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Research Article

Constraints Optimal Control Governing by Triple Nonlinear Hyperbolic Boundary Value Problem

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The focus of this work lies on proving the existence theorem of a unique state vector solution (Stvs) of the triple nonlinear hyperbolic boundary value problem (TNHBVP) when the classical continuous control vector (CCCVE) is fixed by using the Galerkin method (Galm), proving the existence theorem of a unique constraints classical continuous optimal control vector (CCCOCVE) with vector state constraints (equality EQVC and inequality INEQVC). Also, it consists of studying for the existence and uniqueness adjoint vector solution (Advs) of the triple adjoint vector equations (TAEqs) associated with the considered triple state equations (Tsteqs). The Fréchet Derivative (Frde.) of the Hamiltonian (HAM) is found. At the end, the theorems for the necessary conditions and the sufficient conditions of optimality (Necoop and Sucoop) are achieved.

1. Introduction

The subject of optimal control problem (OCP) plays a basic role in many real life problems in different branches of sciences; for example, in, medicine [1], engineering and social sciences [2], biology [3], ecology [4], electric power [5], aerospace [6], and many other branches.

This role encouraged many researchers to go deeply into studying the OCPS governed by differential equations (deqs). Such OCP problems are studied at the beginning for the systems which are controlled by nonlinear ordinary deqs (nodeqs) [7] or by linear deqs (lpdeqs) [8]. Later great interests have been made to study this subject but for systems which are controlled by pdeqs of elliptic type (ET) [9], or of hyperbolic type (HT) [10], or of parabolic type (PT) [11], or by couple of npdeqs of ET [12], or of PT [13], or of HT [14].

Recently, the attention in this subject is magnified to deal with more general types as the studying the CCCOPCVE controlling by TBVP of ET [15], and of PT [16]. All these studies motivated us to look deep inside the CCCOPCVE controlled by TNHBVP.

In this work and at first, we give a mathematical description for the CCCOPCVE, and then the TNHBVP is written in its weak form (wkf), and the existence and uniqueness theorem of the Stvs for the TNHBVP using the Galm with the Aubin compactness theorem is proved under appropriate hypotheses when the CCCVE is given. Under reasonable hypotheses, the objective function and the EQVC and INEQVC are proved continuous. The proof of the existence theorem of a CCCOPCVE governed by the TNHBVP is achieved. Under a certain hypotheses, the study of the existence theorem for a unique Advs of the TAEqs associated with the considered Tsteqs is done. The Fréchet Derivative (Frde.) of the HAM is found. Finally, the Necoop and the Sucoop theorems for the CCCOPCVE are proved.

1.1. Problem Description. Let $I = [0, T]$, $T < \infty$, E be a bounded and open region in $\mathbb{R}^2$ with Lipschitz (Lip.) boundary ∂E, $\Pi = E \times I$, and $\partial\Pi = \partial E \times I$. The considered CCOPCP consists of the Steq which is given by the TNLHPDEqs:

$$\psi_{1tt} - \Delta\psi_1 + \psi_1 - \psi_2 - \psi_3 = k_1(x, t, \psi_1, \omega_1), \quad \text{in } \Pi, \quad (1)$$

$$\psi_{2tt} - \Delta\psi_2 + \psi_2 + \psi_3 + \psi_1 = k_2(x, t, \psi_2, \omega_2), \quad \text{in } \Pi, \quad (2)$$

$$\psi_{3tt} - \Delta\psi_3 + \psi_3 + \psi_1 - \psi_2 = k_3(x, t, \psi_3, \omega_3), \quad \text{in } \Pi, \quad (3)$$

with the BCs and ICs.

$$\psi_1(x,t)=0, \psi_2(x,t)=0, \psi_3(x,t)=0, \quad \text{on } \partial\Pi, \tag{4}$$

$$\psi_1(x,0)=\psi_1^0(x), \psi_2(x,0)=\psi_2^0(x), \psi_3(x,0)=\psi_3^0(x), \quad \text{on } E, \tag{5}$$

$$\psi_{1t}(x,0)=\psi_1^1(x), \psi_{1t}(x,0)=\psi_2^1(x), \psi_{3t}(x,0)=\psi_3^1(x), \quad \text{on } E, \tag{6}$$

where $\overrightarrow{\psi}=(\psi_1,\psi_2,\psi_3)\in(H^1(E))^3$ is the Stvs, corresponding to the CCCVE $\overrightarrow{\omega}=(\omega_1,\omega_2,\omega_3)\in(L^2(\Pi))^3$ and $(k_1,k_2,k_3)\in(L^2(\Pi))^3$ is a function defined on $(\Pi\times\mathscr{R}\times\complement_1)\times(\Pi\times\mathscr{R}\times\complement_2)\times(\Pi\times\mathscr{R}\times\complement_3)$ with $\complement_i\subset\mathscr{R}$, for $i=1,2,3$.

The controls set are $\overrightarrow{\omega}\in\overrightarrow{\widehat{W}}$, $\overrightarrow{\widehat{W}}\subset(L^2(\Pi))^3$ with $\overrightarrow{\widehat{W}}=\{\overrightarrow{\widehat{W}}\in(L^2(\Pi))^3 \mid \overrightarrow{\widehat{W}}\in\overrightarrow{\complement}, \text{a.e. in }\Pi\}$, with $\overrightarrow{\complement}\subset\mathscr{R}^3$.

The cost function is

$$M_0(\overrightarrow{\omega})=\sum_{i=1}^{3}\int_{\Pi}m_{0i}(x,t,\psi_i,\omega_i)\mathrm{d}x\,\mathrm{d}t. \tag{7}$$

The EQVC and INEQVC on the state vectors are

$$M_r(\overrightarrow{\omega})=\sum_{i=1}^{3}\int_{\Pi}m_{ri}(x,t,\psi_i,\omega_i)\mathrm{d}x\,\mathrm{d}t=0, \quad 1\le r\le p, \tag{8}$$

$$M_r(\overrightarrow{\omega})=\sum_{i=1}^{3}\int_{\Pi}m_{ri}(x,t,\psi_i,\omega_i)\mathrm{d}x\,\mathrm{d}t\le 0, \quad p+1\le r\le q. \tag{9}$$

The set of admissible control vector is $\overrightarrow{\widehat{W}}_A=\left\{\overrightarrow{\omega}\in\overrightarrow{\widehat{W}} \mid M_r(\overrightarrow{\omega})=0, M_{r+p}(\overrightarrow{\omega})\le 0, 1\le r\le p\right\}$.

The continuous optimal control problem is to find $\overrightarrow{\omega}\in\overrightarrow{\widehat{W}}_A$ such that $M_0(\overrightarrow{\omega})=\min_{\overrightarrow{\widehat{\omega}}\in\overrightarrow{\widehat{W}}_A}M_0(\overrightarrow{\widehat{\omega}})$.

Let $\overrightarrow{\Upsilon}=.\Upsilon_1\times.\Upsilon_2\times.\Upsilon_3=\{\overrightarrow{v}:\overrightarrow{v}\in(H^1(\Omega))^3$, with $v_1=v_2=v_2=0$ on $\partial E\}$, $\overrightarrow{v}=(v_1,v_2,v_3)$.

We denote by (v,v), $(v,v)_1$, and $(\overrightarrow{v},\overrightarrow{v})_1=\sum_{i=1}^{3}(v_i,v_i)_1$ to the inner products in $L^2(E)$, $H^1(E)$, and $\overrightarrow{\Upsilon}$ respectively while the norms in these spaces is denoted by v_0, v_1, and $\overrightarrow{v}_1^2=\sum_{i=1}^{3}v_{i1}^2$, $\overrightarrow{\Upsilon}^*$ is denoted the dual of $\overrightarrow{\Upsilon}$. Also, the symbol–will be used to indicate that the convergence of a sequence is weak, while the strong convergence of a sequence will be indicated by $\longrightarrow$.

The wkf of problem (1)–(6) when $\overrightarrow{\psi}\in(H_0^1(E))^3$ is given almost everywhere on I for each $v_1\in\Upsilon_1$, $v_2\in\Upsilon_2$, and $v_3\in\Upsilon_3$:

$$\langle\psi_{1tt},v_1\rangle+(\nabla\psi_1,\nabla v_1)+(\psi_1-\psi_2-\psi_3,v_1)=(k_1(\psi_1,\omega_1),v_1), \quad \psi_1(\cdot,t)\in Y_1, \tag{10}$$

$$\begin{aligned}(\psi_1^0,v_1)&=(\psi_1(0),v_1),\\(\psi_1^1,v_1)&=(\psi_{1t}(0),v_1),\end{aligned} \tag{11}$$

$$\langle\psi_{2tt},v_2\rangle+(\nabla\psi_2,\nabla v_2)+(\psi_2+\psi_1+\psi_3,v_2)=(k_2(\psi_2,\omega_2),v_2), \quad \psi_2(\cdot,t)\in Y_2, \tag{12}$$

$$\begin{aligned}(\psi_2^0,v_2)&=(\psi_2(0),v_2),\\(\psi_2^1,\nu_2)&=(\psi_{2t}(0),v_2),\end{aligned} \tag{13}$$

$$\langle\psi_{3tt},v_3\rangle+(\nabla\psi_3,\nabla v_3)+(\psi_3+\psi_1-\psi_2,v_3)=(k_3(\psi_3,\omega_3),v_3), \quad \psi_3(\cdot,t)\in Y_3, \tag{14}$$

$$\begin{aligned}(\psi_3^0,v_3)&=(\psi_3(0),v_3),\\(\psi_3^1,v_3)&=(\psi_{3t}(0),v_3).\end{aligned} \tag{15}$$

The following assumptions (Assums.) are needed to investigate the classical continuous optimal control problem (CCOPCP).

Assumption A. k_i is of the Carathéodory type (Caraty.) on $\Pi\times(\mathscr{R}\times\complement_i)$ and satisfies the following conditions for $(x,t)\in\Pi$ and $\forall i=1,2,3$:

$$\begin{aligned} &\left|k_i(x,t,\psi_i,\omega_i)\right| \le F_i(x,t)+\beta_i\left|\psi_i\right|, \quad \text{where } \psi_i,\omega_i \in \mathscr{R}, \beta_i>0, F_i \in L^2(\Pi), \\ &\left|k_i(x,t,\psi_i,\omega_i)-k_i(x,t,\overline{\psi}_i,\omega_i)\right| \le L_i\left|\psi_i-\overline{\psi}\right|, \quad \text{where } \psi_i, \overline{\psi}_i, \omega_i \in \mathscr{R}, L_i>0. \end{aligned} \tag{16}$$

1.2. The Solution of the State Equations. In this part, the existence theorem of a unique solution for triple nonlinear hyperbolic partial differential equations (TNLHPDEQs) under Assumption A is proved when the control vector is given, and the following proposition will be needed.

