehabilitation of criminals and victims from the classes C_R and S_R, at the rates ψ_1 and ψ_2 respectively. The population of at-risk individuals is reduced due to criminalisation at a rate λ_K, or victimisation, at a rate λ_V, or by natural death at a rate η_D. We assume that natural death occurs in each human population while only core criminals are liable to crime-induced death η_C. The population of individuals whose moral uprightness and integrity has been compromised and now fantasise criminality (that is they contemplate indulging criminal activities), in other words, the population of the criminally exposed members of the community, is assumed to be increased when at-risk individuals, victimised individuals and individuals undergoing rehabilitation are criminalised at rates $\lambda_K, \gamma_1\lambda_K$ and $\gamma_2\lambda_K$ (where the modification parameters $0 < \gamma_1 < 1$ and $0 < \gamma_2 < 1$ account for the risk of the victimised and rehabilitated victims to be criminalised). It is decreased by death and crime-career progression at the rate κ. The population of criminals is generated from the crime fantasising class at a rate κ and decreased by death (both natural and crime-induced at rates η_D and η_C respectively) and enrolment for reformation/correction at a rate τ_1. The population of criminally corrected or reformed individuals is generated by the identification and enrolment of individuals with confirmed disruptive behavioural disorder and its subsequent effective reformation in a competent correction facility at a rate τ_1 and could decrease, in addition to natural death, by the reintegration of reformed criminals in to the susceptible population at a rate ψ_1. The population of individuals in the nearly victimised class is generated from the at-risk and rehabilitated individuals at the rates λ_V and $\alpha\lambda_{V1}$ respectively, where the modification parameter $0 < \alpha < 1$ models the reduced tendency for a repeat victimisation on rehabilitated individuals (it is assumed that they have been exposed to coping strategies against criminal techniques and so are therefore less prone to becoming victims of crime). It is reduced due to eventual progression to the class of victims of crime at a rate υ, or death. The population of the victims of crime is generated by the success of a criminal attempt on the nearly victimised individuals at a victimisation progression rate υ. It is assumed to be decreased by criminalisation (due to discontentment over perceived unfair judicial process and/or an inert desire for revenge) at a rate $\gamma_1\lambda_K$ (where the modification parameter γ_1 models the degree of discontentment which is liable to increase the contemplation for criminalisation). The population of rehabilitated victims of crime is increased by the admittance of traumatic victims of crime for psychotherapy at the rate τ_2. This population is decreased by dead or the successful completion of the rehabilitation cycle on victims (at a rate ψ_2) or criminalisation (at the rate $\gamma_2\lambda_K$) and repeat victimisation (at the rate $\alpha\lambda_{V1}$), where the modification parameters γ_2 and $0 < \alpha < 1$ model the fear and reduced tendencies for criminalisation and repeat victimisation respectively. Finally, it is assumed that individuals in all the aforementioned classes can report criminal conducts at a rate η_T, however, with varying levels of enthusiasm modelled by ϕ_i (for $i = 1, 2, \ldots, 7$) where $\phi_i \in (0, 1), i = 1, \ldots, 7$ corresponds, respectively, to individuals in S, S_E, C, C_R, S_V, V and S_R. We assume that some discontented doubled-crossed criminals would "anonymously" disclose the details of a "contentious" criminal case (since reportage is via SMS) at a forwarding enthusiasm rate given by $0 \le \phi_3 \ll 1$. It is further assumed that the density of forwarded cases is reduced by further legal action at a rate η_I.

Combining the aforementioned assumptions and definitions, the variable and parameter descriptions and transfer diagram for the model of the interactional dynamics between crime, criminality, and victimisation in a community are shown in the Tables 1 and 2, and Figure 1, respectively.

From Figure 1, the mathematical representation of the model is given by the following system of ordinary differential equations:

$$
\begin{aligned}
\frac{dS}{dt} &= \Delta + \psi_1 C_R + \psi_2 S_R - \lambda_K S - \lambda_V S - \eta_D S,\\
\frac{dS_E}{dt} &= \lambda_K S + \gamma_1 \lambda_K V + \gamma_2 \lambda_K S_R - Q_1 S_E,\\
\frac{dC}{dt} &= \kappa S_E - Q_2 C,\\
\frac{dC_R}{dt} &= \tau_1 C - Q_3 C_R,\\
\frac{dS_V}{dt} &= \lambda_V S + \alpha\lambda_{V1} S_R - Q_4 S_V,\\
\frac{dV}{dt} &= \upsilon S_V - \gamma_1 \lambda_K V - Q_5 V,\\
\frac{dS_R}{dt} &= \tau_2 V - \gamma_2 \lambda_K S_R - \alpha\lambda_{V1} S_R - Q_6 S_R,\\
\frac{dI}{dt} &= \eta_T \left(\phi_1 S + \phi_2 S_E + \phi_3 C + \phi_4 C_R + \phi_5 S_V + \phi_6 V + \phi_7 S_R\right) - \eta_I T,
\end{aligned}
\tag{4}
$$

Table 2: Parameters for the crime model and their various descriptions.

Parameter	Description
Δ	Recruitment rate into the population
η_C, η_D	Natural and crime-induced death rates
η_V, η_{V1}	Modification parameters
η_T	Text messages (SMS) rates
ψ_1	Correctional success rate leading to criminal career termination
ψ_2	Psychotherapy (trauma rehabilitation) success rate leading to emotional stabilisation of victims of criminal event
κ, υ	Progression rates into criminal career and victims of crime
τ_1	Probability of identification and investigation of active criminals (reformation rate)
τ_2	Probability of trauma management given victimisation (psychotherapeutic rate for traumatised victims)
$\alpha, \gamma_1, \gamma_2, \phi_i; i = 1, \ldots, 7$	Modification parameters
β_1, β_2	Effective contact rates for criminalisation and victimisation
$\lambda_K, \lambda_V, \lambda_{V1}$	Criminalisation and victimisations rates

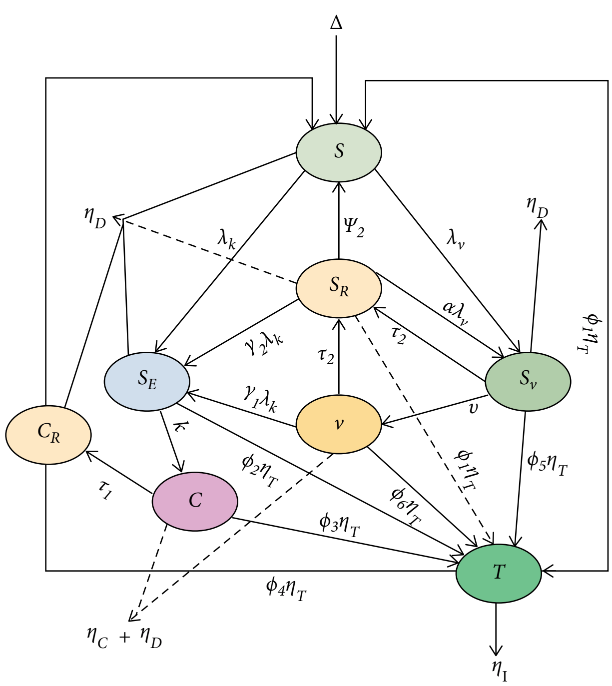

Figure 1: Transfer diagram for the crime model 1.

where

$$Q_1 = \kappa + \eta_D, Q_2 = \tau_1 + \eta_C + \eta_D, Q_3 = \psi_1 + \eta_D, \\ Q_4 = \upsilon + \eta_D, Q_5 = \tau_2 + \eta_D, Q_6 = \psi_2 + \eta_D. \quad (5)$$

2.1.2. Qualitative Properties of Model 1. Since the model is proposed to monitor the dynamics of the interaction between crime, criminality, and victimisation with the impact of case reporting through SMS, it is therefore necessary that all the variables and parameters of the model be nonnegative. Further, the region for the systemic analysis of model 1 should guarantee criminological feasibility. We proceed to establish this by considering only the first seven equations of model 1 since the variable $T(t)$ appear only in the last of the eight equations in system 1. Thus, we propose the region of criminological feasibility for model 1 to be

$$\mathscr{D} = \left\{ (S, S_E, C, C_R, S_V, V, S_R) \in \mathbb{R}_+^7 | P \cdot \leq \frac{\Delta}{\eta_D}, I \leq \frac{\phi_1 \eta_T \Delta}{\eta_D \eta_I} \right\}. \quad (6)$$

Then $\mathscr{D}$ is a positively invariant set for model 1 since adding the first seven equations of model 1 gives:

$$\frac{dP(t)}{dt} = \Delta - \eta_D P(t) - \eta_C C(t). \quad (7)$$

From where we see that $(dP(t))/(dt)$ is bounded by $\Delta - \eta_D P(t)$. Thus,

$$\frac{dP(t)}{dt} \leq \Delta - \eta_D P(t), \quad (8)$$

and so $(dP(t))/(dt) < 0$ if $P(t) > (\Delta/\eta_D)$. Thus from 2 and Gronwall's inequality it follows that

$$P(t) \leq P(0) e^{\eta_D t} + \frac{\Delta}{\eta_D} (1 - e^{-\eta_D t}). \quad (9)$$

Thus $P(t) \leq (\Delta/\eta_D)$ provided $P(0) \leq (\Delta/\eta_D)$. Hence $\mathscr{D}$ is a positively-invariant set and therefore the model is well-posed both mathematically and in criminological sense. Thus, the dynamical flow of the model can be sufficiently considered in $\mathscr{D}$.

3. Equilibria and Stability Analysis

It is easy to see that the model has two equilibria – the crime-free equilibrium (CFE), $\mathscr{E}_0$ and the persistent crime equilibrium (PCE), $\mathscr{E}_1$ which are obtained by setting the right-hand sides of the equations in the model to zero.

3.1. Local Stability of the CFE. The CEF, $\mathscr{E}_0$ of model 1, obtained by setting the right-hand sides of the equations in the model to zero, is given by

$$\mathscr{E}_0 = (S^*, S_E^*, C^*, C_R^*, S_V^*, V^*, S_R^*, T^*) \\ = \left(\frac{\Delta}{\eta_D}, 0, 0, 0, 0, 0, 0, \frac{\phi_1 \eta_T \Delta}{\eta_D \eta_I} \right). \quad (10)$$

The nonnegative matrix, $\mathscr{F}$, of new criminality terms and the M-matrix, $\mathscr{V}$, of transfer terms associated with the model 1, needed for computing the crime generation number, $\mathscr{R}_0$ using the next generation method [51], are given respectively, by

$$\mathscr{F}=\begin{pmatrix}0&0&0&0&0&0&0\\0&0&\beta_1(1+\gamma_1+\gamma_2)&\eta_K\beta_1(1+\gamma_1+\gamma_2)&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&\beta_2(1+\alpha)&\eta_{V1}\beta_2(1+\alpha)&0&\eta_V\beta_2(1+\alpha)&\sigma\eta_V\beta_2\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\end{pmatrix},\tag{11}$$

and

$$\mathscr{V}=\begin{pmatrix}\eta_D&0&0&-\psi_1&0&0&-\psi_2\\0&Q_1&0&0&0&0&0\\0&-\kappa&Q_2&0&0&0&0\\0&0&-\tau_1&Q_3&0&0&0\\0&0&0&0&Q_4&0&0\\0&0&0&0&-\upsilon&Q_5&0\\0&0&0&0&0&0&Q_6\end{pmatrix}.\tag{12}$$

Thus, the *basic reproductive* number denoted by $\mathscr{R}_0=\rho\left(\mathscr{F}\mathscr{V}^{-1}\right)$, with ρ as the spectral radius, is given by

$$\mathscr{R}_0=\max\{\mathscr{R}_K,\mathscr{R}_V\},\tag{13}$$

where,

$$\mathscr{R}_K=\frac{\kappa\beta_1(1+\gamma_1+\gamma_2)(\tau_1\eta_K+Q_3)}{Q_1Q_2Q_3},\ \text{and}$$
$$\mathscr{R}_V=\frac{\upsilon\eta_V\beta_2(\sigma\tau_2+(1+\alpha)Q_6)}{Q_4Q_5Q_6}.\tag{14}$$

The following result is established from Theorem 2 of [51].

