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Research Article

Theoretical Aspect of Diagonal Bregman Proximal Methods

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In this paper, we propose and study a diagonal inexact version of Bregman proximal methods, to solve convex optimization problems with and without constraints. The proposed method forms a unified framework for existing algorithms by providing others.

1. Introduction

Let f_i: $R^d \longrightarrow R$ $(i = 0, 1, \ldots, m)$ convex functions and C the nonempty subset of R^d are defined by

$$C = \left\{ x \in R^d : f_i(x) \le 0, i = 1, \ldots, m \right\}. \quad (1)$$

Let us consider the problem of convex optimization:

$$(P): \min \{ f_0(x), x \in C \}. \quad (2)$$

To solve (P), many authors [1–7] have combined the exterior penalty methods with the proximal method (PM) defined by

$$x^k \in \varepsilon_k - \text{Arg}\min \left\{ f(\cdot) + \frac{1}{2\lambda_k} \left\| \cdot - x^{k-1} \right\|^2 \right\}, \quad (3)$$

where $f \in \Gamma_0(R^d)$ is set of proper closed convex functions on R^d. PM and its variants have been studied by several authors [6, 8–13]. In this labor, we generalize this process by introducing Bregman's distance $D_h(.,.)$ defined by

$$D_h(x, y) := h(x) - h(y) - \langle x - y, \nabla h(y) \rangle, \quad (4)$$

where h is Bregman's function [14].

In order to solve (P), we study the coupling of the methods of the exterior penalty with the diagonal inexact version of the Bregman proximal methods defined by

$$x^k \in \varepsilon_k - \text{Arg}\min \left\{ f(\cdot) + \lambda_k^{-1} D_h\left(\cdot, x^{k-1}\right) \right\}. \quad (5)$$

The exact version PMD is defined by

$$x^k = \arg\min \left\{ f(\cdot) + \lambda_k^{-1} D_h\left(\cdot, x^{k-1}\right) \right\}, \quad (6)$$

has been studied by several authors [15–18].

We propose and study a diagonal inexact version of the Bregman proximal method, which we call DBPM, defined by

$$x^k \in \varepsilon_k - \text{Arg}\min \left\{ f^k(\cdot) + \lambda_k^{-1} D_h\left(\cdot, x^{k-1}\right) \right\}, \quad (7)$$

where the sequence $\{f^k\}_k \subset \Gamma_0(R^d)$ is given and approaches f.

By introducing the penalty functions in DBPM, we deduce a solution of (P).

If $f^k = f \forall k$, the proposed method appears as an inexact version of (6) and solves the problem of convex optimization without constraints:

$$(P'): \min \left\{ f(x), x \in R^d \right\}. \quad (8)$$

For $h(\cdot) = (1/2)\|\cdot\|^2$, DBPM coincides with diagonal proximal method of Alart and Lemaire [1] as well as the penalization method given by Auslender [2].

2. Preliminary

In this section, we remind some theoretical properties of the approximations called entropic studied by Kabbadj in [17]. This study covers the properties of regularity and approximations of the Moreau–Yosida approximations [19]. These results are necessary for the analysis of the methods proposed in Section 3.

Let S be an convex open subset of R^d and $h\colon \overline{S} \longrightarrow R$. We define $D_h(.,.)$ by

$$\forall x \in \overline{S}, \forall y \in S\colon D_h(x,y) := h(x) - h(y) - \langle x-y, \nabla h(y)\rangle. \tag{9}$$

Let us consider the following hypotheses:

H_1: h is continuously differentiable on S.

H_2: h is continuous and strictly convex on $\overline{S}$.

H_3: $\forall r \geq 0, \forall x \in \overline{S}, \forall y \in S$, the sets $L_1(x,r)$ and $L_2(y,r)$ are bounded where

$$\begin{aligned} L_1(x,r) &= \{y \in S / D_h(x,y) \leq r\}, \\ L_2(y,r) &= \{x \in \overline{S} / D_h(x,y) \leq r\}. \end{aligned} \tag{10}$$

(i) H_4: if $\{y^k\}_k \subset S$ is such that $y^k \longrightarrow y^* \in \overline{S}$, then,

$$D_h(y^*, y^k) \longrightarrow 0. \tag{11}$$

H_5: if $\{x^k\}_k$ and $\{y^k\}_k$ are two sequences of S such that $D_h(x^k, y^k) \longrightarrow 0$ and $x^k \longrightarrow x^* \in S$, then

$$y^k \longrightarrow x^*. \tag{12}$$

Definition 1

(i) h: $\overline{S} \longrightarrow R$ is a Bregman type function on S or "D-function" if h verifies H_1, H_2, H_3, H_4, and H_5.

(ii) $D_h(.,.)$ is called entropic distance if h is a Bregman function.

We put

$A(S) = \{h\colon \overline{S} \longrightarrow R$ verifying H_1 and $H_2\}$

$B(S) = \{h\colon \overline{S} \longrightarrow R$ verifying H_1, H_2, H_3, H_4, and $H_5\}$.

Theorem 1 (see [17]). *Let $f \in \Gamma_0(R^d)$ and $h \in A(S)$ such that $\operatorname{dom} f \cap \overline{S} \neq \phi$.*

If one of the two following conditions is verified,

(i) $\inf_{\overline{S}} f > -\infty$ and h verifies H_3

(ii) $\operatorname{Im}\nabla h = R^d$,

then for all $x \in S$ and for all $\lambda > 0$, the function $u \longrightarrow f(u) + \lambda^{-1} D_h(u,v)$ has a unique minimum point on $\overline{S}$.

Definition 2. f and h verify the hypothesis of Theorem 1.