Proposition 1 (see [17]). *Suppose $D \subset \mathscr{R}^s$ $(s=2,3)$, k: $D \times \mathscr{R}^n \longrightarrow \mathscr{R}^m$ is of Caraty. It satisfies $\|k(v,x)\| \le \alpha(v)+\beta(v)\|x\|^a$, for each $(v,x) \in D \times \mathscr{R}^n$, where $x \in L^b(D,\mathscr{R}^n)$, $\alpha \in L^1(D,R)$, $\beta \in L^{(b/b-a)}(D,R)$, $a \in [0,b]$, if $b \ne \infty$, $\alpha=0$ if $b=\infty$. Then, the functional $K(x)=\int_D k(v,x(v))dv$ is cont.*

Theorem 1. *Existence and Uniqueness of the Stvs: with Assumption A, for any given $\vec{\omega} \in (L^2(\Pi))^3$, the wkf of (10)–(15) has a unique solution $\vec{\psi}=(\psi_1,\psi_2,\psi_3)$ s.t. $\vec{\psi} \in (L^2(I,\Upsilon))^3$, $\vec{\psi}_t=(\psi_{1t},\psi_{2t},\psi_{3t}) \in (L^2(\Pi))^3$, and $\vec{\psi}_{tt}=(\psi_{1t},\psi_{2t},\psi_{3t}) \in (L^2(I,\Upsilon^*))^3$.*

Proof. Let $\vec{\Upsilon}_n=\Upsilon_n \times \Upsilon_n \times \Upsilon_n \subset \vec{\Upsilon}$ (for each n) be the set of cont. and piecewise affine function in E. $\left\{\vec{\Upsilon}_n\right\}_{n=1}^{\infty}$ be a sequence of subspaces of $\vec{\Upsilon}$, such that $\forall \vec{v}=(v_1,v_2,v_3) \in \vec{\Upsilon}$, there exists a sequence $\{\vec{v}_n\}$ with $\vec{v}_n=(v_{1n},v_{2n},v_{3n}) \in \vec{\Upsilon}_n, \forall n$, and $\vec{v}_n \longrightarrow \vec{v}$ in $\vec{Y} \Longrightarrow \vec{v}_n \Longrightarrow \vec{v}$ in $(L^2(E))^3$. $\left\{\vec{v}_j=(v_{1j},v_{2j},v_{3j})\text{: } j=1,2,\ldots,n\right\}$ be a finite basis of $\vec{\Upsilon}_n$ (where $\vec{v}_j$ is cont. function in E, with $\vec{v}_j(x)=0$ on the boundary ∂E) and let $\vec{\psi}_n=(\psi_{1n},\psi_{2n},\psi_{3n})$ be the Galerkin approximate solution (Galso) to the exact solution $\vec{\psi}=(\psi_1,\psi_2,\psi_3)$ such that

$$\psi_{in}=\sum_{j=1}^{n} c_{ij}(t)v_{ij}(x), \tag{17}$$

where $c_{ij}(t)$ are unknown functions of t, $\forall j=1,2,\ldots,n$, and. $\forall i=1,2,3$.

The wkf of (10)–(15) is approximated with respect to x using the Galm, and substituting $\psi_{\text{int}}=\zeta_{in}$ $(i=1,2,3)$ in the obtained equations, they become for each $v_1,v_2,v_3 \in \Upsilon_n$:

$$\langle\zeta_{1nt},v_1\rangle+(\nabla\psi_{1n},\nabla v_1)+(\psi_{1n},v_1)-(\psi_{2n},v_1)-(\psi_{3n},v_1)=(k_1(\psi_{1n},\omega_1),v_1), \tag{18}$$

$$\begin{aligned} \left(\psi_{1n}^0,v_1\right)&=\left(\psi_1^0,v_1\right), \\ \left(\psi_{1n}^1,v_1\right)&=\left(\psi_1^1,v_1\right), \end{aligned} \tag{19}$$

$$\langle\zeta_{2nt},v\rangle+(\nabla\psi_{2n},\nabla v_2)+(\psi_{2n},v_2)+(\psi_{3n},v_2)+(\psi_{1n},v_2)=(k_2(\psi_{2n},\omega_2),v_2), \tag{20}$$

$$\begin{aligned} \left(\psi_{2n}^0,v_2\right)&=\left(\psi_2^0,v_2\right), \\ \left(\psi_{2n}^1,\nu_2\right)&=\left(\psi_2^1,v_2\right), \end{aligned} \tag{21}$$

$$\langle\zeta_{3nt},v_3\rangle+(\nabla\psi_{3n},\nabla v_3)+(\psi_{3n},v_3)+(\psi_{1n},v_3)-(\psi_{2n},v_3)=(k_3(\psi_{3n},\omega_3),v_3), \tag{22}$$

$$\begin{aligned} \left(\psi_{3n}^0,v_3\right)&=\left(\psi_3^0,v_3\right), \\ \left(\psi_{3n}^1,v_3\right)&=\left(\psi_3^1,v_3\right), \end{aligned} \tag{23}$$

where $\psi_{in}^0=\psi_{in}^0(x)=\psi_{in}(x,0) \in \Upsilon_n$ (respectively, $\zeta_{in}^0=\psi_{in}^1=\psi_{in}^1(x)=\psi_{\text{int}}(x,0) \in L^2(E)$) be the projection of y_i^0 onto V (be the projection of $\psi_i^1=\psi_{it}$ onto $L^2(E)$), $\forall i=1,2,3$, i.e.,

$$\psi_{in}^0 \longrightarrow \psi_i^0 \text{ in } V, \quad \text{with} \left\|\vec{\psi}_n^0\right\|_1 \le b_0 \text{ and } \left\|\vec{\psi}_n^0\right\|_0 \le b_0, \tag{24}$$

$$\psi_{in}^1 \longrightarrow \psi_i^1, \quad \text{in } L^2(E) \text{ and } \left\|\vec{\psi}_n^1\right\|_0 \le b_1. \tag{25}$$

Substituting (17) for each $i=1,2,3$, respectively, in the pairs (18) and (19), (20) and (21), and in (22) and (23), setting $v_1=v_{1i}, v_2=v_{2i}, v_3=v_{3i}$, the obtained equations will be written as the following 1[st] order nodeqs with their ICs and have a unique solution $\vec{\psi}_n \in C(I,\vec{\Upsilon})$, i.e., for each $l=1,2,3$ and $\overline{m}=0,1$:

$$\begin{aligned}
A_1D_1'(t)+B_1C_1(t)-E_1C_2(t)-H_1C_3(t) &= b_1\left(\overline{Y}_1^T(x)C_1(t)\right),\\
A_1C_1(0) &= b_1^0,\\
A_1D_1(0) &= b_1^1,\\
A_2D_2'(t)+B_2C_2(t)+E_2C_1(t)+H_2C_3(t) &= b_2\left(\overline{Y}_2^T(x)C_2(t)\right),\\
A_2C_2(0) &= b_2^0,\\
A_2D_2(0) &= b_2^1,\\
A_3D_3'(t)+B_3C_3(t)+E_3C_1(t)-H_3C_3(t) &= b_3\left(\overline{Y}_3^T(x)C_3(t)\right),\\
A_3C_3(0) &= b_3^0,\\
A_3D_3(0) &= b_3^1,
\end{aligned} \tag{26}$$

where $C_l(t)=(c_{lj}(t))_{n\times 1}$, $C_2'(t)=(c_{2j}'(t))_{n\times 1}$, $D_l'(t)=(d_{lj}'(t))_{n\times 1}$, $D_l(t)=(d_{lj}(t))_{n\times 1}$, $b_l=(b_{li})_{n\times 1}$, $b_{li}=(k_l(V_l^T c_l(t),\omega_l),v_{li})$, $b_l^m=(b_{lj}^m)$, $b_{lj}^m=(\psi_l^m,v_{lj})$, $A_l=(a_{lij})_{n\times n}$, $a_{lij}=(v_{lj},v_{li})$, $E_1=(e_{1ij})_{n\times n}$, $e_{1ij}=(v_{2j},v_{1i})$, $H_1=(f_{1ij})_{n\times n}$ $f_{1ij}=(v_{3j},v_{1i})$, $E_2=(e_{2ij})_{n\times n}$, $e_{2ij}=(v_{1j},v_{2i})$, $H_2=(f_{2ij})_{n\times n}$, $f_{2ij}=(v_{3j},v_{2i})$, $E_3=(e_{3ij})_{n\times n}$, $e_{3ij}=(v_{1j},v_{3i})$, $H_3=(f_{3ij})_{n\times n}$, $f_{3ij}=(v_{2j},v_{3i})$, $B_l=(b_{lij})_{n\times n}$, and $b_{lij}=[(\nabla v_{lj},\nabla v_{li})+(v_{lj},v_{li})]$.

Then, corresponding to the sequence $\left\{\overrightarrow{Y}_n\right\}$, there exists a sequence of the following approximation problems, i.e., for each $\overrightarrow{v}_n=(v_{1n},v_{2n},v_{3n})\subset\overrightarrow{Y}_n$, and $n=1,2,\ldots$:

$$\langle\psi_{1ntt},v_{1n}\rangle+(\nabla\psi_{1n},\nabla v_{1n})+(\psi_{1n}-\psi_{2n}-\psi_{3n},v_{1n})=(k_1(\psi_{1n},\omega_1),v_{1n}), \tag{27}$$

$$\begin{aligned}
\left(\psi_{1n}^0,v_{1n}\right) &= \left(\psi_1^0,v_{1n}\right),\\
\left(\psi_{1n}^1,v_1\right) &= \left(\psi_1^1,v_{1n}\right),
\end{aligned} \tag{28}$$

$$\langle\psi_{2ntt},v_{2n}\rangle+(\nabla\psi_{2n},\nabla v_{2n})+(\psi_{2n}+\psi_{3n}+\psi_{1n},v_{2n})=(k_2(\psi_{2n},\omega_2),v_{2n}), \tag{29}$$

$$\begin{aligned}
\left(\psi_{2n}^0,v_{2n}\right) &= \left(\psi_2^0,v_{2n}\right),\\
\left(\psi_{2n}^1,v_2\right) &= \left(\psi_2^1,v_{2n}\right),
\end{aligned} \tag{30}$$

$$\langle\psi_{3ntt},v_{3n}\rangle+(\nabla\psi_{3n},\nabla v_{3n})+(\psi_{3n}+\psi_{1n}-\psi_{2n},v_{3n})=(k_3(\psi_{3n},\omega_3),v_{3n}), \tag{31}$$

$$\begin{aligned}
\left(\psi_{3n}^0,v_{3n}\right) &= \left(\psi_3^0,v_{3n}\right),\\
\left(\psi_{3n}^1,v_3\right) &= \left(\psi_3^1,v_{3n}\right),
\end{aligned} \tag{32}$$

which has a sequence of unique solution $\left\{\overrightarrow{\psi}_n\right\}$. Substituting $v_{in}=\psi_{int}$ for $i=1,2,3$ in (25)–(27), respectively, adding the three obtained equations together, and employing Lemma 1.2 in [18] for the first term of the LHS, to get (33) which is given by

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left[\left\|\overrightarrow{\psi}_{nt}(t)\right\|_0^2+\left\|\overrightarrow{\psi}_n\right\|_1^2\right] = {}& 2\left((\psi_{2n},\psi_{1nt})+(\psi_{3n},\psi_{1nt})-(\psi_{1n},\psi_{2nt})-(\psi_{3n},\psi_{2nt})-(\psi_{1n},\psi_{3nt})+(\psi_{2n},\psi_{3nt})\right.\\
&\left.+(k_1(\psi_{1n},\omega_1),\psi_{1nt})+(k_2(\psi_{2n},\omega_2),\psi_{2nt})+(k_3(\psi_{3n},\omega_3),\psi_{3nt})\right).
\end{aligned} \tag{33}$$

