Hindawi
Journal of Applied Mathematics
Volume 2020, Article ID 2360487, 13 pages
https://doi.org/10.1155/2020/2360487

*Research Article*

# Approximation Techniques for Solving Linear Systems of Volterra Integro-Differential Equations

**Ahmad Issa, Naji Qatanani, and Adnan Daraghmeh**

*Department of Mathematics, Faculty of Science, An-Najah National University, Nablus, State of Palestine*

Correspondence should be addressed to Naji Qatanani; nqatanani@najah.edu

Received 22 July 2019; Revised 8 October 2019; Accepted 14 October 2019; Published 4 February 2020

Academic Editor: Yansheng Liu



In this paper, a collocation method using sinc functions and Chebyshev wavelet method is implemented to solve linear systems of Volterra integro-differential equations. To test the validity of these methods, two numerical examples with known exact solution are presented. Numerical results indicate that the convergence and accuracy of these methods are in good a agreement with the analytical solution. However, according to comparison of these methods, we conclude that the Chebyshev wavelet method provides more accurate results.

## 1. Introduction

Systems of integro-differential equations have motivated huge amounts of research in recent years. They arise in many physical phenomena like wind ripple in the desert, nano-hydrodynamics, population growth model, glass-forming process, and oceanography [1–3]. Various numerical methods for solving systems of linear integro-differential equations have been developed by many researchers. Hesameddini and Rahimi [4] used the reconstruction of variational iteration method (RVIM) for solving systems of Volterra integro-differential equations. In [5], Hesameddini and Asadolhifard, implemented the sinc-collocation method to approximate the solution of systems of linear Volterra integro-differential equations with initial conditions. Aminikhah and Hosseini [6] applied the wavelet method for the numerical solution of systems of integro-differential equations. They used the operational matrix of integration to solve these systems. Draidi and Qatanani [7] emplemented product Nystrom and sinc-collocation methods to solve Volterra integral equation with Carleman kernel. Hamaydi and Qatanani [8] used the Taylor expansion and the variational iteration methods to give approximate solution of Volterra integral equation of the second kind. In addition, Issa [9] has employed several numerical techniques for solving systems of Volterra integro-differential equations. Other numerical methods for systems of integro-differential equations are (power) functions and Chebyshev polynomials [10], single term Walsh series [11], Chebyshev collocation [12], rationalized Haar functions [13], differential transform [14], homotopy perturbation [15], power series [16], and finite difference approximation [17]. Regarding the stability of a system of Volterra integro-differential equations, some stability results are proposed for the linear system VIDEs in the 1980s, those of Burton are worthy to mention. His work [18, 19] laid the foundation for a systematic treatment of the basic structure and stability properties of VIDEs via the direct method of Lyapunov. A more recent result is by Elaydi [20], who proposed a type of Lyapunov functional that is also applicable to delay equations. Moreover, Zhang [21] proposed recently a stability result from which certain well-known result could be derived. Also, Vanualailai and Nakagiri [22] have proposed a new stability criteria based on new and known forms of Lyapunov functionals for a system of Volterra integro-differential equations. In this article, we propose two numerical methods, namely, a collocation method using sinc functions and Chebyshev wavelet method to approximate the solution of a system of linear Volterra integro-differential equations given by

$$u_i^{(n)}(x) = f_i(x) + \int_a^x \left( \sum_{j=1}^{N} k_{ij}(x,t) u_j(t) \right) dt, \quad a \le x \le b, \quad 1 \le i \le N \tag{1}$$

subject to the initial conditions

$$u_i^{(s)} = a_{is}, \quad i = 1, 2, 3, \ldots, N, \quad s = 0, 1, 2, \ldots, (n-1). \tag{2}$$

The kernels $k_{ij}(x,t)$ and the function $f_i(x)$ are given real valued functions and the unknown functions $u_i(x)$ are to be determined. A comparison between these methods is carried out by solving some numerical examples.

The paper is organized as follows: In Section 2 the sinc-collocation method based on sinc functions is presented. The Chebyshev wavelet method is addressed in Section 3. In Section 5, the proposed methods are implemented using two numerical examples with known analytical solution by applying MAPLE software. Conclusions are followed in Section 6.

## 2. Sinc Collocation Method Based on Sinc Functions

The sinc collocation method based on sinc functions is widely used for obtaining the approximate solution of ordinary and partial differential equations and integral equations [5]. It is well-known that the sinc approximate solution converges exponentially to the exact solution.

*Definition 1 (see [23]).* The sinc function is defined on the whole real line $-\infty < x < \infty$ by

$$\operatorname{sinc}(x) = \begin{cases} \dfrac{\sin(\pi x)}{\pi x}, & x \neq 0 \\ 1, & x = 0 \end{cases} \tag{3}$$

as shown in Figure 1.

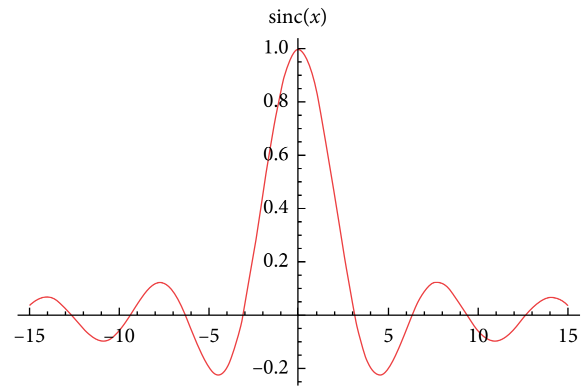



Figure 1: Sinc function.

*Definition 2 (see [23]).* Let $k = 0, \pm 1, \pm 2, \pm 3, \ldots$ then the translated sinc basis functions are defined as

$$s(k,r)(x) = \operatorname{sinc}\left(\frac{x-kr}{r}\right) = \begin{cases} \dfrac{\sin[(\pi/r)(x-kr)]}{(\pi/r)(x-kr)}, & x \neq kr \\ 1, & x = kr \end{cases} \tag{4}$$

which are called the $k^{\text{th}}$ sinc functions.

**Corollary 1** (see [5]). *The sinc function for the interpolating points $x_d = dr$, is given by*

$$s(k,r)(dr) = \mu_{kd} = \begin{cases} 0, & k \neq d \\ 1, & k = d \end{cases} \tag{5}$$

**Corollary 2** (see [5]). *If $p(x)$ is defined on the real axis and $r$ is a positive integer, then the series*

$$a(p,r)(x) = \sum_{k=-\infty}^{\infty} p(kr)s(k,r)(x) \tag{6}$$

is called the Whittaker Cardinal expansion of $p(x)$.

The properties of the Whittaker Cardinal expansion have been extensively studied in [22]. These properties are derived in the infinite strip $D_s$ of the complex $w$–planes where for any $g > 0$,

$$G_s = \left\{ w = t + is : |s| < g \leq \frac{\pi}{2} \right\}. \tag{7}$$

To construct an approximation on the interval $(a,b)$, we use the conformal map:

$$w = \varphi(z) = \ln\left(\frac{z-a}{b-z}\right). \tag{8}$$

This map carries the eye-shaped region

$$G_E = \left\{ z \in \mathbb{C} : \left| \arg\left(\frac{z-a}{b-z}\right) \right| < g \leq \frac{\pi}{2} \right\}. \tag{9}$$

For the sinc method, the basis functions on $(a,b)$ for $z \in G_E$ are derived from the composite translated sinc functions,

$$s(k,r) \circ \varphi(x) = \operatorname{sinc}\left(\frac{\varphi(x) - kr}{r}\right), \tag{10}$$

where $s(k,r) \circ \varphi(x) = s(k,r)(\varphi(x))$.

The inverse map of $w = \varphi(z)$ is

$$z = \varphi^{-1}(w) = \frac{a + be^{w}}{1 + e^{w}}. \tag{11}$$

Also we define the range of $\varphi^{-1}$ on the real line as

$$\tau = \left\{ y < u < \infty \varphi^{-1}(u) \in G_E : -\infty < u < \infty \right\} \tag{12}$$

and the interpolation points $\{x_d\}$ are then given by:

$$x_d = \varphi^{-1}(dr) = \frac{a + be^{dr}}{1 + e^{dr}}, \quad d = 0 \pm 1, \pm 2, \ldots. \tag{13}$$

*Definition 3 (see [23]).* Let $L_\alpha(G_E)$ be the set of all analytic functions. Then, there exists a constant $c$, such that:

$$|u(z)| \leq c \frac{|\beta(z)|^{\alpha}}{\left[1 + |\beta(z)|^{2\alpha}\right]}, \quad z \in G_E, \quad 0 < \alpha \leq 1, \tag{14}$$

where $\beta(z) = e^{\varphi(z)}$.

