Research Article

Game Chromatic Number of Generalized Petersen Graphs and Jahangir Graphs

Ramy Shaheen ⓘ**, Ziad Kanaya, and Khaled Alshehada**

Department of Mathematics, Faculty of Science, Tishreen University, Lattakia, Syria

Correspondence should be addressed to Ramy Shaheen; shaheenramy2010@hotmail.com

Let $G = (V, E)$ be a graph, and two players Alice and Bob alternate turns coloring the vertices of the graph G a proper coloring where no two adjacent vertices are signed with the same color. Alice's goal is to color the set of vertices using the minimum number of colors, which is called game chromatic number and is denoted by $\chi_g(G)$, while Bob's goal is to prevent Alice's goal. In this paper, we investigate the game chromatic number $\chi_g(G)$ of Generalized Petersen Graphs $GP(n, k)$ for $k \geq 3$ and arbitrary n, n-Crossed Prism Graph, and Jahangir Graph $J_{n,m}$.

1. Introduction

n 1981, Brams invented the game as a tool to provide a theoretical proof of the Four Color theorem without using computers [1]. But, it soon became apparent that there were many planar graphs which had game chromatic numbers greater than 4. In 1991, Bodlaender reinvented the game as a two-person game, Alice and Bob [2]. Players take turns coloring the vertices of a given finite graph $G = (V, E)$ and a color set $X = \{1, 2, \ldots, k\}$. Alternately, players take turns over the elements of the set V; Alice starts the game, and her goal is to color the vertices with only $k = |X|$ colors, while Bob's goal is to prevent her. Bob wins the game if at any stage there are some vertices without available colors. The game chromatic number of a graph G, denoted by $\chi_g(G)$, is defined as the smallest integer k such that Alice has a winning strategy on only k colors. In [3], Faigle et al. proved that $\chi_g(T) \leq 4$ for a tree graph. In [4], Guan and Zhu introduced an upper bound to the game chromatic number of outerplanar graphs, which is $\chi_g(OP) \leq 7$. In [5], Bartnicki et al. studied the game chromatic number for the Cartesian product of two graphs. In [6], Sia studied the game for some families of Cartesian product graphs. Raspaud and Wu proved that $\chi_g(C_{2m}C_n) = 5 \longrightarrow \chi_g(C_{2m}\square C_n) = 5$ [7].

In [8], Shaheen et al. investigated the game chromatic number for some special circulant graphs and Generalized Petersen Graphs when $k = 1, 2, 3$.

The Coloring Number was defined in [9] as the following:

Let $\Pi(G)$ be the set of linear orderings on the vertex set of a graph G, and let $L \in \Pi(G)$. For a vertex $v \in V(G)$, *the back degree* of v with respect to L is the number of its neighbors that precede it in L, i.e., $b(v)tn = q|hN_L^+(v)| \longrightarrow b(v) = |N_L^+(v)|$ where $N_L^+(v) = \{u \in N(v) \, and \, u <_L v\}$. The back degree of L is the maximum back degree of the vertices in L. The coloring number of G denoted by $col(G)$ is $1 + b$ where b is the minimum back degree of a linear ordering L over all $L \in \Pi(G)$, i.e., $col(G) = 1 + min_{L \in \Pi(G)} max_{v \in V(G)}(b(v))$. Clearly, $\chi(G) \leq col(G)$.

The Game Coloring Number $col_g(G)$ is the competitive version of the coloring number, which was introduced for the first time by Zhu in [10] to give an improved upper bound of the game chromatic number of planar graphs. The game coloring number is a two-person game, which is played by Alice and Bob, who alternate turns and with Alice starting first. After every move, a vertex is selected (among the remaining vertices) and is added to the end of the linear ordering which was formed by previous moves. After selecting all the vertices, a linear ordering L of $V(G)$ is obtained, let b be its back degree and then $col_g(G) = 1 + b$.

Alice's goal is to minimize the back degree b, while Bob's goal is to maximize it. Both players use their optimal strategies to form L.