Or (33) can be rewritten as (34) which is

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left[\left\|\overrightarrow{\psi}_{nt}(t)\right\|_0^2+\left\|\overrightarrow{\psi}_n\right\|_1^2\right] \le {}& 2\left(\left|(\psi_{2n},\psi_{1nt})\right|+\left|(\psi_{3n},\psi_{1nt})\right|+\left|(\psi_{1n},\psi_{2nt})\right|+\left|(\psi_{3n},\psi_{2nt})\right|+\left|(\psi_{1n},\psi_{3nt})\right|\right.\\
&\left.+\left|(\psi_{2n},\psi_{2nt})\right|+\left|(k_1(\psi_{1n},\omega_1),\psi_{1nt})\right|+\left|(k_2(\psi_{2n},\omega_2),\psi_{2nt})\right|+\left|(k_3(\psi_{3n},\omega_3),\psi_{3nt})\right|\right).
\end{aligned} \tag{34}$$

Using Assumption A for the RHS of (34), integrating both sides on $[0,t]$, using $\|\psi_{in}\|_0 \le \|\psi_{in}\|_1 \le \|\vec{\psi}_n\|_1$, and $\|\psi_{\text{int}}\|_0 \le \|\vec{\psi}_{nt}\|_0$, we get

$$\begin{aligned}&\int_0^t \frac{d}{dt}\left[\|\vec{\psi}_{nt}(t)\|_0^2 + \|\vec{\psi}_n\|_1^2\right]dt \\ &\le \int_0^t \left(\|\vec{\psi}_{nt}\|_0^2 + \|\vec{\psi}_n\|_1^2\right)dt + \int_0^t \sum_{i=1}^3 \|F_i\|_0^2 dt \\ &\quad + \beta_4 \int_0^t \left(\|\vec{\psi}_{nt}\|_0^2 + \|\vec{\psi}_n\|_1^2\right)dt + \int_0^t \|\vec{\psi}_{nt}\|_0^2 dt \\ &\le \sum_{i=1}^3 \|F_i\|_Q^2 + \beta_6 \int_0^t \left(\|\vec{\psi}_{nt}\|_0^2 + \|\vec{\psi}_n\|_1^2\right)dt \\ &\le \beta_8 + \beta_7 \int_0^t \left(\|\vec{\psi}_{nt}\|_0^2 + \|\vec{\psi}_n\|_1^2\right)dt,\end{aligned} \tag{35}$$

where $\beta_4 = \sum_{i=1}^3 \beta_i$, $\beta_5 = 1 + \beta_4$, $\beta_6 = 2 + \beta_4$, $\beta_7 = \max(\beta_5, \beta_6)$, $\beta_8 = \sum_{i=1}^3 b_i'$.

Since $\|\vec{\psi}_n^0\|_1 \le b_1$, and $\|\vec{\psi}_n^1\|_0 \le b_0$, with $\beta_9 = b_0 + b_1 + \beta_7$, then (35) is reduced to

$$\|\vec{\psi}_{nt}(t)\|_0^2 + \|\vec{\psi}_n(t)\|_1^2 \le \beta_9 + \beta_7 \int_0^t \left(\|\vec{\psi}_n\|_0^2 + \|\vec{\psi}_{nt}\|_1^2\right)dt. \tag{36}$$

Appling the Belman–Gronwall (BGin) inequality, the abovementioned inequality gives $\forall t \in [0,T]$:

$$\begin{aligned}&\|\vec{\psi}_{nt}(t)\|_0^2 + \|\vec{\psi}_n(t)\|_1^2 \le \beta_9 e^{\beta_7} = b^2(c) \\ &\Rightarrow \|\vec{\psi}_{nt}(t)\|_0^2 \le b^2(c), \\ &\|\vec{\psi}_n(t)\|_1^2 \le b^2(c), \quad \forall t \in [0,T].\end{aligned} \tag{37}$$

Easily, one can obtain that $\|\vec{\psi}_{nt}(t)\|_Q \le b_1(c)$ and $\|\vec{\psi}_n(t)\|_{L^2(I,Y)} \le b(c)$.

Then, by applying Alaoglu's theorem (Algth), $\{\vec{\psi}_n\}_{n\in N}$ has a subsequence; it is not loss of generality to say $\{\vec{\psi}_n\}_{n\in N}$ such that $\vec{\psi}_{nt} \rightharpoonup \vec{\psi}$ in $(L^2(\Pi))^3$ and $\vec{\psi}_n \rightharpoonup \vec{\psi}$ in $(L^2(I,Y))^3$, and

$$\left(L^2(\mathscr{R},Y)\right)^3 \subset \left(L^2(\mathscr{R},E)\right)^3 \cong \left(\left(L^2(\mathscr{R},E)\right)^*\right)^3 \subset \left(L^2(\mathscr{R},Y^*)\right)^3. \tag{38}$$

Then, the Aubin compactness theorem [18] can be applied here to get that $\vec{\psi}_n \longrightarrow \vec{\psi}$ in $(L^2(\Pi))^3$. Now, multiplying both sides of (27) and (29), and (31) by $\chi_i(t) \in C^2[0,T]$, such that $\chi_i(T) = \chi_i'(T) = 0$, $\chi_i(0) \ne 0$, $\chi_i'(0) \ne 0$, $\forall i = 1,2$, integrating on, finally integrating by parts twice the 1st term of each one of the obtained three equations, led to

$$\begin{aligned}&-\int_0^T \frac{d}{dt}(\psi_{1n}, v_{1n})\chi_1'(t)dt + \int_0^T [(\nabla\psi_{1n}, \nabla v_{1n})\chi_1(t) + ((\psi_{1n}, v_{1n}) - (\psi_{2n}, v_{1n}) - (\psi_{3n}, v_{1n}))\chi_1(t)]dt \\ &= \int_0^T (k_1(\psi_{1n}, \omega_1), v_{1n})\chi_1(t)dt + \left(\psi_{1n}^1, v_{1n}\right)\chi_1(0),\end{aligned} \tag{39}$$

$$\begin{aligned}&\int_0^T (\psi_{1n}, v_{1n})\chi_1''(t)dt + \int_0^T [(\nabla\psi_{1n}, \nabla v_{1n})\chi_1(t) + ((\psi_{1n}, v_{1n}) - (\psi_{2n}, v_{1n}) - (\psi_{3n}, v_{1n}))\chi_1(t)]dt \\ &= \int_0^T (k_1(\psi_{1n}, \omega_1), v_{1n})\chi_1(t)dt + \left(\psi_{1n}^1, v_{1n}\right)\chi_1(0) + \left(\psi_{1n}^0, v_{1n}\right)\chi_1'(0),\end{aligned} \tag{40}$$

$$\begin{aligned}&-\int_0^T \frac{d}{dt}(\psi_{2n}, v_{2n})\chi_2'(t)dt + \int_0^T [(\nabla\psi_{2n}, \nabla v_{2n})\chi_2(t) + ((\psi_{2n}, v_{2n}) + (\psi_{3n}, v_{2n}) + (\psi_{1n}, v_{2n}))\chi_2(t)]dt \\ &= \int_0^T (k_2(\psi_{2n}, \omega_2), v_{2n})\chi_2(t)dt + \left(\psi_{2n}^1, v_{2n}\right)\chi_2(0),\end{aligned} \tag{41}$$

$$\begin{aligned}&\int_0^T (\psi_{2n}, v_{2n})\chi_2''(t)dt + \int_0^T [(\nabla\psi_{2n}, \nabla v_{2n})\chi_2(t) + ((\psi_{2n}, v_{2n}) + (\psi_{3n}, v_{2n}) + (\psi_{1n}, v_{2n}))\chi_2(t)dt] \\ &= \int_0^T (k_2(\psi_{2n}, \omega_2), v_{2n})\chi_2(t)dt + \left(\psi_{2n}^1, v_{2n}\right)\chi_2(0) + \left(\psi_{2n}^0, v_{2n}\right)\chi_2'(0),\end{aligned} \tag{42}$$

$$\begin{aligned}&-\int_0^T \frac{d}{dt}(\psi_{3n}, v_{3n})\chi_3'(t)dt + \left[\int_0^T (\nabla\psi_{3n}, \nabla v_{3n})\chi_3(t) + ((\psi_{3n}, v_{3n}) + (\psi_{1n}, v_{3n}) - (\psi_{2n}, v_{3n}))\chi_3(t)dt\right] \\ &= \int_0^T (k_3(\psi_{3n}, \omega_3), v_{3n})\chi_3(t)dt + \left(\psi_{3n}^1, v_{3n}\right)\chi_3(0),\end{aligned} \tag{43}$$

$$\begin{aligned}&\int_0^T (\psi_{3n}, v_{3n})\chi_3''(t)\mathrm{d}t + \int_0^T [(\nabla\psi_{3n}, \nabla v_{3n})\chi_3(t) + ((\psi_{3n}, v_{3n}) + (\psi_{1n}, v_{3n}) - (\psi_{2n}, v_{3n}))\chi_3(t)\mathrm{d}t] \\ &= \int_0^T (k_3(\psi_{3n}, \omega_3), v_{2n})\chi_3(t)\mathrm{d}t + (\psi_{3n}^1, v_{3n})\chi_3(0) + (\psi_{3n}^0, v_{3n})\chi_3'(0).\end{aligned} \tag{44}$$

Now, for each $i = 1, 2, 3$, we have the following convergences:

First, since

$$v_{in} \longrightarrow v_i \,\mathrm{in}\, Y \longrightarrow \begin{cases} v_{in}\chi_i(t) \longrightarrow v_i\chi_i(t) \\ v_{in}\chi_i'(t) \longrightarrow v_i\chi_i'(t) \end{cases}, \ \mathrm{in}\, L^2(I, Y) \\ \quad v_{in}\chi_i(0) \longrightarrow v_i\chi_i(0), \quad \mathrm{in}\, L^2(E)$$

On the other hand, since

$$v_{in} \longrightarrow v_i \mathrm{in} L^2(E)\} \longrightarrow \begin{cases} v_{in}\chi_i'(t) \longrightarrow v_i\chi_i'(t) \\ v_{in}\chi_i''(t) \longrightarrow v_i\chi_i''(t) \end{cases}, \mathrm{in} L^2(\Pi) \\ \quad v_{in}\chi_i'(0) \longrightarrow v_i\chi_i'(0), \quad \mathrm{in} L^2(E)$$

Second, we have

$$\begin{cases} \psi_{int} \rightharpoonup \psi_{it}, & \mathrm{in}\, L^2(\Pi) \\ \psi_{in} \rightharpoonup \psi_i, & \mathrm{in}\, L^2(I, Y) \\ \psi_{in} \longrightarrow \psi_i, & \mathrm{in}\, L^2(\Pi) \end{cases}$$

Third, let $w_{in} = v_{in}\chi_i$ and $w_i = v_i\chi_i$, then $w_{in} \longrightarrow w_i$ in $L^2(\Pi)$ and w_{in} is measurable in E, so from Assumption (A–I) and Proposition 1, the integral $\int_\Pi k_i(x, t, \psi_{in}, \omega_i)w_{in}\mathrm{d}x\,\mathrm{d}t$ is cont. with respect to $(\psi_{in}, \omega_i, w_{in})$, then

$$\int_0^T (k_i(\psi_{in}, \omega_i), v_{in})\chi_i(t)\mathrm{d}t \longrightarrow \int_0^T (k_i(\psi_i, \omega_i), v_i)\chi_i(t)\mathrm{d}t. \tag{45}$$

□

Now, from these convergences and (24) and (25), for $(i = 1)$, we can passage to the limits in (39) and (40) to get

$$\begin{aligned}&-\int_0^T (\psi_{1t}, v_1)\chi_1'(t)\mathrm{d}t + \int_0^T [(\nabla\psi_1, \nabla v_1) + (\psi_1 - \psi_2 - \psi_3, v_1)]\chi_1(t)\mathrm{d}t \\ &= \int_0^T (k_1(\psi_1, \omega_1), v_1)\chi_1(t)\mathrm{d}t + (\psi_1^1, v_1)\chi_1(0),\end{aligned} \tag{46}$$

$$\begin{aligned}&\int_0^T (\psi_1, v_1)\chi_1''(t)\mathrm{d}t + \int_0^T [(\nabla\psi_1, \nabla v_1) + (\psi_1 - \psi_2 - \psi_3, v_1)]\chi_1(t)\mathrm{d}t \\ &= \int_0^T (k_1(\psi_1, \omega_1), v_1)\chi_1(t)\mathrm{d}t + (\psi_1^1, v_1)\chi_1(0) + (\psi_1^0, v_1)\chi_1'(0),\end{aligned} \tag{47}$$

Case 1. Choose $\chi_i \in C^2[0, T]$, s.t. $\chi_i(0) = \chi_i'(0) = \chi_i(T) = \chi_i'(T) = 0$. Using these values in (47) (for $i = 1$), using integration by parts twice for the first terms in the LHS of the obtained equation, yields to

$$\begin{aligned}&\int_0^T \langle\psi_{1tt}, v_1\rangle\chi_1(t)\mathrm{d}t + \int_0^T [(\nabla\psi_1, \nabla v_1) \\ &\quad + (\psi_1 - \psi_2 - \psi_3, v_1)]\chi_1(t)\mathrm{d}t \\ &= \int_0^T (k_1(\psi_1, \omega_1), v_1)\chi_1(t)\mathrm{d}t,\end{aligned} \tag{48}$$

which give that ψ_1 is a solution of (10) (a.e. on I).

