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*Research Article*

# Effects of Radiation Pressure on the Elliptic Restricted Four-Body Problem

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In this paper, under the effects of the largest primary radiation pressure, the elliptic restricted four-body problem is formulated in Hamiltonian form. Moreover, the canonical equations are obtained which are considered as the equations of motion. The Lagrangian points within the frame of the elliptic restricted four-body problem are obtained. The true anomalies are considered as independent variables. An analytical and numerical approach had been used. A code of Mathematica version 12 is constructed to truncate these considerations and is applied on the Earth-Moon-Sun system. In addition, the stability and periodicity of the motion about the equilibrium points are studied by using the Poincare maps. The motion about the collinear point L2 is presented as an example for the obtained results, and some families of periodic orbits are presented.

## 1. Introduction

Several systems in space dynamics such as two-body, three-body, and four-body are considered. There are many attempts to study the problem of these objects using classical methods or Hamiltonian techniques which are good techniques for solving it (see Budrikis [1] and Stone and Leigh [2]). Many authors (Ershkov [3] and Ito [4]) have used these models with various perturbations such as eccentricity, solar radiation, the variable mass of primes, Coriolis and centrifugal forces to checkpoints of vibration, linear stability, and periodic orbits. Singh and Haruna [5] have studied the locations and linear stability of trigonometric points when two primaries radiate, and they studied their heterogeneity. The constrained two-circle four-body problem (4BP) is the simplest model used in the four-body problem (4BP) field Andreu [6]. In this model, the two primers move in circular orbits around the center of mass while this center of mass also moves in a circular orbit around the largest mass in the same plane. Kumari and Kushvah [7] and Ismail et al. [8] used Poincaré's surface sections to examine the invariance of the constrained four-body problem and applied their study to the Sun-Earth-Moon system.

El-Saftawy and Abd El-Salam [9] obtained the solution of the problem using the technique of Delva. Grebenikov et al. [10] studied the problem of the four constrained bodies, by building the Hamiltonian technique, and obtained equilibrium solutions and found six possible equilibrium configurations. Lakin [11] studied the elliptic restricted three-body problem using different coordinate systems to illustrate the stability. Assadian and Pourtakdoust [12] and Chakraborty and Narayan [13] used the problem of the four finite Belial bodies to examine the influence of the Sun on the points of vibration of the Earth-Moon system. Chakraborty and Narayan [14] investigated the equilibrium points, linear stability, zero velocity curves, and fractal trough of the four-body constrained elliptic problem. The model cited in Xu and Fu [15] is useful as a simple example of nonintegral dynamical systems and is successful in understanding many quasiperiodic phenomena in astronomy. Ibrahim [16] studied the effect of solar radiation pressure onto the motion of many-body problem, and applied the study on the Sun-Earth-Moon-spacecraft system.

In this paper, we study the problem of the four bodies under the influence of the combined forces of gravity of the elementary bodies that revolve in an elliptical orbit around their center of mass and the pressure of radiation of the largest primary. The problem is formulated in the Hamiltonian form. Then, the canonical form is obtained.

The locations of the equilibrium points were determined in terms of the mass ratio of the primaries $\mu$ and $\beta$ the ratio of the radiation strength to the gravitational force of the largest mass. The stability of the motion about the points of vibration under the forces of gravity and radiation is studied. Finally, the canonical equations are applied to the Earth-Moon-Sun system model to calculate the corresponding solutions for both the trigonometric and linear equilibrium points. Finally, the Poincare maps are presented to show these stabilities.

## 2. Description of the Problem

The elliptic restricted four-body problem models represent the motion of the infinitesimal mass, $m$, under the influence of the gravitational field of the three primaries $m_1$, $m_2$, and $m_3$ and radiation pressure of the third primary $m_3$. The three primaries and the infinitesimal mass $m_4$ are moving in the same plane, i.e., coplanar motion. The two smallest primaries $m_1$ and $m_2$ are moving in an elliptic orbit with $(e_2, f_2)$ about the barycenter CM1 orbiting the largest primary $m_3$ in an elliptic orbit with $(e_1, f_1)$, such that $0 < e_1$, and $e_2 < 1$. CM is the barycenter between CM1 and $m_3$. The distance between CM1 and $m_3$ is referred to by $D$, and the distance between $m_1$ and $m_2$ is referred to by $d$ as shown in Figure 1.

Then, we assume the following two hypotheses: (1) $m_3 \gg m_1 + m_2 \gg m_2$ and (2) $D \gg d$. Then, the distance between $m_3$ and CM1 from CM is $D_1$ and $D_2$, respectively, and that of $m_1$ and $m_2$ from CM1 is $d_1$ and $d_2$, respectively. We have to keep in our minds that the above hypotheses are in case that the movement of the infinitesimal mass $m_4$ takes place in space as a spatial elliptic restricted four-body problem [17].