Lemma 1. *The CFE of model (4), given by (10) is locally asymptotically stable (LAS) if $\mathscr{R}_0<1$ and unstable if $\mathscr{R}_0>1$.*

The criminological implication of Lemma 1 is that crime and criminality can be eradicated from the community (with $\mathscr{R}_0<1$) if the initial densities of the model's classes are in the basin of attraction of $\mathscr{E}_0$.

3.1.1. Analyses of the Basic Reproduction Numbers. In this section, analyses of the two threshold quantities $\mathscr{R}_K$ and $\mathscr{R}_V$ are performed to ascertain which of the rehabilitation strategies is most effective in the containment of crime and criminality in the community. It can be observed from the expressions for $\mathscr{R}_K$ and $\mathscr{R}_V$ that

$$\lim_{\tau_1\to\infty}\mathscr{R}_K=\frac{\kappa\eta_K\beta_1(1+\gamma_1+\gamma_2)}{(\kappa+\eta_D)(\psi_1+\eta_D)}>0,\tag{15}$$

and

$$\lim_{\tau_2\to\infty}\mathcal{R}_V=\frac{\upsilon\sigma\eta_V\beta_2}{(\upsilon+\eta_D)(\psi_2+\eta_D)}<0,\tag{16}$$

from the foregoing, sufficiently effective reformation and psychotherapeutic programmes that are implemented, respectively, on criminals (at a rate $\tau_1\to\infty$) and victims of crime (at rate $\tau_2\to\infty$) have the enormous capacity to effectively contain crime and criminality if in each case the right-hand sides of (15) and (16) can be made less than unity.

By computing the partial derivatives of $\mathscr{R}_K$ and $\mathscr{R}_V$, respectively, with respect to τ_2 and τ_1 gives more insights into the impact of the rehabilitation parameters τ_1 and τ_2 on model (1). It can be verified that

$$\frac{\partial\mathscr{R}_K}{\partial\tau_1}=\frac{\kappa\beta_1(1+\gamma_1+\gamma_2)[\eta_KQ_2-(\eta_K\tau_1+Q_3)]}{(\kappa+\eta_D)(\tau_1+\eta_C+\eta_D)^2(\psi_1+\eta_D)}\tag{17}$$

and

$$\frac{\partial\mathscr{R}_V}{\partial\tau_2}=\frac{\upsilon\eta_V\beta_2[\sigma Q_5-(\sigma\tau_2+(1+\alpha)Q_6)]}{(\upsilon+\eta_D)(\psi_2+\eta_D)(\tau_2+\eta_D)^2}\tag{18}$$

From where it can be deduced, respectively, from (17) and (18) that $(\partial\mathscr{R}_K)/(\partial\tau_1)<0$ if $\eta_K(\eta_C+\eta_D)<\psi_1+\eta_D$ and that $(\partial\mathscr{R}_V)/(\partial\tau_2)<0$ if $\sigma\eta_D<(1+\alpha)(\psi_2+\eta_D)$.

3.2. Global Stability of the CFE

Theorem 1. *The CFE of model (4) is globally-asymptotically stable (GAS) in $\mathscr{D}$ when ever $\mathscr{R}_K\le 1$ and $\mathscr{R}_V\le 1$.*

We will perform this analysis on the assumption that $\alpha=\gamma_1=\gamma_2=\psi_1=\psi_2=0$. Further, we follow the procedure in [52] to confirm the present claim. To achieve this, we apply the fluctuation lemma (Lemma 2.1) of [52] which we reproduce below for emphasis.

Lemma 2. *(Thieme [52]) Given a real-valued function f on $0,\infty$. Define $f_\infty=\lim\inf_{t\to\infty}f(t)$ and $f^\infty=\lim\sup_{t\to\infty}f(t)$; so that if $f:[0,\infty)\to\mathbb{R}$ is bounded and twice continuously differentiable with bounded second derivative and that t_n, $t_n\to\infty$ is a sequence such that $f(t_n)$ converges to f^∞ or f_∞ for $n\to\infty$; then $f'(t_n)\to 0$ as $n\to\infty$.*

Proof. Let $\mathscr{R}_K<1$. Choose a sequence $t_n\to\infty$ such that.

$$C(t_n)\to C^\infty\Rightarrow\frac{d}{dt}C(t_n)\to 0.\tag{19}$$

It then follows from the equation for $(dC)/(dt)$ and [52] that

$$C^\infty\le\frac{\kappa}{Q_2}S_E^\infty.\tag{20}$$

Next, we choose a sequence $q_n\to\infty$ with the condition that

$$C_R(q_n)\to C_R^\infty,\frac{d}{dt}C_R(q_n)\to 0,\tag{21}$$

thus using the equation for $(dC_R)/(dt)$ in model (1) gives

$$C_R^\infty\le\frac{\tau_1}{Q_3}C^\infty,\tag{22}$$

and applying the relation (20), we have

$$C_R^{\infty} \le \frac{\kappa\tau_1}{Q_2Q_3} S_E^{\infty}. \quad (23)$$

Finally, for some sequence $g_n \to \infty$ with the condition that

$$S_E(g_n) \to S_E R^{\infty}, \frac{d}{dt} S_E(g_n) \to 0, \quad (24)$$

thus using the equation for $(dS_E)/(dt)$ and model 1 together with (20), (23) and the expression for λ_K, gives

$$C_R^{\infty} \le \frac{\tau_1}{Q_3} C^{\infty}, \quad (25)$$

and applying the relation (20), we have

$$\begin{aligned} 0 &\le \beta_1 \left(\frac{S}{P}\right)^{\infty} \left(C^{\infty} + \eta_K C_R^{\infty}\right) - Q_1 S_E^{\infty} \\ &\le \beta_1 \left(C^{\infty} + \eta_K C_R^{\infty}\right) - Q_1 S_E^{\infty} \\ &\beta_1 \frac{\kappa(\tau_1\eta_K + Q_3)}{Q_2Q_3} S_E^{\infty} - Q_1 S_E^{\infty} \\ &= Q_1 \left[\beta_1 \frac{\kappa(\tau_1\eta_K + Q_3)}{Q_1Q_2Q_3} - 1\right] S_E^{\infty} \\ &= Q_1 (\mathscr{R}_K - 1) S_E^{\infty}. \end{aligned} \quad (26)$$

This implies that $S_E^{\infty} \le 0$ since ($\mathscr{R}_K < 1$). But hypothetically $(S_E)_{\infty} \ge 0$, hence leading to a contradiction. Thus, $S_E = (S_E)_{\infty} = 0$, and therefore

$$S_E(t) \to 0, t \to \infty. \quad (27)$$

Similarly, for $\mathscr{R}_V < 1$. Choose a sequence $a_n \to \infty$ such that

$$V(a_n) \to V^{\infty} \Rightarrow \frac{d}{dt} V(a_n) \to 0. \quad (28)$$

Thus from the equation for $(dV)/(dt)$ and [52] it follows that

$$V^{\infty} \le \frac{\upsilon}{Q_5} S_V^{\infty}. \quad (29)$$

Again, for a sequence $x_n \to \infty$ with the property that

$$S_R(x_n) \to S_R^{\infty}, \frac{d}{dt} S_R(x_n) \to 0. \quad (30)$$

Then using the equation for $(dS_R)/(dt)$ in model (1) gives

$$S_R^{\infty} \le \frac{\tau_2}{Q_6} V^{\infty}. \quad (31)$$

Thus substituting (29) into (31), we have

$$S_R^{\infty} \le \frac{\upsilon\tau_2}{Q_5Q_6} S_V^{\infty}. \quad (32)$$

Further, for a sequence $y_n \to \infty$ such that

$$S_V(y_n) \to S_V^{\infty}, \frac{d}{dt} S_E(g_n) \to 0, \quad (33)$$

then on using the equation for $(dS_V)/(dt)$ from model (1), and the expression for λ_V it follows that

$$\begin{aligned} 0 &\le \beta_2 \left(\frac{S}{P}\right)^{\infty} \left[C^{\infty} + \eta_V\left(V^{\infty} + \sigma S_R^{\infty}\right) + \eta_{V1} C_R^{\infty}\right] - Q_4 S_V^{\infty} \\ &\le \beta_2 \left[C^{\infty} + \eta_V\left(V^{\infty} + \sigma S_R^{\infty}\right) + \eta_{V1} C_R^{\infty}\right] - Q_4 S_V^{\infty}. \end{aligned} \quad (34)$$

Thus, on substituting equations (20), (23), (29), (32), into (1), it will follow that

$$\begin{aligned} 0 &\le \beta_2 \left[C^{\infty} + \eta_V\left(V^{\infty} + \sigma V_R^{\infty}\right) + \eta_{V1} C_R^{\infty}\right] - Q_4 S_V^{\infty} \\ &\le \beta_2 \left[\kappa\left(\frac{\tau_1\eta_{V1} + Q_3}{Q_2Q_3}\right) S_E^{\infty} + \sigma\eta_V \left(\frac{\sigma\tau_2 + Q_6}{Q_5Q_6}\right) S_V^{\infty}\right] - Q_4 S_V^{\infty} \\ &\le \sigma\eta_V\beta_2 \left(\frac{\sigma\tau_2 + Q_6}{Q_5Q_6}\right) S_V^{\infty} - Q_4 S_V^{\infty} \\ &= Q_4 \left[\sigma\eta_V\beta_2 \frac{\upsilon\tau_2 + Q_6}{Q_4Q_5Q_6} - 1\right] S_V^{\infty} \\ &= Q_4 (\mathscr{R}_V - 1) S_V^{\infty}. \end{aligned} \quad (35)$$

This implies that $S_V^{\infty} \le 0$ since ($\mathscr{R}_V < 1$). However, the previous hypothesis that $(S_V)_{\infty} \ge 0$ leads to a contradiction. Thus, $S_V = (S_V)_{\infty} = 0$, and therefore

$$S_V(t) \to 0, t \to \infty. \quad (36)$$

From the foregoing, it thus follows that

$$C(t) \to 0, C_R(t) \to 0, V(t) \to 0, V_R(t) \to 0, \text{as } t \to \infty. \quad (37)$$

In which case it will follow from the equation for $(dP)/(dt)$ in equation (1) that

$$P_{\infty} \ge \frac{\Delta}{\eta_D}\left(\Delta - \eta_C C_C^{\eta}\right) = \frac{\Delta}{\eta_D}. \quad (38)$$

However, with $P^{\infty} \le 1/\eta_D$, then

$$P_{\infty} = P^{\infty} = \frac{\Delta}{\eta_D}. \quad (39)$$

Finally, from the equation for dI/dt in model (1), we get that

$$\begin{aligned} I_{\infty} \ge \frac{\eta_T}{\eta_I} &\left(\phi_1 S^{\infty} + \phi_2 S_E^{\infty} + \phi_3 C^{\infty} + \phi_4 C_R^{\infty} + \phi_5 S_V^{\infty}\right. \\ &\left. + \phi_6 K^{\infty} + \phi_7 V_R^{\infty}\right) \ge \frac{\phi_1\eta_T}{\eta_D\eta_I}. \end{aligned} \quad (40)$$