Similar way can be used for $i = 2, 3$, with (41)–(44) respectively to get that ψ_2 and ψ_3 are solutions of and (12) and (14) respectively (a.e. on I).

Case 2. Choose $\chi_i \in C^2[0, T]$, such that $\chi_i(T) \neq 0$ and $\chi_i(0) \neq 0$. For $i = 1$, integrating both sides of (10) on $[0, T]$ after multiplying it by $\chi_1(t)$, using integrating by parts for the first term in the LHS of the obtained equation, then subtracting the obtained equation from (41), we get $(\psi_i^1, v_i)\chi_i(0) = (\psi_{it}(0), v_i)\chi_i(0)$.

Also, similar way can be used but for $i = 2, 3$ with the pairs (12) and (47) and (14) and (48), respectively, to get the same result.

Case 3. Choose $\chi_i \in C^2[0, T]$, such that $\chi_i(0) = \chi_i(T) = \chi_i'(T) = 0$, $\chi_i'(0) \neq 0$. For $i = 1$, integrating both sides of (10) on $[0, T]$ after multiplying it by $\chi_1(t)$, using integrating by parts for the first term in the LHS of the obtained equation, then subtracting this obtained equation from (47), we get $(\psi_i^0, v_i)\chi_i'(0) = (\psi_i(0), v_i)\chi_i'(0)$. Also, for $i = 2, 3$ and by using (12) and (14), we can use a similar way to get the same result.

From the last two cases, easily we can get the ICs (11) and (13), and (15).

Uniqueness of the solution: let $\vec{\psi} = (\psi_1, \psi_2, \psi_3)$ and $\vec{\overline{\psi}} = (\overline{\psi}_1, \overline{\psi}_2, \overline{\psi}_3)$ be two solutions of the wkf (10)–(15), i.e., ψ_i and $\overline{\psi}_i$ (for each $i = 1, 2, 3$) are satisfied the wkf (10)–(15), subtracting each equality from the other and letting $v_i = \psi_i - \overline{\psi}_i$, yields to

$$\begin{aligned}\langle(\psi_i-\psi_i)_{tt},\psi_i-\overline{\psi}_i\rangle+\|\psi_i-\overline{\psi}_i\|_1^2&=(k_i(\psi_i,\omega_i)-k_i(\overline{\psi}_i,\omega_i),\psi_i-\overline{\psi}_i),\\ ((\psi_i-\overline{\psi}_i)(0),\psi_i-\overline{\psi}_i(0))&=0,\\ ((\psi_i-\overline{\psi}_i)_t(0),(\psi_i-\overline{\psi}_i)_t(0))&=0.\end{aligned}\tag{49}$$

Adding these three equations, using Lemma 1.2 in ref. [18] on the first term in LHS of the obtained equation which will be positive, integrating both sides from 0 to t, using the initial conditions, the Lipshctiz property on the RHS, and lastly applying the B–G inequality, to get

$$\begin{aligned}&\int_0^t\left[\frac{d}{dt}\left\|(\vec{\psi}-\vec{\overline{\psi}})_t(t)\right\|_0^2+2\|(\vec{\psi}-\vec{\overline{\psi}})\|_1^2\right]dt\le 2L\int_0^t\left(\left\|(\vec{\psi}-\vec{\overline{\psi}})_t(t)\right\|_0^2+\|(\vec{\psi}-\vec{\overline{\psi}})\|_1^2\right)dt\\ &\left\|(\vec{\psi}-\vec{\overline{\psi}})_t(t)\right\|_0^2+\|(\vec{\psi}-\vec{\overline{\psi}})\|_1^2\le\Rightarrow\|(\vec{\psi}-\vec{\overline{\psi}})(t)\|_1^2=0,\quad\forall t\in I\Rightarrow\\ &\|(\vec{\psi}-\vec{\overline{\psi}})(t)\|_{L^2(I,Y)}=0\Rightarrow,\quad\text{the solution is unique.}\end{aligned}\tag{50}$$

Lemma 1. *In addition to Assumption A, if the functions k_i (for each $i=1,2,3$) is Lip. with respect to y_i and ω_i, and if the control vector is bounded, then the operator $\vec{\omega}\mapsto\vec{\psi}_{\vec{\omega}}$ from $(L^2(\Pi))^3$ into $(L^\infty(I,L^2(E)))^3$ or in to $(L^2(I,Y))^3$ or in to $(L^2(\Pi))^3$ is cont.*

Proof. Let $\vec{\omega}=(\omega_1,\omega_2,\omega_2)$, $\vec{\overline{\omega}}=(\overline{\omega}_1,\overline{\omega}_2,\overline{\omega}_3)\in(L^2(\Pi))^3$, $\overrightarrow{\Delta\omega}=\vec{\overline{\omega}}-\vec{\omega}\in(L^2(\Pi))^3$, then by Theorem 1, $\vec{\psi}=\vec{\psi}_{\vec{\omega}}=(\psi_1,\psi_2,\psi_3)$ and $\vec{\overline{\psi}}=\vec{\overline{\psi}}_{\vec{\omega}}=(\overline{\psi}_1,\overline{\psi}_2,\overline{\psi}_3)$ are their corresponding states' solutions which satisfy the wkf of (10)–(15), setting $\overrightarrow{\Delta\psi}=(\Delta\psi_1,\Delta\psi_2,\Delta\psi_3)=\vec{\overline{\psi}}-\vec{\psi}$, then

$$\begin{aligned}&\langle\Delta\psi_{1tt},v_1\rangle+(\nabla\Delta\psi_1,\nabla v_1)+(\Delta\psi_1-\Delta\psi_2-\Delta\psi_3,v_1)\\ &\quad=(k_1(\psi_1+\Delta\psi_1,\omega_1+\Delta\omega_1)-k_1(\psi_1,\omega_1),v_1),\end{aligned}\tag{51}$$

$$\begin{aligned}\Delta\psi_1(x,0)&=0,\\ \Delta\psi_{1t}(x,0)&=0,\end{aligned}\tag{52}$$

$$\begin{aligned}&\langle\Delta\psi_{2tt},v_2\rangle+(\nabla\Delta\psi_2,\nabla v_2)+(\Delta\psi_2+\Delta\psi_1+\Delta\psi_3,v_2)\\ &\quad=(k_2(\psi_2+\Delta\psi_2,\omega_2+\Delta\omega_2)-k_2(\psi_2,\omega_2),v_2),\end{aligned}\tag{53}$$

$$\begin{aligned}\Delta\psi_2(x,0)&=0,\\ \Delta\psi_{2t}(x,0)&=0,\end{aligned}\tag{54}$$

$$\begin{aligned}&\langle\Delta\psi_{3tt},v_3\rangle+(\nabla\Delta\psi_3,\nabla v_3)+(\Delta\psi_3+\Delta\psi_1-\Delta\psi_2,v_3)\\ &\quad=(k_3(\psi_3+\Delta\psi_3,\omega_3+\Delta\omega_3)-k_3(\psi_3,\omega_3),v_3),\end{aligned}\tag{55}$$

$$\begin{aligned}\Delta\psi_3(x,0)&=0,\\ \Delta\psi_{3t}(x,0)&=0.\end{aligned}\tag{56}$$

Substituting $v_i=\Delta\psi_{it}$ for each $i=1,2,3$ in (51), (53), and (55), respectively, adding the obtained three equations together, and using the same way that we used to get (32), a similar equation can be obtained but with $\overrightarrow{\Delta\psi}$ in state of $\vec{\psi}_n$, and then integration of both sides on $[0,t]$, using the Lip. property on k_i, $i=1,2,3$, with respect to each dependent variable, yields

$$\begin{aligned}\int_0^t\frac{d}{dt}\left[\left\|\overrightarrow{\Delta\psi}_t(t)\right\|_0^2+\|\overrightarrow{\Delta\psi}\|_1^2\right]&\le 2\int_0^t\left[\left(|\Delta\psi_2|+|\Delta\psi_3|\right)|\Delta\psi_{1t}|+\left(|\Delta\psi_1|+|\Delta\psi_3|\right)|\Delta\psi_{2t}|\right]dt\\ &+2\int_0^t\left[\left(|\Delta\psi_1|+|\Delta\psi_2|\right)|\Delta\psi_{3t}|+\left(\overline{L}_1|\Delta\psi_1|+\overline{\overline{L}}_1|\Delta\omega_1|\right)|\Delta\psi_{1t}|\right]dt\\ &+2\int_0^t\left[\overline{L}_2\left(|\Delta\psi_2|+\overline{\overline{L}}_2|\Delta\omega_2|\right)|\Delta\psi_{2t}|+\overline{L}_3\left(|\Delta\psi_3|+\overline{\overline{L}}_3|\Delta\omega_3|\right)|\Delta\psi_{3t}|\right]dt.\end{aligned}\tag{57}$$

Using the definitions of the norms and the relations between them, we get

$$\begin{aligned}\left\|\overrightarrow{\Delta\psi_t}(t)\right\|_0^2+\|\overrightarrow{\Delta\psi}\|_1^2 &\le 2\int_0^t\left(\|\overrightarrow{\Delta\psi}\|_0^2+\left\|\overrightarrow{\Delta\psi_t}\right\|_1^2\right)dt+\tilde{L}_1\int_0^t\left(\|\overrightarrow{\Delta\psi}\|_0^2+\left\|\overrightarrow{\Delta\psi_t}\right\|_1^2\right)dt\\&+\bar{L}^2\int_0^T\|\overrightarrow{\Delta\omega}\|_0^2dt+\bar{L}^2\int_0^t\left\|\overrightarrow{\Delta\psi_t}\right\|_1^2dt\\&\le\bar{L}^2\|\overrightarrow{\Delta\omega}(t)\|_\Pi^2+L_1\int_0^t\left(\|\overrightarrow{\Delta\psi}\|_0^2+\left\|\overrightarrow{\Delta\psi_t}\right\|_1^2\right)dt,\end{aligned}\tag{58}$$

where $\tilde{L}_1=\max(\bar{L}_1,\bar{L}_2,\bar{L}_3)$, $\bar{L}^2=\max(\bar{\bar{L}}_1,\bar{\bar{L}}_2,\bar{\bar{L}}_3)$, and $L_1=\max(2+\tilde{L}_1,2+\tilde{L}_1+\bar{L}^2)$.