The main idea in this research is to study the effect of radiation pressure from the more massive body $m_3$ and to identify the disturbance which will affect the smallest mass $m_4$, so that $m_3$ is considered as perturbing body and the formation of the Hamiltonian system is needed to obtain the canonical form which is considered as the equations of motion.

To introduce the dimensionless system, let the distance between $m_1$ and $m_2$ equals the unity [18], that means

$$\begin{aligned} G(m_1 + m_2) &= 1, \\ \mu &= \frac{m_2}{m_1 + m_2}, \\ 1 - \mu &= \frac{m_1}{m_1 + m_2}, \\ \mu_3 &= \frac{m_3}{m_1 + m_2}. \end{aligned} \tag{1}$$

Additionally, let the value of the orbital angular momentum with respect to the motion of the primaries $m_1$ and $m_2$ as unity (see Zebehely [17] and Ismail et al. [18, 19]).

The coordinates of $m_1$, $m_2$, $m_3$, and $m_4$ are given as follows:

$$\begin{gathered} m_1 : m_1(\xi_1, \eta_1, \zeta_1)\text{: } (D_2 \cos f_1 - d_1 \cos f_2, D_2 \sin f_1 - d_1 \sin f_2, 0), \\ m_2 : m_2(\xi_2, \eta_2, \zeta_2)\text{: } (D_2 \cos f_1 + d_1 \cos f_2, D_2 \sin f_1 + d_1 \sin f_2, 0), \\ m_4 : m_4 (\xi, \eta, \zeta)\text{: } (\xi, \eta, \zeta), \\ m_3 : m_3 (\xi_3, \eta_3, \zeta_3)\text{: } (-D_1 \cos f_1, -D_1 \sin f_1, 0). \end{gathered} \tag{2}$$

Relative to the approximation involved in the assumption of elliptic motion, the distances $D$ and $d$ are defined by

$$D = \frac{a_1(1 - e_1^2)}{1 + e_1 \cos f_1}, d = \frac{a_2(1 - e_2^2)}{1 + e_2 \cos f_2}, \tag{3}$$

where $a_1$ and $a_2$ are the semimajor axis of the elliptic orbits; thus, $D_1 = a_1 m_3, D_2 = d_1(m_1 + m_2)$, and $D = D_1 + D_2$. Similarly, $d_1 = a_2 m_1$, $d_2 = a_2 m_2$, and $d = d_1 + d_2$ and $f_1, f_2$ are the true anomalies of the bodies $m_3$ and $m_1$, respectively.

Now, the system $(\xi, \eta, \zeta)$ will be transformed to the system $(x, y, z)$. The new coordinates of $m_1$, $m_2$, $m_3$, and $m_4$ become

$$\begin{gathered} m_1 : m_1(x_1, y_1, z_1)\text{: } (-d_1, 0, 0), \\ m_2 : m_2(x_2, y_2, z_2)\text{: } (d_2, 0, 0), \\ m_3\, m_3(x_3, y_3, z_3)\text{: } (-Ds \cos (f_2 - f_1), -Ds \sin (f_2 - f_1), 0), \end{gathered} \tag{4}$$

where $Ds = (R_s(1 - e_1^2))/(1 + e_1 \cos f_1)$.

## 3. The Hamiltonian of the Problem

The equations of motion of the small particle $m_4$ are formulated in Hamiltonian form under the effects of the gravitational potential of small primaries $m_1$ and $m_2$ taking into consideration the perturbing gravitational and radiation of the third large primary $m_3$. The system has a Lagrangian function represented by

$$L\left(x, y, z, \dot{x^2}, \dot{y^2}, \dot{z}^2\right) = T - V, \tag{5}$$

where $T$ is the total kinetic energy and $V$ is the total potential energy of the system. The potential is defined as

$$V = \frac{1 - \mu}{r1} + \frac{\mu}{r2} + \frac{(1 - \beta)\, \mu_3}{r_3}, \tag{6}$$

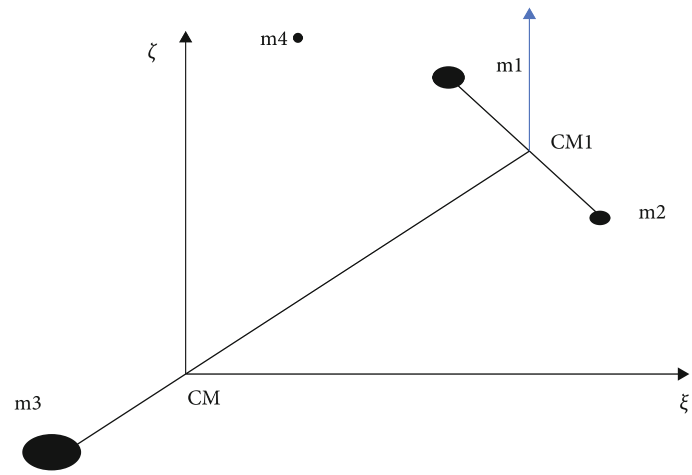


Figure 1: Present the motion of $m_1$, $m_2$, $m_3$, and $m_4$.