(i) The entropic approximation of f compared to h, of parameter $\lambda\,(\lambda > 0)$, is the function defined by

$$f_{h\lambda}(x) := \inf_{y \in \overline{S}} \{f(y) + \lambda^{-1} D_h(y,x)\}, \quad \forall x \in S. \tag{13}$$

(ii) The application entropic proximal of f comparing to h, of parameter λ, is the operator defined by

$$h_\lambda^f(x) := \operatorname{prox}_{\lambda f}^h(x) := \arg\min_{y \in \overline{S}} \{f(y) + \lambda^{-1} D_h(y,x)\}, \quad \forall x \in S. \tag{14}$$

Proposition 1 (see [17]). *Let $h \in A(S)$ and $f \in \Gamma_0(R^d)$ such that*

(a) $ri\ (dom\ f) \cap S \neq \phi$

(b) $\operatorname{Im}\nabla h = R^d$

Then, $\forall x \in S, \forall \lambda > 0$.

$$h_\lambda^f(x) \in S, \tag{15}$$

$$\inf_S f_{h\lambda} = \inf_S f, \tag{16}$$

$$\frac{\nabla h(x) - \nabla h\left(h_\lambda^f(x)\right)}{\lambda} \in \partial f\left(h_\lambda^f(x)\right), \tag{17}$$

$$f_{h\lambda}(x) \leq f_{h\mu}(x) \leq f(x), \quad \forall \mu\colon 0 < \mu \leq \lambda. \tag{18}$$

Proposition 2 (see [17]). *We suppose that h and f verify the conditions of Proposition 1.*

If $\inf(f) > -\infty$ and h verify H_3, then $h_\lambda^f\colon S \longrightarrow S$ is a continuous application.

Proposition 3 (see [17]). *We suppose that h and f verify the hypothesis of Proposition 2.*

If h is twice continuously differentiable on S and $D_h(.,.)$ and jointly convex, then $f_{h\lambda}$ is continually differentiable and convex such that $\forall x \in S$:

$$\nabla f_{h\lambda}(x) = \lambda^{-1} H(x)\left(x - h_\lambda^f(x)\right), \tag{19}$$

where $H = \nabla^2 h$.

Proposition 4. *We suppose that h and f verify the hypothesis of the Proposition 3. If H is defined positive, then*

$$\operatorname{Arg}\min_S f = \operatorname{Arg}\min_S f_{h\lambda}. \tag{20}$$

Proof. Let $u^* \in Arg\min_S f_{h\lambda}$.

$$\begin{aligned} f_{h\lambda}(u^*) = \inf_S f_{h\lambda} &\Longleftrightarrow 0 \in \partial f_{h\lambda}(u^*) \\ &\Longleftrightarrow 0 = \nabla f_{h\lambda}(u^*) \\ &\Longleftrightarrow H(u^*)\left(u^* - h_\lambda^f(u^*)\right). \end{aligned} \tag{21}$$

Since H is defined positive, we deduct then that $u^* = h_\lambda^f(u^*)$. From (17), we have

$$u^* = h_\lambda^f(u^*) \Longrightarrow 0 \in \partial f(u^*) \Longrightarrow u^* \in \arg\min_S f. \tag{22}$$

We get then $\operatorname{Arg}\min_S f_{h\lambda} \subset \operatorname{Arg}\min_S f$.

Reciprocally, let x^* such that $f(x^*) = \inf_S f$. From (16) and (18), we have

$$f(x^*) = \inf_S f_{h\lambda} \le f_{h\lambda}(x^*) \le f(x^*); \quad (23)$$

thus, we have $f(x^*) = \inf_S f_{h\lambda} = f_{h\lambda}(x^*)$, which completes the demonstration.

Some examples of Bregman functions are given below. □

Example 1. If $S_0 = R^d$ and $h_0(x) = (1/2)\|x\|^2$, then

$$D_{h_0}(x, y) = \frac{1}{2}\|x - y\|^2. \quad (24)$$

Example 2. If $S_1 = R^d_{++} := \{x \in R^d / x_i > 0, i = 1, \dots, d\}$ and

$$h_1(x) = \sum_{i=1}^{i=d} x_i \log x_i - x_i; \quad \forall x \in \overline{S}_1, \quad (25)$$

with the convention $0 \log 0 = 0$, then

$$D_{h_1}(x, y) = \sum_{i=1}^{d} x_i \log \frac{x_i}{y_i} + y_i - x_i, \quad \forall (x, y) \in \overline{S}_1 X S_1. \quad (26)$$

Example 3. If $S_2 = [-1, 1]^d$ and $h_2(x) = -\sum_{i=1}^{i=d} \sqrt{1 - x_i^2}$, then

$$D_{h_2}(x, y) = h_2(x) + \sum_{i=1}^{d} \frac{1 - x_i y_i}{\sqrt{1 - y_i^2}}, \quad \forall (x, y) \in \overline{S_2} X S_2. \quad (27)$$

We easily verify that $h_i \in B(S_i), i = 0, 1, 2$.

3. Analysis of the Diagonal Bregman Proximal Method

In this paragraph, we assume the following:

(A): $h \in B(S)$: $\operatorname{Im}\nabla h = R^d$ and $\overline{\operatorname{domf}} \subset S$

(B): $f, f^k \in \Gamma_0(R^d)$: $\overline{\operatorname{domf}^k} \subset S$, $k = 1, 2, \dots$

(C): $\lim\inf (\inf f^k) > -\infty$

From (15), we can then construct the sequence $\{x^k\}_k$ defined by (Algorithm 1):

In what follows, we will derive a convergence result (Theorem 2) for the DPMD framework. First, we need to establish a few technical results.

Lemma 1 (see [20]). *Let f_1, f_2 be two functions of $\Gamma_0(R^d)$ if there exists $\overline{x} \in domf_1$ in which f_2 is finite and continuous, then for $\varepsilon > 0$, for all $y \in domf_1 \cap domf_2$,*

$$\partial_\varepsilon (f_1 + f_2)(y) = \bigcup_{\varepsilon_1 + \varepsilon_2 = \varepsilon, \varepsilon_1 \ge 0, \varepsilon_2 \ge 0} \partial_{\varepsilon_1} f_1(y) + \partial_{\varepsilon_2} f_2(y). \quad (28)$$

Definition 3. The sequence $\{(\lambda_k; a_k; b_k; c_k; d_k)\}_k \in R^{+*} X S^4$ verifies the K-property only if the following properties are verified:

K_1 : $\exists \underline{\lambda} > 0, \forall k, \lambda_k \ge \underline{\lambda}$.

K_2 : $\{a_k\}$ is bounded and $\operatorname{Adh}\{a_k\} \subset S$.

K_3 : $D_h(a_k, b_k) \longrightarrow 0$.

K_4 : $D_h(a_k, c_k) \longrightarrow 0$.

K_5 : $d_k = (\nabla h(b_k) - \nabla h(c_k))/\lambda_k$.

