Hindawi
Journal of Applied Mathematics
Volume 2021, Article ID 8244000, 10 pages
https://doi.org/10.1155/2021/8244000

Research Article

Approximations of Tangent Polynomials, Tangent –Bernoulli and Tangent – Genocchi Polynomials in terms of Hyperbolic Functions

Cristina B. Corcino,[1,2] **Roberto B. Corcino**,[1,2] **and Jay M. Ontolan**[1,2]

[1]*Research Institute for Computational Mathematics and Physics, Cebu Normal University, Cebu City 600, Philippines*
[2]*Mathematics Department, Cebu Normal University, Cebu City 600, Philippines*

Correspondence should be addressed to Roberto B. Corcino; rcorcino@yahoo.com

Received 30 September 2021; Revised 13 November 2021; Accepted 15 November 2021; Published 29 November 2021

Academic Editor: Saeid Abbasbandy

Asymptotic approximations of Tangent polynomials, Tangent-Bernoulli, and Tangent-Genocchi polynomials are derived using saddle point method and the approximations are expressed in terms of hyperbolic functions. For each polynomial there are two approximations derived with one having enlarged region of validity.

1. Introduction

Several well-known special functions, numbers and polynomials (e.g. zeta functions, Bernoulli, Euler and Genocchi numbers and polynomials and derivative polynomials [1–6]) have been studied by many researchers in recent decade due to their wide-ranging applications from number theory and combinatorics to other fields of applied mathematics. With these, different variations and generalizations of these functions, numbers and polynomials have been constructed and investigated. For instance, papers in [7, 8] have introduced and investigated q-analogues of zeta functions and Euler polynomials. Some variations are constructed by mixing the concept of two special functions, numbers or polynomials. For example, poly-Bernoulli numbers and polynomials in [9–11] are constructed by mixing the concepts of polylogarithm and Bernoulli numbers and polynomials. Moreover, the Apostol-Genocchi polynomials, Frobenius-Euler polynomials, Frobenius-Genocchi polynomials and Apostol-Frobenius-type poly-Genocchi polynomials in [12–14] are constructed by mixing the concepts of Apostol, Frobenius, Genocchi and Euler polynomials.

Another interesting mixture of special polynomials can be constructed by joining the concept of Tangent polynomials with Bernoulli and Genocchi polynomials. The Tangent polynomials $T_n(z)$, Tangent – Bernoulli $(TB)_n(z)$ and Tangent – Genocchi $(TG)_n(z)$ polynomials are defined by the generating functions

$$\frac{2e^{wz}}{e^{2w}+1} = \sum_{n=0}^{\infty} T_n(z)\frac{w^n}{n!}, |w| < \frac{\pi}{2} \tag{1}$$

$$\frac{\mathrm{w}e^{wz}}{e^{2w}-1} = \sum_{n=0}^{\infty} (TB)_n(z)\frac{w^n}{n!}, |w| < \pi \tag{2}$$

$$\frac{2\mathrm{w}e^{wz}}{e^{2w}+1} = \sum_{n=0}^{\infty} (TG)_n(z)\frac{w^n}{n!}, |w| < \frac{\pi}{2}. \tag{3}$$

When $z = 0$, (1.1) reduces to the generating function of the tangent numbers T_n given by

$$\frac{2}{e^{2w}+1} = \sum_{n=0}^{\infty} T_n\frac{w^n}{n!}, |w| < \frac{\pi}{2}. \tag{4}$$

In [15], the tangent polynomials can be determined explicitly using the tangent numbers T_n which is given by

$$T_n(z) = \sum_{k=0}^{n} T_k \binom{n}{k} z^{n-k}. \tag{5}$$

We are interested in finding asymptotic approximations of the Tangent polynomials