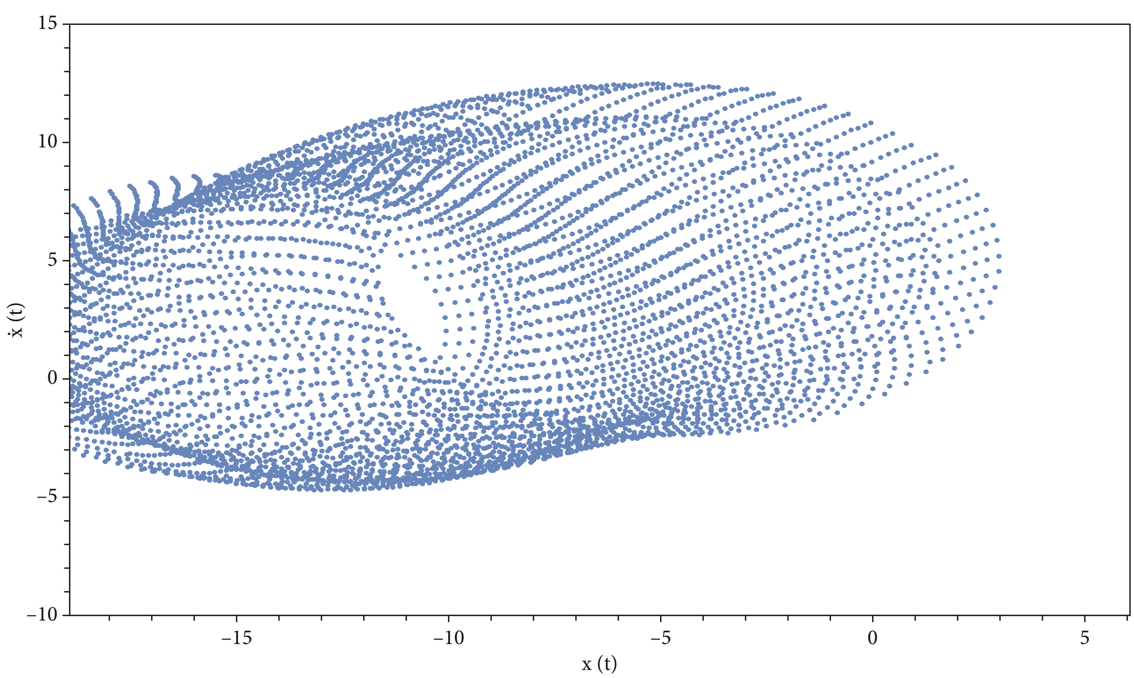


Figure 2: Poincare map about Lx2 with $f_1 = 10^{\circ}$.

where

$$r_1 = \sqrt{(x+\mu)^2 + +y^2 + z^2},$$
$$r_2 = \sqrt{(x-1+\mu)^2 + y^2 + z^2},$$
$$r_3 = \sqrt{(x - Ds\cos(f_1 - f_2))^2 + (y - Ds\sin(f_1 - f_2))^2 + z^2}. \tag{7}$$

$\beta$ is the radiation pressure coefficient of the third body.

Then,

$$L = \frac{1}{2}(\dot{x} - y)^2 + (\dot{y} + x)^2 + \dot{z}^2 + \frac{1-\mu}{r_1} + \frac{\mu}{r_2} + \frac{(1-\beta)\mu_3}{r_3}. \tag{8}$$