Lemma 2. *If the sequence $\{(\lambda_k; a_k; b_k; c_k; d_k)\}_k$ verifies the K-property, then $d_k \longrightarrow 0$.*

Proof. If the sequence $\{d_k\}$ does not tend to zero, then it exists that $M > 0$ and the subsequence $\{d_{k_i}\}$ of $\{d_k\}$ such that

$$\forall k_i, \left\| d_{k_i} \right\| > M. \quad (29)$$

The sequence $\{a_{k_i}\}$ is bounded and Adh $\{a_{k_i}\} \subset S$; it exists that the subsequence $\{a_{k_j}\}$ of $\{a_{k_i}\}$ and $u^* \in S$ such that $a_{k_j} \longrightarrow u^*$. $D_h(a_{k_j}, b_{k_j}) \longrightarrow 0$ and $D_h(a_{k_j}, c_{k_j}) \longrightarrow 0$ allow to write, from H_5, $b_{k_j} \longrightarrow u^*$ and $c_{k_j} \longrightarrow u^*$. On the other hand,

$$0 \le \left\| d_{k_j} \right\| = \left\| \frac{\nabla h\left(b_{k_j}\right) - \nabla h\left(c_{k_j}\right)}{\lambda_{k_j}} \right\| \le \underline{\lambda}^{-1} \left\| \nabla h\left(b_{k_j}\right) - \nabla h\left(c_{k_j}\right) \right\|, \quad (30)$$

∇h is continuous on S, then $\nabla h(b_{k_j}) - \nabla h(c_{k_j}) \longrightarrow 0$. It follows that $\|d_{k_j}\| \longrightarrow 0$. $\{d_{k_j}\}$ is a subsequence of $\{d_{k_i}\}$, from with the entropic proximal method (29), we have $0 \ge M > 0$, so $d_k \longrightarrow 0$.

Lets consider now the function $h_{u,\lambda}$ defined by $h_{u,\lambda}$: $\overline{S} \longrightarrow R$, $\forall \lambda > 0, \forall u \in S$.

$$h_{u,\lambda}(x) = \lambda^{-1} D_h(x, u), \quad \forall x \in \overline{S}. \quad (31)$$

□

Proposition 5. $\forall \varepsilon > 0, \forall \lambda > 0, \forall u \in S, \forall x^* \in \overline{S}$.

$$\partial_\varepsilon h_{u,\lambda}(x^*) = \left\{ \frac{z}{z} = \frac{\nabla h(\overline{x}) - \nabla h(u)}{\lambda} \text{ with } \overline{x} \in S \text{ and } D_h(x^*, \overline{x}) \le \lambda\varepsilon \right\}. \quad (32)$$

Proof. $z \in \partial_\varepsilon h_{u,\lambda}(x^*)$.

$$\begin{aligned}
&\Longleftrightarrow h_{u,\lambda}(x) - h_{u,\lambda}(x^*) \ge \langle z, x - x^* \rangle - \varepsilon, \quad \forall x \in \overline{S}, \\
&\Longleftrightarrow \lambda^{-1} [h(x) - h(u) - \langle x - u, \nabla h(u) \rangle - h(x^*) + h(u) \\
&\quad + \langle x^* - u, \nabla h(u) \rangle] \ge \langle z, x - x^* \rangle - \varepsilon, \quad \forall x \in \overline{S}, \\
&\Longleftrightarrow h(x^*) - h(x) - \langle x^* - x, \nabla h(u) \rangle \\
&\quad \le \langle \lambda z, x^* - x \rangle + \lambda\varepsilon, \quad \forall x \in \overline{S},
\end{aligned} \quad (33)$$

which is equivalent to

$$h(x^*) - h(x) - \langle x^* - x, \nabla h(u) + \lambda z \rangle \le \lambda\varepsilon. \quad (34)$$

According to (A), it exists that $\overline{x} \in S$ such that

(1) Input: $x^0 \in S$
(2) Choose $\overline{\lambda} \geq \lambda_k \geq \underline{\lambda} > 0$ and $\varepsilon_k \geq 0$, and find $x^k \in S$, such that $x^k \in \varepsilon_k - \operatorname{Arg\,min}\{f^k(\cdot) + \lambda_k^{-1} D_h(\cdot, x^{k-1})\}$.
(3) Set $k \longleftarrow k+1$ and go to step 2

ALGORITHM 1: DPMD.

$$\nabla h(u) + \lambda z = \nabla h(\overline{x}), \tag{35}$$

which means

$$z = \frac{\nabla h(\overline{x}) - \nabla h(u)}{\lambda} \quad \exists \overline{x} \in S. \tag{36}$$

Replacing in (34) x by $\overline{x}$, we get

$$D_h(x^*, \overline{x}) \leq \lambda\varepsilon. \tag{37}$$

Finally

$$\partial_\varepsilon h_{u,\lambda}(x^*) \subset \left\{ \frac{z}{z} = \frac{\nabla h(\overline{x}) - \nabla h(u)}{\lambda} \text{ with } \overline{x} \in S \text{ and } D_h(x^*, \overline{x}) \leq \lambda\varepsilon \right\}. \tag{38}$$

Conversely, let z such as

$$z = \frac{\nabla h(\overline{x}) - \nabla h(u)}{\lambda},$$

$$D_h(x^*, \overline{x}) \leq \lambda\varepsilon,$$

$$\begin{aligned} D_h(x^*, \overline{x}) \leq \lambda\varepsilon &\Longrightarrow h(x^*) - h(\overline{x}) - \langle x^* - \overline{x}, \nabla h(\overline{x})\rangle \leq \lambda\varepsilon \leq \lambda\varepsilon \\ &\quad + D_h(x, \overline{x}), \\ &\Longrightarrow h(x^*) - h(\overline{x}) - \langle x^* - \overline{x}, \nabla h(\overline{x})\rangle - h(x) \\ &\quad + h(\overline{x}) + \langle x - \overline{x}, \nabla h(\overline{x})\rangle \leq \lambda\varepsilon, \\ &\Longrightarrow h(x^*) - h(x) - \langle x^* - x, \nabla h(\overline{x})\rangle \leq \lambda\varepsilon. \end{aligned} \tag{39}$$