A *combinatorial game* is a finite two-player game with perfect information and no chance. The game coloring number and the game chromatic number are combinatorial games [11].

In 2009, Kierstead and Kostochka were the first to present an application of coloring games to a nongame problem, which was the graph packing problem [12].

Several studies were presented in 2019 and discussed the game chromatic number of some specific graphs [11, 13, 14]. In [14], C. Chamberlin et al. investigated the game chromatic number of G which is a graph produced by subdividing each edge of $GP(n, k)$ with n large enough and $\gcd(n, k) = 1$.

The uncolored vertex v is called *critical vertex* if there is just one available color c and there are some uncolored neighbors of v which can be signed with c. In any stage of the game, if there are some critical vertices in Bob's turn, then he wins the game. Also, if there are critical vertices in Alice's turn, then she must defend them by her turn [7].

We denote the ith turn of Alice by A_i and denote the process of assigning the vertex v with the color c by $v \longleftarrow c$.

Tow vertices are adjacent if there is an edge joining them together. For a vertex v in the graph $G(V, E)$, *the open neighbor set* is defined as $N(v) = \{u \in V ; \exists uv \in E\}$ and *the degree* of v is $\rho(v) = |N(v)| \longrightarrow \rho(v) = |N(v)|$. A graph $G = (V, E)$ is *r-regular* if $\rho(v) = r$ for every $v \in V$.

Definition 1. Let the integer k, n satisfy $k < n/2$; Generalized Petersen Graph $GP(n, k)$ is the graph whose vertex set is $V \cup U$ where $V = \{v_1, ..., v_n\}$ and $U = \{u_1, ..., u_n\}$ and its edge set is $E = \{v_i v_{i+1}, v_i u_i, u_i u_{i+k}\}$, where $i = 1, 2, ..., n$ and subscripts are reduced modulo n.

Definition 2. For a positive even number $n \geq 4$, n-crossed prism graph $G = (V, E)$ is a graph obtained by taking two distinct cycles, outer cycle C_n^1 where $V(C_n^1) = \{u_1, ..., u_n\}$, and the inner one C_n^2 where $V(C_n^2) = \{v_1, ..., v_n\}$ and adding edges between them as $\{u_i v_{i+1}, u_j v_{j-1}\}$; $i \in \{1, 3, ..., n - 1\}$, $j \in \{2, 4, ..., n\}$.

Definition 3. Jahangir graph, $J_{n,m}$ for $m \geq 3$, is a cycle C_{nm} with adding one vertex v_0 adjacent to m vertices of C_{nm} which are at distance n from each other in C_{nm}.

Definition 4. The n-sunlet graph S_n is a cycle C_n where $V(C_n) = \{u_1, u_2, ..., u_n\}$ with n additional vertex $\{v_1, v_2, ..., v_n\}$, and $v_i u_i \in E(S_n)$ for $i = 1, ..., n$.

Proposition 1 (see [10]). $\chi_g(G) \leq col_g(G)$.

Proposition 2 (see [15]). *The game chromatic number of the wheel graph W_n is $\chi_g(W_n) = 4$.*

Theorem 1 (see [8]). *For $n \geq 6$ and $k \in \{1, 2, 3\}$, we have $\chi_g(GPt(n, k)) = 4 \longrightarrow \chi_g(GP(n, k)) = 4$.*

2. Main Results

In this section, we investigate the game chromatic number of generalized Petersen graph, n-crossed prism graph, and Jahangir graph.

Lemma 1. *For $n \geq 9$ and $k \geq 2$, every Generalized Petersen graph $GP(n, k)$ has S_8 as a subgraph like the graph in Figure 1.*

The proof comes clearly from the definition of the Generalized Petersen graph.

Theorem 2. *For $n \geq 9$ and $k \geq 4$, the game chromatic number of the Generalized Petersen Graph $GP(n, k)$ is $\chi_g(GP(n, k)) = 4$.*

Proof. Generalized Petersen Graphs are 3-regular graphs; therefore, $\chi_g(GP(n, k)) \leq \Delta + 1 = 4$. For $k \in \{1, 2, 3\}$, $\chi_g(GP(n, k)) = 4$ is proven in Theorem 1.