The Hamiltonian function defined as

$$H = P^T \dot{X} - L, \tag{9}$$

$$P = \frac{\partial L}{\dot{X}}. \tag{10}$$

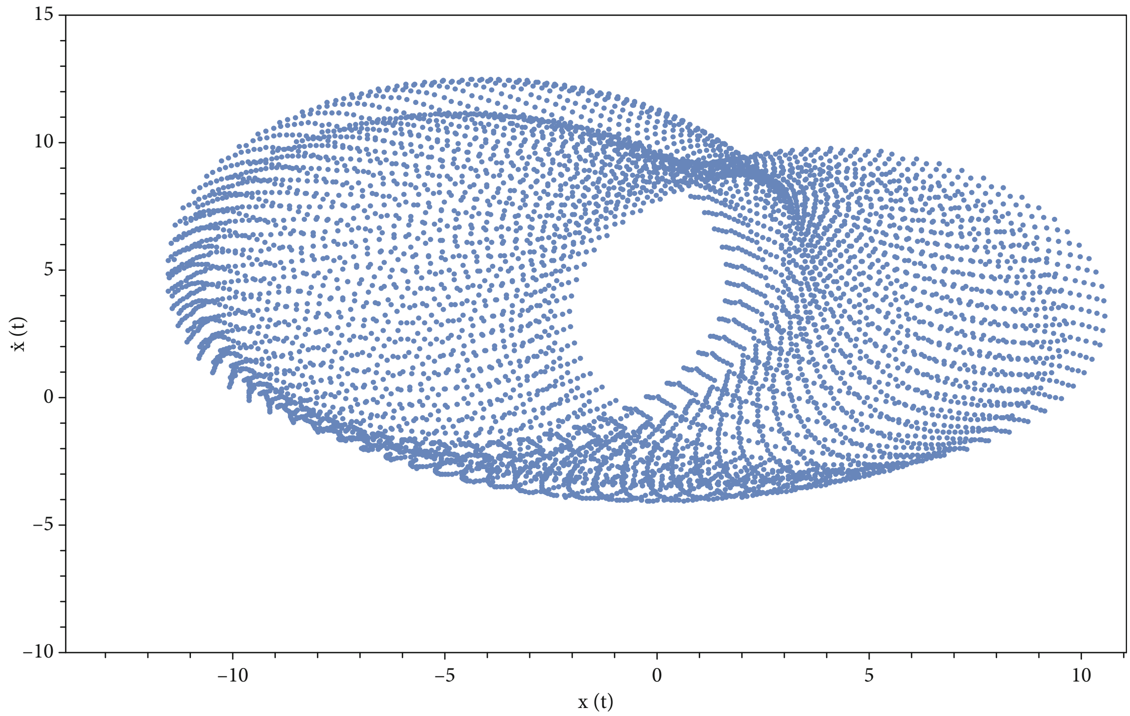


Figure 3: Poincare map about Lx2 with $f_1 = 170^{\circ}$.

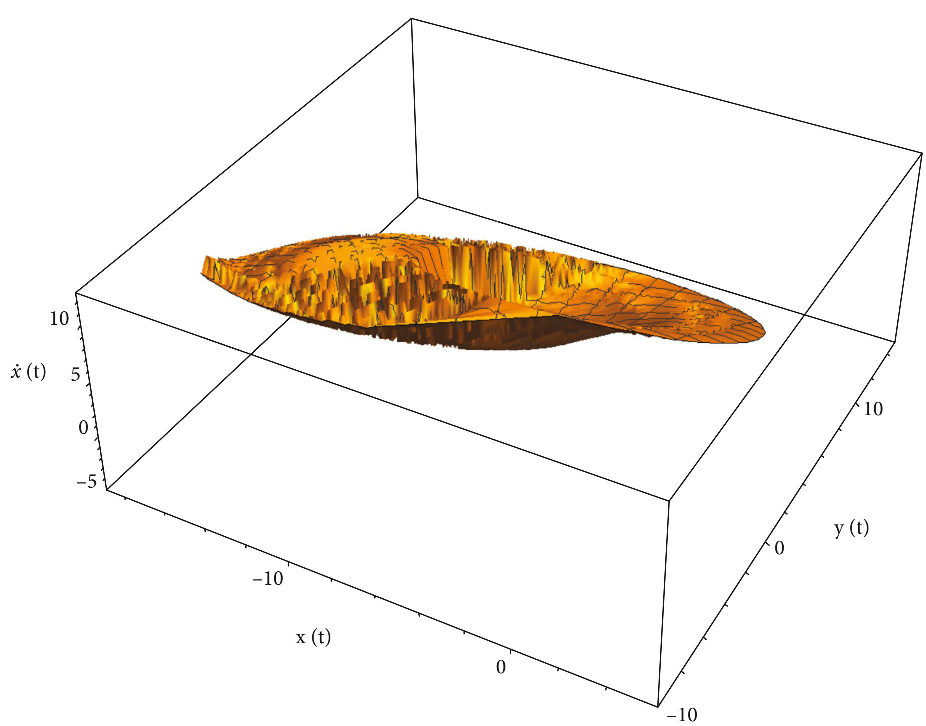


Figure 4: Poincare map about Lx2 with $f_1 = 10^{\circ}$ in three dimensions.