Replacing $\nabla h(\overline{x})$ by $\nabla h(u) + \lambda z$, we get (34). According to what precedes,

$$(34) \Longleftrightarrow z \in \partial_\varepsilon h_{u,\lambda}(x^*), \tag{40}$$

which establishes the desired equality. □

Definition 4.

$$\begin{gathered} \forall \lambda > 0, \forall \rho \geq 0, \forall f, g \in \Gamma_0(R^d), \\ d^\rho_{h,\lambda}(f, g) := \sup_{\|x\| \leq \rho, x \in S} \left| f_{h\lambda}(x) - g_{h\lambda}(x) \right|. \end{gathered} \tag{41}$$

Theorem 2. *We assume that*

(i) $\sum_k \varepsilon_k + 2d_{k,\rho} < +\infty, \forall \rho \geq 0$, *where*

$$d_{k,\rho} := d^\rho_{h,\underline{\lambda}}(f^k, f). \tag{42}$$

(ii) The sequence $\{x^k\}$ *generated by DPMD is bounded. Then*

(a) $f^k(x^k) \longrightarrow \inf f$

(b) Moreover, if f *and* h *verify the conditions of Proposition 4, then*

$$\operatorname{Adh}\{x^k\} \subset \operatorname{Arg\,min} f. \tag{43}$$

Proof.

$$\begin{aligned} &x^k \in \varepsilon_k - \operatorname{Arg\,min}\{f^k(u) + \lambda_k^{-1} D_h(u, x^{k-1})\} \\ &\quad \Longrightarrow f^k(x^k) + \lambda_k^{-1} D_h(x^k, x^{k-1}) \leq f^k_{h\lambda_k}(x^{k-1}) + \varepsilon_k, \end{aligned} \tag{44}$$

according to (18), we can write

$$f^k_{h\underline{\lambda}}(x^k) + \lambda_k^{-1} D_h(x^k, x^{k-1}) \leq f^k_{h\underline{\lambda}}(x^{k-1}) + \varepsilon_k. \tag{45}$$

The sequence $\{x^k\}$ is bounded; let $\rho \geq 0$ such that

$$\forall k, \left\|x^k\right\| \leq \rho. \tag{46}$$

Considering (45),

$$\lambda_k^{-1} D_h(x^k, x^{k-1}) + f_{h\underline{\lambda}}(x^k) \leq f_{h\underline{\lambda}}(x^{k-1}) + \varepsilon_k + 2d_{k,\rho}. \tag{47}$$

Therefore,

$$f_{h\underline{\lambda}}(x^k) \leq f_{h\underline{\lambda}}(x^{k-1}) + \varepsilon_k + 2d_{k,\rho}. \tag{48}$$

So, from (i), we have

$$\lim f_{h\underline{\lambda}}(x^k) = l \geq \inf f_{h\underline{\lambda}} = \inf f \geq -\infty. \tag{49}$$

On one hand,

$$\begin{gathered} f^k(x^k) \leq f^k_{h\underline{\lambda}}(x^{k-1}) + \varepsilon_k \\ \Downarrow \\ f^k(x^k) \leq f_{h\underline{\lambda}}(x^{k-1}) + d_{k,\rho} + \varepsilon_k, \end{gathered} \tag{50}$$

on the other hand, we have

$$f_{h\underline{\lambda}}(x^k) - d_{k,\rho} \leq f^k(x^k); \tag{51}$$

finally, the two previous inequalities make it possible to write

$$f_{h\underline{\lambda}}(x^k) - d_{k,\rho} \leq f^k(x^k) \leq f_{h\underline{\lambda}}(x^{k-1}) + d_{k,\rho} + \varepsilon_k. \tag{52}$$

If $l = -\infty$, then $\inf f = -\infty$ and $f^k(x^k) \longrightarrow -\infty$. So,

$$f^k(x^k) \longrightarrow \inf f. \tag{53}$$

If $l > -\infty$, then, from (52),

$$\lim f^k(x^k) = \lim f_{h\underline{\lambda}}(x^k) = l. \tag{54}$$

Let us show that $l = \inf f$, from (45), $\lambda_k^{-1} D_h(x^k, x^{k-1}) \longrightarrow 0$ when $k \longrightarrow +\infty$. As $\underline{\lambda} \leq \lambda_k \leq \overline{\lambda}$, we have

$$D_h\left(x^k, x^{k-1}\right) \longrightarrow 0. \tag{55}$$

On the other hand,

$$\begin{aligned} x^k &\in \varepsilon_k - \operatorname{Arg\,min}\left\{f^k(u) + \lambda_k^{-1} D_h\left(u, x^{k-1}\right)\right\} \\ &\Longleftrightarrow 0 \in \partial_{\varepsilon_k}\left[f^k(\cdot) + \lambda_k^{-1} D_h\left(\cdot, x^{k-1}\right)\right]\left(x^k\right). \end{aligned} \tag{56}$$

From Lemma 1, there exists $\varepsilon_{k_1}, \varepsilon_{k_2} \geq 0$ such that $\varepsilon_{k_1} + \varepsilon_{k_2} = \varepsilon_k$ and

$$0 \in \partial_{\varepsilon_{k_1}} f^k\left(x^k\right) + \partial_{\varepsilon_{k_2}}\left(\lambda_k^{-1} D_h\left(\cdot, x^{k-1}\right)\right)\left(x^k\right). \tag{57}$$

Since $\partial_\varepsilon f^k$ increases with ε, we have

$$0 \in \partial_{\varepsilon_k} f^k\left(x^k\right) + \partial_{\varepsilon_k}\left(\lambda_k^{-1} D_h\left(\cdot, x^{k-1}\right)\right)\left(x^k\right). \tag{58}$$

Therefore, there exists $z_k \in \partial_{\varepsilon_k} f^k(x^k)$ such that

$$-z_k \in \partial_{\varepsilon_k}\left(\lambda_k^{-1} D_h\left(\cdot, x^{k-1}\right)\right)\left(x^k\right). \tag{59}$$

From Proposition 5, there exits $\overline{x}^k \in S$ such that

$$\begin{aligned} -z_k &= \frac{\nabla h\left(\overline{x}^k\right) - \nabla h\left(x^{k-1}\right)}{\lambda_k}, \\ D_h\left(x^k, \overline{x}^k\right) &\leq \lambda_k \varepsilon_k. \end{aligned} \tag{60}$$