Now, by contradiction, we will prove that Player A does not have any winning strategy for any integer $k \geq 4$.

Suppose that we have a color set $C = \{1, 2, 3\}$; by using Lemma 2.1, we notice that, in $GP(n, k)$, there is the subgraph as the graph in Figure 1.

Whatever the vertex is chosen by player A to start with, we will get the same state, so without loss of generality assume that player A starts the game with v_1, i.e., $A_1: v_1 \longleftarrow 1$ then $B_1: u_{k+1} \longleftarrow 2$; now, u_1 is a critical vertex so player A will defend it by coloring it with the third color or coloring its unique uncolored neighbor with a legal color, and whatever his choice for A_2 is, then $B_2: v_k \longleftarrow 1$ which means that the vertex v_{k+1} will become a critical vertex and also player A will have to defend it so whatever his choice is for A_3 then $B_3: u_n \longleftarrow 2$ and now there are two critical vertices $\{u_k, v_n\}$ and player A cannot defend them together so three colors are not enough for the first player to have a winning strategy. Then, $3 < \chi_g(GP(n, k)) = 4$. □

Theorem 3. *For $n \geq 6$, let G be a n-crossed prism graph, and the game chromatic number of G is $\chi_g(G) = 4$.*

Proof. $\chi_g(G) \leq 4$ since $\Delta(G) = 3$; therefore, we are going to contradict that three colors are not sufficient. Let $X = \{1, 2, 3\}$ and $A_1: v_1 \longleftarrow 1$; then, $B_1: v_3 \longleftarrow 2$, and now, v_2 is a critical vertex so Alice is forced to defend it in only two ways. We will discuss Alice's second turn A_2 as follows:

> *Case 1.* $A_2: v_2 \longleftarrow 3$ then $B_2: u_2 \longrightarrow u_3$, that gives two critical vertices u_2, v_4 and Alice cannot defend them at the same time
>
> *Case 2.* $A_2: u_1 \longleftarrow c$ where $c \in \{1, 2\}$
>
> > *Subcase 2.1.* $c = 1$ then $B_2: v_{n-1} \longleftarrow 2$ then u_n, v_n will be critical vertices and Alice cannot defend them at the same time
> >
> > *Subcase 2.2.* $c = 2$ then u_2 will become a critical vertex during Bob's turn who will color u_3 with the color 3, and Alice loses the game

That means $\chi_g(G) > 3$, so $\chi_g(G) = 4$. □

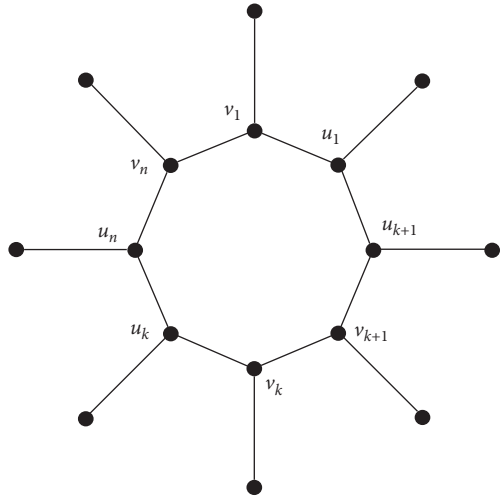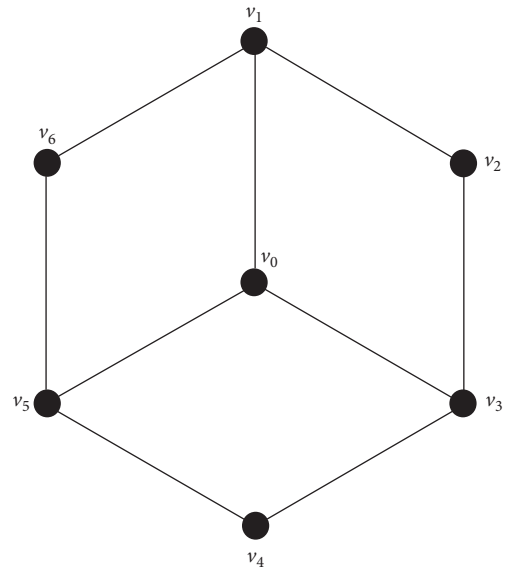Figure 1: S_8 as subgraph of $GP(n, k)$.