Applying the BGin, with $L^2=\bar{L}^2e^{L_1}$, we get

$$\begin{aligned}&\left\|\overrightarrow{\Delta\psi_t}(t)\right\|_0^2+\|\overrightarrow{\Delta\psi}\|_1^2\le L^2\|\overrightarrow{\Delta\omega}(t)\|_\Pi^2,\quad\forall t\in\bar{I}\Rightarrow\\&\|\overrightarrow{\Delta\psi}(t)\|_1^2\le L^2\|\overrightarrow{\Delta\omega}(t)\|_\Pi^2,\quad\forall t\in\bar{I}\Rightarrow\\&\|\overrightarrow{\Delta\psi}\|_{L^\infty(I,L^2(\mathrm{E}))}\le L\|\overrightarrow{\Delta\omega}\|_\Pi,\\&\|\overrightarrow{\Delta\psi}\|_{L^2(I,\mathrm{Y})}\le L\|\overrightarrow{\Delta\omega}\|_\Pi\text{ and }\|\overrightarrow{\Delta\psi}\|_\Pi\le L\|\overrightarrow{\Delta\omega}\|_\Pi.\end{aligned}\tag{59}$$

□

From the abovementioned three inequalities, the Lip. continuity of the operator $\vec{\omega}\mapsto\vec{\psi}$ easily obtained.

1.3. The Existence of a Classical Optimal Control. This section concerned with proving the existence theorem with a CCOPCV satisfying the EQVC and INEQVC is studied. Hence, the following assumption and lemmas will be needed.

Assumption B. Consider m_{ri} (for $r=0,\dots,q$ and $i=1,2,3$) is of Caraty. on $\Pi\times(\mathscr{R}\times\complement_i)$ and satisfies $\psi_i\in\mathscr{R}$ and $\omega_i\in\complement_i$:

$$|m_{ri}(x,t,\psi_i,\omega_i)|\le M_{ri}(x,t)+c_{ki}\psi_i^2,\quad\text{where }M_{ri}\in L^1(\Pi),\ \forall i=1,2,3,\ \forall r=0,\dots,q.\tag{60}$$

Lemma 2. *With Assumption B, the functional* $\vec{\omega}\mapsto M_r(\vec{\omega})$ *is cont. on* $(L^2(\Pi))^3$.

Proof. Using Assumption B and Proposition 1 gives $\int_Q m_{ri}(x,t,\psi_i,\omega_i)dx\,dt$ is cont. on $L^2(\Pi)$, $\forall i=1,2,3$, and $\forall r=0,\dots,q$; hence, $M_l(\vec{\omega})$ is cont. on $(L^2(\Pi))^3$. □

Lemma 3 (see [13]). *Let* $m:Q\times\mathscr{R}^2\longrightarrow\mathscr{R}$ *is of Caraty. on* $\Pi\times(\mathscr{R}\times\mathscr{R})$ *and satisfies* $|m(x,\psi,\omega)|\le m(x,t)+c\psi^2$, *where* $(x,t)\in L^1(\Pi)$, $\omega\in\complement$, $c\ge0$, $\complement\subset\mathscr{R}$ *is compact. Then,* $\int_\Pi m(x,\psi,\omega)dx$ *is cont. on* $L^2(\Pi)$, *with respect to* ψ.

Theorem 2. *With the Assumptions A and B, if the set* $\vec{\complement}$ *is convex (cox.) and compact (com),* $\vec{W}_A\neq\varnothing$, *and the function* k_i $(\forall i=1,2,3)$ *has the form*

$$k_i(x,t,\psi_i,\omega_i)=k_{i1}(x,t,\psi_i)+k_{i2}(x,t)\omega_i,\tag{61}$$

where $|k_{i1}(x,t,\psi_i)|\le\eta_i(x,t)+c_i|\psi_i|$, $|k_{i2}(x,t)|\le k_i$, $\eta_i\in L^2(\Pi)$, $c_i\ge0$.

In addition m_{ri} is independent of ω_i $(\forall i=1,2,3$, and $r=1,\dots,p)$, m_{ri} $(\forall i=1,2,3$, and $r=1,\dots,p)$ is cox. with respect to ω_i for fixed (x,t,ψ_i), there exists a CCOPCV.

Proof. From the Assumption on $\complement_i\subset\mathscr{R}$ $\forall i=1,2,3$ and Egorov's theorem, one obtains that $\vec{W}_1\times\vec{W}_2\times\vec{W}_3=\vec{W}$ is weakly com. Since $\vec{W}_A\neq\varnothing$, hence there is $\vec{\vec{\omega}}\in\vec{W}_A$ s.t. $r(\vec{\vec{\omega}})=0, 1\le r\le p$, $M_k(\vec{\vec{\omega}})\le0$, for $p+1\le r\le q$ and there is a minimizing sequence $\{\vec{\omega}_\rho\}$ s.t. $\vec{\omega}_\rho\in\vec{W}_A,\forall\rho$, which satisfies $\rho\overset{\lim}{\longrightarrow}\infty M_0(\vec{\omega}_\rho)=\vec{\vec{\omega}}\overset{\inf}{\in}\vec{W}_A M_0(\vec{\vec{\omega}})$. Since $\vec{\omega}_\rho\in\vec{W}_A,\forall\rho$ and $\vec{W}$ is weakly com., then $\{\vec{\omega}_\rho\}$ has a subsequence say again $\{\vec{\omega}_\rho\}$ which converges weakly to some $\vec{\omega}$ in $\vec{W}$, i.e., $\vec{\omega}_\rho\rightharpoonup\omega$ in $(L^2(\Pi))^3$ and $\|\vec{\omega}_\rho\|_\Pi\le c,\forall\rho$. From Theorem 1, for any given control $\vec{\omega}_\rho$, then $\vec{\psi}_\rho=\vec{\psi}_{\vec{\omega}_\rho}$ is a unique solution for the Tsteqs, and $\|\vec{\psi}\|_{\rho L^2(I,Y)}$,

$\|\overrightarrow{\psi}\|_{\rho t L^2(\Pi)}$ are bounded, then by Algth, the sequences $\{\overrightarrow{\psi}_\rho\}$ and $\{\overrightarrow{\psi}_{\rho t}\}$ have a weakly converging subsequences, say for simplicity, $\{\overrightarrow{\psi}_\rho\}$ and $\{\overrightarrow{\psi}_{\rho t}\}$, i.e.,

$$\begin{aligned}\overrightarrow{\psi}_\rho &\rightharpoonup \overrightarrow{\psi}, \quad \text{in} \left(L^2(I, Y)\right)^3, \\ \overrightarrow{\psi}_{\rho t} &\rightharpoonup \overrightarrow{\psi}_t, \quad \text{in} \left(L^2(\Pi)\right)^3.\end{aligned} \tag{62}$$

Then, by applying the Aubin compactness theorem [18], the sequence $\{\overrightarrow{\psi}_\rho\}$ has a strongly converging subsequence, say for simplicity, $\{\overrightarrow{\psi}_\rho\}$ such that $\overrightarrow{\psi}_\rho \longrightarrow \overrightarrow{\psi}$ in $(L^2(\Pi))^3$.

Now, for each ρ, substitute the solution $(\psi_{1\rho}, \psi_{2\rho}, \psi_{3\rho})$ in the wkf of (18), (20), and (22), then multiply both sides of each one by $\chi_i(t)$ (with $\chi_i \in C^2[0, T]$, such that $\chi_i(T) = \chi_i'(T) = 0$, $\chi_i(0) \neq 0, \chi_i'(0) \neq 0$, for $i = 1, 2, 3$). After rewriting the first terms in the LHS in each one of them, integrating both sides on $[0, T]$, and then by applying integration by parts for these first terms, we get

$$\begin{aligned}&\int_0^T \frac{d}{dt}\left(\psi_{1\rho t}, v_1\right)\chi_1(t)dt + \int_0^T \left[\left(\nabla\psi_{1\rho}, \nabla v_1\right) + \left(\psi_{1\rho}, v_1\right) - \left(\psi_{2\rho}, v_1\right) - \left(\psi_{3\rho}, v_1\right)\right]\chi_1(t)dt \\ &\quad = \int_0^T \left(k_{11}\left(x, t, \psi_{1\rho}\right) + k_{12}(x, t)\omega_{1\rho}, v\right)\chi_1(t)dt,\end{aligned} \tag{63}$$

$$\begin{aligned}&\int_0^T \frac{d}{dt}\left(\psi_{2\rho t}, v_2\right)\chi_2(t)dt + \int_0^T \left[\left(\nabla\psi_{2\rho}, \nabla v_2\right) + \left(\psi_{2\rho}, v_2\right) + \left(\psi_{1\rho}, v_2\right) + \left(\psi_{3\rho}, v_2\right)\right]\chi_2(t)dt \\ &\quad = \int_0^T \left(k_{21}\left(x, t, \psi_{2\rho}\right) + k_{22}(x, t)\omega_{2\rho}, v_2\right)\chi_2(t)dt,\end{aligned} \tag{64}$$

$$\begin{aligned}&\int_0^T \frac{d}{dt}\left(\psi_{3\rho t}, v_3\right)\chi_3(t)dt + \int_0^T \left[\left(\nabla\psi_{3\rho}, \nabla v_3\right) + \left(\psi_{3\rho}, v_3\right) + \left(\psi_{1\rho}, v_3\right) - \left(\psi_{2\rho}, v_3\right)\right]\chi_3(t)dt \\ &\quad = \int_0^T \left[\left(k_{31}\left(x, t, \psi_{3\rho}\right) + k_{32}(x, t)\omega_{3\rho}, v_3\right)\chi_3(t)\right]dt.\end{aligned} \tag{65}$$

One can passage the limits in the LHS of (50)–(52) by applying the same manner which is applied in the proof of Theorem 1 to passage the limits in RHS of these equations; we suppose $(\forall i = 1, 2, 3)$, $v_i \in C[\overline{\mathrm{E}}]$, $w_i = v_i\chi_i(t)$, then $w_i \in C[\overline{\Pi}] \in L^\infty(I, V) \subset L^2(\Pi)$, set $\overline{k}_{i1}(\psi_{1\rho}) = k_{i1}(\psi_{i\rho})w_i$, then $\overline{k}_{i1}: \Pi \times \mathbb{R} \longrightarrow \mathbb{R}$ is of Caraty., using Proposition 1, to get the integral $\int_\Pi k_{i1}(\psi_{i\rho})w_i dx\, dt$ is cont. with respect to $\psi_{i\rho}$, but $\psi_{i\rho} \longrightarrow \psi_i$ in $L^2(\Pi)$ and $\omega_{i\rho} \rightharpoonup \omega_i$ in $L^2(\Pi)$, then

$$\int_\Pi k_{i1}\left(\psi_{i\rho}\right)w_i dx\, dt \longrightarrow \int_\Pi k_{i1}(\psi_i)w_i dx\, dt, \quad \forall w_i \in C[\overline{\Pi}] \text{ for } i = 1, 2, 3, \tag{66}$$

$$\int_\Pi k_{i2}(x, t)\omega_{i\rho}w_i dx\, dt \longrightarrow \int_\Pi k_{i2}(x, t)\omega_i w_i dx\, dt, \quad \forall w_i \in C[\overline{\Pi}], \text{ for } i = 1, 2, 3. \tag{67}$$

Thus, (66) and (67) are holded for every $v_i \in \Upsilon$, since $C(\overline{E})$ is dense in Υ; hence, we get the wkf (10), (12), and (14).