Finally, there exists $\left\{\overline{x}^k\right\}$ such that

$$z_k = \frac{\nabla h\left(x^{k-1}\right) - \nabla h\left(\overline{x}^k\right)}{\lambda_k} \in \partial_{\varepsilon_k} f^k\left(x^k\right) \text{with}\, D_h\left(x^k, \overline{x}^k\right) \leq \lambda_k \varepsilon_k. \tag{61}$$

From (55) and (61), we have

$$\begin{cases} z_k = \dfrac{\nabla h\left(x^{k-1}\right) - \nabla h\left(\overline{x}^k\right)}{\lambda_k} \in \partial_{\varepsilon_k} f^k\left(x^k\right), \\ D_h\left(x^k, \overline{x}^k\right) \longrightarrow 0, \\ D_h\left(x^k, x^{k-1}\right) \longrightarrow 0. \end{cases} \tag{62}$$

Since $\operatorname{Adh}\ \{x^k\} \subset \cup \overline{\operatorname{dom} f^k} \subset S$, the sequence $\left\{(\lambda_k; x^k; \overline{x}^k; x^{k-1}; z_k)\right\}_k$ verifies then the K-property. From Lemma 2, $z_k \longrightarrow 0$. On the other hand, for all $y \in S$,

$$\begin{aligned} f^k(y) \geq f^k\left(x^k\right) + \langle z_k, y - x^k\rangle - \varepsilon_k &\Longrightarrow f^k(y) + \underline{\lambda}^{-1} D_h(y, x) \\ &\geq f^k\left(x^k\right) + \langle z_k, y - x^k\rangle - \varepsilon_k, \end{aligned} \tag{63}$$

there exists $\widehat{x}^k$ such that for all $k \in N$,

$$\begin{aligned} \inf\left\{f^k(y) + \underline{\lambda}^{-1} D_h(y, x)\right\} &= f^k_{h\underline{\lambda}}(x) \\ &= f^k\left(\widehat{x}^k\right) + \underline{\lambda}^{-1} D_h\left(\widehat{x}^k, x\right). \end{aligned} \tag{64}$$

By replacing y by $\widehat{x}^k$ in (63), we get

$$f^k\left(\widehat{x}^k\right) + \underline{\lambda}^{-1} D_h\left(\widehat{x}^k, x\right) \geq f^k\left(x^k\right) + \langle z_k, \widehat{x}^k - x^k\rangle - \varepsilon_k. \tag{65}$$

It is still

$$f^k_{h\underline{\lambda}}(x) \geq f^k\left(x^k\right) + \langle z_k, \widehat{x}^k - x^k\rangle - \varepsilon_k. \tag{66}$$

$\left\{\widehat{x}^k\right\}$ is bounded. Indeed,

$$-\infty < \inf_k \inf_S f^k \leq f^k\left(\widehat{x}^k\right), \tag{67}$$

so it exists $K_1 \in R$ such that $K_1 \leq f^k(\widehat{x}^k)$.

From (i),

$$d_{k,\rho} \longrightarrow 0 \Longrightarrow \left\{f^k_{h\underline{\lambda}}(x)\right\} \text{is convergent}, \tag{68}$$

so

$$\begin{aligned} \exists K_2 \in R&: K_2 \geq K_1, \\ f^k_{h\underline{\lambda}}(x) &\leq K_2. \end{aligned} \tag{69}$$

From (64), we have

$$D_h\left(\widehat{x}^k, x\right) \leq \underline{\lambda}\left(K_2 - K_1\right). \tag{70}$$

From H_3, $\left\{\widehat{x}^k\right\}$ is bounded. Going to the limit in (66), we have

$$f_{h\underline{\lambda}}(x) \geq \lim f^k\left(x^k\right) = l, \tag{71}$$

then,

$$\begin{aligned} f(x) \geq l, \quad \forall x &\Longrightarrow \inf f \geq l, \\ (52) &\Longrightarrow f_{h\underline{\lambda}}\left(x^k\right) \leq f^k\left(x^k\right) + d_{k,\rho}, \\ &\Longrightarrow \inf f = \inf f_{h\underline{\lambda}} \leq f^k\left(x^k\right) + d_{k,\rho}, \\ &\Longrightarrow \inf f \leq \lim f^k\left(x^k\right) = l. \end{aligned} \tag{72}$$

Finally, we have

$$\lim f^k\left(x^k\right) = \inf f. \tag{73}$$

(b) Let $x^* \in \operatorname{Adh}\{x^k\}$, there exists then the subsequence $\{x^{k_i}\}$ of $\{x^k\}$ such that $x^{k_i} \longrightarrow x^*$, we have

$$\begin{aligned} \inf f_{h\underline{\lambda}} &\leq f_{h\underline{\lambda}}(x^*) \leq \lim f_{h\underline{\lambda}}\left(x^{k_i}\right) \leq \lim f^{k_i}\left(x^{k_i}\right) \\ &= \inf f = \inf f_{h\underline{\lambda}} \Longrightarrow \inf f_{h\underline{\lambda}} = f_{h\underline{\lambda}}(x^*). \end{aligned} \tag{74}$$

From (20), we have

$$x^* \in \operatorname{Arg\,min} f_{h\underline{\lambda}} = \operatorname{Arg\,min} f. \tag{75}$$

□

4. Exterior Penalty Coupled with Bregman Proximal Method

Let f_i: $R^d \longrightarrow R, i = 1, \ldots, m$, be the convex function and let C the set of constraints given by

$$C = \left\{x \in R^d \colon f_i(x) \leq 0, i = 1, \ldots, m\right\}. \tag{76}$$

We suppose that C verifies the condition of Slater:

$$\exists x^* \in R^d \colon f_i(x^*) < 0, \quad i = 1, \dots, m. \tag{77}$$

Let us consider the functions of the linear penalty defined by

$$\varphi_1^n(x) = r_n \sum_{i=1}^{m} f_i^+(x), \quad \forall x \in R^d, \forall n \in N^*, \tag{78}$$

and the quadratic exterior penalty defined by

$$\varphi_2^n(x) = r_n \sum_{i=1}^{m} [f_i^+(x)]^2, \quad \forall x \in R^d, \forall n \in N^*, \tag{79}$$

where $a^+ = \max\{0, a\}$ and $\{r_n\}_n$ is an increasing sequence of strictly positive real numbers which tends to $+\infty$.