$T_n(z)$, Tangent–Bernoulli $(TB)_n(z)$ and Tangent–Genocchi $(TG)_n(z)$ polynomials for large n which are uniformly valid in some unbounded region of the complex variable z. Equation Equation (1) yields a recurrence relation

$$T_n(z) = z^n - \sum_{k=1}^{n} 2^{k-1} \binom{n}{k} T_{n-k}(z), \tag{6}$$

with initial value $T_0(z) = 1$. The Tangent polynomials $T_n(z)$ can be determined explicitly using (6). The first few values are

$$\begin{aligned} &T_0(z) = 1, T_1(z) = z - 1, T_2(z) = z^2 - 2z, T_3(z) = z^3 - 3z^2 + 2, \\ &T_4(z) = z^4 - 4z^3 + 8z, T_5(z) = z^5 - 5z^4 + 20z^2 - 16. \end{aligned} \tag{7}$$

The Tangent-Bernoulli polynomials and Tangent-Genocchi polynomials satisfy the relations

$$(TB)_n(z) = 2^{n-1} B_n\left(\frac{z}{2}\right) \tag{8}$$

$$(TG)_n(z) = 2^{n-1} G_n\left(\frac{z}{2}\right), \tag{9}$$

where Bernoulli and Genocchi polynomials were defined in [7, 16], respectively. Some specific values are given below.

$$\begin{aligned} &(TB)_0(z) = \frac{1}{2}, (TB)_1(z) = -\frac{1}{2} + \frac{z}{2}, (TB)_2(z) = \frac{1}{6}\left(2 - 6z + 3z^2\right), \\ &(TB)_3(z) = \frac{1}{2} z\left(2 - 3z + z^2\right), (TB)_4(z) = -\frac{4}{15} + 2z^2 - 2z^3 + \frac{z^4}{2}, \\ &(TB)_5(z) = \frac{1}{6} z\left(-8 + 20z^2 - 15z^3 + 3z^4\right), \\ &(TG)_0(z) = 0, (TG)_1(z) = 1, (TG)_2(z) = -2 + 2z, (TG)_3(z) = -6z + 3z^2 \\ &(TG)_4(z) = 8 - 12z^2 + 4z^3, (TG)_5(z) = 40z - 20z^3 + 5z^4. \end{aligned} \tag{10}$$

Applications of Bernoulli polynomials can be found in [16] while new formulas for Genocchi polynomials involving Bernoulli polynomials can be found in [17]. The Bernoulli polynomials and Genocchi polynomials were expressed in terms of hyperbolic function in [7, 16] as follows

$$\begin{aligned} B_n\left(z + \frac{1}{2}\right) &= \frac{n!}{2\pi i} \int_C \frac{w e^{wz}}{2 \sinh (w/2)} \frac{dw}{w^{n+1}} \\ G_n\left(z + \frac{1}{2}\right) &= \frac{n!}{2\pi i} \int_C \frac{w e^{wz}}{\cosh (w/2)} \frac{dw}{w^{n+1}}. \end{aligned} \tag{11}$$

where C is a circle about 0 with radius $<2\pi$ (resp. $<\pi$). These integral representations were used to establish the asymptotic approximations of Bernoulli and Genocchi numbers.

In this paper, the Tangent polynomials, Tangent-Bernoulli, and Tangent-Genocchi polynomials will be given asymptotic approximations using the method used in [19, 20].

2. Uniform Approximations

First, let us consider the approximation of Tangent polynomials. Using the saddle point method introduced we can establish the following theorem.

Theorem 1. *For $z \in \mathbb{C} \setminus \{0\}$ such that $|I_m z^{-1}| < \pi/2$ and $|z \pm (2i/\pi)| > 2/\pi$ and $n \geq 1$,*

$$T_n(nz + 1) = \frac{(nz)^n}{\cosh (1/z)} \cdot \left\{1 + \frac{1}{2nz^2}\left(\operatorname{sech}^2\left(\frac{1}{z}\right) - \tanh^2\left(\frac{1}{z}\right)\right) + O\left(\frac{1}{n^2}\right)\right\}. \tag{12}$$