However, with $I^{\infty} \le (\phi_1\eta_T)/(\eta_D\eta_I)$, it follows that

$$I_{\infty} = I^{\infty} = \frac{\phi_1\eta_T}{\eta_D\eta_I}. \quad (41)$$

Thus the greatest compact invariant set in

$$\mathscr{D} = \left\{\left(S, S_E, C, C_R, S_V, V, V_R\right) \in \mathbb{R}_7^+ | P \le \frac{\Delta}{\eta_D}, (I) \in \mathbb{R}^+ | I \le \frac{\phi_1\Delta}{\eta_D\eta_I}\right\} \quad (42)$$

is the singleton set $\{\mathscr{E}_0\}$. Thus

$$\left(S(t), S_E(t), C(t), C_R(t), S_V(t), V(t), V_R(t), I(t)\right) \to (0,0,0,0,0,0,0,0), \quad (43)$$

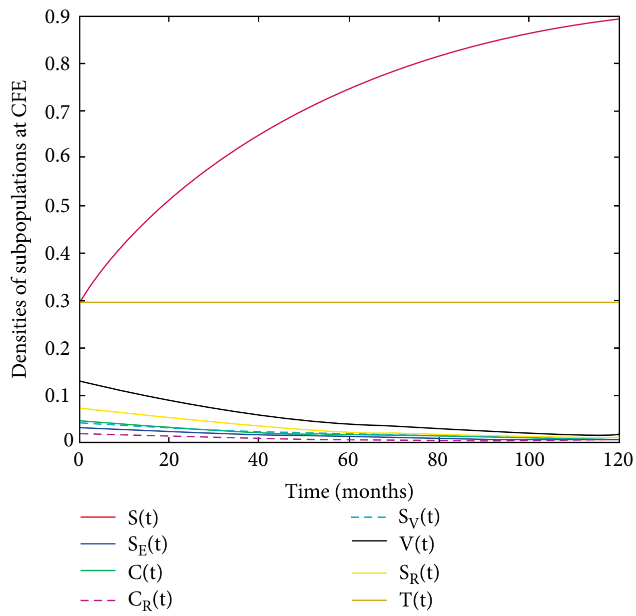

FIGURE 2: CFE, $\mathscr{E}_0$, is GAS when $\mathscr{R}_0 < 1$.

as $t \to \infty$. Therefore, every solution of the equations in the model (4) with initial conditions in $\mathscr{D}$ approaches $\mathscr{E}_0$ as $t \to \infty$, whenever $\mathscr{R}_0 \leq 1$. Thus, $\mathscr{E}_0$ is GAS in $\mathscr{D}$ provided that $\mathscr{R}_0 \leq 1$ and $\mathscr{R}_0 \leq 1$ simultaneously.

The criminological implication of this theorem is that if $\mathscr{R}_0 = \max\{\mathscr{R}_K, \mathscr{R}_V\}$ has a value less than unity, then a small influx of criminal individuals into the community is not sufficient to result into criminal tendencies and so criminality would be substantially contained to such levels that would not result in the escalation of crime and criminality.

In Figure 2, the respective densities of the model's subpopulations confirm that the CFE, $\mathscr{E}_0$, of the model 4 is indeed globally asymptotically stable with $\mathscr{R}_0 = \max\{\mathscr{R}_K, \mathscr{R}_V\} = 0$, thus verifying Theorem 1.

3.3. Existence and Stability of Persistent-Crime Equilibria. Existence The presence of crime, criminality and victimisation in the community will guarantee the existence of a persistent-crime equilibrium, $\mathscr{E}_1 = (S^{**}, S_E^{**}, C^{**}, C_R^{**}, S_V^{**}, V^{**}, V_R^{**}, T^{**})$. Again, we will perform this analysis on the assumption that $\alpha = \gamma_1 = \gamma_2 = \psi_1 = \psi_2 = 0$. That is, that persistence does not require any form of modification. Further, as previously stated, it is sufficient to exclude the eighth equation of system (4). It can easily be seen also from model (4) that

$$P^{**} = \frac{\Delta - \eta_C}{\eta_D}, \tag{44}$$

thus reducing the model (4) to:

$$\begin{aligned}
\frac{dS}{dt} &= \Delta - (\eta_D + \lambda_K + \lambda_V)S, \\
\frac{dS_E}{dt} &= \lambda_K S - (\kappa + \eta_D)S_E, \\
\frac{dC}{dt} &= \kappa S_E - (\tau_1 + \eta_C + \eta_D)C, \\
\frac{dC_R}{dt} &= \tau_1 C - \eta_D C_R, \\
\frac{dS_V}{dt} &= \lambda_V S - (\upsilon + \eta_D)S_R, \\
\frac{dV}{dt} &= \upsilon S_V - (\tau_2 + \eta_D)V, \\
\frac{dV_R}{dt} &= \tau_2 K - \eta_D V_R, \\
\frac{dt}{dt} &= \eta_T(\phi_1 S + \phi_2 S_E + \phi_3 C + \phi_4 C_R + \phi_5 S_V + \phi_6 V + \phi_7 V_R) - \eta_I T.
\end{aligned} \tag{45}$$

The unique persistence equilibrium of the reduced system (45) is given by $\mathscr{E}_1 = (S^{**}, S_E^{**}, C^{**}, C_R^{**}, S_V^{**}, V^{**}, V_R^{**}, T^{**})$, where

$$\begin{aligned}
S^{**} &= \frac{\Delta + \psi_1 C_R^{**} + \psi_2 S_R^{**}}{\eta_D + \lambda_K^{**} + \lambda_V^{**}}, S_E^{**} = \frac{\lambda_K^{**}(S^{**} + \gamma_1 V^{**} + \gamma_2 S_R^{**})}{\kappa + \eta_D}, \\
C^{**} &= \frac{\kappa S_E^{**}}{\tau_1 + \eta_C + \eta_D}, C_R^{**} = \frac{\tau_1 C^{**}}{\psi_1 + \eta_D}, S_V^{**} = \frac{\lambda_V^{**} S^{**} + \alpha \lambda_{V1}^{**} S_R^{**}}{\upsilon + \eta_D}, \\
V^{**} &= \frac{\upsilon S_V^{**}}{\gamma_1 \lambda_K^{**} + \tau_2 + \eta_D}, S_R^{**} = \frac{\tau_2 V^{**}}{\psi_2 + \eta_D + \gamma_2 \lambda_K^{**} + \alpha \lambda_{V1}^{**}}, \\
T^{**} &= \frac{\eta_T}{\eta_I}(\phi_1 S^{**} + \phi_2 S_E^{**} + \phi_3 C^{**} + \phi_4 C_R^{**} \\
&\quad + \phi_5 S_V^{**} + \phi_6 V^{**} + \phi_7 S_R^{**}).
\end{aligned} \tag{46}$$

The corresponding forces of criminalisation and victimisation become, respectively

$$\lambda_K^{**} = \frac{\eta_D \beta_1 (C^{**} + \eta_K C_R^{**})}{\Delta - \eta_C C^{**}}, \tag{47}$$

and

$$\lambda_V^{**} = \frac{\eta_D \beta_2 [C^{**} + \eta_V (V^{**} + \sigma V_R^{**}) + \eta_{V1} C_R^{**}]}{\Delta - \eta_C C^{**}}. \tag{48}$$

Simplifying (47) and (48) using (46) gives

$$\lambda_K^{**} = \frac{\kappa \beta_1 [\eta_D + \tau_1 \eta_K Q_1 Q_2 (\eta_D + \lambda_K^{**} + \lambda_V^{**})] \lambda_K^{**}}{a_1 \lambda_K^{**} + Q_1 Q_2 (\eta_D + \lambda_V^{**})}, \tag{49}$$

and

$$\lambda_K^{**} = \frac{\kappa \beta_2 [\eta_D + \tau_1 \eta_{V1} Q_4 Q_5 (\eta_D + \lambda_K^{**} + \lambda_V^{**})] \lambda_K^{**} + a_2 (\eta_D + \lambda_K^{**} + \lambda_V^{**}) \lambda_V^{**}}{a_1 \lambda_K^{**} + Q_4 Q_5 [Q_1 Q_2 (\eta_D + \lambda_V^{**})]}, \tag{50}$$

where $a_1 = \kappa(\tau_1 + \eta_D) + Q_2 Q_3$ and $a_2 = \upsilon \eta_V \beta_2 Q_1 Q_2 (\eta_D + \sigma \tau_2)$.

The positive equilibrium of the reduced system (45) can be obtained by simultaneously solving for λ_K^{**} and λ_V^{**} in (49) and (50) and using the result in (46). Obviously $\lambda_K^{**} = \lambda_V^{**} = 0$ is a fixed point of (49) and (50). On the other hand, the values $\lambda_K^{**} \neq 0$ and $\lambda_V^{**} \neq$ in (49) and (50) gives the nontrivial solutions of λ_K^{**} and λ_V^{**} as

$$\lambda_K^{**} = \frac{\eta_D Q_1 Q_2 Q_3(\mathscr{R}_V - 1)(\mathscr{R}_K - \mathscr{R}_V)}{a_1 Q_3(\mathscr{R}_K - \mathscr{R}_V) + \kappa\beta_2(\tau_1\eta_{V1} + Q_3)}, \quad (51)$$

and

$$\lambda_V^{**} = \frac{\kappa\eta_D\beta_2(\tau_1\eta_{V1} + Q_3)(\mathscr{R}_K - 1)}{a_1 Q_3(\mathscr{R}_K - \mathscr{R}_V) + \kappa\beta_2(\tau_1\eta_{V1} + Q_3)}. \quad (52)$$

Since all parameters are necessarily nonnegative, it follows that both (51) and (52) are also nonnegative for $\mathscr{R}_K > 1$, $\mathscr{R}_V > 1$ and $\mathscr{R}_K > \mathscr{R}_V$, while for $\mathscr{R}_K < 1$ and $\mathscr{R}_V < 1$ both (51) and (52) are negative, which is meaningless in criminological sense. Finally, $\mathscr{R}_K = \mathscr{R}_K = 1$, will imply that both (51) and (52) are zero, which corresponds to the CFE $\mathscr{E}_0$ Thus, we have establish the following result.

Lemma 3. *The unique PCE $\mathscr{E}_0$ of the reduced crime and victimisation model (68) is locally asymptotically stable if and only if $\mathscr{R}_0 > 1$.*

Theorem 2. *The PCE of the crime model (4) is GAS whenever .*

$\mathscr{R}_0 > 1$.

Proof. The prove is achieved using the Goh-Volterra type nonlinear. Lyapunov function.