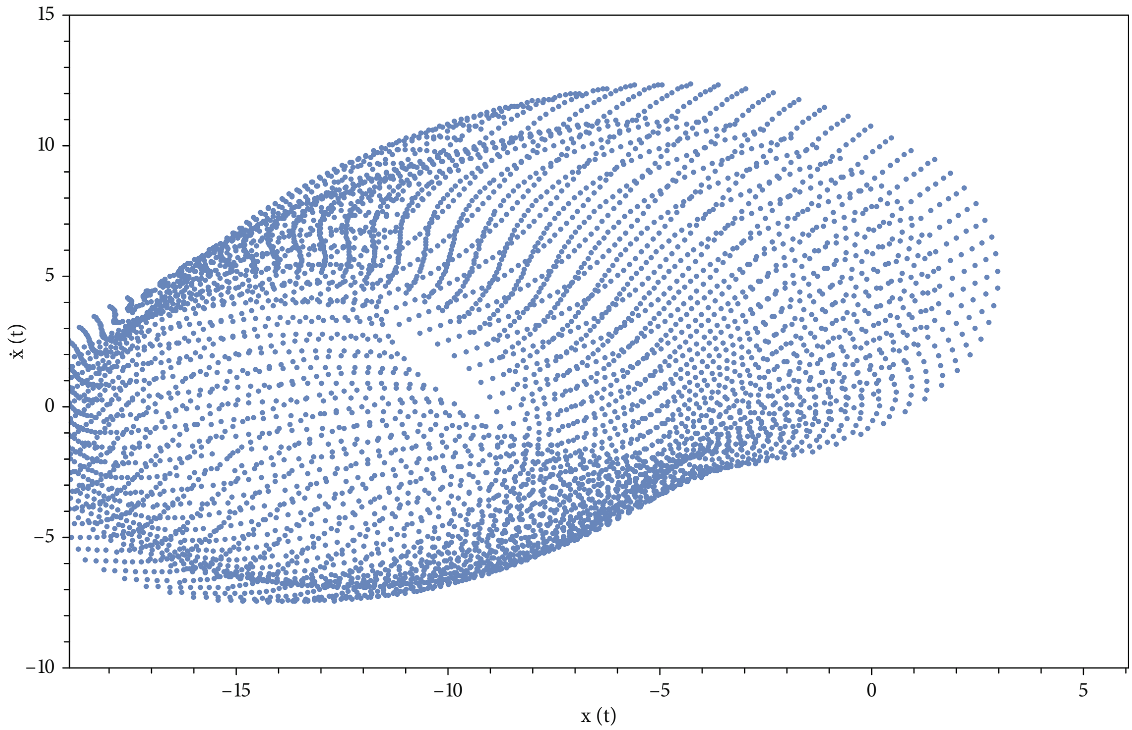


FIGURE 5: Poincare map about Lx2 with $F = 30$.

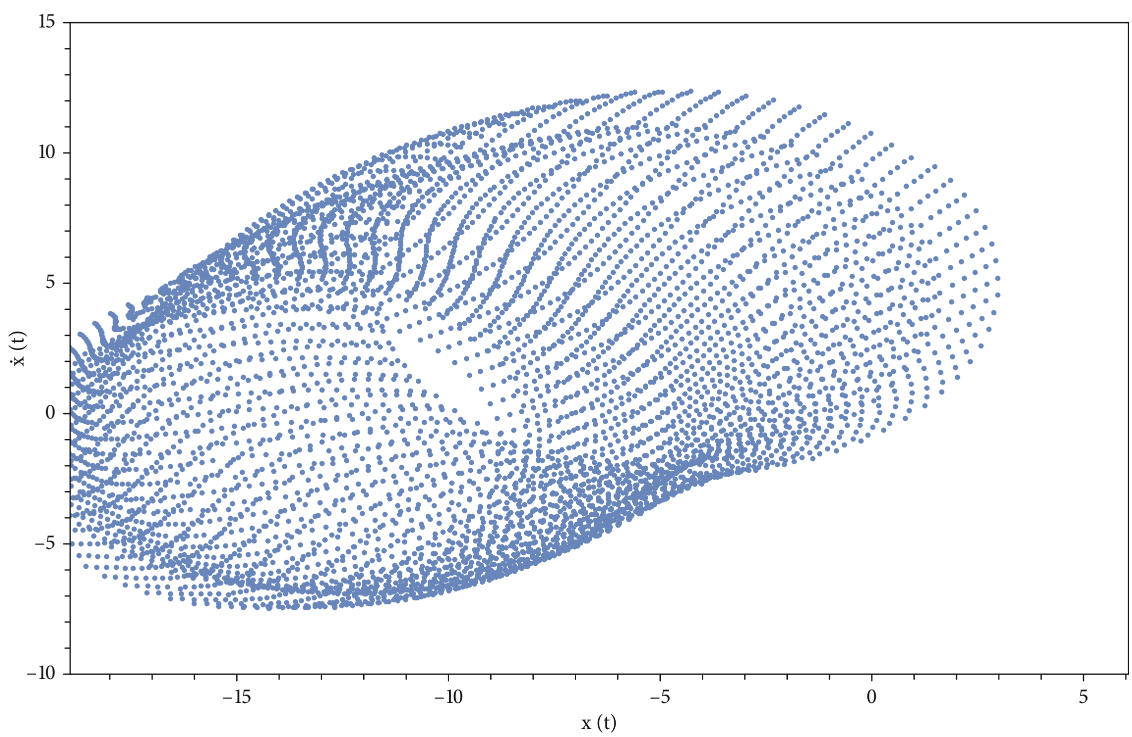


FIGURE 6: Poincare map about Lx2 with $F = 150$.