Proof. Applying the Cauchy-Integral Formula to (1)

$$\frac{T_n(z)}{n!} = \frac{1}{2\pi i} \int_C \frac{2e^{wz}}{e^{2w} + 1} \frac{dw}{w^{n+1}}, \tag{13}$$

where C is a circle about 0 with radius $<\pi/2$. It follows from (13) that

$$T_n(z + 1) = \frac{n!}{2\pi i} \int_C \frac{2e^{(z+1)w}}{e^{2w} + 1} \frac{dw}{w^{n+1}}. \tag{14}$$

With $2e^w \cosh w = e^{2w} + 1$, (14) can be written as

$$T_n(z + 1) = \frac{n!}{2\pi i} \int_C \frac{e^{wz}}{\cosh w} \frac{dw}{w^{n+1}}. \tag{15}$$

Let $f(w) = 1/\cosh w$. The singularities of $f(w)$ are the zeros of $\cosh w$, which are $w_j = (2j + 1)(\pi/2)i$, $j \in \mathbb{Z}$. Each of these singularities is a simple pole of $f(w)$ while 0 is a pole of order $n + 1$ of the integrand of (15).

Take $z \longmapsto nz$ and let $nz \longmapsto \infty$ by letting $n \longrightarrow \infty$ with z fixed. Then

$$T_n(nz + 1) = \frac{n!}{2\pi i} \int_C f(w) e^{wnz} \frac{dw}{w^{n+1}} = \frac{n!}{2\pi i} \int_C f(w) e^{n(wz - \log w)} \frac{dw}{w}. \tag{16}$$

The main contribution of the integrand to the integral above originates at the saddle point of the argument of the exponential (see [18]). This saddle point is at w such that

$$\frac{d}{dw}(n(wz - \log w)) = 0 \Longleftrightarrow w = \frac{1}{z} = z^{-1}. \tag{17}$$

Assume that z^{-1} is not a pole of $f(w)$. Approximations of $T_n(nz + 1)$ can be obtained by expanding $f(w)$ around the saddle point [19]. With z^{-1} not a pole of $f(w)$, we can expand $f(w)$ around z^{-1} . That is,

$$f(w)=\sum_{k=0}^{\infty}\frac{f^{(k)}\left(z^{-1}\right)}{k!}\left(w-z^{-1}\right)^k,\ \left|w-z^{-1}\right|<r \tag{18}$$

where r is the distance from z^{-1} to the nearest singularity of $f(w)$. For $w\in\mathbb{C}$, the above series is absolutely convergent if the saddle point z^{-1} is closer to the origin than to any of the singularities w_j. That is, if z^{-1} is in the strip $|\mathrm{Im}\ z^{-1}|<\pi/2$ and $|z^{-1}|<|z^{-1}-w_j|$ for all $j=0,\pm1,\pm2,\cdots$. It follows from Lemma 1, Lemma 2 and Theorem 1 of [19] that

$$T_n(nz+1)=(nz)^n\sum_{k=0}^{\infty}\frac{f^{(k)}\left(z^{-1}\right)}{k!}\frac{p_k(n)}{(nz)^k}, \tag{19}$$