**Theorem 1** (see [23]). *Let $u \in L_\alpha(G_E)$ and $N$ is a natural number, and $r$ be selected by the formula*

$$r = \left(\frac{\pi g}{\alpha N}\right)^{1/2}, \tag{15}$$

where

$$0 < \alpha \leq 1, \quad g \leq \frac{\pi}{2}. \tag{16}$$

Then, there exists a positive constant $c_1$, independent of $N$, such that

$$\sup_{z\in\tau}\left|u(z) - \sum_{d=-N}^{N} u(z_d)s(d,r)\circ\varphi(z)\right| \leq c_1 e^{-(\pi g\alpha N)^{1/2}}. \tag{17}$$

**Theorem 2** (see [5]). *Let $u/\varphi \in L_\alpha(G_E)$ and $N$ is a natural number, and $r$ be given as*

$$r = \left(\frac{\pi g}{\alpha N}\right)^{1/2}, \tag{18}$$

*where*

$$0 < \alpha \leq 1, \quad g \leq \frac{\pi}{2}. \tag{19}$$

*Moreover, let $\mu_{kd}^{(-1)}$ be defined as*

$$\mu_{kd}^{(-1)} = \frac{1}{2} + \int_0^{k-d}\frac{\sin(\pi t)}{\pi t}dt. \tag{20}$$

*Then, there exists a positive constant $c_2$, independent of $N$, such that*

$$\left|\int_a^{z_k} u(t)dt - r\sum_{d=-N}^{N}\mu_{kd}^{(-1)}\frac{u(z_d)}{\varphi(z_d)}\right| \leq c_2 e^{-(\pi g\alpha N)^{1/2}}. \tag{21}$$

**Theorem 3** (see [5]). *Let $\varphi$ be a conformal injective map of the simply connected domain $G_E$ onto $G_E$. Then*

$$\begin{aligned}
\mu_{kd}^{(0)} &= \left[s(k,d)\circ\varphi(x)\right]\big|_{x=x_d} = \begin{cases} 1, & k = d \\ 0, & k \neq d \end{cases} \\
\mu_{kd}^{(1)} &= r\frac{d}{d\varphi}\left[s(k,d)\circ\varphi(x)\right]\big|_{x=x_d} = \begin{cases} 0, & k = d \\ \dfrac{(-1)^{d-k}}{d-k}, & k \neq d \end{cases} \\
\mu_{kd}^{(2)} &= r^2\frac{d^2}{d\varphi^2}\left[s(k,d)\circ\varphi(x)\right]\big|_{x=x_d} = \begin{cases} \dfrac{-\pi^2}{3}, & k = d \\ \dfrac{-2(-1)^{d-k}}{(d-k)^2}, & k \neq d \end{cases}
\end{aligned} \tag{22}$$

We consider the system of linear Volterra integro-differential equations of the form:

$$u_i^{(n)}(x) = f_i(x) + \int_a^x\left(\sum_{j=1}^{N} k_{ij}(x,t)u_j(t)\right)dt, \quad 1 \leq i \leq N. \tag{23}$$

Subject to the initial conditions:

$$u_i^{(s)}(0) = a_{is}, \quad i = 1, 2, \ldots, N, \quad s = 0, 1, 2, \ldots, (n-1), \tag{24}$$

in the domain $[0, 1]$, and let $u_i(x) \in L_\alpha(G_E)$. By using Theorem 1, $u_i(x)$ is approximated as follows:

$$u_i(x) = R_i(x) + A_i(x), \tag{25}$$

where

$$\begin{aligned}
R_i(x) &= \sum_{k=-N}^{N} c_k^i w(x)\operatorname{sinc}\left(\frac{\varphi(x) - kr}{r}\right), \\
A_i(x) &= \sum_{j=0}^{n} a_j^i x^j,
\end{aligned} \tag{26}$$

where $c_k$ are unknown coefficients and $w(x) = x^n(x-1)^n$.

Integrating both sides of Equation (25) from 0 to $x$ we get

$$\int_0^x u_i(t)dt = \int_0^x R_i(t)dt + \int_0^x A_i(t)dt \tag{27}$$

and by differentiating both sides of Equation (25) with respect to $x$ we get

$$u_i^{(n)}(x) = R_i^{(n)}(x) + A_i^{(n)}(x), \tag{28}$$

where

$$R_i^{(n)}(x) = \sum_{k=-N}^{N} c_k \frac{d^n(w(x)\operatorname{sinc}((\varphi(x) - kr)/r))}{dx^n}. \tag{29}$$

Substituting Equations (27) and (29) into Equation (23), and by evaluating the result at the sinc points

$$x_j = \frac{e^{jr}}{1 + e^{jr}}, \tag{30}$$

where $j = -N-1, \ldots, N$, and using Theorems 2 and 3, we obtain a system of algebraic equations. Solving this system we obtain the unknown coefficients

$$\left\{c_k^i\right\}_{k=-N}^{N} \tag{31}$$

and

$$\left\{a_j^i\right\}_{j=0}^{n}. \tag{32}$$

## 3. Chebyshev Wavelets Method (CWM)

The main idea of using Chebyshev basis is that the problem under study reduces to a system of linear or nonlinear algebraic equations. This may be done by truncated series of orthogonal basis functions for the solution of the problem and using the operational matrices [6]. Wavelets constitute a family of functions constructed from dilation and translation of a single function called the mother wavelet [24–26]. When dilation $p$ and translation $q$ vary continuously, we have the following family of continuous wavelets as

$$\psi_{p,q}(x) = \frac{1}{\sqrt{|p|}}\psi\left(\frac{x-q}{p}\right), \quad p, q \in \mathbb{R}, \; p \neq 0. \tag{33}$$

If we choose the dilation and translation $p^{-a}$, and $bqp^{-a}$, respectively, where $p > 1$, $q > 0$, then we have the following family of continuous wavelets as

$$\psi_{a,b}(x) = \sqrt{|p|^a}\,\psi(p^{-a}x - bq), \quad a, b \in \mathbb{Z}^+, \tag{34}$$

where $\psi_{a,b}$ forms a wavelet basis for

$$L^2(\mathbb{R}) = \left\{ |g : \mathbb{R} \to \mathbb{C} \,|\, \int_{-\infty}^{\infty} |g(x)|^2 dx < \infty \right\}, \tag{35}$$

where $L^2(\mathbb{R})$: set of all square integrable functions equipped with norm

$$\|f\|_{L^2[a,b]} = \left( \int_a^b |f(x)|^2 dx \right)^{1/2}. \tag{36}$$

For the particular case, when $p = 2$ and $q = 1$, then $\psi_{a,b}(x)$ forms an orthogonal basis.

Chebyshev wavelets $\psi_{b,c}(x) = \psi_{a,b,c}(x)$ have four parameters, $b = 1, 2, 3, \dots, 2^{(a-1)}$, $a \in \mathbb{Z}^+$ and $c$ is the degree of Chebyshev polynomials of the first kind. They are defined on the interval $0 \le x \le 1$ by:

$$\psi_{b,c}(x) = \psi_{a,b,c}(x) = \begin{cases} 2^{a/2}\tilde{T}_c\left(2^a x - 2b + 1\right), & \text{for } \dfrac{b-1}{2^a - 1} \le x \le \dfrac{b}{2^a - 1}, \\ 0, & \text{otherwise,} \end{cases} \tag{37}$$

where

$$\tilde{T}_c(x) = \begin{cases} \dfrac{1}{\sqrt{\pi}}, & c = 0 \\ \sqrt{\dfrac{1}{\pi}}\, T_c(x), & c > 0 \end{cases} \tag{38}$$

and $c = 0, 1, 2, \dots, C - 1$, and $b = 1, 2, 3, \dots, 2^{a-1}$.

$T_c(x)$ are the famous Chebyshev polynomials of the first kind of degree $c$ which are orthogonal with respect to the weight function

$$w(x) = \sqrt{1 - x^2}, \quad -1 \le x \le 1 \tag{39}$$

and satisfy the following recurrence relation

$$T_0(x) = 1, \quad T_1(x) = x, \quad T_{c+1}(x) = 2xT_c(x) - T_{c-1}(x), \quad c = 1, 2, 3, \dots \tag{40}$$

**Corollary 3** (see [6]). *The set of Chebyshev wavelets is an orthogonal set with respect to the weight function*

$$w_b(x) = w\left(2^a x - 2b + 1\right). \tag{41}$$

*Definition 4 (see [6]).* A function $h(x)$ defined on the interval $0 \le x \le 1$ is called the wavelet series if this function is written in the following form

$$h(x) = \sum_{b=1}^{\infty} \sum_{c=0}^{\infty} d_{bc} \psi_{bc}(x), \tag{42}$$

where

$$d_{bc} = \left(h(x), \psi_{bc}(x)\right) w_b(x) \tag{43}$$

is the inner product in $L^2_{w_b}[0, 1]$.