Figure 2: Jahangir graph $J_{2,3}$.

Theorem 4. *For $m \geq 3$, let $G \cong J_{n,m}$ be the Jahangir graph, and the game chromatic number*

$$\chi_g(G) = \begin{cases} 4, & if\ n = 1\ or\ (n = 2\ and\ m \neq 3), \\ 3, & if\ (n = 2\ and\ m = 3)\ or\ (n \geq 3\ and\ m \geq 3). \end{cases}$$

(1)

Proof. First, because the Jahangir graph can be divided into cycles, then it is easy to show that $\chi_g(J_{n,m}) > 2$. Now, we will discuss the following cases.

Case 1. $n = 1\ or\ (n = 2\ and\ m \neq 3)$

Subcase 1.1. $n = 1$ then $G \cong W_m$, and using Proposition 1, $\chi_g(G) = \chi_g(W_m) = 4$.

Subcase 1.2. $n = 2$ and $m \neq 3$ then three colors are not sufficient because if we suppose the color set $X = \{1, 2, 3\}$ and $A_1: v_0 \longleftarrow 1$ then $B_1: v_2 \longleftarrow 2$, then there will be two critical vertices v_1, v_3, and Alice cannot defend them together on her next turn, so she loses the game.

Case 2. $(n = 2\ and\ m = 3)\ or\ (n \geq 3\ and\ m\ is\ arbitrary)$

Subcase 2.1. When $n = 2$ and $m = 3$, as it is shown in Figure 2.

If $n = 2$ and $m = 3$, Alice can defend any critical vertices with three colors even with Bob's strategy which was mentioned in Subcase 1.2. We can explain Alice's winning strategy as follows: let the color set be $X = \{1, 2, 3\}$ and $A_1: v_0 \longleftarrow 1$ then $B_1: v_2 \longleftarrow 2$, so there will be two critical vertices v_1, v_3, and then when Alice chooses the next move to be $A_2: v_5 \longleftarrow 3$; therefore, Bob cannot use the color 3 on the vertices v_4, v_6 to win the game and Alice wins with only three colors.

Subcase 2.2. When $n \geq 3$ and $m \geq 3$.

We will use the game coloring number as a tool to investigate the upper bound of $\chi_g(J_{n,m})$.

Let $D_2 = \{v \in V | \rho(v) = 2\}$ and $D_3 = \{v \in V | \rho(v) = 3\}$. The chosen vertex set is $C = \{v \in D_3 | v\ is\ a\ added\ to\ L\}$.

At the beginning, Alice chooses the central vertex v_0. Bob chooses a vertex $v \notin L$. Alice follows the next strategy to respond.

(1) If $v \in D_2$ and $\exists u \in N(v) \cap D_3$ and $u \notin L$ then Alice adds u to L

(2) Else, if $v \in D_2$ and $\exists u \in D3C \longrightarrow \exists u \in D_3 \backslash C$ then Alice chooses u and adds it to L

(3) Else, Alice chooses any unchosen vertex u

Using this strategy on $J_{n,m}$ in case $n \geq 3$ produces an optimal linear order L with back degree $b = max_{v \in L} |N_L^+(v)| = 2$. So, $col_g(J_{n,m}) = b + 1 = 3$, and then, $\chi_g(J_{n,m}) \leq 3$ by using Proposition 1. Therefore, $\chi_g(J_{n,m}) = 3$ since $\chi_g(J_{n,m}) > 2$. □

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

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