Let us put $\forall n \in N^*$:

$$\begin{aligned} f &= f_0 + \Psi_C, \\ f_1^n &= f_0 + \varphi_1^n, \\ f_2^n &= f_0 + \varphi_2^n. \end{aligned} \tag{80}$$

In what follows, we assume

(A'): $h \in B(R^d)$: $\mathrm{Im}\nabla h = R^d$

(B'): $\inf_{R^d} f > -\infty$,

so conditions (A), (B), and (C) of Section 3 are verified for f and f_j^n, $j = 1, 2$; $n \in N$.

We give below an estimate of $d_{\lambda,\rho}^h(f_j^n, f)$, $j = 1, 2$.

Proposition 6

(a) $\forall \underline{\lambda} > 0, \forall \rho \geq 0, \exists r_h \geq 0,$

$$d_{\lambda,\rho}^h(f_1^n, f) = 0, \quad \forall n \colon r_n \geq r_h, \forall \lambda \geq \underline{\lambda}. \tag{81}$$

(b) $\forall \underline{\lambda} > 0, \forall \rho \geq 0, \exists \mu_h \geq 0,$

$$d_{\lambda,\rho}^h(f_2^n, f) \leq \frac{\mu_h}{r_n}, \quad \forall n \geq 1, \forall \lambda \geq \underline{\lambda}. \tag{82}$$

Proof. Let $\lambda > 0$ and $x \in R^d$.

$$f_{h\lambda}(x) = \inf\{f + \lambda^{-1} D_h(\cdot, x)\} = \inf_C\{f_0 + \lambda^{-1} D_h(\cdot, x)\}. \tag{83}$$

Since Slater's condition is verified, there exists from Ekeland-Temam [21] (chap 3, Theorem 5.2) multiplicators of Lagrange

$$p_j(\lambda, x) \geq 0; \quad j = 1, \dots, m, \tag{84}$$

such that $\forall y \in R^d$.

$$f_{h\lambda}(x) \leq f_0(y) + \lambda^{-1} D_h(y, x) + \sum_{j=1}^{m} p_j(\lambda, x) f_j(y). \tag{85}$$

From (18), we have $f_{h\lambda}(x) \geq \inf_C f_0$, by replacing y with x^*, we obtain

$$\inf_C f_0 \leq f_0(x^*) + \lambda^{-1} D_h(x^*, x) + \sum_{j=1}^{m} p_j(\lambda, x) f_j(x^*), \tag{86}$$

where x^* verifies (3). On the other hand

$$\sum_{j=1}^{m} p_j(\lambda, x) f_j(x^*) \leq \left[\sum_{j=1}^{m} p_j(\lambda, x)\right] \sup_j f_j(x^*). \tag{87}$$

Let us put

$$\sup_j f_j(x^*) = -c, \quad \text{where } c > 0. \tag{88}$$

It follows that

$$\sum_{j=1}^{m} p_j(\lambda, x) f_j(x^*) \leq -c \|p(\lambda, x)\|_1, \tag{89}$$

where $p(\lambda, x) = (p_1(\lambda, x), \dots, p_m(\lambda, x))$. Therefore,

$$\begin{aligned} D_h(x^*, x) &\leq D_h(x^*, x) + D_h(x, x^*) \\ &= \langle x^* - x, \nabla h(x^*) - \nabla h(x) \rangle \\ &\leq \|x^* - x\| \cdot \|\nabla h(x) - \nabla h(x^*)\|, \end{aligned} \tag{90}$$

which leads to

$$D_h(x^*, x) \leq (\|x\|^* + \|x\|)(\|\nabla h(x^*)\| + \|\nabla h(x)\|). \tag{91}$$

From (86), (89), and (91), we obtain

$$\begin{aligned} \|p(\lambda, x)\|_1 \leq \frac{1}{c} \Big[& f_0(x^*) + \underline{\lambda}^{-1}(\|x^*\| + \|x\|)(\|\nabla h(x^*)\| \\ & + \|\nabla h(x)\|) - \inf_C f_0 \Big]. \end{aligned} \tag{92}$$

(a) From (85),

$$f_{h\lambda}(x) \leq f_0(y) + \underline{\lambda}^{-1} D_h(y, x) + \|p(\lambda, x)\|_1 \sum_{i=1}^{m} f_i^+(y). \tag{93}$$

For $x \in B_\rho$, from (92), we have

$$\begin{aligned} \|p(\lambda, x)\|_1 \leq \frac{1}{c} \Big[& f_0(x^*) + \underline{\lambda}^{-1}(\|x^*\| + \rho) \\ & \cdot \left(\|\nabla h(x^*)\| + \sup_{x \in B_\rho} \|\nabla h(x)\| \right) - \inf_C f_0 \Big] =: r_h. \end{aligned} \tag{94}$$

Thus, for n such as $r_n \geq r_h$,

$$\begin{gathered} f_{h\lambda}(x) \leq f_1^n(y) + \underline{\lambda}^{-1} D_h(y, x), \quad \forall y \in R^p \\ \Downarrow \\ f_{h\lambda}(x) \leq (f_1^n)_{h\lambda}(x), \quad \forall x \in B_\rho. \end{gathered} \tag{95}$$

Conversely,

$$\begin{aligned}(f_1^n)_{h\lambda}(x) &= \inf_{y\in R^p}\left\{f_0(y) + r_n\sum_{i=1}^{m} f_i^+(y) + \lambda^{-1}D_h(y,x)\right\}\\ &\le \inf_{y\in C}\left\{f_0(y) + \lambda^{-1}D_h(y,x)\right\}\\ &= \inf\left\{f(y) + \lambda^{-1}D_h(y,x)\right\}.\end{aligned} \tag{96}$$

Therefore,

$$(f_1^n)_{h\lambda}(x) \le f_{h\lambda}(x). \tag{97}$$

(b) If $\|\cdot\|_m$ indicates the Euclidean norm on R^d, so

$$\sum_{j=1}^{m} p_j(\lambda,x) f_j(x) \le \frac{1}{2r_n}\|p(\lambda,x)\|_m^2 + \frac{r_n}{2}\sum_{i=1}^{m}\left[f_i^+(y)\right]^2; \tag{98}$$

by proceeding as below, we deduce the result.