**Corollary 4** (see [6]). *The wavelet series in $L^2[0,1]$ is convergent if*

$$\lim_{u_1, u_2 \to \infty} \left\| h(x) - \sum_{b=1}^{u_1} \sum_{c=0}^{u_2} d_{bc} \psi_{bc}(x) \right\| = 0. \tag{44}$$

If the wavelet series is truncated, then it can be written as

$$h(x) \cong \sum_{b=1}^{2^{a-1}} \sum_{c=0}^{c-1} d_{bc} \psi_{bc}(x) = D^T \psi(x), \tag{45}$$

where $D$ and $\psi(x)$ are $2^{a-1}c \times 1$ matrices given by

$$\begin{aligned} D &= \left[ d_{1,0}, d_{1,1}, \dots, d_{1,c-1}, d_{2,0}, d_{2,1}, \dots, d_{2,c-1}, \dots, d_{2^{a-1},0}, \dots, d_{2^{a-1},c-1} \right]^T \\ &= \left[ d_1, d_2, \dots, d_c, d_{c+1}, \dots, d_{2^{a-1},c} \right]^T, \\ \psi(x) &= \left[ \psi_{1,0}(x), \psi_{1,1}(x), \dots, \psi_{1,c-1}(x), \psi_{2,0}(x), \psi_{2,1}(x), \dots, \psi_{2,c-1}(x), \dots, \psi_{2^{a-1},0}(x), \dots, \psi_{2^{a-1},c-1}(x) \right]^T \\ &= \left[ \psi_1(x), \psi_2(x), \dots, \psi_c(x), \psi_{c+1}(x), \dots, \psi_{2^{a-1},c}(x) \right]^T. \end{aligned} \tag{46}$$

**Corollary 5** (see [27]). *The integral of the multiple of two Chebyshev wavelets vector functions with respect to $w_b(x)$ from 0 to 1 is an identity matrix. Moreover, a function $h(x, y)$ defined on $[0, 1] \times [0, 1]$ can be approximated as:*

$$h(x, y) \cong \sum_{i=1}^{2^{a-1}c} \sum_{j=1}^{2^{a-1}c} a_{ij} \psi_i(x) \psi_j(y) = \psi^T(x) A \psi(y), \tag{47}$$

*where $A = \left[a_{ij}\right]$ is a matrix of the entries $2^{a-1}c \times 2^{a-1}c$, that can be determined by*:

$$a_{ij} = \left(\psi_i(x), \left(h(x, y), \psi_j(y)\right) w_b(y)\right) w_b(x), \tag{48}$$

$i = 1, 2, 3, \dots, 2^{a-1}c$ *and* $j = 1, 2, 3, \dots, 2^{a-1}c$.

The integral of the vector $\psi(x)$ defined in Equation (46), is given as:

$$\int_0^x \psi(t) dt = B\psi(x), \tag{49}$$

where $B$ is the $2^{a-1}c \times 2^{a-1}c$ operational matrix of integration [2].

This matrix has the form:

$$B = 2^{-a} \begin{pmatrix} M & E & E & \cdots & E \\ O & M & E & \ddots & \vdots \\ O & O & M & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & E \\ O & \dots & O & O & M \end{pmatrix}, \tag{50}$$

where $M$, $E$, and $O$ are $C \times C$ matrices given by

$$M = \begin{pmatrix} 1 & \dfrac{1}{\sqrt{2}} & 0 & 0 & 0 & \cdots & 0 \\ -\dfrac{\sqrt{2}}{4} & 0 & \dfrac{1}{4} & 0 & 0 & \cdots & 0 \\ -\dfrac{\sqrt{2}}{3} & -\dfrac{1}{2} & 0 & \dfrac{1}{6} & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \dfrac{\sqrt{2}(-1)^{\lambda}}{2}\left(\dfrac{1\lambda - 1}{-}\dfrac{1}{\lambda}\right) & \cdots & \dfrac{-1}{2(\lambda - 1)} & 0 & \dfrac{1}{2\lambda} & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \dfrac{\sqrt{2}(-1)^{c}}{2}\left(\dfrac{1}{c-1} - \dfrac{1}{c}\right) & 0 & 0 & 0 & \cdots & \dfrac{-1}{2(c-1)} & 0 \end{pmatrix} \tag{51}$$

$$E = \begin{pmatrix} 2 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ -\frac{2\sqrt{2}}{3} & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\sqrt{2}}{2}\left(\frac{1-(-1)^{\lambda}}{\lambda} - \frac{1-(-1)^{\lambda-2}}{\lambda-2}\right) & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\sqrt{2}}{2}\left(\frac{1-(-1)^{C}}{C} - \frac{1-(-1)^{C-2}}{C-2}\right) & 0 & 0 & \cdots & 0 \end{pmatrix} \text{ and}$$

$$O = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix}. \tag{52}$$

The product characteristic of two Chebyshev wavelets vector functions are given as

$$\psi(x)\psi^T Z \approx \tilde{Z}\psi(x), \tag{53}$$

where $Z$ is a given vector and $\tilde{Z} = \left[\tilde{Z}_{ij}\right]_{2^{a-1}c\times 2^{a-1}c}$ is an operational matrix. We consider the system of linear Volterra integro-differential equations of the form:

$$u_i^{(n)}(x) = f_i(x) + \sum_{j=1}^{c} \int_0^x k_{ij}(x,t)u_j(t)dt \tag{54}$$

with the following conditions

$$u_i^{(r)}(0) = a_{ir}, \quad i = 1,2,3,\ldots,n, \quad r = 0,1,2,\ldots,(n-1), \quad c = 1,2,3,\ldots. \tag{55}$$

Now we approximate $u_i^{(n)}(x)$ by using Chebyshev wavelet space as follows

$$u_i^{(n)}(x) = D_i^T\psi(x), \quad i = 1,2,3,\ldots,n. \tag{56}$$

Therefore we have

$$u_i^{(r)}(x) = D_i^T B^{n-r}\psi(x) + \sum_{j=0}^{n-r-1} a_{ir}\frac{x^j}{j!}, \quad i = 1,2,3,\ldots,n, \tag{57}$$

where $D$ and $\psi(x)$ are $2^{a-1}c \times 1$ matrices given by

$$\begin{aligned} D &= \left[d_{1,0}^i, d_{1,1}^i, \ldots, d_{1,c-1}^i, d_{2,0}^i, d_{2,1}^i, \ldots, d_{2,c-1}^i, \ldots, d_{2^{a-1},0}^i, \ldots, d_{2^{a-1},c-1}^i\right]^T \\ &= \left[d_{i,1}, d_{i,2}, \ldots, d_{i,c}, d_{i,c+1}, \ldots, d_{i,2^{a-1},c}\right]^T, \\ \psi(x) &= \left[\psi_{1,0}(x), \psi_{1,1}(x), \ldots, \psi_{1,c-1}(x), \psi_{2,0}(x), \psi_{2,1}(x), \ldots, \psi_{2,c-1}(x), \ldots, \psi_{2^{a-1},0}(x), \ldots, \psi_{2^{a-1},c-1}(x)\right]^T \\ &= \left[\psi_1(x), \psi_2(x), \ldots, \psi_c(x), \psi_{c+1}(x), \ldots, \psi_{2^{a-1},c}(x)\right]^T. \end{aligned} \tag{58}$$

In virtue of Equations (55) and (56) we have the following approximations:

$$\begin{aligned} f_i(x) &\cong F_i^T\psi(x), \\ u_i(x) &\cong D_i^T B\psi(x) + E_i^T\psi(x), \\ k_{ij}(x,t) &\cong \psi^T k_{ij}\psi(t), \end{aligned} \tag{59}$$