where

$$p_0(n)=1, p_1(n)=0, p_2(n)=-n, p_3(n)=2n, \tag{20}$$

$$\mathrm{p_k(n)=(1-k)\,p_{k-1}\,(n)+n\,p_{k-2}(n).} \tag{21}$$

Writing the first few terms of (19), we have

$$\begin{aligned}T_n(nz+1)&=(nz)^n\left\{f^0\left(z^{-1}\right)p_0\left(n\right)+f^1\left(z^{-1}\right)\frac{p_1(n)}{nz}+\frac{f^{(2)}\left(z^{-1}\right)}{2!}\frac{p_2(n)}{(nz)^2}+O\left(\frac{1}{n^2}\right)\right\}\\&=(nz)^n\left\{\frac{1}{\cosh\left(1/z\right)}\frac{-n}{2(nz)^2}\left(-\mathrm{sech}^3\left(\frac{1}{z}\right)+\mathrm{sech}\left(\frac{1}{z}\right)\tanh^2\left(\frac{1}{z}\right)\right)+O\left(\frac{1}{n^2}\right)\right\}\\&=\frac{(nz)^n}{\cosh\left(1/z\right)}\left\{1+\frac{1}{2nz^2}\left(\frac{1}{\cosh^2(1/z)}-\tanh^2\left(\frac{1}{z}\right)+O\left(\frac{1}{n^2}\right)\right)\right\}\\&=\frac{(nz)^n}{\cosh\left(1/z\right)}\left\{1+\frac{1}{2nz^2}\left(\mathrm{sech}^2\left(\frac{1}{z}\right)-\tanh^2\left(\frac{1}{z}\right)\right)+O\left(\frac{1}{n^2}\right)\right\}.\end{aligned} \tag{22}$$

□

Figure 1 below depicts the graphs of $T_n(nz+1)$ (in solid lines) generated using relation (6) and the graphs of the approximate values of $T_n(nz+1)$ (in dashed lines) generated using the expansion at the right-hand side of (12). The graphs below are generating using the software Mathematica.

The graphs show the accuracy of approximation (12) for several values of n for real values of the uniform parameter z. For a real argument, the oscillatory region of $T_n(nz+1)$ is also contained in $|x|\leq 2\pi^{-1}$, whereas the monotonic region contains $|x|>2\pi^{-1}$. Therefore, the accuracy of approximation (12) is restricted to the monotonic region.

The next theorem contains the approximation formula for Tangent-Bernoulli polynomials.

Theorem 2. *For $z\in\mathbb{C}\setminus\{0\}$ such that $|I_m z^{-1}|<\pi$ and $|z\pm(1/\pi)|>1/\pi$ and $n\geq1$,*

$$\begin{aligned}(TB)_n(nz+1)=&\frac{(nz)^{n-1}\ \mathrm{csch}\ (1/z)}{8z^2}\\&\cdot\left\{4nz^2-1+4z\coth\left(\frac{1}{z}\right)-\coth^2\left(\frac{1}{z}\right)\right.\\&\left.-3\,\mathrm{csch}^2\left(\frac{1}{z}\right)+O\left(\frac{1}{n^2}\right)\right.\end{aligned} \tag{23}$$

Proof. Applying the Cauchy-Integral Formula to (2)

$$\frac{(TB)_n(z)}{n!}=\frac{1}{2\pi i}\int_C\frac{we^{wz}}{e^{2w}-1}\frac{dw}{w^{n+1}}, \tag{24}$$

where C is a circle about 0 with radius $<\pi$. It follows from (24) that

$$(TB)_n(z+1)=\frac{n!}{2\pi i}\int_C\frac{we^{(z+1)w}}{e^{2w}-1}\frac{dw}{w^{n+1}}. \tag{25}$$

With $2e^w\sinh w=e^{2w}-1$, (25) can be written as

$$(TB)_n(z+1)=\frac{n!}{2\pi i}\int_C\frac{we^{wz}}{2\sinh w}\frac{dw}{w^{n+1}}. \tag{26}$$

Let $f(w)=w/2\sinh w$. The singularities of $f(w)$ are the zeros of $\sinh w$, which are.

$w_j=j\pi$, $j\in\mathbb{Z}$. Each of these singularities is a simple pole of $f(w)$ while 0 is a pole of order $n+1$ of the integrand of (26).