Consider the following Lyapunov function

$$\begin{aligned}\mathscr{U} &= S - S^{**} - S^{**}\ln\frac{S}{S^{**}} + S_E - S_E^{**} \\ &- S_E^{**}\ln\frac{S_E}{S_E^{**}} + A_1\left(C - C^{**} - C^{**}\ln\frac{C}{C^{**}}\right) \\ &+ A_2\left(C_R - C_R^{**} - C_R^{**}\ln\frac{C_R}{C_R^{**}}\right) + S_V - S_V^{**} \\ &- S_V^{**}\ln\frac{S_V}{S_V^{**}} + A_3\left(V - V^{**} - V^{**}\ln\frac{V}{V^{**}}\right) \\ &+ A_4\left(S_R - S_R^{**} - S_R^{**}\ln\frac{S_R}{S_R^{**}}\right) + T - T^{**} - T^{**}\ln\frac{T}{T^{**}}.\end{aligned} \quad (53)$$

Thus the time derivative of the Lyapunov function is

$$\begin{aligned}\dot{\mathscr{U}} &= \dot{S}\left(1 - \frac{S^{**}}{S}\right) + \dot{S}_E\left(1 - \frac{S_E^{**}}{S_E}\right) + A_1C^{**}\left(1 - \frac{C^{**}}{C}\right) \\ &+ A_2C_R^{**}\left(1 - \frac{C_R^{**}}{C_R}\right) + \dot{S}_V\left(1 - \frac{S_V^{**}}{S_V}\right) + A_3V^{**}\left(1 - \frac{V^{**}}{V}\right) \\ &+ A_4S_R^{**}\left(1 - \frac{S_R^{**}}{S_R}\right) + \dot{T}\left(1 - \frac{T^{**}}{T}\right),\end{aligned} \quad (54)$$

where

$$A_1 = \frac{\eta_T(\phi_3\eta_D + \phi_4\tau_1) + \left[\tilde{\beta}_1(\eta_D + \tau_1\eta_K) + \tilde{\beta}_2(\eta_D + \tau_1\eta_{V1})\right]S^{**}}{\eta_D Q_2},$$

$$A_2 = \frac{\phi_4\eta_T + \left(\eta_K\tilde{\beta}_1 + \eta_{V1}\tilde{\beta}_2\right)S^{**}}{\eta_D}, \quad (55)$$

$$\begin{aligned}A_3 &= \frac{\eta_T(\phi_6\eta_D + \phi_7\tau_2) + \eta_V\tilde{\beta}_2(\eta_D + \sigma\tau_2)S^{**}}{\eta_D Q_5}, \\ A_4 &= \frac{\phi_7\eta_T + \sigma\eta_V\tilde{\beta}_2 S^{**}}{\eta_D}.\end{aligned} \quad (56)$$

with $\tilde{\beta} = \beta/\Delta$. The forces of criminalisation and victimisation therefore become $\hat{\lambda}_K = \tilde{\beta}_1(C + \eta_K C_R)$ and $\hat{\lambda}_V = [C + \eta_{V1}C_R + \eta_V(V + \sigma S_R)]$, respectively. Further, it can easily be shown from model *(4)* that at the persistent steady-state,

$$\begin{aligned}Q_1S_E^{**} &= \tilde{\beta}_1 S^{**}(C^{**} + \eta_K C_R^{**}), \\ Q_4S_V^{**} &= S^{**}\tilde{\beta}_2[C^{**} + \eta_V(V^{**} + \sigma V_R^{**}) + \eta_{V1}C_R^{**}], \\ \Delta &= \eta_D + Q_1S_E^{**} + Q_4S_V^{**}, Q_2 = \kappa S_E^{**}, \\ \eta_D C_R^{**} &= \tau_1 C^{**}, Q_5V^{**} = \upsilon S_V^{**}, \eta_D S_R^{**} = \tau_2 V^{**}, \\ \eta_I T^{**} &= \eta_T(\phi_1 S^{**} + \phi_2 S_E^{**} + \phi_3 C^{**} \\ &+ \phi_4 C_R^{**} + \phi_5 S_V^{**} + \phi_6 S^{**} + \phi_7 V_R^{**}).\end{aligned} \quad (57)$$

Substituting these expressions together with the right hand sides of the equations in model (1) into the time derivative of the Lyapunov function (54) and simplifying give

$$\begin{aligned}\dot{\mathscr{U}} &= \eta_D S^{**}\left(2 - \frac{S}{S^{**}} - \frac{S^{**}}{S}\right) + \tilde{\beta}_1 S^{**}C^{**}\left(3 - \frac{S^{**}}{S} - \frac{S}{S^{**}}\frac{S_E^{**}}{S_E} - \frac{C^{**}}{C}\frac{S_E}{S_E^{**}}\right) \\ &+ \eta_T\phi_1 S^{**}\left(1 - \frac{S}{S^{**}}\frac{T^{**}}{T}\right) + \eta_K\tilde{\beta}_1 S^{**}C_R^{**}\left(4 - \frac{S^{**}}{S} - \frac{S}{S^{**}}\frac{S_E^{**}}{S_E} - \frac{C^{**}}{C}\frac{S_E}{S_E^{**}} - \frac{C}{C^{**}}\frac{C_R^{**}}{C_R}\right) \\ &+ \phi_3\eta_T C^{**}\left(2 - \frac{C^{**}}{C}\frac{S_E}{S_E^{**}} - \frac{C}{C^{**}}\frac{T^{**}}{T}\right) + \sigma\eta_V\tilde{\beta}_2 S^{**}S_R^{**}\left(4 - \frac{S^{**}}{S} - \frac{S_R^{**}}{S_R} - \frac{S}{S^{**}}\frac{S_V^{**}}{S_V} - \frac{V^{**}}{V}\frac{S_V}{S_V^{**}}\right) \\ &+ \phi_6\eta_T V^{**}\left(2 - \frac{V^{**}}{V}\frac{S_V}{S_V^{**}} - \frac{T^{**}}{T}\frac{V}{V^{**}}\right) + \eta_{V1}\tilde{\beta}_2 S^{**}C_R^{**}\left(4 - \frac{S^{**}}{S} - \frac{C}{C^{**}}\frac{C_R^{**}}{C_R} - \frac{C^{**}}{C}\frac{S_E}{S_E^{**}} - \frac{S}{S^{**}}\frac{S_V^{**}}{S_V}\right) \\ &+ \phi_7\eta_T S_R^{**}\left(3 - \frac{S_R^{**}}{S_R} - \frac{S_R}{S_R^{**}}\frac{T^{**}}{T} - \frac{V^{**}}{V}\frac{S_V}{S_V^{**}}\right) + \phi_4\eta_T C_R^{**}\left(3 - \frac{C}{C^{**}}\frac{C_R^{**}}{C_R} - \frac{C_R}{C_R^{**}}\frac{T^{**}}{T} - \frac{C^{**}}{C}\frac{S_E}{S_E^{**}}\right) \\ &+ \tilde{\beta}_2 S^{**}C^{**}\left(3 - \frac{S^{**}}{S} - \frac{S}{S^{**}}\frac{S_V^{**}}{S_V} - \frac{C^{**}}{C}\frac{S_E}{S_E^{**}}\right) + \eta_V\tilde{\beta}_2 S^{**}V^{**}\left(3 - \frac{S^{**}}{S} - \frac{S}{S^{**}}\frac{S_V^{**}}{S_V} - \frac{V^{**}}{V}\frac{S_V}{S_V^{**}}\right) \\ &+ \phi_5\eta_T S_V^{**}\left(1 - \frac{S_V}{S_V^{**}}\frac{T^{**}}{T}\right) + \phi_2\eta_T S_E^{**}\left(1 - \frac{S_E}{S_E^{**}}\frac{T^{**}}{T}\right).\end{aligned} \quad (58)$$

Since the arithmetic mean exceeds the geometric mean, the following inequalities hold:

$$\begin{aligned}
&2-\frac{S}{S^{**}}-\frac{S^{**}}{S}\leq 0, 3-\frac{S^{**}}{S}-\frac{S}{S^{**}}\frac{S_E^{**}}{S_E}-\frac{C^{**}}{C}\frac{S_E}{S_E^{**}}\leq 0,\\
&1-\frac{S}{S^{**}}\frac{T^{**}}{T}\leq 0, 3-\frac{S^{**}}{S}-\frac{S}{S^{**}}\frac{S_V^{**}}{S_V}-\frac{C^{**}}{C}\frac{S_E}{S_E^{**}}\leq 0,\\
&3-\frac{S^{**}}{S}-\frac{S}{S^{**}}\frac{S_V^{**}}{S_V}-\frac{V^{**}}{V}\frac{S_V}{S_V^{**}}\leq 0,\\
&4-\frac{S^{**}}{S}-\frac{S}{S^{**}}\frac{S_E^{**}}{S_E}-\frac{C^{**}}{C}\frac{S_E}{S_E^{**}}-\frac{C}{C^{**}}\frac{C_R^{**}}{C_R}\leq 0,\\
&2-\frac{C^{**}}{C}\frac{S_E}{S_E}-\frac{C}{C^{**}}\frac{T^{**}}{T}\leq 0,\\
&4-\frac{S^{**}}{S}-\frac{S_R^{**}}{S_R}-\frac{S}{S^{**}}\frac{S_V^{**}}{S_V}-\frac{V^{**}}{V}\frac{S_V}{S_V^{**}}\leq 0,\\
&2-\frac{V^{**}}{V}\frac{S_V}{S_V^{**}}-\frac{V}{V^{**}}\frac{T^{**}}{T}\leq 0,\\
&4-\frac{S^{**}}{S}-\frac{C}{C^{**}}\frac{C_R^{**}}{C_R}-\frac{C^{**}}{C}\frac{S_E}{S_E^{**}}-\frac{S}{S^{**}}\frac{S_V^{**}}{S_V}\leq 0\\
&1-\frac{S_E}{S_E^{**}}\frac{T^{**}}{T}\leq 0, 3-\frac{S_R^{**}}{S_R}-\frac{S_R}{S_R^{**}}\frac{T^{**}}{T}-\frac{V^{**}}{V}\frac{S_V}{S_V^{**}}\leq 0,\\
&3-\frac{C}{C^{**}}\frac{C_R^{**}}{C_R}-\frac{C_R}{C_R^{**}}\frac{T^{**}}{T}-\frac{S_E}{S_E^{**}}-\frac{C^{**}}{C}\leq 0, 1-\frac{S_V}{S_V^{**}}\frac{T^{**}}{T}\leq 0.
\end{aligned} \tag{59}$$

We observe that at the crime persistent steady-state, since $t\to\infty$, then

$$\begin{aligned}
&\lim_{t\to\infty}S(t)=S^{**}, \lim_{t\to\infty}S_E(t)=S_E^{**}, \lim_{t\to\infty}C(t)=C^{**},\\
&\lim_{t\to\infty}C_R(t)=C_R^{**}, \lim_{t\to\infty}S_V(t)=S_V^{**}, \lim_{t\to\infty}V(t)=V^{**},\\
&\lim_{t\to\infty}S_R(t)=S_R^{**}, \lim_{t\to\infty}T(t)=T^{**}.
\end{aligned} \tag{60}$$

Hence

$$\begin{aligned}
&2-\frac{S}{S^{**}}-\frac{S^{**}}{S}=3-\frac{S^{**}}{S}-\frac{S}{S^{**}}\frac{S_E^{**}}{S_E}-\frac{C^{**}}{C}\frac{S_E}{S_E^{**}}=1-\frac{S}{S^{**}}\frac{T^{**}}{T}\\
&=3-\frac{S^{**}}{S}-\frac{S}{S^{**}}\frac{S_V^{**}}{S_V}-\frac{C^{**}}{C}\frac{S_E}{S_E^{**}}=0,\\
&3-\frac{S^{**}}{S}-\frac{S}{S^{**}}\frac{S_V^{**}}{S_V}-\frac{V^{**}}{V}\frac{S_V}{S_V^{**}}=4-\frac{S^{**}}{S}-\frac{S}{S^{**}}\frac{S_E^{**}}{S_E}\\
&\quad-\frac{C^{**}}{C}\frac{S_E}{S_E^{**}}-\frac{C}{C^{**}}\frac{C_R^{**}}{C_R}=2-\frac{C^{**}}{C}\frac{S_E}{S_E}-\frac{C}{C^{**}}\frac{T^{**}}{T}=0,\\
&4-\frac{S^{**}}{S}-\frac{S_R^{**}}{S_R}-\frac{S}{S^{**}}\frac{S_V^{**}}{S_V}-\frac{V^{**}}{V}\frac{S_V}{S_V^{**}}=2-\frac{V^{**}}{V}\frac{S_V}{S_V^{**}}-\frac{V}{V^{**}}\frac{T^{**}}{T}\\
&=4-\frac{S^{**}}{S}-\frac{C}{C^{**}}\frac{C_R^{**}}{C_R}-\frac{C^{**}}{C}\frac{S_E}{S_E^{**}}-\frac{S}{S^{**}}\frac{S_V^{**}}{S_V}=0\\
&1-\frac{S_E}{S_E^{**}}\frac{T^{**}}{T}=3-\frac{S_R^{**}}{S_R}-\frac{S_R}{S_R^{**}}\frac{T^{**}}{T}-\frac{V^{**}}{V}\frac{S_V}{S_V^{**}}=3-\frac{C}{C^{**}}\frac{C_R^{**}}{C_R}\\
&\quad-\frac{C_R}{C_R^{**}}\frac{T^{**}}{T}-\frac{S_E}{S_E^{**}}-\frac{C^{**}}{C}=1-\frac{S_V}{S_V^{**}}\frac{T^{**}}{T}=0\\
&C^{**}-C=C_R^{**}-C_R=S_V^{**}-S_V=V^{**}-V=S_R^{**}-S_R=0.
\end{aligned} \tag{61}$$

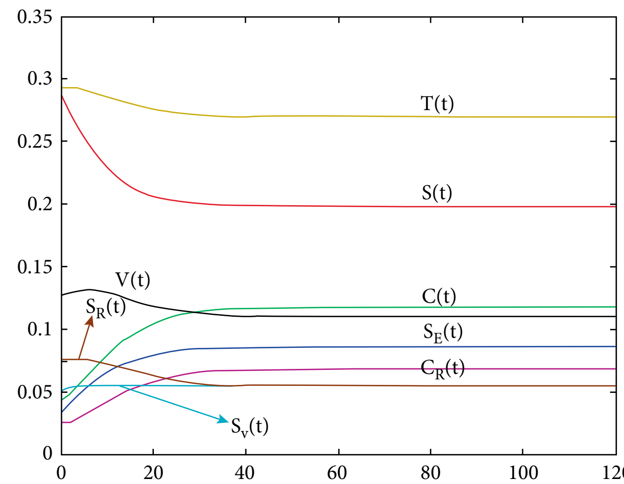

FIGURE 3: CFE, $\mathscr{E}_1$, is GAS when $\mathscr{R}_0=\max\{\mathscr{R}_K,\mathscr{R}_V\}>1$.