Let us consider now the methods of the coupled exterior penalties with the entropic proximal method (Algorithm 2): □

Theorem 3. *Let us suppose*

(i) $\sum_{n=1}^{\infty}\varepsilon_n < +\infty$, *and* $\sum_{n=1}^{\infty}(1/r_n) < +\infty$

(ii) $\exists i_0 \ge 1$: f_{i_0} *is coercive*

Then the sequence $\{x_j^n\}_n$ *generated by* $(DBPM)_j$ $j = 1, 2$ *is bounded,* $Adh\{x_j^n\} \subset Arg\min f$, *and*

$$f_j^n\{x_j^n\} \longrightarrow \inf f, \quad j = 1, 2 \ldots. \tag{99}$$

Proof. Let us show that $\{x_1^n\}_n$ is bounded.

$$f_1^n(x_1^n) + \lambda_n^{-1}D_h\left(x_1^n, x_1^{n-1}\right) \le f_1^n(u) + \lambda_n^{-1}D_h\left(u, x_1^{n-1}\right) + \varepsilon_n. \tag{100}$$

By replacing u by x_1^{n-1} in (100), we obtain

$$f_0(x_1^n) + r_n\sum_{i=1}^{m} f_i^+(x_1^n) \le f_0\left(x_1^{n-1}\right) + r_n\sum_{i=1}^{m} f_i^+\left(x_1^{n-1}\right) + \varepsilon_n. \tag{101}$$

Let $R = \inf_{R^d} f_0$, we have

$$\frac{f_0(x_1^n) - R}{r_n} + \sum_{i=1}^{m} f_i^+(x_1^n) \le \frac{f_0(x_1^{n-1}) - R}{r_n} + \sum_{i=1}^{m} f_i^+\left(x_1^{n-1}\right) + \frac{\varepsilon_n}{r_n}, \tag{102}$$

$\{r_n\}$ is increasing, so we put

$$A_n = \frac{f_0(x_1^n) - R}{r_n} + \sum_{i=1}^{m} f_i^+(x_1^n), \tag{103}$$

we deduce

$$A_n \le A_{n-1} + \frac{\varepsilon_n}{r_n}. \tag{104}$$

which leads to

$$A_n \le A_0 + \sum_{k=1}^{n}\frac{\varepsilon_k}{r_k}. \tag{105}$$

Let ε such as $\varepsilon_k \le \varepsilon$, since $(f_0(x_1^n) - R/r_n) > 0$, we have

$$f_{i_0}(x_1^n) \le A_0 + \sum_{k=1}^{\infty}\frac{\varepsilon}{r_k}. \tag{106}$$

From (i) and (ii), we deduce that the sequence $\{x_1^n\}_n$ is bounded, so by application of Theorem 2 and Proposition 6, the result is immediate.

In a similar way, we deduce the result for $j = 2$. □

5. Example

Let us consider the following optimization problem:

$$(P_1): \begin{cases}\min\langle a, x\rangle,\\ \|x\|^2 \le b,\end{cases} \tag{107}$$

where $a \in R^d$ and $b \in R^{+*}$.

The $(DBPM)_2$ algorithm can be applied to solve (P_1). We take

$$\begin{aligned}&\forall n, \varepsilon_n = 0,\\ &r_n = 2^n,\\ &\bar{\lambda} \ge \lambda_n \ge \underline{\lambda} > 0.\end{aligned} \tag{108}$$

Let us consider the function h: $R^d \longrightarrow R$ defined by

$$h(x) = \frac{1}{p}\sum_{i=1}^{i=d}|x_i|^p; \quad p \ge 2; \forall x \in R^d. \tag{109}$$

We easily show that $h \in B(R^d)$ and which checks (A') Let us put $\forall x \in R^d$:

$$\begin{aligned}&f_0(x) = \langle a, x\rangle,\\ &f_1(x) = \left\|x^2\right\| - b,\\ &\varphi^n(x) = \varphi_2^n(x) = r_n\left[f_1^+(x)\right]^2, \quad \forall n \in N^*.\end{aligned} \tag{110}$$

$C = \{x \in R^d: f_1(x) \le 0\}$ is compact and f_0 is continuous, so

$$\begin{aligned}&\operatorname{Arg}\min f \ne \emptyset,\\ &\inf_{R^d} f = \inf_{C} f_0 > -\infty.\end{aligned} \tag{111}$$

We have

$$\begin{aligned}f &= f_0 + \Psi_C,\\ f^n &:= f_2^n = f_0 + \varphi^n.\end{aligned} \tag{112}$$

The sequence $\{x^n\}$ generated by the $(DBPM)_2$ algorithm is defined by $x^0 \in R^d$ and

$$x^n \in \operatorname{Arg}\min\left\{f^n(\cdot) + \lambda_n^{-1}D_h\left(\cdot, x^{n-1}\right)\right\}, \quad n \ge 1. \tag{113}$$

By writing the condition of optimality, we have

(1) Input: $x_j^0 \in R^d$
(2) Choose $\overline{\lambda} \geq \lambda_n \geq \underline{\lambda} > 0$ and $\varepsilon_n \geq 0$, and find $x_j^n \in S$, such that $x_j^n \in \varepsilon_n - \operatorname{Arg\,min}\left\{f_j^n(\cdot) + \lambda_n^{-1} D_h(\cdot, x_j^{n-1})\right\}$.
(3) Set $n \longleftarrow n+1$ and go to step 2

Algorithm 2: $(\text{DBPM})_j$, $j = 1,2$.

$$\nabla f^n(x^n) + \lambda_n^{-1}\left(\nabla h(x^n) - \nabla h\left(x^{n-1}\right)\right) = 0. \tag{114}$$

On the other hand,

$$\nabla\left[f_1^+\right]^2(x) = 2f_1^+(x)\nabla f_1(x) = 4f_1^+(x)x. \tag{115}$$

Then,

$$2^{n+2} f_1^+(x^n)x^n + \lambda_n^{-1}\nabla h(x^n) = \lambda_n^{-1}\nabla h\left(x^{n-1}\right) - a, \tag{116}$$

where

$$(\nabla h(x^n))_i = \operatorname{sign}(x_i^n)\left|x_i^n\right|^{p-1}, \quad i = 1, \ldots, d, \tag{117}$$

f_1 is coercive, so by applying Theorem 3, we have

$$\begin{aligned} \operatorname{Adh}\{x^n\} \subset \operatorname{Arg\,min} f &= \left\{x \in R^d\colon f_0(x) = \inf_C f_0\right\}, \\ f^n\{x^n\} &\longrightarrow \inf_C f_0. \end{aligned} \tag{118}$$