Take $z\longmapsto nz$ and let $nz\longmapsto\infty$ by letting $n\longrightarrow\infty$ with z fixed. Then in view of Theorem 1, we can write

$$(TB)_n(nz+1)=\frac{n!}{2\pi i}\int_C f(w)e^{n(wz-\log w)}\frac{dw}{w}. \tag{27}$$

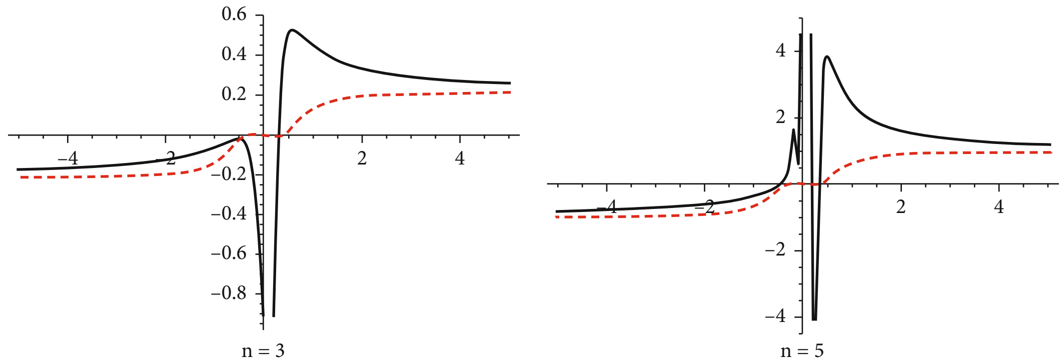

FIGURE 1: Solid lines represent $T_n(nz+1)$ for several values of n, whereas dashed lines represent the right-hand side of (12) with $z \equiv x$, both normalized by the factor $(1+|x/\alpha|^n)^{-1}$ where $\alpha = 0.2$.

With z^{-1} not a pole of $f(w)$, we can expand $f(w)$ around z^{-1}. That is, if z^{-1} is in the strip $|\text{Im } z^{-1}| < \pi$ and $|z^{-1}| < |z^{-1} - w_j|$ for all $j = 0, \pm 1, \pm 2, \cdots$. It follows from Lemma 1, Lemma 2 and Theorem 1 of [19] that

$$(TB)_n(nz+1) = (nz)^n \sum_{k=0}^{\infty} \frac{f^{(k)}(z^{-1})}{k!} \frac{p_k(n)}{(nz)^k}, \qquad (28)$$

where $p_k(n)$ are given in (20). Writing the first few terms of (28), we have

$$\begin{aligned} (TB)_n(nz+1) &= (nz)^n \left\{ f^0(z^{-1})\, p_0(n) + f^1(z^{-1}) \frac{p_1(n)}{nz} \right. \\ &\quad \left. + \frac{f^{(2)}(z^{-1})}{2!} \frac{p_2(n)}{(nz)^2} + O\left(\frac{1}{n^2}\right) \right\} \\ &= \frac{(nz)^{n-1} \operatorname{csch}(1/z)}{8z^2} \left\{ 4nz^2 - 1 + 4z \coth\left(\frac{1}{z}\right) \right. \\ &\quad \left. - \coth^2\left(\frac{1}{z}\right) - 3\operatorname{csch}^2\left(\frac{1}{z}\right) + O\left(\frac{1}{n^2}\right) \right\}. \end{aligned} \qquad (29)$$

□

The following figure depicts the graphs of $(TB)_n(nz+1)$ (in solid lines) generated using relation (8) and the graphs of the approximate values of $(TB)_n(nz+1)$ (in dashed lines) generated using the expansion at the right-hand side of (23). The graphs are generated using Mathematica.

The graphs show the accuracy of approximation (23) for several values of n for real values of the uniformity parameter z. For a real argument, the oscillatory region of $(TB)_n(nz+1)$ is also contained in $|x| \le \pi^{-1}$, whereas the monotonic region contains $|x| > \pi^{-1}$. Therefore, the accuracy of approximation (23) is restricted to the monotonic region.

The following theorem contains the approximation formula for Tangent-Genocchi polynomials This theorem is proved similarly as the first two theorems so the proof is omitted.