1: Input:
 (i) $N, M, \alpha, d, n$
 (ii) $f_i(x)$ *for* $i = 1,2,\ldots,N$
 (iii) $k_{ij}(x)$ *for* $i = 1,2,\ldots,N$, $j = 1,2,\ldots,N$
 (iv) Initial Condition $U_i(0)$ *for* $i = 1,2,\ldots,N$
 (v) Initial Condition $U_i^d(0)$ *for* $i = 1,2,\ldots,N$, $d = 1,2,\ldots,n-1$
2: Define:
 (i) Sinc Function $S(k,h,x)$
 (ii) $P_i(x) = \sum_{j=0}^{n} a_{ij}x^j$ *for* $i = 1,2,\ldots,n$
 (iii) $Y_i(x) = \sum_{j=-M}^{M} C_{ij}S(k,h,x)$ *for* $i = 1,2,\ldots,N$
3: Replace $u_i(x) = Y_i(x) + P_i(x)$ *for* $i = 1,2,\ldots,N$, in each equation
4: Calculate Sinc point $x_p$ *for* $p = -M-1, -M, -M+1, \ldots, 0, 1, \ldots, M$
5: Evaluate $u_i(x)_p = f_i(x)_p + \int_0^{x_p}\sum_{j=1}^{N} k_{ij}(x_p,t) * u_j(t)dt$ *for* $p = -M-1, -M, -M+1, \ldots, 0, 1, \ldots, M$, *for* $i = 1,2,\ldots,N$
 From this step, we get $(2(M+1)*N)$ equation
6: Evaluate $u_i(0) = U_i(0)$ *for* $i = 1,2,\ldots,N$. From this step, we get $(N)$ equation
7: Evaluate $u_i^d(0) = U_i^d(0)$ *for* $i = 1,2,\ldots,N$, $d = 1,2,\ldots,n-1$. From this step, we get $(N*(n-1))$ equation
8: Solving the algebraic system to get $c_{ij}$ and $a_{ik}$ *for* $i = 1,2,\ldots,N$, $j = 1,2,\ldots,N$, $k = 1,2,\ldots,n$
9: Set $u_{(aprroxi)}(x) = \sup c_{ij}$ and $a_{ik}$ in $u_i(x)$
10: Input $u_{(exacti)}(x)$
11: Plot $u_{(aprroxi)}(x); u_{(exacti)}(x)$
12: Define the error $|u_{(exacti)}(x) - u_{(aprroxi)}(x)|$
13: Plot the error

Algorithm 1: Numerical Realization Using the Sinc Collocation Method.

where $k_{ij}$ and $F_i$ are known matrices for $i = 1,2,3,\ldots,n$, $j = 1,2,3,\ldots,c$.

By substituting the approximations Equations (56) and (59) into the system equation (54), we obtain:

$$\begin{aligned} D_i^T\psi(x) &\cong F_i^T\psi(x) + \sum_{j=1}^{c}\int_0^x \left(\psi^T(x)k_{ij}\psi(t)\right)\left(D_i^T B\psi(t) + E_i^T\psi(t)\right)dt \\ &= F_i^T\psi(x) + \sum_{j=1}^{c}\left(\psi^T(x)k_{ij}\right)\left(\int_0^x \psi(t)\left(D_i^T B\psi(t) + E_i^T\psi(t)\right)dt\right) \\ &= F_i^T\psi(x) + \sum_{j=1}^{c}\left(\psi^T(x)k_{ij}\right)\left(\int_0^x \psi(t)\left(D_i^T B + E_i^T\right)\psi(t)dt\right). \end{aligned} \tag{60}$$

Therefore,

$$D_i^T\psi(x) \cong F_i^T\psi(x) + \sum_{j=1}^{c}\psi^T(x)k_{ij}\tilde{Z}_i B\psi(t), \quad i = 1,2,3,\ldots,n \tag{61}$$

1: Input:

- (i) $N, M, k, b, n$
- (ii) $f_i(x)$ *for* $i = 1, 2, \ldots, N$
- (iii) $k_{ij}(x)$ *for* $i = 1, 2, \ldots, N,\ j = 1, 2, \ldots, N$
- (iv) Initial Condition $U_i(0)$ *for* $i = 1, 2, \ldots, N$
- (v) Initial Condition $U_i^r(0)$ *for* $i = 1, 2, \ldots, N$, $r = 1, 2, \ldots, n-1$

2: Define:

- (i) Chebyshev Function $T(k, b, m, x)$
- (ii) weight function $W(x)$
- (iii) $D_i(x) = [d_{i1}, d_{i2}, \ldots, d_{iM}]$ *for* $i = 1, 2, \ldots, N$
- (iv) $\psi(x) = [T(k, 0, m, x), \ldots, T(k, b, M-1, x)]$
- (v) $DM[h(x)]$ as definition (4)
- (vi) operator $DM[h(x, y)]$ as remark (5)

3: Calculate $F_i = DM[F_i(x)]$, *for* $i = 1, 2, \ldots, N$

4: Calculate $Q_i = DM[U_i(0)]$, *for* $i = 1, 2, \ldots, N$

5: Calculate $k_i = DM[k_{ij}(x)]$, *for* $i = 1, 2, \ldots, N$, $j = 1, 2, \ldots, N$

6: Define operation matrix $B$

7: Define $u_i^{(n)}(x) = D_i^T \cdot \psi(x)$, *for* $i = 1, 2, \ldots, N$

8: Define $u_i^{(r)}(x) = D_i \cdot B^{n-r} \cdot \psi(x) + \sum_{j=0}^{n-r-1} a_{ir(x^j/j!)}$, for $i = 1, 2, \ldots, N, r = 1, 2, \ldots, n-1$

9: Substituting $u_i^{(r)}(x)$, *for* $i = 1, 2, \ldots, N$ *for* $r = 1, 2, \ldots, n$ in the system

10: Multiplying each equation by $W(x) \cdot (x)\psi(x)$

11: Applying $\int_0^1 \cdot\, dx$ for all equations

12: From this step, we get $M * N$ equation

13: Solving the algebraic system to get $d_{ij}$, *for* $i = 1, 2, \ldots, N,\ j = 1, 2, \ldots, N$

14: Set $u_{(aprroxi)}(x) = \sup d_{ij}$ in $u_i(x)$

15: Input $u_{(exacti)}(x)$

16: Plot $u_{(aprroxi)}(x); u_{(exacti)}(x)$

17: Define the error $|u_{(exacti)}(x) - u_{(aprroxi)}(x)|$

18: Plot the error

Algorithm 2: Numerical Realization Using the Chebyshev Wavelets Method.

$c = 1, 2, 3, \ldots$, where $B$ is the $2^{a-1}c \times 2^{a-1}c$ operational matrix of integration and $\tilde{Z}_i$ are $2^{a-1}c \times 1$ matrices.

We multiply both sides of Equation (61) by $w_n(x)\psi^T(x)$ and integrating with respect to $x$ from 0 to 1, we obtain a linear system in terms of input $D_i, i = 1, 2, 3, \ldots, n$. Consequently, the vector functions $D_i$ elements are calculated by solving this system.

## 4. Stability of Systems of Volterra Integro-Differential Equations (VIDEs)

In this section, we present some important results on the stability of VIDEs (1) (for more details see [22]).

*Definition 5 (see [22]).* If $\mu : [0, x_0] \to \mathbb{R}^n$ is a continuous initial function, then $u(x, x_0, \mu)$ will denote the solution of (1) on $[x_0, \infty]$. Frequently, it is sufficient to write $u(x)$. If $f_i(0) = u_i(0) = 0$, then $u(x) \equiv 0$ is a solution of (1) called the zero solution. The norm on the initial function $\mu(x) = (\mu_1(x), \mu_2(x), \cdots, \mu_n(x))$ is given by

$$\|\mu\| = \sup\left\{ |\mu(x)| = \sum_{i=1}^{n} \mu(i) : 0 \le x \le x_0 \right\}. \tag{62}$$

The definition of stability of the zero solution is given in Burton [18] and is restated below.

*Definition 6 (see [22]).* The zero solution of (1) is stable if for each $\epsilon > 0$ and each $x_0 \ge 0$, there exists $\delta$ such that $|\mu(x)| \le \delta$ on $[0, x_0]$ and $x \ge x_0$ imply $|u(x, x_0, \mu)| \le \epsilon$.

We next define the statement "a Lyapunov functional for system (1)". Let $V(x, \Psi(.))$ be defined for $x \ge 0$ and $\Psi \in ([0, x]; \mathbb{R}^n)$ and let $V$ be locally Lipschitz in $\Psi$. For each $x \ge 0$ and every $\Psi \in ([0, x]; \mathbb{R}^n)$, we define the derivative $V$ along a solution of (1) by

$$V'_{(1)}(x, \Psi(.)) = \lim_{\Delta x \to 0^+} \sup \frac{V(x + \Delta x, u(\cdot, x, \Psi)) - V(x, \Psi(.))}{\Delta x}, \tag{63}$$

where $u(\xi; x, \Psi)$ is the unique solution of (1) with initial conditions $x$ and $\Psi$. Then the following result by Driver ([28]) gives a definition of the Lyapunov functional.