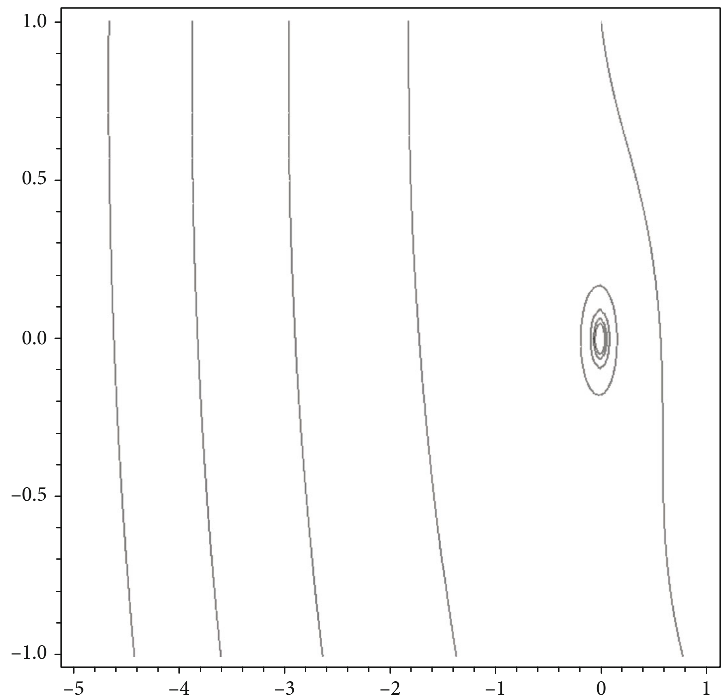


Figure 7: ZVC about Lx2 with $f_1 = 10^\circ$.

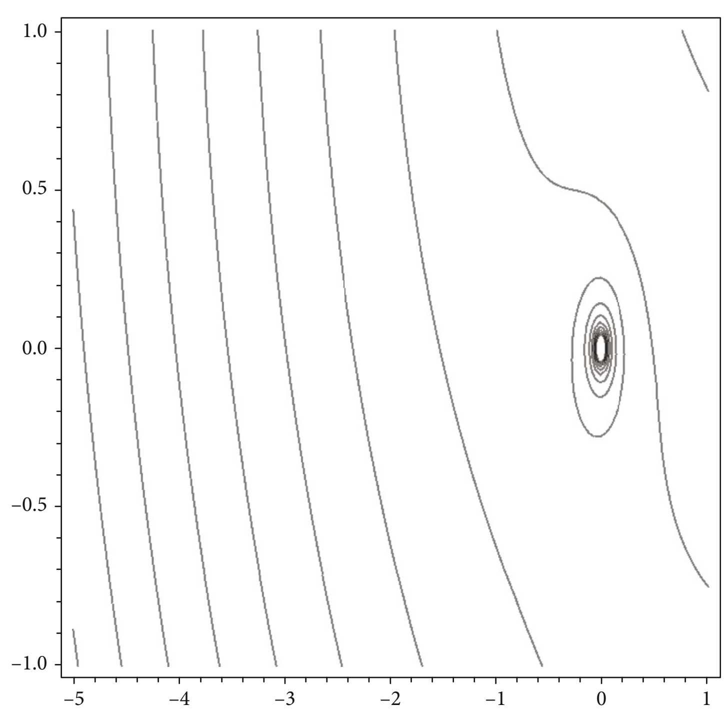


Figure 8: ZVC about Lx2 with $f_1 = 30^\circ$.

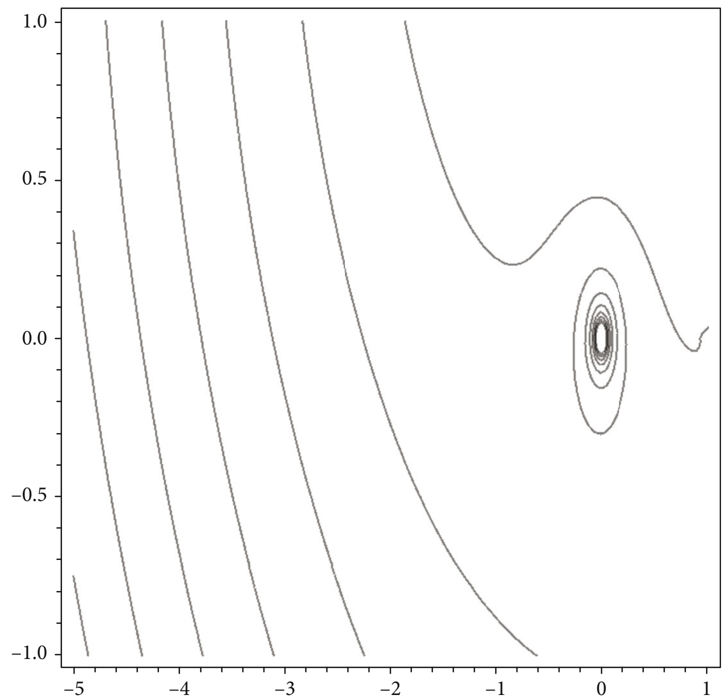


FIGURE 9: ZVC about Lx2 with $f_1 = 150^\circ$.