Thus, since all the model parameters are nonnegative, it follows that $\dot{\mathscr{U}}\leq 0$ for $\mathscr{R}_0>1$. Hence $\mathscr{U}$ is a Lyapunov function on $\mathscr{D}$. The proof is thus completed following through from LaSelle's Invariance Principle [53] that every solution to the equation in model (4) with initial conditions in $\mathscr{D}$, approaches $\mathscr{E}_1$ as $t\to\infty$, so that $\mathscr{E}_1$ is GAS in $\mathscr{D}$ whenever $\mathscr{R}_0=\max\{\mathscr{R}_K,\mathscr{R}_V\}>1$.

In Figure 3, the respective densities of the model's subpopulations confirm that the PCE, $\mathscr{E}_1$, of model 1 is globally asymptotically stable with $\mathscr{R}_0=\max\{\mathscr{R}_K=2.2534, \mathscr{R}_V=1.7181\}=2.2534<1$, thus verifying Theorem 2. The sociological implication of this theorem is further buttressed by Figures 9, 10, 11 and 12. Here, the convergence of the densities of crime/criminality and victimisation to the PCE is observed despite the increasing contact rates of criminalisation and victimisation. It can be seen from Figures 9 and 11 that though an increasing criminalisation contact rate increases crime and criminality, the model always converges to the PCE. Similarly, from (Figures 10 and 12), it is noted that though the cumulative density of victimisation always increases with increasing rate of victimisation, these trajectories will, in each case, converge to the PCE.

3.3.1. Backward Bifurcation Analysis. In the following, system 1 is considered for the investigation of the phenomenon of backward bifurcation involving the CFE, $\mathscr{E}_0$, for $\mathscr{R}_0=1$. More precisely, the conditions on the parameter values that cause the bifurcation (either forward or backward) are sought. The Centre Manifold Theory (CMT), [3, 49, 54], will be explored to investigate the phenomenon of backward bifurcation. To describe the CMT, consider a general system of an ODE with a parameter ϕ given by

$$\frac{dx}{dt}=f(x,\phi), f:\mathbb{R}^n\times\mathbb{R}\to\mathbb{R} \text{and} f\in\mathbb{C}^2(\mathbb{R},\mathbb{R}), \tag{62}$$

where 0 is an equilibrium point of the system. That is, $f(x,\phi)\equiv 0, \forall\phi$ and assume that:

(1) $A = \mathscr{D}_x f(0,0) = \left(\left(\partial f_i/\partial x_j\right)(0,0)\right)$ is the linearisation matrix of system (62) around the equilibrium point 0 with ϕ evaluated at zero. Zero is a simple eigenvalue of A and other eigenvalues of A have negative real parts;

(2) A has a right and left eigenvectors, W and V, respectively; each corresponding to the zero eigenvalue.

Let f_k be the kth component of f and

$$a = \sum_{k,i,j=1}^{n} v_k w_i w_j \frac{\partial^2 f_k}{\partial x_i \partial x_j}(0,0), \text{ and } b = \sum_{k,i=1}^{n} v_k w_i \frac{\partial^2 f_k}{\partial x_i \partial \phi}(0,0). \tag{63}$$

Then the local dynamics of system (63) around the point 0 is totally determined by the signs of a and b as follows.

(i) $a > 0, b > 0$. When $\phi < 0$ with $|\phi| \ll 1$, 0 is locally asymptotically stable and there exists a positive unstable equilibrium; when $0 < \phi \ll 1$, 0 is unstable and there exists a negative locally asymptotically stable equilibrium.

(ii) $a < 0, b < 0$; When $\phi < 0$ with $|\phi| \ll 1$, 0 is unstable; when $0 < \phi \ll 1$, 0 is a locally asymptotically stable equilibrium, and there exists a positive unstable equilibrium.

(iii) $a > 0, b < 0$. When $\phi < 0$ with $|\phi| \ll 1$, 0 is unstable, and there exists a locally asymptotically stable negative equilibrium; when $0 < \phi \ll 1$, 0 is stable and there exists a positive unstable negative equilibrium.

(iv) $a < 0, b > 0$. When $\phi < 0$ changes from negative to positive, 0 changes its stability from stable to unstable. Correspondingly, a negative equilibrium becomes positive and locally asymptotically stable.

Note: if $a > 0$ and $b > 0$, the a backward bifurcation occurs at $\phi = 0$.

Pursuant to the forgoing and preparatory to systemic transformation for computational convenience, the following simplifications and change of variables is made. Let $S = x_1$, $S_E = x_2$, $C = x_3$, $C_R = x_4$, $S_V = x_5$, $V = x_6$, $S_R = x_7$ and $T = x_8$; so that on using the vector notation $\boldsymbol{x} = (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8)^T$, system (1) can be rewritten in the form $d\boldsymbol{x}/dt = F(\boldsymbol{x})$, where $F = (f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8)^T$, as follows:

$$\begin{aligned}
\frac{dx_1}{dt} &= f_1 = \Delta + \psi_1 x_4 + \psi_2 x_7 - \frac{\beta_1(x_3 + \eta_K x_4) + \beta_2[x_3 + \eta_V(x_6 + \sigma x_7) + \eta_{V1}x_4]}{x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7} x_1 - \eta_D x_1, \\
\frac{dx_2}{dt} &= f_2 = \frac{\beta_1(x_3 + \eta_K x_4)(x_1 + \gamma_1 x_6 + \gamma_2 x_7)}{x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7} - Q_1 x_2, \\
\frac{dx_3}{dt} &= f_3 = \kappa x_2 - Q_2 x_3, \\
\frac{dx_4}{dt} &= f_4 = \tau_1 x_3 - Q_3 x_4, \\
\frac{dx_5}{dt} &= f_5 = \frac{\beta_2(x_1 + \alpha x_7)[x_3 + \eta_V(x_6 + \sigma x_7) + \eta_{V1}x_4]}{x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7} - Q_4 x_5 \\
\frac{dx_6}{dt} &= f_6 = \upsilon x_5 - \frac{\gamma_2\beta_1(x_3 + \eta_K x_4)x_6}{x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7} - Q_5 x_6, \\
\frac{dx_7}{dt} &= f_7 = \tau_2 x_6 - \frac{\gamma_2\beta_1(x_3 + \eta_K x_4) + \alpha\beta_2[x_3 + \eta_V(x_6 + \sigma x_7) + \eta_{V1}x_4]}{x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7} x_7 - Q_6 x_7, \\
\frac{dx_8}{dt} &= f_8 = \eta_T(\phi_1 x_1 + \phi_2 x_2 + \phi_3 x_3 + \phi_4 x_4 + \phi_5 x_5 + \phi_6 x_6 + \phi_7 x_7) - \eta_I x_8
\end{aligned} \tag{64}$$

The CFE is $\{[x_1^{**} = \Delta/\eta_D,\ x_2^{**} = 0,\ x_2^{**} = 0,\ x_2^{**} = 0,\ x_2^{**} = 0,\ x_2^{**} = 0,\ x_2^{**} = 0,\ x_8^{**} = (\phi_1\eta_T\Delta)/(\eta_D\eta_I)]\}$. Consider the case $\mathscr{R}_0 = 1$ (that is $\mathscr{R}_0 = \{\mathscr{R}_K, \mathscr{R}_V\} = 1$). Also, suppose that $\beta_1 = \beta_1^*$ is chosen as the bifurcation parameter. Solving for $\beta_1 = \beta_1^*$ from $\mathscr{R}_0 = 1$ in the transformed system (64) above gives

$$\beta_1^* = \frac{Q_1 Q_2 Q_3}{\kappa(1 + \gamma_1 + \gamma_2)(\tau_1\eta_K + Q_3)}. \tag{65}$$

The linearisation matrix of system (64), evaluated at the CFE with $\beta_1 = \beta_1^*$ is

$$\mathscr{J}(\mathscr{E}_0) = \begin{pmatrix}
-\eta_D & 0 & -(\beta_1 + \beta_2) & \psi_1 - (\eta_K\beta_1 + \eta_{V1}\beta_2) & 0 & -\eta_V\beta_2 & \psi_2 - \sigma\eta_V\beta_2 & 0 \\
0 & -Q_1 & (1 + \gamma_1 + \gamma_2)\beta_1 & \eta_K(1 + \gamma_1 + \gamma_2)\beta_1 & 0 & 0 & 0 & 0 \\
0 & \kappa & -Q_2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \tau_1 & -Q_3 & 0 & 0 & 0 & 0 \\
0 & 0 & (1 + \alpha)\beta_2 & \eta_{V1}(1 + \alpha)\beta_2 & -Q_4 & \eta_V(1 + \alpha)\beta_2 & \sigma\eta_V\beta_2 & 0 \\
0 & 0 & -\gamma_1\beta_1 & -\gamma_1\eta_K\beta_1 & \upsilon & -Q_5 & 0 & 0 \\
0 & 0 & (\gamma_2\beta_1 + \alpha\beta_2) & -(\gamma_2\eta_K\beta_1 + \alpha\eta_{V1}\beta_2) & 0 & \tau_2 & -Q_6 & 0 \\
\eta_T\phi_1 & \eta_T\phi_2 & \eta_T\phi_3 & \eta_T\phi_4 & \eta_T\phi_5 & \eta_T\phi_6 & \eta_T\phi_7 & -\eta_I
\end{pmatrix}, \tag{66}$$

Following from [2], zero is a simple eigenvalue (with all other eigenvalues having negative real parts) of $\mathscr{J}(\mathscr{E}_0)$. Therefore, the CMT can be used to analyse the dynamics of the transformed system (64) near $\beta_1 = \beta_1^*$.

Eigenvectors of $\mathcal{J}(\mathcal{E}_0)$, corresponding to $\beta_1 = \beta_1^*$. For the case $\mathcal{R}_0 = 1$, a left eigenvector of $\mathcal{J}(\mathcal{E}_0)$ that is associated with the zero eigenvalue denoted by $\boldsymbol{v} = [v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8]$, is such that

$$v_2 = \frac{\kappa}{Q_1}v_3, v_5 = \frac{\upsilon}{Q_6}v_6, v_7 = \frac{\sigma\eta_V\beta_2}{Q_6}v_5 \text{ and} \qquad (67)$$
$$v_6 = \frac{\eta_V(1+\alpha)\beta_2 v_5 + \tau_2 v_7}{Q_5}.$$

Thus, if $v_2 > 0$ then $v_3 > 0$ and if $v_5 > 0$, then $v_7 > 0$ and $v_6 > 0$, while $v_4 > 0$ provided $Q_1Q_2 > \kappa(1+\gamma_1+\gamma_2)$ and $\gamma_1\beta_1Q_4Q_6 + \sigma\upsilon\eta_V\beta_2(\gamma_2+\alpha\beta_2) < \upsilon(1+\alpha)\beta_2Q_6$ with $v_8 = v_1 = 0$.