**Theorem 4** (see [28]). *If $V(x, \Psi(.))$ is defined for $x \ge 0$ and every $\Psi \in ([0, x]; \mathbb{R}^n)$ with*

(1) $V(x, 0) \equiv 0$.

(2) *$V$ continuous in $x$ and Lipschitz in $\Psi$.*

(3) *$V(x, \Psi(.)) \ge W(|\Psi(.)|)$, where $W : [0, \infty] \to [0, \infty]$ is a continuous function with*

$$W(0) = 0,\ W(r) > 0,\ r > 0 \tag{64}$$

*and $W$ strictly increasing (positive definiteness).*

(4) $V'_{(1)}(x, \Psi(.)) \le 0$.

*then the zero solution of (1) is stable, and*

$$V(x, \Psi(.)) = V(x, \Psi(t)) : 0 \le t \le x \tag{65}$$

*is called a Lyapunov functional for system (1).*

Finally, we assumed that the functions in (1) are well behaved, that continuous initial functions generate solutions, and that solutions which remain bounded can be continued.

## 5. Numerical Examples and Results

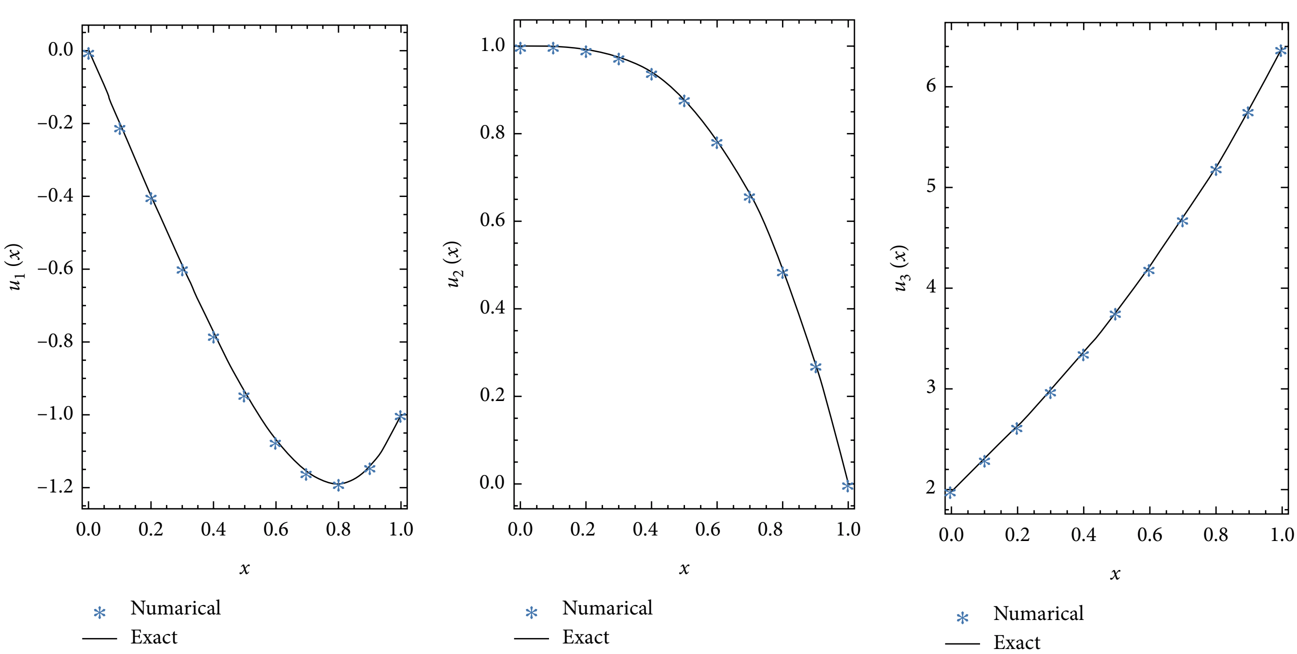
In this section, some numerical examples are presented to show the validity of the proposed methods. In addition, the numerical results are compared with exact solution.

Table 1: The exact and numerical solutions of applying Algorithm 1 for system (66) with $M = 8$.

| $x$ | Exact solution $u_1(x) = x^4 - 2x$ | Numerical solution $u_{1app}$ | Exact solution $u_2(x) = 1 - x^3$ | Numerical solution $u_{2app}$ | Exact solution $u_3(x) = x + 2e^x$ | Numerical solution $u_{1app}$ |
|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 1 | 1 | 2 | 2 |
| 0.1 | −0.1999 | −0.1998999986410037 | 0.999 | 0.9989999996164405 | 2.3103418361512955 | 2.310341836911406 |
| 0.2 | −0.3984 | −0.39840000469382025 | 0.992 | 0.992000001268595 | 2.64280551632034 | 2.642805513660997 |
| 0.3 | −0.5919 | −0.5919000795392859 | 0.973 | 0.9730000216099085 | 2.999717615152006 | 2.9997175706314225 |
| 0.4 | −0.7744 | −0.77440022803092 | 0.936 | 0.9360000615945837 | 3.3836493952825406 | 3.3836492678242998 |
| 0.5 | −0.9375 | −0.9375003156846542 | 0.875 | 0.8750000858865024 | 3.7974425414002564 | 3.7974423632300143 |
| 0.6 | −1.0704 | −1.0704001918522101 | 0.784 | 0.784000056734032 | 4.244237600781018 | 4.244237484652817 |
| 0.7 | −1.1599 | −1.1598998894031196 | 0.657 | 0.6569999842789827 | 4.727505414940953 | 4.72750545433764 |
| 0.8 | −1.1904 | −1.190399643757478 | 0.488 | 0.4879999282244343 | 5.251081856984936 | 5.251082019406921 |
| 0.9 | −1.1439 | −1.1438996142131692 | 0.271 | 0.2709999270259329 | 5.8192062223139 | 5.819206393371196 |

Table 2: The resulting error for the numerical solution.

| $x$ | Absolute error $\lvert u_1 - u_{1app}\rvert$ | Absolute error $\lvert u_2 - u_{2app}\rvert$ | Absolute error $\lvert u_3 - u_{3app}\rvert$ |
|---|---|---|---|
| 0 | 0 | −1 | 0 |
| 0.1 | $1.358996248868038e-9$ | $3.835595174805917e-10$ | $7.601106410959346e-10$ |
| 0.2 | $4.693820221390865e-9$ | $1.268595006820305e-9$ | $2.659342968058808e-9$ |
| 0.3 | $7.953928582438152e-8$ | $2.160990852928535e-8$ | $4.452058366410938e-8$ |
| 0.4 | $2.280309290281224e-7$ | $6.159458376675531e-8$ | $1.274582408505864e-7$ |
| 0.5 | $3.156846541951807e-7$ | $8.588650235452633e-8$ | $1.781702421155273e-7$ |
| 0.6 | $1.91852210118526e-7$ | $5.673403202788307e-8$ | $1.161282012773767e-7$ |
| 0.7 | $1.10596880320557e-7$ | $1.572101726576846e-8$ | $3.939668680175146e-8$ |
| 0.8 | $3.562425212599862e-7$ | $7.177556560211684e-8$ | $1.624219851947828e-7$ |
| 0.9 | $3.857868307033385e-7$ | $7.297406701134435e-8$ | $1.710572963276035e-7$ |




(a) (b) (c)

Figure 2: A comparison between exact and numerical solutions by applying Algorithm 1 for system (66) with $M = 8$. (a) The exact and numerical solutions of $u_1$. (b) The exact and numerical solutions of $u_2$. (c) The exact and numerical solutions of $u_3$.