Also, the same manner which is applied in the proof of Theorem 2 can be used here to passage the limits in the ICs. Hence, (ψ_1, ψ_2, ψ_3) is a solution of the wkf of (10)–(15).

On the other hand, since m_{ri} (for $i = 1, 2, 3$ and $r = 1, 2, \ldots, p$) is independent of ω_i and cont. with respect to ψ_i, by Lemma 2, $\int_\Pi m_{ri}(x, t, \psi_{i\rho})dx\, dt$ is cont. with respect to y_i, $\overrightarrow{\psi}_\rho \longrightarrow \overrightarrow{\psi}$ in $(L^2(\Pi))^3$, then

$$\int_\Pi m_{ri}\left(x, t, \psi_{i\rho}\right)dx\, dt \longrightarrow \int_\Pi m_{ri}(x, t, \psi_i)dx\, dt. \tag{68}$$

Hence, $M_r(\overrightarrow{\omega}) = \rho \overset{\lim}{\longrightarrow} \infty M_r(\overrightarrow{\omega}_\rho) = 0$.

Now, since $m_{ri}(x, t, \psi_i, \omega_i)$ (for each $r = 0, 1 + p, 2 + p, \ldots, q$ and $i = 1, 2, 3$) is cont. with respect to (ψ_i, ω_i) and since C_i is compact, then using Lemma 3, we get

$$\int_\Pi m_{ri}\left(x, t, \psi_{i\rho}, \omega_{i\rho}\right)dx\, dt \longrightarrow \int_\Pi m_{ri}\left(x, t, \psi_i, \omega_{i\rho}\right)dx\, dt. \tag{69}$$

On the other hand, $\int_\Pi m_{ri}(x, t, \psi_i, \omega_{i\rho})dx\, dt$ is cox. and cont. with respect to ω_i (since $m_{ri}(x, t, \psi_i, \omega_i)$ is cox. and cont. with respect to ω_i), then $\int_\Pi m_{ri}(x, t, \psi_i, \omega_i)dx\, dt$ Necessary and Sufficient is weakly lower semicont. (welsc) with respect to ω_i, i.e.,

$$\begin{gathered}\int_{\Pi} m_{ri}(x,t,\psi_i,\omega_i)\mathrm{d}x\,\mathrm{d}t \le \rho \overset{\lim}{\longrightarrow} \infty\inf \int_{\Pi}\left(m_{ri}\left(x,t,\psi_i,\omega_{i\rho}\right)-m_{ri}\left(x,t,\psi_{i\rho},\omega_{i\rho}\right)\right)\mathrm{d}x\,\mathrm{d}t \\ +\rho \overset{\lim}{\longrightarrow} \infty\inf \int_{\Pi} m_{ri}\left(x,t,\psi_{i\rho},\omega_{i\rho}\right)\mathrm{d}x\,\mathrm{d}t \le \rho \overset{\lim}{\longrightarrow} \infty\inf \int_{\Pi} m_{ri}\left(x,t,\psi_{i\rho},\omega_{i\rho}\right)\mathrm{d}x\,\mathrm{d}t, \quad \text{by (55)}\Rightarrow \\ \sum_{i=1}^{2}\int_{\Pi} m_{ri}(x,t,\psi_i,\omega_i)\mathrm{d}x\,\mathrm{d}t \le k \overset{\lim}{\longrightarrow} \infty\inf \sum_{i=1}^{2}\int_{\Pi} m_{ri}\left(x,t,\psi_{ik},\omega_{i\rho}\right)\mathrm{d}x\mathrm{d}t \Rightarrow \\ M_r(\overrightarrow{\omega}) \le \lim_{\rho\longrightarrow\infty} \inf M_r\left(\overrightarrow{\omega}_\rho\right),\end{gathered} \tag{70}$$

then $M_r(\overrightarrow{\omega}) \le 0$ $(r = p+1,\dots,q)$ since $\overrightarrow{\omega}_\rho \in \overrightarrow{W}_A$, $\forall\rho$, and we get that

$$\begin{aligned} M_0(\overrightarrow{\omega}) &\le \rho \overset{\lim}{\longrightarrow} \infty \inf M_0\left(\overrightarrow{\omega}_\rho\right) \\ &= \lim_{\rho\longrightarrow\infty} M_0\left(\overrightarrow{\omega}_\rho\right) = \overset{\inf}{\overrightarrow{\omega} \in \overrightarrow{W}_A} M_0(\overrightarrow{\omega}) \\ &= \overset{\text{Min}}{\overrightarrow{\omega} \in \overrightarrow{W}_A} M_0(\overrightarrow{\omega}). \end{aligned} \tag{71}$$

Thus, $\overrightarrow{\omega}$ is a CCOPCV.

Assumption C. Assume for $r = 0,\dots,q$ and $i = 1,2,3$, the functions k_i, $k_{i\psi_i}$, $k_{i\omega_i}$, $m_{r_i\psi_i}$, and $m_{r_i\omega_i}$ are defined and are of Caraty. on $\Pi \times (\mathscr{R} \times \complement')$ (where $\complement'$ is an open subset in $\complement$) and satisfy $|k_{i\psi_i}(x,t,\psi_i,\omega_i)| \le L_i$, $|k_{i\omega_i}(x,t,\psi_i,\omega_i)| \le L_i'$:

$$\begin{aligned} \left|m_{r_i\psi_i}(x,t,\psi_i,\omega_i)\right| &\le M_{r_{i5}}(x,t) + c_{r_{i5}}\left|\psi_i\right|, \\ \left|m_{r_i\omega_i}(x,t,\psi_i,\omega_i)\right| &\le M_{r_{i6}}(x,t) + c_{r_{i6}}\left|\psi_i\right|, \end{aligned} \tag{72}$$

where $(x,t) \in \Pi$, $\psi_i,\omega_i \in \mathbb{R}$, $M_{r_{i5}}, M_{r_{i6}} \in L^2(\Pi)$. $L_i, L_i', c_{r_{i5}}, c_{r_{i6}} \ge 0$.

Note: for simplicity, in the following theorem, we will drop the index k from the functions m_{li} and M_l. Also, we assume the Assums. (A), (B), and (C) are considered.

Theorem 3. *Consider the TAEqs* $\overrightarrow{\xi} = (\xi_1,\xi_2,\xi_3)$ *of Tseqs (1)–(6) are defined by*

$$\begin{aligned} \xi_{1tt} - \Delta\xi_1 + \xi_1 + \xi_2 + \xi_3 &= \xi_1 k_{1\psi_1}(x,t,\psi_1,\omega_1) + m_{1\psi_1}(x,t,\psi_1,\omega_1), \quad \text{on } \Pi, \\ \xi_1 = 0, \quad \text{on } \Sigma \quad & \xi_1(x,T) = \xi_{1t}(x,T) = 0, \quad \text{on } E, \\ \xi_{2tt} - \Delta\xi_2 + \xi_2 - \xi_1 - \xi_3 &= \xi_2 k_{2\psi_2}(x,t,\psi_2,\omega_2) + m_{2\psi_2}(x,t,\psi_2,\omega_2), \quad \text{on } \Pi, \\ \xi_2 = 0, \quad \text{on } \Sigma, \quad & \xi_2(x,T) = \xi_{2t}(x,T) = 0, \quad \text{on } E, \\ \xi_{3tt} - \Delta\xi_3 + \xi_3 - \xi_1 + \xi_2 &= \xi_{13} k_{3\psi_3}(x,t,\psi_3,\omega_3) + m_{3\psi_3}(x,t,\psi_3,\omega_3), \quad \text{on } \Pi, \\ \xi_3 = 0, \quad \text{on } \Sigma \quad & \xi_3(x,T) = \xi_{3t}(x,T) = 0, \quad \text{on } E, \end{aligned} \tag{73}$$

and the Ham is given by $\mathscr{H}(x,t,\overrightarrow{\psi},\overrightarrow{\omega},\overrightarrow{\xi}) = \sum_{i=1}^{3}\zeta_i(x,t,\psi_i,\omega_i)$, *where* $\zeta_i = \xi_i k_i(x,t,\psi_i,\omega_i) + m_i(x,t,\psi_i,\omega_i)$, *for each* $i = 1,2,3$.

Then, the Frde. of G is defined by

$$M'(\overrightarrow{\omega})\Delta\overrightarrow{\omega} = \int_{\Pi} \mathscr{H}_{\overrightarrow{\omega}}(x,t,\overrightarrow{\psi},\overrightarrow{\omega},\overrightarrow{\xi})\Delta\overrightarrow{\omega}\,\mathrm{d}x\,\mathrm{d}t, \tag{74}$$

where $\mathscr{H}_{\overrightarrow{\omega}}(x,t,\overrightarrow{\psi},\overrightarrow{\xi},\overrightarrow{\omega}) = (\zeta_{1\omega_1},\zeta_{1\omega_1},\zeta_{1\omega_1})$, *and* $\Delta\overrightarrow{\omega} = (\Delta\omega_1,\Delta\omega_2,\Delta\omega_3)$, *with* $\zeta_{i\omega_i} = \xi_i k_{i\omega_i}(x,t,\psi_i,\omega_i) + m_{i\omega_i}(x,t,\psi_i,\omega_i)$, $m_i = \sum_{r=0}^{q}\kappa_r m_{ri}$ *and* $\xi_i = \sum_{r=0}^{q}\kappa_r\xi_{li}$, *for each* $i = 1,2,3$.

Proof. at first, let the wkf of the TAEqs are given $\forall v_1,v_2,v_3 \in Y$, by

$$\langle\xi_{1tt},v_1\rangle + (\nabla\xi_1,\nabla v_1) + (\xi_1+\xi_2+\xi_3,v_1) = \left(\xi_1 k_{1\psi_1} + m_{1\psi_1},v_1\right), \quad \forall v_1 \in Y \text{ a.e. on } I, \tag{75}$$

$$(\xi_1(T),v_1) = (\xi_{1t}(T),v_1) = 0, \tag{76}$$

$$\langle z\xi_{2t},v_2\rangle + (\nabla\xi_2,\nabla v_2) + (\xi_2-\xi_1-\xi_3,v_2) = \left(\xi_2 k_{2\psi_2} + m_{2\psi_2},v_2\right), \quad \forall v_2 \in Y \text{ a.e. on } I, \tag{77}$$

$$(\xi_2(T),v_2) = (\xi_{2t}(T),v_2) = 0 \tag{78}$$

$$\langle\xi_{3t},v_3\rangle + (\nabla\xi_3,\nabla v_3) + (\xi_3-\xi_1+\xi_2,v_3) = \left(\xi_3 k_{3\psi_3} + m_{3\psi_3},v_3\right), \quad \forall v_2 \in Y \text{ a.e. on } I, \tag{79}$$

$$(\xi_3(T),v_3) = (\xi_{3t}(T),v_3) = 0. \tag{80}$$

From the assumptions and using the same manner which is applied in the proof of Theorem 1, once can prove that the wkf (75)–(80) has a unique solution $\vec{\xi} = (\xi_1, \xi_2, \xi_3) \in (L^2(\Pi))^3$.