$(p_x, p_y, p_z)$ are the components of the momentum $P$ and $(\dot{x}, \dot{y}, \dot{z})$ are the components of the velocity $\dot{X}$;since the motion is in the plane $x-y$, then $\dot{z}=0$ and $p_z=0$; from Equations (8), (9), and (10), the Hamiltonian is given by

$$H = \frac{1}{2}\left(p_x{}^2 + p_y{}^2\right) - \left(p_x y - p_y x\right) - \frac{1-\mu}{r_1} + \frac{\mu}{r_2} + \frac{(1-\beta)\mu_3}{r_3} \tag{11}$$

## 4. The Equations of Motion and Libration Points

The equations of motions are obtained by considering the canonical form of the Hamiltonian which are

$$\dot{X} = \frac{\partial H}{\partial P}, \tag{12}$$

$$\dot{P} = -\frac{\partial H}{\partial X}. \tag{13}$$

Now, the libration points are obtained by applying the following conditions:

$$\dot{x} = 0, \tag{14}$$

$$\dot{y} = 0, \tag{15}$$

$$\dot{p_x} = 0, \tag{16}$$

$$\dot{p_y} = 0. \tag{17}$$

For collinear libration points, Equations (12), (13), and (14) and (15), (16), and (17) are solved together for $x$ and $y=0$, while for the nonlinear libration points, Equations (12), (13), and (14) and (15), (16), and (17) are solved together for $(x, y)$.

## 5. Results and Discussion

*5.1. Stability about Equilibrium Points.* Since the primaries are moving in elliptical orbits, then to study the motion about any of the libration points and its stability, it is more convenient to introduce the independent variable $f_2$ instead of the independent variable $t$, so that let

$$\begin{aligned}
\epsilon &= \frac{1-e_2{}^2}{1+e_2\cos f_2}, \\
x &= \epsilon\bar{x}, \\
y &= \epsilon\bar{y}, \\
z &= \epsilon\bar{z}, \\
p_x &= \epsilon\overline{p_x}, \\
p_y &= \epsilon\overline{p_y}, \\
p_z &= \epsilon\overline{p_z}, \\
H &= \epsilon\bar{H}.
\end{aligned} \tag{18}$$

The new Hamiltonian is given by

$$\bar{H} = \epsilon\left[\frac{1}{2}\left(\overline{p_x}^2 + \overline{p_y}^2\right) - \left(\overline{p_x}\bar{y} - \overline{p_y}\bar{x}\right)\right] - \frac{1}{\epsilon^2}\left[\frac{1-\mu}{\bar{r}_1} + \frac{\mu}{\bar{r}_2} + \frac{(1-\beta)\mu_3}{\bar{r}_3}\right]. \tag{19}$$

Then, the equations of motion in the new coordinates are given by

$$\begin{aligned}
\acute{x} &= \sigma(\overline{p_x} - \bar{y}) - \sigma_1 \bar{x}, \acute{y} = \sigma\left(\overline{p_y} + \bar{x}\right) - \sigma_1 \bar{y}, \acute{\overline{p_x}} = -\sigma\left(\overline{p_y} + \bar{x}\right) - \sigma 1 \overline{p_x} - \frac{\sigma}{\epsilon^3}\left[\frac{\mu(-1+\bar{x}+\mu)}{\left(\bar{y}^2 + \bar{z}^2 + (-1+\bar{x}+\mu)^2\right)^{3/2}}\right. \\
&\quad \left. + \frac{(1-\mu)(\bar{x}+\mu)}{\left(\bar{y}^2 + \bar{z}^2 + (\bar{x}+\mu)^2\right)^{3/2}} + \frac{(-Ds\cos(f_1 - f_2) + x)\mu_3(1-\beta)}{\left((\bar{x} - Ds\cos(f_1 - f_2))^2 + (\bar{y} - Ds\sin(f_1 - f_2))^2 + \bar{z}^2)\right)^{3/2}}\right], \acute{\overline{p_y}} \\
&= -\sigma(\overline{p_x} - \bar{y}) - \sigma 1 \overline{p_y} - \frac{\sigma}{\epsilon^3}\left[\frac{\bar{y}\mu}{\left(\bar{y}^2 + \bar{z}^2 + (-1+\bar{x}+\mu)^2\right)^{3/2}} + \frac{\bar{y}(1-\mu)}{\left(\bar{y}^2 + \bar{z}^2 + (\bar{x}+\mu)^2\right)^{3/2}}\right. \\
&\quad \left. + \frac{(\bar{y} - Ds\sin(f_1 - f_2))\mu_3(1-\beta)}{\left((\bar{x} - Ds\cos(f_1 - f_2))^2 + (\bar{y} - Ds\sin(f_1 - f_2))^2 + \bar{z}^2)\right)^{3/2}}\right],
\end{aligned} \tag{20}$$