Table 3: The exact and numerical solutions of applying Algorithm 2 for system (66).

| $x$ | Exact solution $u_1(x) = x^4 - 2x$ | Numerical solution $u_{1app}$ | Exact solution $u_2(x) = 1 - x^3$ | Numerical solution $u_{2app}$ | Exact solution $u_3(x) = x + 2e^x$ | Numerical solution $u_{1app}$ |
|---|---|---|---|---|---|---|
| 0 | 0 | 2.690833666996184$e$ − 13 | 1 | 0.9999999999543171 | 2 | 1.9999999974820484 |
| 0.1 | −0.1999 | −0.19990000000022942 | 0.999 | 0.9990000000727636 | 2.3103418361512955 | 2.3103418351947798 |
| 0.2 | −0.3984 | −0.39839999999984066 | 0.992 | 0.991999999990548 | 2.64280551632034 | 2.6428055151638277 |
| 0.4 | −0.7744 | −0.7744000000009819 | 0.936 | 0.9360000000694738 | 3.3836493952825406 | 3.3836493955712172 |
| 0.5 | −0.9375 | −0.9374999999997702 | 0.875 | 0.875000000131186 | 3.7974425414002564 | 3.7974425389296056 |
| 0.6 | −1.0704 | −1.0703999999956941 | 0.784 | 0.7840000001191981 | 4.244237600781018 | 4.244237600772698 |
| 0.7 | −1.1599 | −1.1598999999883794 | 0.657 | 0.6570000000729469 | 4.727505414940953 | 4.727505417496077 |
| 0.8 | −1.1904 | −1.190399999977122 | 0.488 | 0.4880000000954549 | 5.251081856984936 | 5.2510818561222665 |
| 0.9 | −1.1439 | −1.1438999999585011 | 0.271 | 0.27100000021849296 | 5.8192062223139 | 5.819206221276965 |

Table 4: The resulting error for the numerical solution.

| $x$ | Absolute error $\lvert u_1 - u_{1app}\rvert$ | Absolute error $\lvert u_2 - u_{2app}\rvert$ | Absolute error $\lvert u_3 - u_{3app}\rvert$ |
|---|---|---|---|
| 0 | 2.690833666996184$e$ − 13 | 4.568290190576363$e$ − 11 | 2.517951624980696$e$ − 9 |
| 0.1 | 2.293998324631729$e$ − 13 | 7.276357294472291$e$ − 11 | 9.565157554902726$e$ − 10 |
| 0.2 | 1.593725151849412$e$ − 13 | 9.451994742448733$e$ − 12 | 1.156512219324668$e$ − 9 |
| 0.3 | 4.383160501220118$e$ − 13 | 9.397482791939638$e$ − 12 | 2.494912720862885$e$ − 9 |
| 0.4 | 9.818812429784884$e$ − 13 | 6.947387110045611$e$ − 11 | 2.886766381493544$e$ − 10 |
| 0.5 | 2.298161660974074$e$ − 13 | 1.311859509911528$e$ − 10 | 2.470650795061146$e$ − 9 |
| 0.6 | 4.305888978706207$e$ − 12 | 1.191982068604602$e$ − 10 | 8.319567257331073$e$ − 12 |
| 0.7 | 1.162048235414658$e$ − 11 | 7.294698178839099$e$ − 11 | 2.555123224112776$e$ − 9 |
| 0.8 | 2.287792177924075$e$ − 11 | 9.545503276697787$e$ − 11 | 8.62669047307918$e$ − 10 |
| 0.9 | 4.14988043928588$e$ − 11 | 2.184930569804066$e$ − 10 | 1.0369349823236$e$ − 9 |

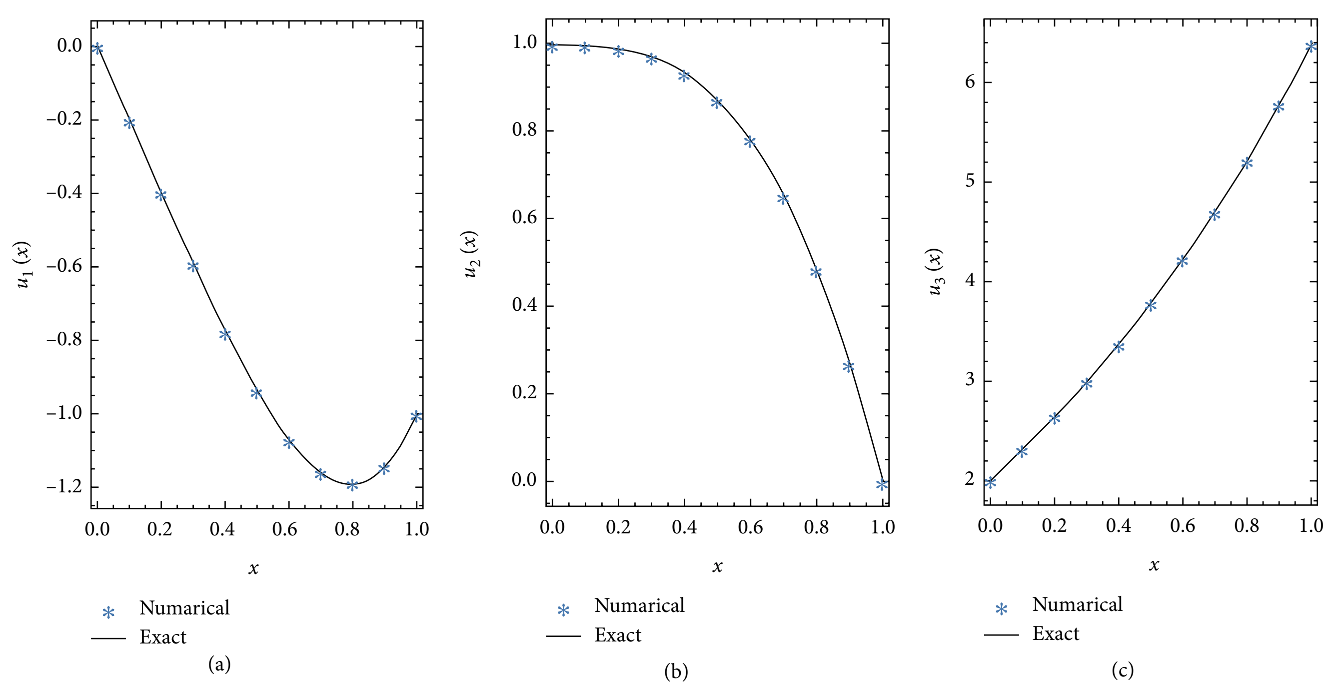



Figure 3: A comparison between exact and numerical solutions by applying Algorithm 2 for system (66) with $M = 8$. (a) The exact and numerical solutions of $u_1$. (b) The exact and numerical solutions of $u_2$. (c) The exact and numerical solutions of $u_3$.

Table 5: The exact and numerical solutions of applying Algorithm 1 for system (69).

| $x$ | Exact solution $u_1(x) = e^x + 1$ | Numerical solution $u_{1app}$ | Absolute error $\lvert u_1 - u_{1app}\rvert$ | Exact solution $u_2(x) = \cos x$ | Numerical solution $u_{2app}$ | Absolute error $\lvert u_2 - u_{2app}\rvert$ |
|---|---|---|---|---|---|---|
| 0 | 2 | 2 | 0 | 1 | 1 | 0 |
| 0.1 | 2.1051709180756477 | 2.1051397636262448 | 0.00003115444940293699 | 0.9950041652780258 | 0.9949953813521188 | 0.0000087839259069741 |
| 0.2 | 2.2214027581601696 | 2.2212621771873557 | 0.00014058097281388626 | 0.9800665778412416 | 0.9800269511507578 | 0.00003962669048385159 |
| 0.3 | 2.349858807576003 | 2.353150577162296 | 0.0032917695862928475 | 0.955336489125606 | 0.9562638272781463 | 0.0009273381525403135 |
| 0.4 | 2.4918246976412703 | 2.5101191909529916 | 0.018294493311721283 | 0.9210609940028851 | 0.9262143139859589 | 0.005153319983073779 |
| 0.5 | 2.648721270700128 | 2.694898115680843 | 0.046176844980714726 | 0.8775825618903728 | 0.890587846895059 | 0.013005285004686229 |
| 0.6 | 2.822118800390509 | 2.8955876162197285 | 0.07346881582921938 | 0.8253356149096782 | 0.8460218395509642 | 0.020686224641286 |
| 0.7 | 3.0137527074704766 | 3.094820636647796 | 0.08106792917731953 | 0.7648421872844884 | 0.7876576255740133 | 0.022815438289524925 |
| 0.8 | 3.225540928492468 | 3.289093529586907 | 0.06355260109443917 | 0.6967067093471654 | 0.7145821371224699 | 0.017875427775304487 |
| 0.9 | 3.45960311115695 | 3.4945194299151083 | 0.03491631875815848 | 0.6216099682706644 | 0.631444245918761 | 0.009834277648096634 |

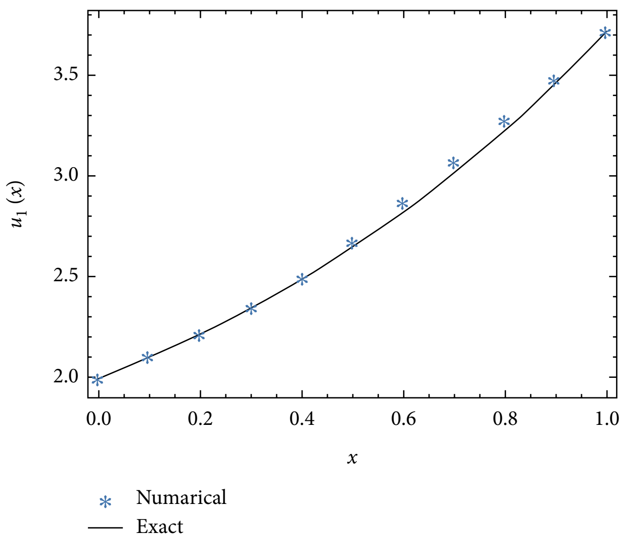



FIGURE 4: The exact and numerical solutions of $u_1$ using Algorithm 1 for system (69).