Substituting $v_i = \Delta\psi_i$ for each $i = 1, 2, 3$ in (75), (77), and (79) and integrating both sides on $[0, T]$, we get

$$\int_0^T [\langle \Delta\psi_1, \xi_{1tt}\rangle + (\nabla\xi_1, \nabla\Delta\psi_1) + (\xi_1 + \xi_3 + \xi_2, \Delta\psi_1)]dt = \int_0^T \left(\xi_1 k_{1\psi_1} + g_{1\psi_1}, \Delta\psi_1\right) dt, \tag{81}$$

$$\int_0^T [\langle \Delta\psi_2, \xi_{2tt}\rangle + (\nabla\xi_2, \nabla\Delta\psi_2) + (\xi_2 - \xi_1 - \xi_3, \Delta\psi_2)]dt = \int_0^T \left(\xi_2 k_{2\psi_2} + g_{2\psi_2}, \Delta\psi_2\right) dt, \tag{82}$$

$$\int_0^T [\langle \Delta\psi_3, \xi_{3tt}\rangle + (\nabla\xi_3, \nabla\Delta\psi_3) + (\xi_3 - \xi_1 + \xi_2, \Delta\psi_3)]dt = \int_0^T \left(\xi_3 k_{3\psi_3} + g_{3\psi_3}, \Delta\psi_3\right) dt. \tag{83}$$

Now, let $\vec{\omega}, \vec{\overline{\omega}} \in (L^2(\Pi))^3$, $\overrightarrow{\Delta\omega} = \vec{\overline{\omega}} - \vec{\omega} \in (L^2(Q))^3$, and then by Theorem 1, $\vec{\psi} = \vec{\psi}_{\vec{\omega}}$ and $\vec{\overline{\psi}} = \vec{\psi}_{\vec{\overline{\omega}}}$ are their corresponding solutions. Set $\overrightarrow{\Delta\psi} = (\Delta\psi_1, \Delta\psi_2, \Delta\psi_3) = \vec{\overline{\psi}} - \vec{\psi}$, substitute $v_i = \xi_i$ for each $i = 1, 2, 3$ in (51), (53), and (55), integrate both sides on $[0, T]$, and then integrate by parts twice the first term in the LHS of each equation. Finding for each $i = 1, 2, 3$ the Frde. of k_i in the RHS of each equation which are exist from the Assumption C, then by Lemma 1, and the inequality of Minkowiski, one has

$$\begin{aligned} &\int_0^T [\langle \Delta\psi_1, \xi_{1tt}\rangle + (\nabla\Delta\psi_1, \nabla\xi_1) + (\Delta\psi_1 - \Delta\psi_2 - \Delta\psi_3, \xi_1)]dt \\ &\quad = \int_0^T \left(k_{1\psi_1}\Delta\psi_1 + k_{1\omega_1}\Delta\omega_1, \xi_1\right)dt + O_1(\overrightarrow{\Delta\omega})\|\overrightarrow{\Delta\omega}\|_\Pi, \end{aligned} \tag{84}$$

$$\begin{aligned} &\int_0^T [\langle \Delta\psi_2, \xi_{2tt}\rangle + (\nabla\Delta\psi_2, \nabla\xi_2) + (\Delta\psi_2 + \Delta\psi_1 + \Delta\psi_3, \xi_2)dt] \\ &\quad = \int_0^T \left(k_{2\psi_2}\delta\psi_2 + k_{2\omega_2}\Delta\omega_2, \xi_2\right)dt + O_2(\overrightarrow{\Delta\omega})\|\overrightarrow{\Delta\omega}\|_\Pi, \end{aligned} \tag{85}$$

$$\begin{aligned} &\int_0^T [\langle \Delta\psi_3, \xi_{3tt}\rangle + (\nabla\delta\psi_3, \nabla\xi_3) + (\Delta\psi_{2\varepsilon} + \Delta\psi_1 - \Delta\psi_2, \xi_3)]dt \\ &\quad = \int_0^T \left(k_{3\psi_3}\Delta\psi_3 + k_{3\omega_3}\Delta\omega_3, \xi_3\right)dt + O_3(\overrightarrow{\Delta\omega})\|\overrightarrow{\Delta\omega}\|_\Pi. \end{aligned} \tag{86}$$

Subtracting (84)–(86) from, respectively, adding the obtain equations, we get

$$\int_0^T \sum_{i=1}^3 \left(k_{i\omega_i}\Delta\omega_i, \xi_i\right)dt + O_4(\overrightarrow{\Delta\omega}) = \int_0^T \sum_{i=1}^3 \left(m_{i\psi_i}, \Delta\psi_i\right)dt, \tag{87}$$

where. $O_4(\varepsilon) \longrightarrow 0$, as $\overrightarrow{\Delta\omega} \longrightarrow 0$, with $O_4(\overrightarrow{\Delta\omega}) = \sum_{i=1}^3 O_i(\overrightarrow{\Delta\omega})\overrightarrow{\Delta\omega}_\Pi$.

On the other hand, from the Assumption B on m_i (for $i = 1, 2, 3$), the Frde definition, Lemma 1, and by applying the inequality of Minkowiski, we obtain

$$\begin{aligned} M(\vec{\omega} + \overrightarrow{\Delta\omega}) - M(\vec{\omega}) &= \int_\Pi \sum_{i=1}^3 \left(m_{i\psi_i}\Delta\psi_i + m_{i\omega_i}\Delta\omega_i\right)dxdt \\ &\quad + O_5(\overrightarrow{\Delta\omega})\overrightarrow{\Delta\omega}_\Pi, \end{aligned} \tag{88}$$

where $O_5(\overrightarrow{\Delta\omega}) \longrightarrow 0$, as $\overrightarrow{\Delta\omega} \longrightarrow 0$.

Now, by substituting (87) in (88), one has

$$\begin{aligned} M(\vec{\omega} + \overrightarrow{\Delta\omega}) - M(\vec{\omega}) &= \int_\Pi \sum_{i=1}^3 \zeta_{i\omega_i}(x, t, \psi_i, \omega_i)\Delta\omega_i dx\, dt \\ &\quad + O_6(\overrightarrow{\Delta\omega})\overrightarrow{\Delta\omega}_\Pi, \end{aligned} \tag{89}$$

where $O_6(\overrightarrow{\Delta\omega}) = O_5(\overrightarrow{\Delta\omega}) + O_5(\overrightarrow{\Delta\omega}) \longrightarrow 0$, as $\overrightarrow{\Delta\omega} \longrightarrow 0$.

Finally, from the Frde. of, we get

$$M'(\vec{\omega})\Delta\vec{\omega} = \int_\Pi \mathscr{H}_{\vec{\omega}}(x, t, \vec{\psi}, \vec{\omega}, \vec{\xi})\Delta\vec{\omega}\, dx\, dt. \tag{90}$$

1.4. Necessary and Sufficient Conditions for Optimality. This section deals with the theorems for the Necoop necessary under certain hypotheses which are proved as follows:

Theorem 4. *Necoop (multipliers theorem):*

(a) *with Assumptions A, B, and C, if* $\overrightarrow{W}$ *is cox. and the* $\overrightarrow{\omega} \in \overrightarrow{W}_A$ *is optimal, then there are multipliers* $\kappa_r \in \mathbb{R}$, $r = 0, 1, \ldots, p, p+1, \ldots, q$ *with* $\kappa_r \geq 0$, *for* $r = 0, p+1, \ldots, q$, $\sum_{r=0}^{q} |\kappa_r| = 1$ *such that the following Kuhn-Tucker-Lagrange (K. T. L.) conditions are holded:*

$$\int_{\Pi} \mathscr{H}_{\overrightarrow{\omega}}(x, t, \overrightarrow{\psi}, \overrightarrow{\omega}, \overrightarrow{\xi})\Delta\overrightarrow{\omega}\,dx\,dt \geq 0, \quad \forall \overrightarrow{\overline{\omega}} \in \overrightarrow{W}, \overrightarrow{\Delta\omega} = \overrightarrow{\overline{\omega}} - \overrightarrow{\omega}, \tag{91}$$

$$\kappa_r M_r(\overrightarrow{\omega}) = 0, \quad \text{for } r = p+1, \ldots, q. \tag{92}$$

(b) *Inequality (91) is equivalent to the (weak) piecewise minimum principle*

$$\mathscr{H}_{\overrightarrow{\omega}}(x, t, \overrightarrow{\psi}, \overrightarrow{\xi}, \overrightarrow{\omega}) \cdot \overrightarrow{\omega}(t) = \min_{\overrightarrow{\overline{\omega}} \in \overrightarrow{\mathbb{C}}} \mathscr{H}_{\overrightarrow{\omega}}(x, t, \overrightarrow{\psi}, \overrightarrow{\xi}, \overrightarrow{\omega}) \cdot \overrightarrow{\overline{\omega}}(t), \quad \text{a.e. on } \Pi. \tag{93}$$

Proof (a) From Lemma 2, the functional $M_r(\overrightarrow{\omega})$ (for $r = 0, 1, \ldots, q$) is cont., and from Theorem 3, the functional M_r' (for $r = 0, 1, \ldots, q$) is cont. with respect to $\overrightarrow{\overline{\omega}} - \overrightarrow{\omega}$ and linear in $\overrightarrow{\overline{\omega}} - \overrightarrow{\omega}$, then M_k' is $L-$ differential for every L, and then applying the K. T. L. theorem [5], there are multipliers $\kappa_r \in \mathbb{R}$, $r = 0, 1, \ldots, q$ with $\kappa_r \geq 0$, for $r = 0, p+1, \ldots, q$, $\sum_{r=0}^{q} |\kappa_r| = 1$, such that (91)–(93) are satisfied, by using Theorem 3, then (91) becomes $\sum_{r=0}^{q} \kappa_r \int_{\Pi} \sum_{i=1}^{3} \zeta_{i\omega_i}(x, t, \psi_i, \omega_i)\Delta\omega_i dx\,dt \geq 0$, which can be rewritten as

$$\int_{\Pi} \overrightarrow{\zeta}_{\overrightarrow{\omega}} \cdot \left(\overrightarrow{\overline{\omega}}_k - \overrightarrow{\omega}\right) dx\,dt \geq 0, \quad \forall \overrightarrow{\overline{\omega}} \in \overrightarrow{W} \text{ with } \overrightarrow{\zeta}_{\overrightarrow{\omega}} = \left(\zeta_{1\omega_1}, \zeta_{2\omega_2}, \zeta_{3\omega_3}\right). \tag{94}$$

(b) Let $\left\{\overrightarrow{\overline{\omega}}_l\right\}$ be a dense sequence (dse) in $\overrightarrow{W}$, m denotes the Lebesgue measure on Π, and $\Gamma \subset \Pi$ be a measurable subset with property $\overrightarrow{\overline{\omega}}(x, t) = \begin{cases} \overrightarrow{\overline{\omega}}_l(x, t), & \text{if } (x, t) \in \Gamma \\ \overrightarrow{\omega}(x, t), & \text{if } (x, t) \notin \Gamma \end{cases}$.