Remarks 1

(i) The convergence performance of the $\{x^n\}$ can be discussed according to the parameter p. We take note that for $p = 2$,

$$h(x) = \frac{1}{2}\|x\|^2. \tag{119}$$

(ii) Let

$$f_0(x) = \frac{1}{2} < Ax, x > - < c, x >, \quad \forall x \in R^d, \tag{120}$$

where A is matrix symmetric definite positive and $c \in R^d$. Previously developed methods can solve optimization problems of the type

$$(P_2)\colon \begin{cases} \min \dfrac{1}{2}\langle Ax, x\rangle - \langle c, x\rangle, \\ \|x\|^2 \leq b. \end{cases} \tag{121}$$

6. Conclusion

The class of the methods studied in this work constitutes a unified framework for several existing methods that solve convex optimization problems with and without constraints while providing others, more precisely.

(i) For $h(\cdot) = (1/2)\|\cdot\|^2$, DBPM coincides with the diagonal proximal method DPM studied by Alart and Lemaire [1].

(ii) If $h(\cdot) = (1/2)\|\cdot\|^2$ in DBPM, $j = 1, 2$, we find then the methods of penalization studied by Auslender [2].

(iii) If $f^k = f \forall k$, DBPM appears as an inexact version of BPM and solves the problem of convex optimization without constraints:

$$(P')\colon \min\left\{f(x), x \in R^d\right\}, \tag{122}$$

the convergence of this version is included in our analysis and responds to the question asked by Eckstein in [15].

(i) If $f^k = f$ and $\varepsilon_k = 0, \forall k$ in DBPM, we find then BPM studied by [15–18].

(ii) If $f^k = f$ and $h(\cdot) = (1/2)\|\cdot\|^2$ in DBPM, we find then PM studied by [6, 8–13].

(iii) If $f^k = f$ and $h_1(x) = \sum_{i=1}^{i=d} x_i \log x_i - x_i$; $\forall x \in \overline{S_1}$, DBPM allows to minimize f on

$$S_1 = R_{++}^d := \left\{x \in R^d / x_i > 0, i = 1, \ldots, d\right\}. \tag{123}$$

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares that there are no conflicts of interest.

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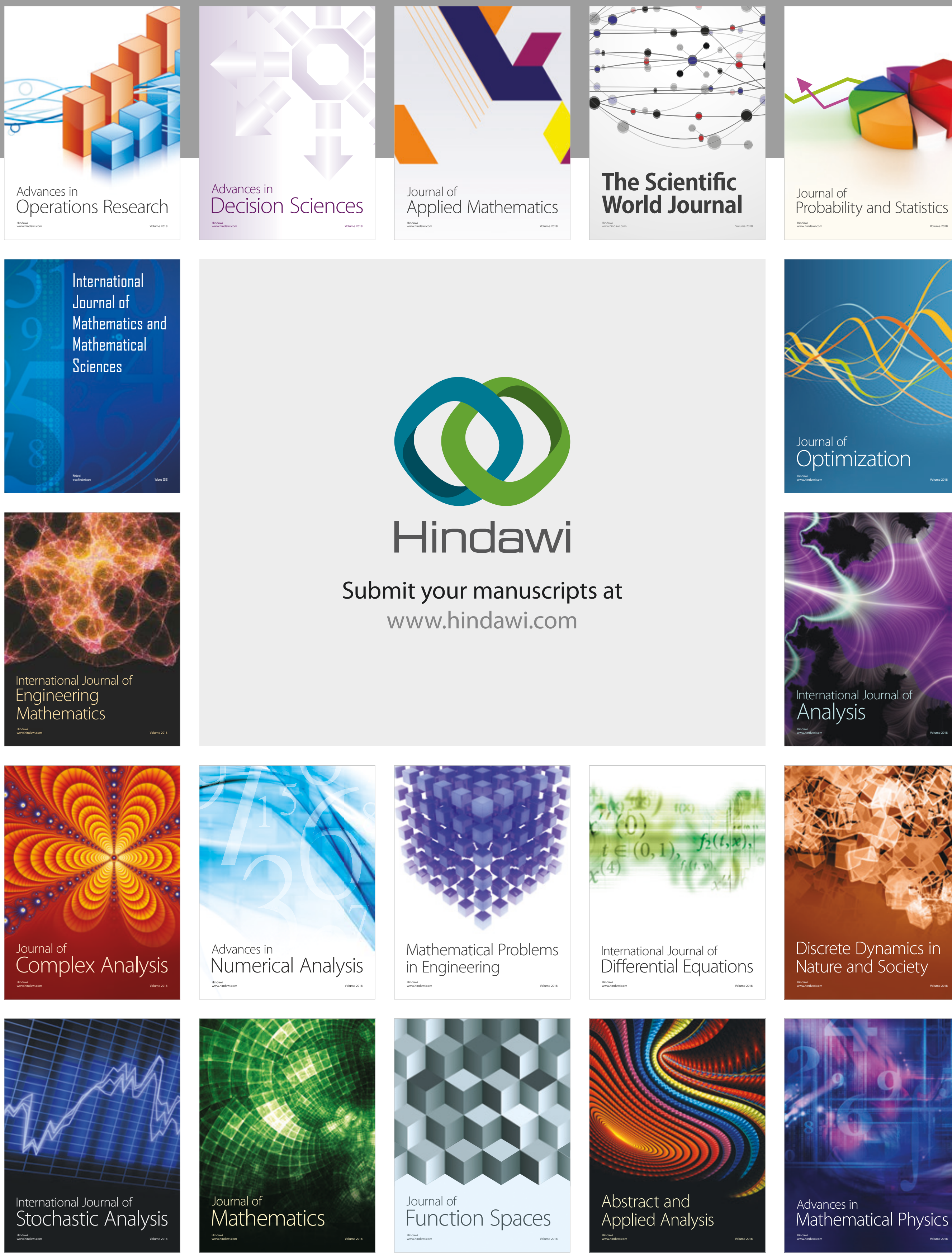
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