Similarly, the components of the right eigenvector (corresponding to the zero eigenvalue), denoted by $\overline{w} = [w_1, w_2, w_3, w_4, w_5, w_6, w_7, w_8]^T$, is such that

$$\begin{aligned}
w_3 &= \frac{\kappa}{Q_2}w_2, w_4 = \frac{\tau_1}{Q_3}w_3,\\
w_5 &= \frac{(1+\alpha)\beta_2w_3 + \eta_{V1}(1+\alpha)\beta_2w_4 + \eta_V(1+\alpha)\beta_2w_6 + \sigma\eta_V\beta_2w_7}{Q_4}\\
w_6 &= \frac{(\gamma_2\beta_1+\alpha\beta_2)w_3 + (\gamma_2\eta_K\beta_1 + \alpha\eta_{V1}\beta_2)w_4 + Q_6w_7}{\tau_2},\\
w_8 &= \frac{\eta_T}{\eta_I}(\phi_1w_1 + \phi_2w_2 + \phi_3w_3 + \phi_4w_4 + \phi_5w_5 + \phi_6w_6 + \phi_7w_7).
\end{aligned} \qquad (68)$$

So that if $w_2 < 0$, then $w_3 < 0$ and it will follow immediately that $w_4 < 0$, then $w_7 < 0$, $w_5 < 0$, $w_6 < 0$ and $w_8 < 0$.

Computation of a and b.

The associated nonzero second order partial derivatives of F (at the CFE $(\mathcal{E}_0)$) are given by

$$\begin{aligned}
&\frac{\partial^2 f_1}{\partial x_2\partial x_3} = \frac{\eta_D(\beta_1^*+\beta_2)}{\Delta}, \frac{\partial^2 f_1}{\partial x_2\partial x_4} = \frac{\eta_D(\eta_K\beta_1^* + \eta_{V1}\beta_2)}{\Delta}, \frac{\partial^2 f_1}{\partial x_2\partial x_6} = \frac{\eta_D\eta_V\beta_2}{\Delta}, \frac{\partial^2 f_1}{\partial x_2\partial x_7} = \frac{\sigma\eta_D\eta_V\beta_2}{\Delta},\\
&\frac{\partial^2 f_1}{\partial x_3\partial x_3} = \frac{2\eta_D(\beta_1^*+\beta_2)}{\Delta}, \frac{\partial^2 f_1}{\partial x_3\partial x_4} = \eta_D\frac{(\eta_K\beta_1^* + \eta_{V1}\beta_2) + \beta_1^* + \beta_2}{\Delta}, \frac{\partial^2 f_1}{\partial x_3\partial x_5} = \frac{\eta_D(\beta_1^*+\beta_2)}{\Delta},\\
&\frac{\partial^2 f_1}{\partial x_3\partial x_6} = \eta_D\frac{\beta_1^* + (1+\eta_V)\beta_2}{\Delta}, \frac{\partial^2 f_1}{\partial x_3\partial x_7} = \eta_D\frac{\beta_1^* + (1+\sigma\eta_V)\beta_2}{\Delta}, \frac{\partial^2 f_1}{\partial x_4\partial x_4} = \frac{2\eta_D(\eta_K\beta_1^* + \eta_{V1}\beta_2)}{\Delta},\\
&\frac{\partial^2 f_1}{\partial x_4\partial x_5} = \frac{\eta_D(\eta_K\beta_1^* + \eta_{V1}\beta_2)}{\Delta}, \frac{\partial^2 f_1}{\partial x_4\partial x_6} = \eta_D\frac{\eta_K\beta_1^* + (\eta_{V1}1 + \eta_V)\beta_2}{\Delta}, \frac{\partial^2 f_1}{\partial x_4\partial x_7} = \eta_D\frac{\eta_K\beta_1^* + (\eta_{V1} + \sigma\eta_V)\beta_2}{\Delta},\\
&\frac{\partial^2 f_1}{\partial x_5\partial x_6} = \frac{\eta_D\eta_V\beta_2}{\Delta}, \frac{\partial^2 f_1}{\partial x_5\partial x_7} = \frac{\sigma\eta_D\eta_V\beta_2}{\Delta}, \frac{\partial^2 f_1}{\partial x_6\partial x_6} = \frac{2\eta_D\eta_V\beta_2}{\Delta}, \frac{\partial^2 f_1}{\partial x_6\partial x_7} = \frac{\eta_D\eta_V(1+\sigma)\beta_2}{\Delta},\\
&\frac{\partial^2 f_1}{\partial x_7\partial x_7} = \frac{2\sigma\eta_D\eta_V\beta_2}{\Delta}, \frac{\partial^2 f_1}{\partial x_5\partial x_7} = \frac{\sigma\eta_D\eta_V\beta_2}{\Delta}, \frac{\partial^2 f_2}{\partial x_2\partial x_3} = -\frac{\eta_D\beta_1^*}{\Delta}, \frac{\partial^2 f_2}{\partial x_2\partial x_4} = -\frac{\eta_D\eta_K\beta_1^*}{\Delta}, \frac{\partial^2 f_2}{\partial x_3\partial x_3} = -\frac{2\eta_D\beta_1^*}{\Delta},\\
&\frac{\partial^2 f_2}{\partial x_3\partial x_4} = -\frac{\eta_D(1+\eta_K)\beta_1^*}{\Delta}, \frac{\partial^2 f_2}{\partial x_3\partial x_5} = -\frac{\eta_D\beta_1^*}{\Delta}, \frac{\partial^2 f_2}{\partial x_3\partial x_6} = -\frac{\eta_D\beta_1^*(1-\gamma_1)}{\Delta}, \frac{\partial^2 f_2}{\partial x_3\partial x_7} = -\frac{\eta_D\beta_1^*(1-\gamma_2)}{\Delta},\\
&\frac{\partial^2 f_2}{\partial x_4\partial x_2} = -\frac{\eta_D\eta_K\beta_1^*}{\Delta}, \frac{\partial^2 f_2}{\partial x_4\partial x_4} = -\frac{2\eta_D\eta_K\beta_1^*}{\Delta}, \frac{\partial^2 f_2}{\partial x_4\partial x_5} = -\frac{\eta_D\eta_K\beta_1^*}{\Delta}, \frac{\partial^2 f_2}{\partial x_4\partial x_6} = -\frac{\eta_D\eta_K\beta_1^*(1-\gamma_1)}{\Delta},\\
&\frac{\partial^2 f_2}{\partial x_4\partial x_7} = -\frac{\eta_D\eta_K\beta_1^*(1-\gamma_2)}{\Delta}, \frac{\partial^2 f_5}{\partial x_2\partial x_3} = -\frac{\eta_D\beta_2}{\Delta}, \frac{\partial^2 f_5}{\partial x_2\partial x_4} = -\frac{\eta_D\eta_{V1}\beta_2}{\Delta}, \frac{\partial^2 f_5}{\partial x_2\partial x_6} = -\frac{\eta_D\eta_V\beta_2}{\Delta},\\
&\frac{\partial^2 f_5}{\partial x_2\partial x_7} = -\frac{\sigma\eta_D\eta_V\beta_2}{\Delta}, \frac{\partial^2 f_5}{\partial x_3\partial x_3} = -\frac{2\eta_D\beta_2}{\Delta}, \frac{\partial^2 f_5}{\partial x_3\partial x_4} = -\frac{\eta_D(1+\eta_{V1})\beta_2}{\Delta},\\
&\frac{\partial^2 f_5}{\partial x_3\partial x_5} = -\frac{\eta_D\beta_2}{\Delta}, \frac{\partial^2 f_5}{\partial x_3\partial x_6} = -\frac{\eta_D(1+\eta_V)\beta_2}{\Delta}, \frac{\partial^2 f_5}{\partial x_4\partial x_4} = -\frac{2\eta_D\eta_{V1}\beta_2}{\Delta}, \frac{\partial^2 f_5}{\partial x_3\partial x_7} = -\frac{\eta_D(1+\sigma\eta_V-\alpha)\beta_2}{\Delta},\\
&\frac{\partial^2 f_5}{\partial x_4\partial x_5} = -\frac{\eta_D\eta_{V1}\beta_2}{\Delta}, \frac{\partial^2 f_5}{\partial x_4\partial x_6} = -\frac{\eta_D(\eta_V+\eta_{V1})\beta_2}{\Delta}, \frac{\partial^2 f_5}{\partial x_5\partial x_6} = -\frac{\eta_D\eta_V\beta_2}{\Delta}, \frac{\partial^2 f_5}{\partial x_5\partial x_7} = -\frac{\sigma\eta_D\eta_V\beta_2}{\Delta},\\
&\frac{\partial^2 f_5}{\partial x_4\partial x_7} = -\frac{\eta_D[\sigma\eta_V + \eta_{V1}(1-\alpha)]\beta_2}{\Delta}, \frac{\partial^2 f_5}{\partial x_6\partial x_6} = -\frac{2\eta_D\eta_V\beta_2}{\Delta}, \frac{\partial^2 f_5}{\partial x_6\partial x_7} = -\frac{\eta_D\eta_V(1+\sigma-\alpha)\beta_2}{\Delta},\\
&\frac{\partial^2 f_5}{\partial x_7\partial x_7} = -\frac{2\eta_D\eta_V(1-\alpha)\beta_2}{\Delta}, \frac{\partial^2 f_6}{\partial x_3\partial x_6} = -\frac{\gamma_1\eta_D\beta_1^*}{\Delta}, \frac{\partial^2 f_6}{\partial x_4\partial x_6} = -\frac{\gamma_1\eta_D\eta_K\beta_1^*}{\Delta}, \frac{\partial^2 f_7}{\partial x_3\partial x_7} = -\frac{\eta_D(\gamma_2\beta_1^*+\alpha\beta_2)}{\Delta},\\
&\frac{\partial^2 f_7}{\partial x_4\partial x_7} = -\frac{\eta_D(\gamma_2\eta_K\beta_1^* + \alpha\eta_{V1}\beta_2)}{\Delta}, \frac{\partial^2 f_7}{\partial x_6\partial x_7} = -\frac{\alpha\eta_D\eta_V\beta_2}{\Delta}, \frac{\partial^2 f_7}{\partial x_7\partial x_7} = -\frac{2\sigma\alpha\eta_D\eta_V\beta_2}{\Delta}.
\end{aligned} \qquad (69)$$

Following from above and (63), the resulting expression for a is

$$\begin{aligned} a = &-\frac{2\eta_D\beta_1^*}{\Delta}v_2\{w_2(w_3+\eta_K w_4)+(1+\eta_K)w_3w_4 \\ &+(\eta_K w_4+w_3)[w_5+(1-\gamma_1)w_6+(1-\gamma_2)w_7+]\} \\ &-\frac{2\eta_D\beta_1^*}{\Delta}v_2\left(w_3^2+\eta_K w_4^2\right)-\frac{2\eta_D\beta_2}{\Delta}v_5\{w_3[(1+\eta_{V1})w_4 \\ &+w_5+(1+\eta_V)w_6+(1+\sigma\eta_V-\alpha)w_7] \\ &+w_5(\eta_V w_6+\sigma\eta_V w_7)w_4\{\eta_V w_5+(\eta_V+\eta_{V1})w_6 \\ &+[\eta_V(1-\alpha)]w_7\}+\eta_V(1+\sigma-\alpha)w_6w_7 \\ &+w_2(w_3+\eta_{V1}w_4+\eta_V w_6+\sigma\eta_V w_7)\} \\ &-\frac{2\eta_D\beta_2}{\Delta}v_5\left\{w_3^2+\eta_{V1}w_4^2+\sigma\eta_V w_7^2\right\}-\frac{2\eta_D\beta_1^*}{\Delta}(w_3+\eta_K w_4) \\ &-\frac{2\eta_D}{\Delta}[(\gamma_2\beta_1^*+\alpha\beta_2)w_3+(\gamma_2\eta_K\beta_1^*+\alpha\eta_{V1}\beta_2)w_4+\alpha\eta_V\beta_2w_6]. \end{aligned} \tag{70}$$

It can easily be seen that $a<0$.