Theorem 3. *For $z \in \mathbb{C} \setminus \{0\}$ such that $|I_m z^{-1}| < \pi/2$ and $|z \pm (2i/\pi)| > 2/\pi$ and $n \ge 1$,*

$$\begin{aligned} (TG)_n(nz+1) = (nz)^n \left\{ \frac{\operatorname{sech}(1/z)}{z} \right. & \\ + \frac{\operatorname{sech}^3(1/z)(3 - \cosh(2/z) + 2z \sinh(2/z))}{4nz^3} & \\ \left. + O\left(\frac{1}{n^2}\right) \right\}. & \end{aligned} \qquad (30)$$

Figure 2 below depicts the graphs of $(TG)_n(nz+1)$ (in solid lines) generated using relation (9) and the graphs of the approximate values of $(TG)_n(nz+1)$ (in dashed lines) generated using the expansion at the right-hand side of (30).

3. Expansion of Tangent, Tangent-Bernoulli, and Tangent-Genocchi Polynomials with Enlarged Region of Validity

The validity of the approximations obtained in Theorems 1, 2, and 3, are restricted to the region $|z^{-1}| < |z^{-1} - w_j|$ for all $j = 1, 2, \cdots$. However, the region $|z^{-1}| < |z^{-1} - w_j|$ may be enlarged by isolating the contribution of the poles w_j's of $f(w)$. We will follow similar procedure done in [16, 19] to prove our next three results. A detailed discussion can be seen in Lemma 3.2 in [16] that allows us to write the polynomial $P_n(nz)$ defined by

$$P_n(nz) = \frac{n!}{2\pi i} \int_C f(w) e^{wz} \frac{dw}{w^{n+1}} \qquad (31)$$

with a meromorphic function $f(w)$ analytic in the origin

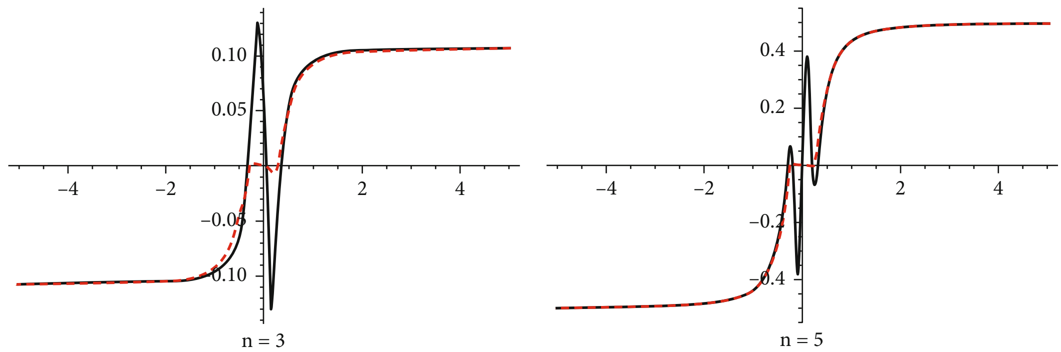

Figure 2: Solid lines represent $(TG)_n(nz+1)$ for several values of n, whereas dashed lines represent the right-hand side of (30) with $z \equiv x$, both normalized by the factor $(1+|x/\alpha|^n)^{-1}$ where $\alpha = 0.2$.

with simple poles $w_1, w_2, \cdots$ represented for each integer $m > 0$ as

$$P_n(nz) = -\sum_{k=1}^{m} \frac{r_k e^{w_k nz}}{w_k^{n+1}} \Gamma(n+1, w_k nz) + (nz)^n \sum_{k=0}^{\infty} \frac{f^{(k)}(z^{-1}) + h_m^{(k)}(z^{-1}) p_k(n)}{k!(nz)^k}, \tag{32}$$

which is valid for all complex number z satisfying $|z^{-1}| < |z^{-1} - w_j|$ for all $j = m+1, m+2, \cdots$, such that C is a circle whose center is at the origin and contains no poles of $f(w)$ inside, $\Gamma(n+1, w_k nz)$ is the incomplete gamma function, $p_k(n)$ are polynomials defined by the relation (20) and $h_m^{(k)}(z^{-1})$ is the kth derivative at z^{-1} of the function

$$h_m(w) = -\sum_{l=1}^{m} \frac{r_l}{w - w_l}. \tag{33}$$

The following theorem contains the asymptotic expansion of Tangent polynomials with enlarged region of validity.