where

$$\begin{aligned}
\dot{f} &= \frac{(1 + e_2\cos f_2)^2}{(1 - e_2{}^2)^{3/2}}, \\
\acute{\epsilon} &= \frac{d\epsilon}{df} = \frac{e_2 \sin f_2 \quad (1 - e_2{}^2)}{(1 + e_2\cos f_2)^2}, \sigma = \frac{1}{\dot{f}}, \sigma_1 = \frac{\acute{\epsilon}}{\epsilon}.
\end{aligned} \tag{21}$$

It is well known that the characteristic equation for the linearized system is given by

$$\lambda^4 + \left(4 - V_{xx} - V_{yy}\right)\lambda^2 + V_{xx}V_{yy} = 0, \tag{22}$$

where $V_{xx}$ is the second derivative of $V$ with respect to $x$ and $V_{yy}$ is the second derivative of $V$ with respect to $y$. The four roots of Equation (22) illustrate the stability of motion which depends on the kind of the root if it is real or imaginary or complex numbers. This was treated in more details in many pervious works. Now, in this work, the Poincare map is used to illustrate the stability.

*5.2. Poincare Map.* To apply the obtained equations of motion, some assumptions must be considered: (i) the relation between $f_1$ and $f_2$ and (ii) the method used to solve the equations of motion. In this study, the numerical method is used using the explicit Runge-Kutta method. A cod was constructed to do this, and the Poincare map is shown to illustrate the region of the stability of motion about any libration. So that if the Sun-Earth-Moon system is chosen, then it is known that for Sun $Ds = \left(R_s(1 - e_1{}^2)\right)/(1 + e_1 \cos f_1)$, $R_s = 328900.48$, $e = 0.016$, the true anomaly of orbit Earth-Moon about Sun $F$ takes the values 0 to $2\pi$, $\mu_s = 328900.48$, $\mu = 0.012$, and $e_1 = 0.0549$, the true anomaly of the orbit of Earth-Moon $f_1$ takes the values 0 to $2\pi$. Notice that when the Earth-Moon complete one revolution about the Sun, the Earth-Moon system completes 12 revolutions about their center of mass. That means the true anomaly $f_2$ is considered as a faster variable, so that the change in the Sun's true anomaly $f_1$ is so slow in comparison with the change in the Earth-Moon true anomaly $f_2$. These notes are taken into consideration when the cod is truncated. $\beta$ is the radiation pressure coefficient of the Sun and is obtained from $\beta = F_{rad}/F_{gr}$, where $F_{gr}$ is the gravitational force of the Sun and $F_{rad}$ the radiation force of the Sun. Now, let $f_1 = 10^\circ$, $f_2$ truncated from 0 to $100\pi$, and $\beta = 10^{-6}$; these values are used in the cod and are applied on the canonical system (16) and (17), so Figure 2 illustrates the Poincare maps about the libration point $Lx_2 = 1.06$. In Figure 3, the Poincare map is obtained with $f_1 = 170^\circ$, to compare the results with those of $f_1 = 10^\circ$ obtained in Figure 2, while Figure 4 illustrates the three dimensions of the Poincare map which appears as elliptical trajectories centered at the libration point Lx, and many periodic trajectories about it are included. In Figure 5, the Poincare map with $F = 30^\circ$ is compared with that in Figure 6 $f_1 = 150^\circ$. It is noticed that the change in the value of the true anomaly $f_1$ does not affect the stability of motion about the libration point just changing the phase of map by$2\pi$. The same was done for the zero velocity curves which are shown in Figures 7–9 with $f_1 = 30^\circ$, $f_1 = 30^\circ$, and $f_1 = 150^\circ$, respectively. It is noted that the regions about Lx are still unchanged while the energy levels out of these regions are changed.

## 6. Conclusion

In this work, the Hamiltonian of the elliptic four-body problem is studied. The new coordinates which depend on the true anomaly as independent variable are used. The effects of the true anomaly on the stability of motion have a great rule through this study. The sizes of gaps between regions of motion about the libration point are affected by the value of the true anomaly, which means that the position of the primaries in their orbits must be taken into consideration when treating these problems. Also, the energy levels are affected with the values of the true anomaly.

## Conflicts of Interest

The authors declare that they have no conflicts of interest.

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