Therefore, (94) becomes

$$\int_{\Gamma} \overrightarrow{\zeta}_{\overrightarrow{\omega}} \cdot (\overrightarrow{\overline{\omega}} - \overrightarrow{\omega}) dx dt \geq 0, \text{which implies to } \overrightarrow{\zeta}_{\overrightarrow{\omega}} \cdot \left(\overrightarrow{\overline{\omega}}_l - \overrightarrow{\omega}\right) \geq 0, \text{a.e. on } \prod. \tag{95}$$

This means the inequality is satisfied on the whole region Q except in a subset Π_l such that $m(\Pi_l) = 0, \forall l$, where m represents the Lebesgue measure; thus, the inequality holds on Π except in the union $\cup_l \Pi_l$ with $m(\cup_l \Pi_l) = 0$, but $\left\{\overrightarrow{\overline{\omega}}_l\right\}$ is a dse in $\overrightarrow{W}$, then there is $\overrightarrow{\overline{\omega}} \in \overrightarrow{W}$ such that $\int_{\Pi} \overrightarrow{\zeta}_{\overrightarrow{\omega}} \cdot (\overrightarrow{\overline{\omega}}_l - \overrightarrow{\omega}) \geq 0$, a.e. on $\Pi, \forall \overrightarrow{\overline{\omega}} \in \overrightarrow{W}$.

That is, (91) gives (94). The converse is clear.

Theorem 5. *Sucoop: besides the Assumptions A, B, and C, suppose* $\overrightarrow{W}$ *is cox., with* $\overrightarrow{\mathbb{C}}$ *cox.,* k_i *and* m_{ri} ($\forall r = 1, 2, \ldots, p$) *are affine with respect to* (ψ_i, ω_i) *for each* (x, t), m_{ri} ($\forall r = 0, p+1, \ldots, q$) *are cox. with respect to* (ψ_i, ω_i) $\forall (x, t)$, *for* $i = 1, 2, 3$. *Then, the Necoop for Theorem 4 with* $\kappa_0 > 0$ *are sufficient.*

Proof. Assume $\overrightarrow{\omega} \in \overrightarrow{W}_A$ satisfies the condition (91) and (92). Let $M(\overrightarrow{\omega}) = \sum_{r=0}^{q} \kappa_r M_r(\overrightarrow{\omega})$, then using Theorem 3, we get

$$M'(\overrightarrow{\omega})\Delta\overrightarrow{\omega} = \sum_{r=0}^{q} \kappa_r M_r'(\overrightarrow{\omega})\Delta\overrightarrow{\omega} = \sum_{r=0}^{q} \kappa_r \int_{\Pi} \sum_{i=1}^{3} \zeta_{i\omega_i} \cdot \delta\omega_i dx\,dt \geq 0. \tag{96}$$

Since $k_i(x, t, \psi_i, \omega_i) = k_{i1}(x, t)\psi_i + k_{i2}(x, t)\omega_i + k_{i3}(x, t)$, $i = 1, 2, 3$.

Let $\omega = (\omega_1, \omega_2, \omega_3)$ and $\overrightarrow{\overline{\omega}} = (\overline{\omega}_1, \overline{\omega}_2, \overline{\omega}_3)$ be two given control vectors, then $\overrightarrow{\psi} = (\psi_{\omega 1}, \psi_{\omega 2}, \psi_{\omega 3}) = (\psi_1, \psi_2, \psi_3)$ and $\overrightarrow{\overline{\psi}} = (\overline{\psi}_{\omega 1}, \overline{\psi}_{\overline{\omega}2}, \overline{\psi}_{\overline{\omega}3}) = (\overline{\psi}_1, \overline{\psi}_2, \overline{\psi}_3)$ represent the corresponding stats solutions. Substituting $(\overrightarrow{\omega}, \overrightarrow{y})$ in (1)–(6), multiplying all the obtained equalities by $\alpha \in [0, 1]$ once and on the other hand substituting $(\overrightarrow{\overline{\omega}}, \overrightarrow{\overline{y}})$ in (1)–(6), multiplying all the obtained equalities by $(1 - \alpha)$ once again, and lastly adding each pair from the corresponding equalities together, we obtain

$$(\alpha\psi_1 + (1-\alpha)\overline{\psi}_1)_{tt} - \Delta(\alpha\psi_1 + (1-\alpha)\overline{\psi}_1) + \alpha(\psi_1 - \psi_2 - \psi_3) + (1-\alpha)(\overline{\psi}_1 - \overline{\psi}_2 - \overline{\psi}_3) \\ = \alpha(k_{11}(x,t)\psi_1 + k_{12}(x,t)\omega_1) + (1-\alpha)(k_{11}(x,t)\overline{\psi}_1 + k_{12}(x,t)\overline{\omega}_1) + k_{13}(x,t), \tag{97}$$

$$\alpha\psi_1(x,t) + (1-\alpha)\overline{\psi}_1(x,0) = 0, \tag{98}$$

$$\alpha\psi_1(x,0) + (1-\alpha)\overline{\psi}_1(x,0) = \psi_1^0(x), \\ \alpha\psi_{1t}(x,0) + (1-\alpha)\overline{\psi}_{1t}(x,0) = \psi_1^1(x), \tag{99}$$

$$(\alpha\psi_2 + (1-\alpha)\overline{\psi}_2)_{tt} - \Delta(\alpha\psi_2 + (1-\alpha)\overline{\psi}_2) + \alpha(\psi_2 + \psi_1 + \psi_3) + (1-\alpha)(\overline{\psi}_2 + \overline{\psi}_1 + \overline{\psi}_3) = \alpha \\ (k_{21}(x,t)\psi_2 + k_{22}(x,t)\omega_2) + (1-\alpha)(k_{21}(x,t)\overline{\psi}_2 + k_{22}(x,t)\overline{\omega}_2) + k_{23}(x,t), \tag{100}$$

$$\alpha\psi_2(x,t) + (1-\alpha)\overline{\psi}_2(x,0) = 0, \tag{101}$$

$$\alpha\psi_2(x,0) + (1-\alpha)\overline{\psi}_2(x,0) = \psi_2^0(x), \\ \alpha\psi_{2t}(x,0) + (1-\alpha)\overline{\psi}_2(x,0) = \psi_2^1(x), \tag{102}$$

$$(\alpha\psi_3 + (1-\alpha)\overline{\psi}_3)_{tt} - \Delta(\alpha\psi_3 + (1-\alpha)\overline{\psi}_3) + \alpha(\psi_3 + \psi_1 - \psi_2) + (1-\alpha)(\overline{\psi}_3 + \overline{\psi}_1 - \overline{\psi}_2) \\ = \alpha(k_{31}(x,t)\psi_3 + k_{32}(x,t)\omega_3) + (1-\alpha)(k_{31}(x,t)\overline{\psi}_3 + k_{32}(x,t)\overline{\omega}_3) + k_{23}(x,t), \tag{103}$$

$$\alpha\psi_3(x,t) + (1-\alpha)\overline{\psi}_3(x,0) = 0, \tag{104}$$

$$\alpha\psi_3(x,0) + (1-\alpha)\overline{\psi}_3(x,0) = \psi_3^0(x), \\ \alpha\psi_{3t}(x,0) + (1-\alpha)\overline{\psi}_{3t}(x,0) = \psi_3^1(x). \tag{105}$$

Equations (97)–(105) show that if the control vector is $\overrightarrow{\widetilde{\omega}} = (\widetilde{\omega}_1, \widetilde{\omega}_2, \widetilde{\omega}_3)$ with $\overrightarrow{\widetilde{\omega}} = \overrightarrow{\omega} + (1-\alpha)\overrightarrow{\overline{\omega}}$ then its corresponding state vector is $\overrightarrow{\widetilde{\psi}} = (\widetilde{\psi}_1, \widetilde{\psi}_2, \widetilde{\psi}_3)$ with $\widetilde{\psi}_i = \psi_{i\widetilde{\omega}_i} = \psi_{i(\alpha\omega_i + (1-\alpha)\overline{\omega}_i)} = \alpha\psi_i + (1-\alpha)\overline{\psi}_i, \forall i = 1,2,3$. This gives the operator $\overrightarrow{\omega} \mapsto \overrightarrow{y}_{\overrightarrow{\omega}}$ is convex–linear with respect to $(\overrightarrow{\psi}, \overrightarrow{\omega})$ for any $(x,t) \in \Pi$.

On the other hand, the function $M_l(\overrightarrow{\omega})$ (for $k = 1, \ldots, p$) is convex–linear with respect to $(\overrightarrow{\psi}, \overrightarrow{\omega})$ for any $(x,t) \in \Pi$, this backs to the fact that the sum of two affine functions $m_{ri}(x, t, \psi_i, \omega_i)$ (for each $i = 1,2,3$, and $(r = 1, \ldots, p)$ with respect to (ψ_i, ω_i) and $\forall (x,t) \in \Pi$ is affine and the operator $\overrightarrow{\omega} \mapsto \overrightarrow{\psi}_{\overrightarrow{\omega}}$ is convex-linear.

The functions $M_r(\overrightarrow{\omega})$, $\forall r = 0,\ p+1, \ldots, q$ are cox. with respect to $(\overrightarrow{\psi}, \overrightarrow{\omega})$, $\forall (x,t) \in \Pi$ (from the assumptions on the functions m_{r1} and m_{r2}, $\forall r = 0,\ p+1, \ldots, q$). Hence, $M(\overrightarrow{\omega})$ is cox. with respect to $(\overrightarrow{\psi}, \overrightarrow{\omega}), \forall (x,t) \in \Pi$ in the cox. set $\overrightarrow{\widehat{W}}$, and has a cont. Fréd satisfies $M'(\overrightarrow{\omega}) \cdot (\overrightarrow{\overline{\omega}} - \overrightarrow{\omega}) \geq 0 \Rightarrow M(\overrightarrow{\omega})$ has a minimum at $\overrightarrow{\omega} \Rightarrow M(\overrightarrow{\omega}) \leq M(\overrightarrow{\overline{\omega}}), \forall \overrightarrow{\overline{\omega}} \in \overrightarrow{\widehat{W}} \Rightarrow$

$$\sum_{r=0}^{p} \kappa_r M_r(\overrightarrow{\omega}) + \sum_{r=p+1}^{q} \kappa_r M_r(\overrightarrow{\omega}) \leq \sum_{r=0}^{p} \kappa_r M_r(\overrightarrow{\overline{\omega}}) + \sum_{r=p+1}^{q} \kappa_r M_r(\overrightarrow{\overline{\omega}}), \quad \forall \overrightarrow{\overline{\omega}} \in \overrightarrow{\widehat{W}}. \tag{106}$$

Let $\overrightarrow{\overline{\omega}} \in \overrightarrow{\widehat{W}}_A$, with $k_k \geq 0$ (for $r = p+1, \ldots, q$), then from (8), (9), and (77), the abovementioned inequality becomes

$$\kappa_0 M_0(\overrightarrow{\omega}) \leq \kappa_0 M_0(\overrightarrow{\overline{\omega}}), \quad \forall \overrightarrow{\overline{\omega}} \in \overrightarrow{\widehat{W}} \Rightarrow G_0(\overrightarrow{\omega}) \leq G_0(\overrightarrow{\overline{\omega}}), \quad \forall \overrightarrow{\overline{\omega}} \in \overrightarrow{\widehat{W}} \Rightarrow \therefore \overrightarrow{\omega} \text{ is a CCOCVE.} \tag{107}$$

2. Conclusions

The Galm with the Aubin compactness theorem are applied successfully to prove the existence of a unique "continuous state vector" solution for the TNLHPDEqs for a given cont. CCOPCVE. The existence theorem of governing by the considered the TNLHPDEqs with EQVC and INEQVC is proved. The existence of a unique solution of ATEqs associated with the considered Tsteqs is studied. The Frde. of the Ham is derived. The theorems of the

Necoop and the Sucoop of the constrained problem are proved.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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