Theorem 4. *For $z \in \mathbb{C} \setminus \{0\}$ such that $|z^{-1}| < |z^{-1} \pm (2k+1)(\pi/2)i|$ for $k = 0, 1, 2, \cdots, m-1$. Then, as $n \longrightarrow \infty$,*

$$\begin{aligned} T_n(nz+1) = {} & i \sum_{k=0}^{m-1} (-1)^k \left[\frac{e^{(2k+1/2)\pi i nz}}{((2k+1/2)\pi i)^{n+1}} \Gamma\left(n+1, \frac{2k+1}{2}\pi i nz\right) \right. \\ & \left. - \frac{e^{-(2k+1/2)\pi i nz}}{(-(2k+1/2)\pi i)^{n+1}} \Gamma\left(n+1, -\frac{2k+1}{2}\pi i nz\right) \right] \\ & + (nz)^n \left\{ \operatorname{sech}\left(\frac{1}{z}\right) + \sum_{k=0}^{m-1} \frac{4(-1)^{k+1}(2k+1)\pi z^2}{4+(2k+1)^2\pi^2 z^2} \right. \\ & + \frac{n}{2(nz)^2} \left[\operatorname{sech}^3\left(\frac{1}{z}\right) - \operatorname{sech}\left(\frac{1}{z}\right) \tanh^2\left(\frac{1}{z}\right) \right. \\ & \left. + \sum_{k=0}^{m-1} \frac{32(-1)^{2k}(\pi+2k\pi)^2 \left[z^2(\pi+2k\pi)^2 - 12\right] z^4}{\left[4+z^2(\pi+2k\pi)^2\right]^3} \right] \\ & \left. + O\left(\frac{1}{n^2}\right) \right\}. \end{aligned} \tag{34}$$

Proof. To obtain the corresponding residues, we first let

$$f(w) = \frac{1}{\cosh w} = \frac{p(w)}{q(w)}. \tag{35}$$

Then

$$r_k = \frac{p(w_k)}{q'(w_k)} = \frac{1}{\sinh(w_k)}, k = 0, 1, 2 \cdots., m-1. \tag{36}$$

Now, for $k = 0, 1, 2 \cdots., m-1$, $\sinh w_k = (-1)^k i$. Hence, $r_k = (-1)^{k+1} i$.

On the other hand, for $k = 0, 1, 2 \cdots., m-1$, $\sinh w_{-(k)} = (-1)^{k+1} i$. Thus, for $k = 0, 1, 2 \cdots., m-1$, $r_{-k} = (-1)^k i$. Hence, for $k = 0, 1, 2 \cdots., m-1$ we have

$$\begin{aligned} r_k &= (-1)^{k+1} i \text{ for } w_k = \frac{2k+1}{2}\pi i \\ r_{-k} &= (-1)^k i \text{ for } w_{-k} = -\frac{2k+1}{2}\pi i. \end{aligned} \tag{37}$$

We then evaluate some derivatives of the function $h_m(w)$ defined in (33) at the saddle point z^{-1}. Then for $k = 0, 1, 2 \cdots., m-1$, we write

$$\begin{aligned} h_m(w) &= -\sum_{k=0}^{m-1} \frac{r_k}{w - w_k} - \sum_{k=0}^{m-1} \frac{r_{-k}}{w - w_{-k}} \\ &= \sum_{k=0}^{m-1} \frac{(-1)^k i}{w - (2k+1/2)\pi i} - \sum_{k=0}^{m-1} \frac{(-1)^k i}{w + (2k+1/2)\pi i} \\ &= \sum_{k=0}^{m-1} \frac{(-1)^{k+1}(2k+1)\pi}{w^2 + \left((2k+1)^2/4\right)\pi^2}. \end{aligned} \tag{38}$$