Similarly, the associated nonzero derivatives of F needed for the computation of the corresponding sign of b are

$$\frac{\partial^2 f_1}{\partial x_3\partial\beta_1^*}=-1,\quad \frac{\partial^2 f_1}{\partial x_4\partial\beta_1^*}=-\eta_K, \tag{71}$$

$$\frac{\partial^2 f_2}{\partial x_3\partial\beta_1^*}=1,\quad \frac{\partial^2 f_1}{\partial x_4\partial\beta_1^*}=\eta_K. \tag{72}$$

So that,

$$b=\sum_{k,i=1}^{8}v_k w_i\frac{\partial^2 f_k}{\partial x_i\partial\beta_1^*}=v_2(w_3+\eta_K w_4)<0. \tag{73}$$

Thus, the transformed system (64), or equivalently, system (1) undergoes a forward bifurcation at $\mathscr{R}_0=\max\{\mathscr{R}_K,\mathscr{R}_V\}=1$.

4. Numerical Simulation

We subject system (1) to numerical analysis primarily to monitor its dynamics and to further illustrate some of the theoretical results arrived at in the paper as well as to provide evidences that our results are likely to provide insight in a more general situation. To achieve this, we have considered the following parameter values from Table 2. $\Delta=0.019$, $\eta_C=0.3$, $\eta_D=0.02$, $\eta_I=0.6$, $\eta_K=0.12$, $\eta_T=0.36$, $\eta_V=0.56$, $\eta_{V1}=0.35$, $\sigma=0.5$, $\psi_1=0.44$, $\psi_2=0.16$, $\kappa=0.5$, $\upsilon=0.65$, $\tau_1=0.3$, $\tau_2=0.15$, $\alpha=0.46$, $\gamma_1=0.25$, $\gamma_2=0.145$, $\phi_1=0.89$, $\phi_2=0.64$, $\phi_3=0.025$, $\phi_4=0.52$, $\phi_5=0.745$, $\phi_6=0.87$, $\phi_7=0.64$, $\beta_1=0.8$, $\beta_2=0.21$, This is summarised in Table 3. The simulations are carried out as follows.

The three solution trajectories for model (1) in Figure 4 describes the impact of an effective psychotherapy ($\tau_2\neq0$, $\psi_2\neq0$) on the cumulative density of criminals/criminal cases as a result of a situational combinations of the parameters α, γ_1 and γ_2, so that: the trajectory with $\alpha=0$ identifies empowering and facilitating victims to avert repeat victimisation as having the most effective containment effect on crime and criminality while guarding against criminalising victims under psychotherapy has the least containment effect.

Figure 5 depicts the projected decrease in the overall criminal population and the corresponding criminal cases that could be averted due to the effective implementation of reformation programme on identified criminals.

Figure 6 shows the solution trajectories of model (1) where we observe enormous benefit of implementing an effective psychotherapy ($\tau_2>0$, $\psi_2>0$) for victims of crime. It can be

Table 3: Nominal values for system's parameters.

Parameter	Nominal value	Source
Δ	0.019	[42]
η_C	0.045	Estimated
η_D	0.02	[45]
η_I	0.6	[35]
η_K	0.12	Estimated
η_T	0.36	[23]
η_V	0.56	[42]
η_{V1}	0.35	[28]
σ	0.5	Estimated
ψ_1	0.5	[40]
ψ_2	0.35	Estimated
κ	0.5	[53]
υ	0.62	[23]
τ_1	0.3	[40]
τ_2	0.175	Estimated
α	0.46	[42]
γ_1,γ_2	0.75, 0.145	Estimated
$\phi_1,\ldots,\phi_7$	0.89, 0.64, 0.025, 0.52, 0.745, 0.87, 0.64	Estimated
β_1	0.8	Estimated
β_2	0.21	[40]

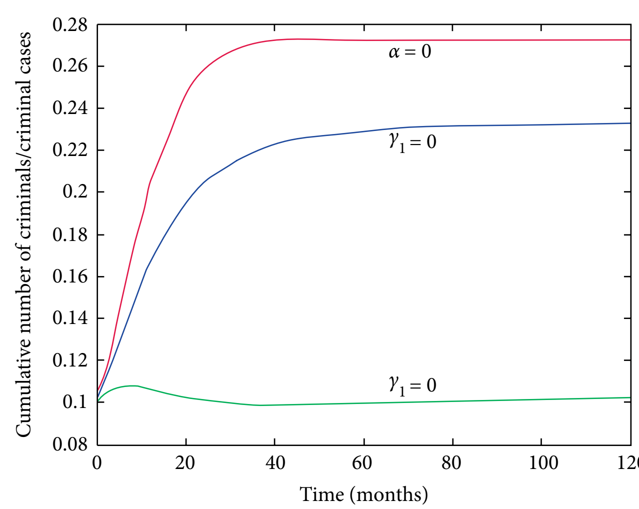

Figure 4: Simulation of model (4) showing the cumulative number of new crimes/criminals (S_E+C+C_R) as a function of time for the values of α, γ_1, and γ_2 changing between zero and their respective values in Table 3. Other parameter values used are as in Table 3.

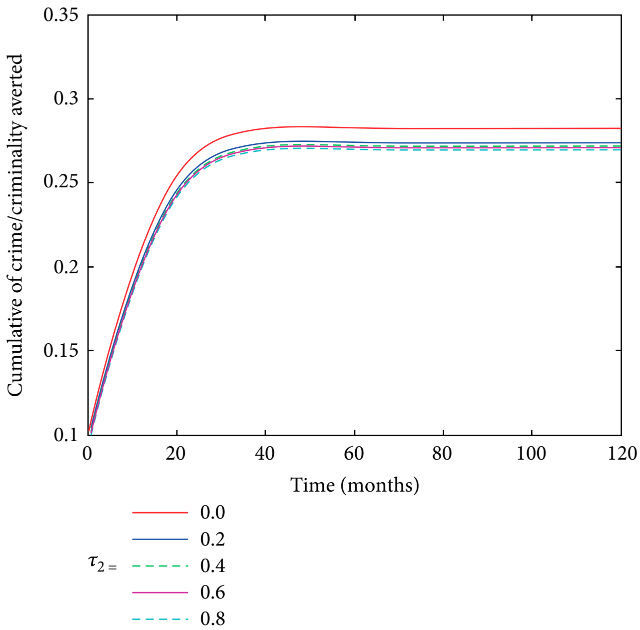

FIGURE 5: Simulation of model (4) showing the cumulative number of criminals and criminal cases averted as a function of time for different values of τ_2. Other parameter values used are as in Table 3.

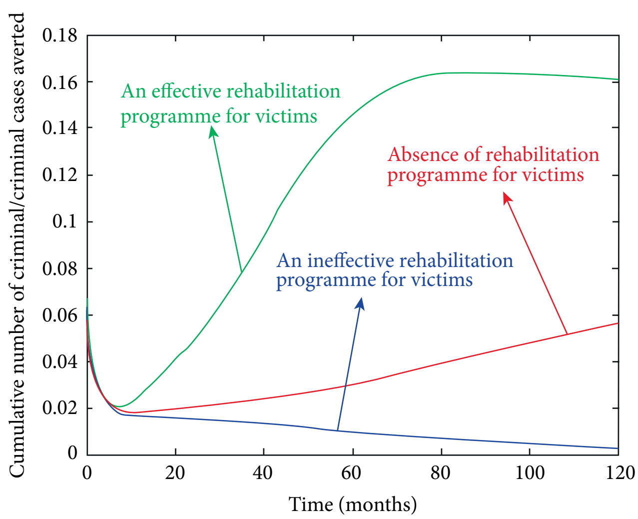

FIGURE 6: Simulation of model (4) showing the cumulative number of quitting criminals and the corresponding averted criminal cases as a function of time for a combination of variable values of τ_2 and ψ_2. Other parameter values used are as in Table 3.

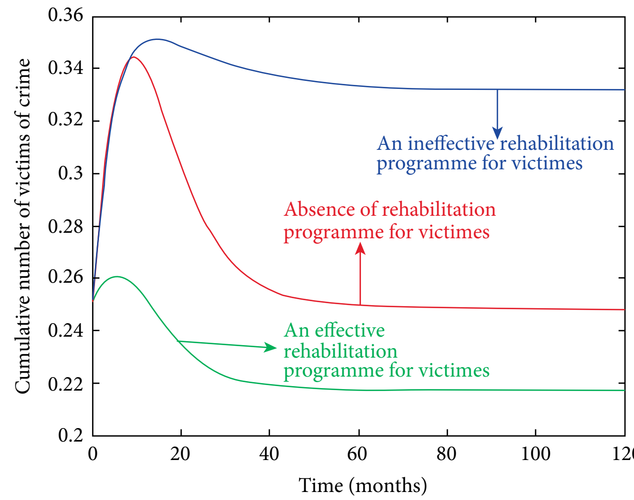

FIGURE 7: Simulation of model (4) showing the impact of a victim rehabilitation programme for victims of crime on criminality as observed from a combination of variable values of τ_2 and ψ_2 on the density of victimisation/victims of crime as a function of time. Other parameter values used are as in Table 3.

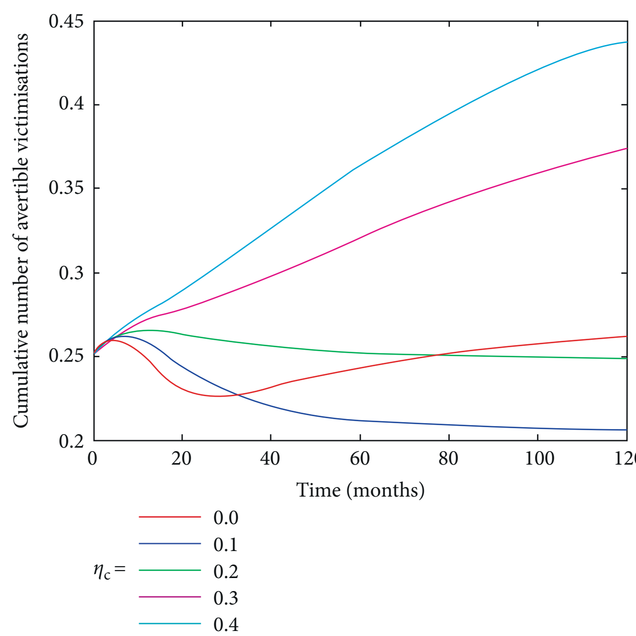

FIGURE 8: Simulation of model (4) showing the effect of varying the values of η_C (*Crime-induced death*) on the density of victimisation/victims as a function of time. Other parameter values used are as in Table 3.

observed that the ineffectiveness of such a programme ($\tau_2 > 0, \psi_2 = 0$) potents a detrimental consequence as compared even to an absolute absence of the programme ($\tau_2 = \psi_2 = 0$) to note the detrimental consequence for such a programme to be ineffective. This implication is further confirmed by the trajectories in Figure 7.

Figure 8 indicates that the removal of criminals (*death of criminals as a result of criminal activities or quiting criminality even without formal reformation*) can significantly reduce the cumulative number of new victimisation cases.