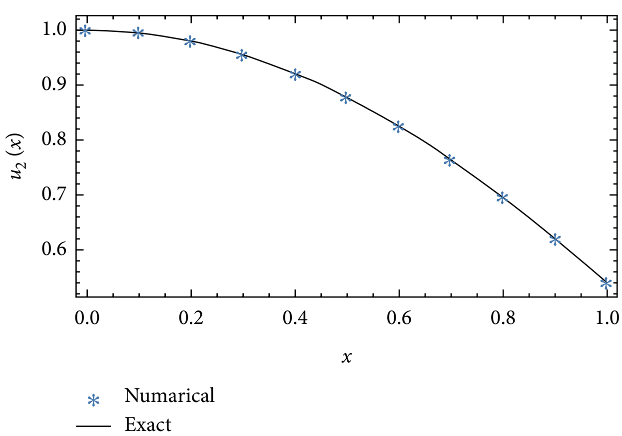



FIGURE 5: The exact and numerical solutions of $u_2$ using Algorithm 1 for system (69).

*Example 1.* Consider the system of Volterra integro-differential equations:

$$\begin{aligned} u_1'(x) &= -2 + x^2 - x^4 + \frac{3x^5}{20} + 2x^6 + \frac{x^7}{5} - \frac{x^8}{8} \\ &\quad + \int_0^x \left(\left(t^3 - x^2\right)u_1 + \left(12t^2 - x\right)u_2\right)dt \\ u_2'(x) &= 4 - 8x - x^4 - \frac{x^3}{3} + 2x^4 - \frac{8x^5}{5} + \frac{x^6}{30} - 4e^x \\ &\quad + \int_0^x ((t-x)u_1 + 8(1-t)u_2 + 2u_3)dt \\ u_3'(x) &= 3 - \frac{7x^2}{2} + \frac{4x^3}{3} + \frac{6x^5}{5} - \frac{7x^6}{30} \\ &\quad + \int_0^x ((2x-t)u_1 + 6tu_2 + u_3)dt \end{aligned} \tag{66}$$

together with the initial conditions

$$u_1(0) = 0, \quad u_2(0) = 1, \quad u_3(0) = 2. \tag{67}$$

The exact solution of system (66) is

$$u_1(x) = x^4 - 2x, \quad u_2(x) = 1 - x^3, \quad u_3(x) = x + 2e^x. \tag{68}$$

We start by implementing Algorithm 1 to solve system (66) using the Sinc collocation method based on sinc functions.

Tables 1 and 2 contain the exact and numerical solutions together with the resulting error with $M = 8$.

Figure 2 shows a comparison between the exact and numerical solutions for system (66). The maximum error corresponding to $u_1, u_2$, and $u_3$ is $E_1 \approx 3.9e^{-7}$, $E_2 \approx 8.5e^{-8}$, and $E_3 \approx 1.79e^{-7}$, respectively.

Next, we implement Algorithm 2 to solve system (66) using the Chebyshev wavelets method. Tables 3 and 4 contain the exact and numerical solutions for system (66) together with the resulting error with $M = 8$.

The maximum error corresponding to $u_1, u_2$ and $u_3$ is $E_1 \approx 7.7e^{-11}$, $E_2 \approx 2.5e^{-10}$, and $E_3 \approx 2.6e^{-9}$, respectively. Figure 3 displays a comparison between the exact and numerical solutions for systems (66) using Algorithm 2.

*Example 2.* Consider the system of Volterra integro-differential equations:

$$\begin{aligned} u_1''(x) &= -1 - x + \cosh x - \frac{\sin^3 x}{3} - \sinh x + e^x \\ &\quad + \int_0^x \left(\left(e^{-t}\right)u_1 + \left(\sin^2 t\right)u_2\right)dt \\ u_2''(x) &= -3 + x^2 - \frac{2x^3}{3} - 2e^x(x-1) \\ &\quad + \int_0^x \left(\left(x^2 - t^2\right)u_1 + (x-t)u_2\right)dt \end{aligned} \tag{69}$$

together with the initial conditions

$$u_1(0) = 2, \quad u_1'(0) = 1, \quad u_2(0) = 1, \quad u_2'(0) = 0. \tag{70}$$

The exact solution of system (69) is

$$u_1(x) = e^x + 1, \quad u_2(x) = \cos x. \tag{71}$$

We implement Algorithm 1 to solve system (69) using the sinc collocation method based on sinc functions. Table 5 contains the exact and numerical solutions using Algorithm 1 for system (69) together with the resulting error with $M = 8$.

Figure 4 compares the exact and numerical solutions $u_1(x) = e^x + 1$ and the approximate solution with $M = 8$. The maximum error corresponding to $u_1$ and $u_2$ is $E_1 \approx 0.081$ and $E_2 \approx 0.229$.

Figure 5 compares the exact solution $u_2(x) = \cos x$ and the approximate solution with $M = 8$.

Next, we implement Algorithm 2 to solve system (69) using the Chebyshev wavelets method. Table 6 contains the exact and numerical solutions with $M = 8$.

Figures 6 and 7 compare the exact solutions using Chebyshev wavlet method with $M = 8$. The maximum error corresponding to $u_1$ and $u_2$ is $E_1 \approx 1.7e^{-9}$ and $E_2 \approx 9.57e^{-9}$ respectively.

Figure 7 compares the exact solution $u_2(x) = \cos x$ and the approximate solution with $M = 8$.

Table 6: The exact and numerical solutions of applying Algorithm 2 for system (69).

| $x$ | Exact solution $u_1(x) = e^x + 1$ | Numerical solution $u_{1app}$ | Absolute error $\lvert u_1 - u_{1app}\rvert$ | Exact solution $u_2(x) = \cos\ x$ | Numerical solution $u_{2app}$ | Absolute error $\lvert u_2 - u_{2app}\rvert$ |
|---|---|---|---|---|---|---|
| 0 | 2 | 1.999999998740645 | 1.25935506467556$e-9$ | 1 | 0.9999999993413167 | 6.586833301014394$e-10$ |
| 0.1 | 2.1051709180756477 | 2.105170917580036 | 4.956115517984472$e-10$ | 0.9950041652780258 | 0.9950041649575281 | 3.204977394588582$e-10$ |
| 0.2 | 2.2214027581601696 | 2.22140275754089 | 6.192797386006532$e-10$ | 0.9800665778412416 | 0.9800665775316376 | 3.096040090966312$e-10$ |
| 0.3 | 2.349858807576003 | 2.349858808759274 | 1.183270814664183$e-9$ | 0.955336489125606 | 0.9553364897343352 | 6.087291781753379$e-10$ |
| 0.4 | 2.4918246976412703 | 2.491824697676863 | 3.559286199106282$e-11$ | 0.9210609940028851 | 0.9210609939384683 | 6.441680522328852$e-10$ |
| 0.6 | 2.822118800390509 | 2.822118800114823 | 2.756861405828203$e-10$ | 0.8253356149096782 | 0.8253356147660844 | 1.435938035143635$e-10$ |
| 0.7 | 3.0137527074704766 | 3.01375270838393 | 9.134533129895317$e-10$ | 0.7648421872844884 | 0.764842187697923 | 4.134346198725325$e-10$ |
| 0.8 | 3.225540928492468 | 3.2255409275988933 | 8.935745476890133$e-10$ | 0.6967067093471654 | 0.6967067087823426 | 5.64822744131277$e-10$ |
| 0.9 | 3.45960311115695 | 3.459603110051884 | 1.105065816631167$e-9$ | 0.6216099682706644 | 0.6216099676731845 | 5.974798433783235$e-10$ |

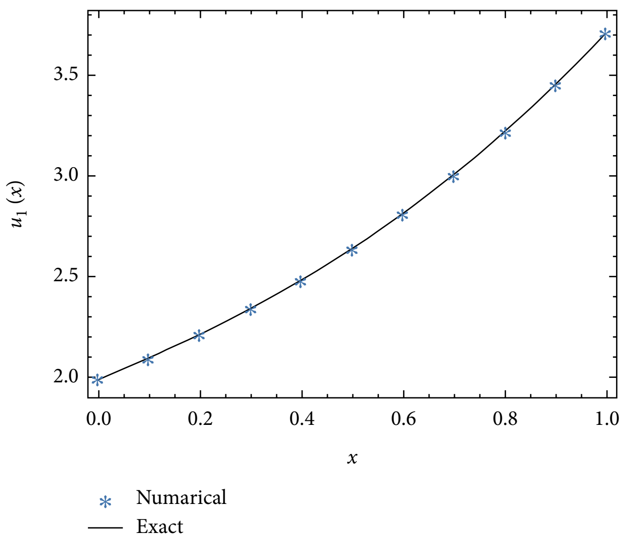



FIGURE 6: The exact and numerical solutions of $u_1$ using Algorithm 2 for system (69).