It is noted from Figure 13 that an effective reformation programme will result in the reduction of texted criminal cases, supposedly due to the obvious fact that the decriminalisation of core criminals would result in reducing the number of criminal activities.

Like the scenario depicted in Figure 13, a similar reduction in the cumulative density of texted criminal cases is observed from Figure 14 and we conclude that an effective psychotherapy will result in the reduction of texted criminal cases,

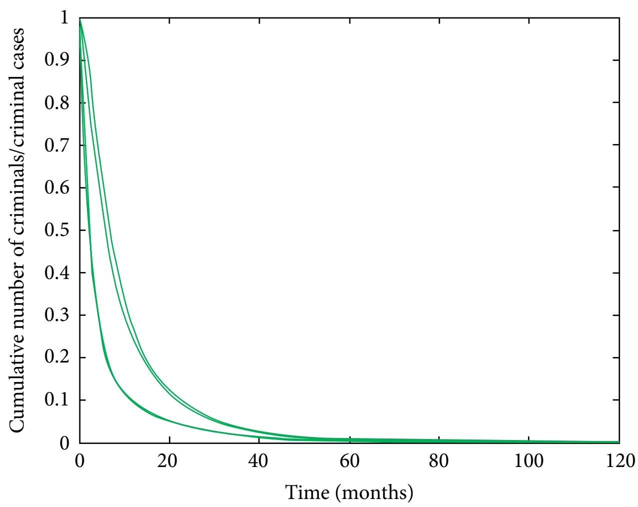

Figure 9: Simulation of model (4) using various initial conditions, showing cumulative number of crime/criminal cases when $\beta_1 = 0.25$ and $\beta_2 = .05$. Other parameter values used are as given in in Table 3.

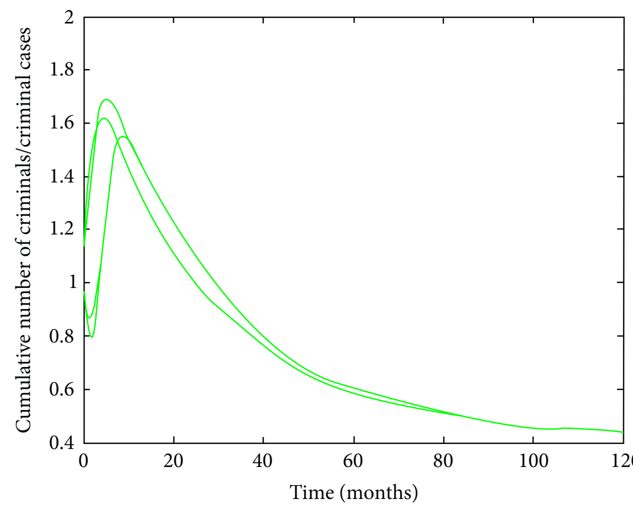

Figure 11: Simulation of model (4) using various initial conditions, showing cumulative number of crime/criminal cases when $\beta_1 = 2.512$ and $\beta_2 = 1.25$. Other parameter values used are as given in in Table 3.

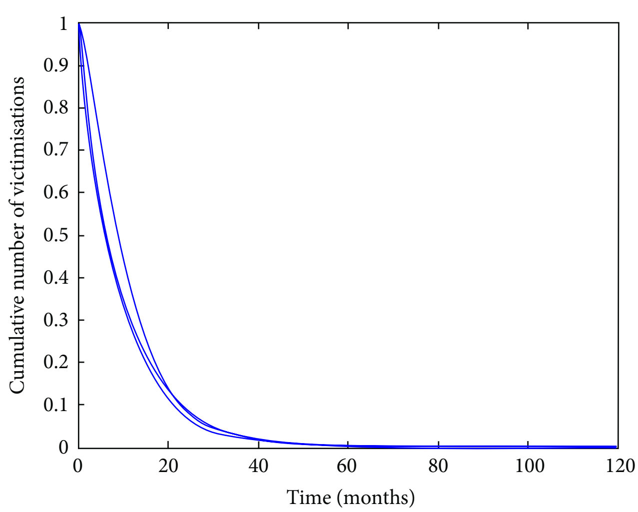

Figure 10: Simulation of model (4) using various initial conditions, showing cumulative number of victimisation cases when $\beta_1 = 0.25$ and $\beta_2 = 0.05$. Other parameter values used are as given in in Table 3.

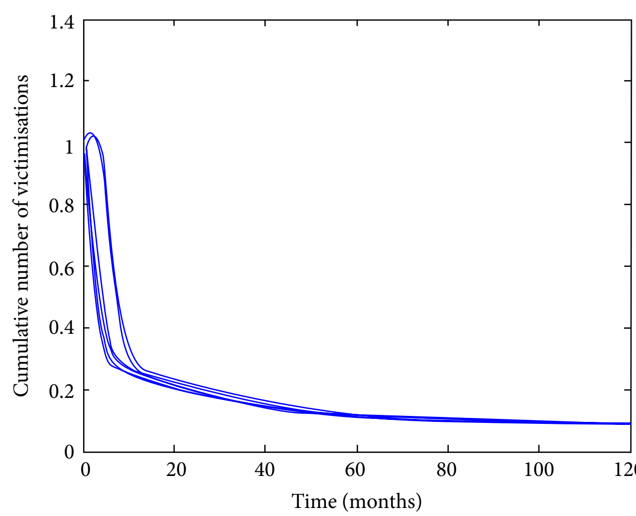

Figure 12: Simulation of model (4) using various initial conditions, showing cumulative number of victimisation cases when $\beta_1 = 2.512$ and $\beta_2 = 1.25$. Other parameter values used are as given in in Table 3.

probably because rehabilitated victims may not so easily fall victims of repeat victimisation.

The advantages of prioritising psychotherapy is captured in Figure 15. Here, we note that the absolute deployment of psychotherapeutic techniques against the prevention of the criminalisation of rehabilitated victims would be less effective in reducing the number of criminal activities as compared to emphasising such deployments to guarding against the criminalisation of non-rehabilitated victims, talk less of when repeat victimisation is completely prevented.

5. Conclusion

A deterministic model for monitoring the interactional dynamics of crime, criminality and victimisation in a population is designed and analysed. The model includes two levels of rehabilitation--a reformation programme for criminal individuals and a psychotherapy for individuals who are victims of crime. The model is shown to have a globally-asymptotically stable crime-free equilibrium whenever the largest of the two associated reproduction numbers is less than unity; and has a unique and locally-asymptotically stable endemic equilibrium when the number exceeds unity. Using Centre Manifold theory, the model was shown to undergo the a forward bifurcation, when the the largest reproduction number is less than unity. Further, the model is shown to have a crime free equilibrium which is locally-asymptotically stable whenever the largest reproduction number is less than unity. By analysing the various associated reproduction numbers, it was shown that by effectively implementing both the reformation and

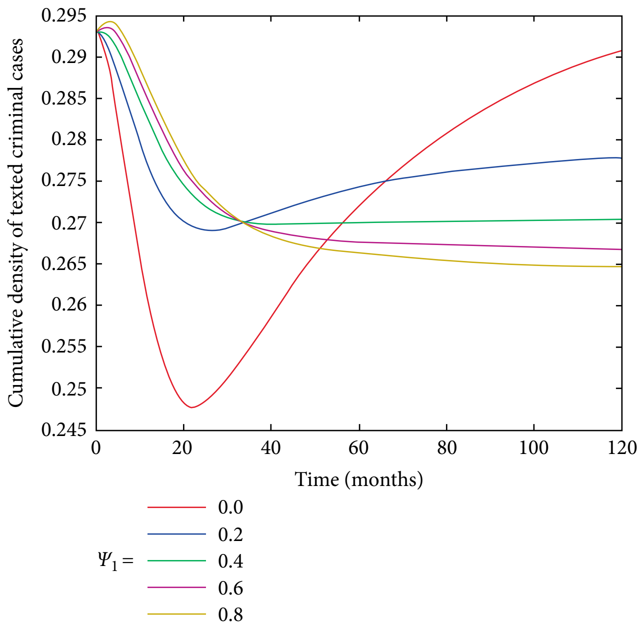

Figure 13: Simulation of model (4) showing the effect of varying the values of ψ_1 on the cumulative density of texted criminal cases as a function of time. Other parameter values used are as given in Table 3.

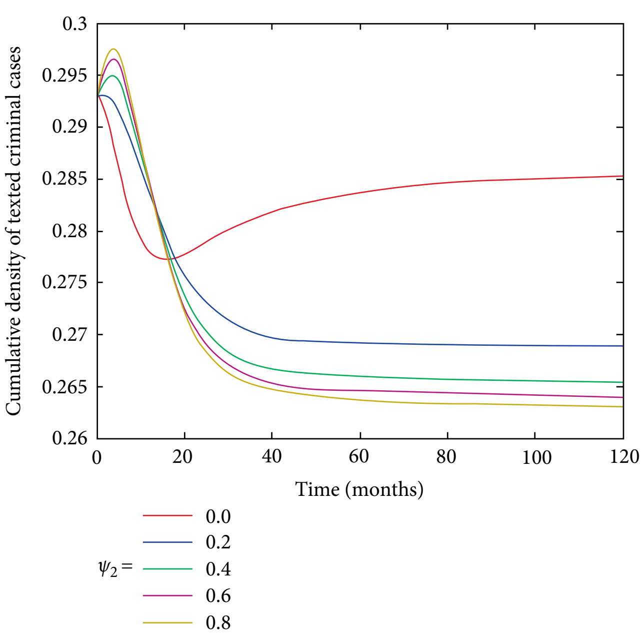

Figure 14: Simulation of model (4) showing the effect of varying the values of ψ_2 on the cumulative density of texted criminal cases as a function of time. Other parameter values used are as given in Table 3.

psychotherapy programmes, respectively for criminals and victims, the menace of crime and criminality would be sufficiently contained in a criminally prone community. Some major findings from numerical simulations of the model include:

(i) The none implementation of rehabilitation is more beneficial than when it is done ineffectively;

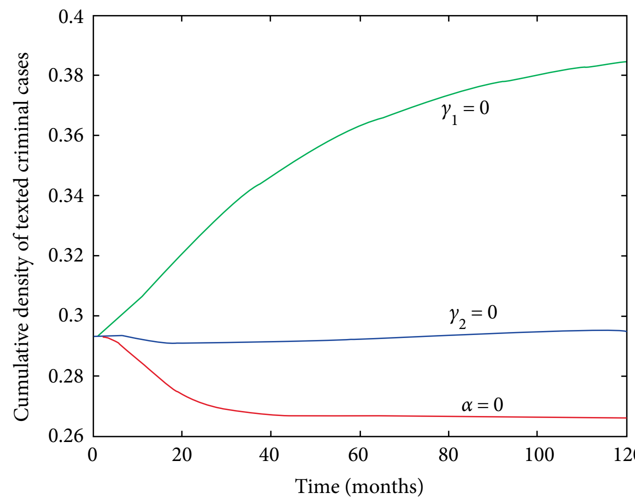

Figure 15: Simulation of model (4) showing the effect of combinations of variable values of α, γ_1 and γ_2 on the cumulative density of texted criminal cases as a function of time. Other parameter values used are as given in Table 3.

(ii) Focusing psychotherapy on the enlightenment of victims against the phenomenon of repeat victimisation is most effective in containing crime and criminality while priotising enlightenment against criminalising victims is least effective;

(iii) The removal of criminals, either through criminally-induced deaths or the willingness to quit crime even without formal reformation, can significantly lead to the containment of crime and criminality;

(iv) The effective reformation of criminals can fundamentally reduce criminals and criminal activities;

(v) The decreases in texted crime cases as a result of the eventual outcome of implementing psychotherapy is a function of the priority direction of psychotherapy. It is noted, just like in (ii), that the absence of repeat victimisations will have the most containment effect on crime as obvious in the minimal number of texted cases while the none criminalisation of victims will have the least of such effects.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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