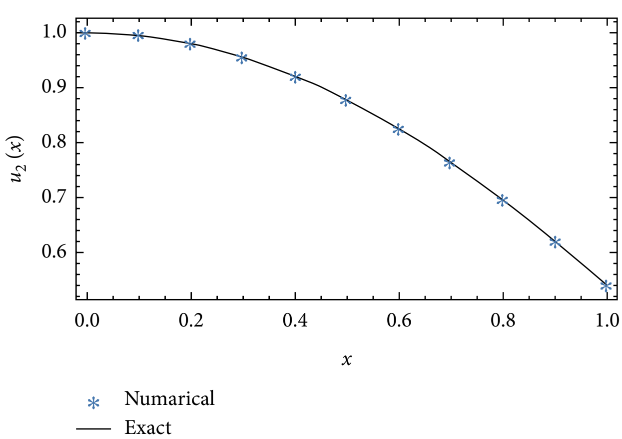



FIGURE 7: The exact and numerical solutions of $u_2$ using Algorithm 2 for system (69).

## 6. Conclusions

In this article, a collocation method using sinc functions and Chebyshev wavelets method is proposed to solve linear system of integro-differential equations. The numerical results show that the convergence and accuracy of both methods were in good agreement with the analytical solution. Comparison of numerical results mentioned in tables and figures shows clearly that the Chebyshev wavelets method provides more accurate results and is therefore more effective than any other methods for solving systems of integro-differential equations.

## Data Availability

We do not have any objection of sharing the data, the findings and the results of our article with other authors in different means of communication.

## Conflicts of Interest

The authors declare that they have no conflicts of interest.

## References

[1] F. Bloom, "Concavity arguments and growth estimates for damped linear integrodifferential equations with applications to a class of holohedral isotropic dielectrics," *Zeitschrift für angewandte Mathematik und Physik ZAMP*, vol. 29, no. 4, pp. 644–663, 1978.

[2] T.-L. Bo, L. Xie, and X. J. Zheng, "Numerical approach to wind ripple in desert," *International Journal of Nonlinear Sciences and Numerical Simulation*, vol. 8, no. 2, pp. 223–228, 2007.

[3] K. Holmåker, "Global asymptotic stability for a stationary solution of a system of integro-differential equations describing the formation of liver zones," *SIAM Journal on Mathematical Analysis*, vol. 24, no. 1, pp. 116–128, 1993.

[4] E. Hesameddini and A. Rahimi, "A new numerical scheme for solving systems of integro-differential equations," *Computational Methods for Differential Equations*, vol. 1, no. 2, pp. 108–119, 2013.

[5] E. Hesameddini and E. Asadolhifard, "Solving systems of linear Volterra integro-differential equations by using sinc-collocation method," *International Journal of Mathematical Engineering and Science*, vol. 2, no. 7, pp. 1–9, 2013.

[6] H. Aminikhah and S. Hosseini, "Numerical solution of linear system of integro-differential equations by using Chebyshev wavelet method," *Mathematical Sciences Letters*, vol. 4, no. 1, pp. 45–50, 2015.

[7] W. Draidi and N. Qatanani, "Numerical schemes for solving Volterra integral equations with Carleman kernel," *International Journal of Applied Mathematics*, vol. 31, no. 5, pp. 647–669, 2018.

[8] J. Hamaydi and N. Qatanani, "Computational methods for solving linear fuzzy Volterra integral equation," *Journal of Applied Mathematics*, vol. 2017, Article ID 2417195, 12 pages, 2017.

[9] A. J. Issa, "Approximate solutions of systems of integro-differential equations," An-Najah National University, 2019, MSc Thesis.

[10] O. M. Al-Faour and R. K. Saeed, "Solution of a system of linear Volterra integral and integro-differential equations by spectral method," *Journal for Al-Nahrain University Science*, vol. 6, no. 2, pp. 30–46, 2006.

[11] R. Chandra Guru Sekar and K. Murugesan, "System of linear second order Volterra integro-differential equations using single term Walsh series technique," *Applied Mathematics and Computation*, vol. 273, pp. 484–492, 2016.

[12] A. Akyüz-Daşcıoğlu and M. Sezerb, "Chebyshev polynomial solutions of systems of higher-order linear Fredholm–Volterra integro-differential equations," *Journal of the Franklin Institute*, vol. 342, no. 6, pp. 688–701, 2005.

[13] K. Maleknejad, F. Mirzaee, and S. Abbasbandy, "Solving linear integro-differential equations system by using rationalized Haar functions method," *Applied mathematics and computation*, vol. 155, no. 2, pp. 317–328, 2004.

[14] A. Arikoglu and I. Ozkol, "Solutions of integral and integro-differential equation systems by using differential transform method," *Computers and Mathematics with Applications*, vol. 56, no. 9, pp. 2411–2417, 2008.

[15] J. Biazar, H. Ghazvini, and M. Eslami, "He's homotopy perturbation method for systems of integro-differential equations," *Chaos, Solitons and Fractals*, vol. 39, no. 3, pp. 1253–1258, 2009.

[16] M. Gachpazan, "Numerical scheme to solve integro-differential equations system," *Journal of Advanced Research in Scientific Computing*, vol. 1, no. 1, 2009.

[17] T. Jangveladze, Z. Kiguradze, and B. Neta, "Finite difference approximation of a nonlinear integro-differential system," *Applied Mathematics and Computation*, vol. 215, no. 2, pp. 615–628, 2009.

[18] T. A. Burton, *Volterra Integral and Differential Equations*, Academic Press, New York, 1983.

[19] T. A. Burton, *Stability and Periodic Solutions of Ordinary and Functional Differential Equations*, Elsevier, New York, 1985.

[20] S. Elaydi, "Stability of integrodifferential systems of nonconvolution type," *Mathematical Inequalities and Applications*, vol. 1, no. 3, pp. 423–430, 1998.

[21] B. Zhang, "Construction of Liapunov functionals for linear Volterra integrodifferential equations and stability of delay systems," *Electronic Journal of Qualitative Theory of Differential Equations*, vol. 30, pp. 1–17, 2000.

[22] J. Vanualailai and S.-i. Nakagiri, "Stability of a system of Volterra integro-differential equations," *Journal of Mathematical Analysis and Applications*, vol. 281, no. 2, pp. 602–619, 2003.

[23] O. H. Mohammed, F. S. Fadhel, and M. G. S. AL-Safi, "Sinc-jacobi collocation algorithm for solving the time-fractional diffusion-wave equations," *International Journal of Mathematics and Statistics Studies*, vol. 3, no. 1, pp. 28–37, UK, 2015.

[24] D. Gottlieb and S. A. Orszag, "Numerical analysis of spectral methods: theory and applications," *Society of Indian Automobile Manufacturers*, vol. 26, 1977.

[25] O. Christensen and K. L. Christensen, *Approximation Theory: From Taylor Polynomials to Wavelets*, Springer Science & Business Media, 2004.

[26] I. Daubechies, "Ten lectures on wavelets," *SIAM Review*, vol. 61, 1992.

[27] J. Biazar and H. Ebrahimi, "A strong method for solving systems of integro-differential equations," *Applied Mathematics*, vol. 2, no. 9, pp. 1105–1113, 2011.

[28] R. D. Driver, "Existence and stability of solutions of a delay-differential system," *Archive for Rational Mechanics and Analysis*, vol. 10, no. 11, pp. 401–426, 1962.