Using (37) we can compute some derivatives of $h_m(w)$ and evaluate it at the saddle point z^{-1}. We obtain the results below.

$$h_m^{(1)}(w) = \sum_{k=0}^{m-1} \frac{32(-1)^{2k+1}(\pi+2k\pi)^2 w}{\left[4w^2 + (\pi+2k\pi)^2\right]^2}, \tag{39}$$

$$h_m^{(2)}(w) = \sum_{k=0}^{m-1} \frac{32(-1)^{2k+1}(\pi+2k\pi)^2\left[(\pi+2k\pi)^2-12w^2\right]}{\left[4w^2+(\pi+2k\pi)^2\right]^3}0 \tag{40}$$

Equations (37) and (39) yield

$$h_m^{(0)}\left(z^{-1}\right) = \sum_{k=0}^{m-1} \frac{4(-1)^{k+1}(2k+1)\pi z^2}{4+(2k+1)^2\pi^2 z^2} \tag{41}$$

$$h_m^{(2)}\left(z^{-1}\right) = \sum_{k=0}^{m-1} \frac{32(-1)^{2k+1}(\pi+2k\pi)^2\left[z^2(\pi+2k\pi)^2-12\right]z^4}{\left[4+z^2(\pi+2k\pi)^2\right]^3}. \tag{42}$$

Using (16) and (32) for $k = 0, 1, 2 \cdots ., m-1$, we have

$$\begin{aligned} T_n(nz+1) = &-\sum_{k=0}^{m-1} \frac{(-1)^{k+1} i\, e^{(2k+1/2)\pi i nz}}{((2k+1/2)\pi i)^{n+1}} \Gamma\left(n+1, \frac{2k+1}{2}\pi i nz\right) \\ &- \sum_{k=0}^{m-1} \frac{(-1)^{k} i\, e^{-(2k+1/2)\pi i nz}}{(-(2k+1/2)\pi i)^{n+1}} \Gamma\left(n+1, -\frac{2k+1}{2}\pi i nz\right) \\ &+ (nz)^n \left\{ \frac{f\left(z^{-1}\right)+h_m\left(z^{-1}\right)p_0(n)}{0!} \right. \\ &+ \frac{f^{(1)}\left(z^{-1}\right)+h_m^{(1)}\left(z^{-1}\right)p_1(n)}{1!(nz)} \\ &\left. + \frac{f^{(2)}\left(z^{-1}\right)+h_m^{(2)}\left(z^{-1}\right)p_2(n)}{2!(nz)^2} + O\left(\frac{1}{n^2}\right) \right\}. \end{aligned} \tag{43}$$

In Theorem 1, we have $p_0(n) = 1, p_1(n) = 0, p_2(n) = -n$, $f(z^{-1}) = 1/\cosh(1/z)$, $f^{(2)}(z^{-1}) = -\mathrm{sech}^3(1/z) + \mathrm{sech}(1/z)\tanh^2(1/z)$ and using (40) and (41), we obtain (34). □

Figure 3 below depicts the graphs of $T_n(nz+1)$ (in solid lines) generated using relation (6) and the graphs of the approximate values of $T_n(nz+1)$ (in dashed lines) generated using the expansion at the right-hand side of (34). We use Mathematica to graph the left-hand and right-hand side of (34).

The graphs show the accuracy of approximation (34) for several values of n for real values of the uniformity parameter z. For a real argument, the approximation of $T_n(nz+1)$ with enlarged validity in the monotonic region is better than the approximation in (12). It can also be observed that, in the oscillatory region, even if the accuracy is not that good the approximation is better compared to the approximation in (12).

Now, let us consider the asymptotic expansion of Tangent-Bernoulli polynomials with enlarged region of validity.

Theorem 5. *For $z \in \mathbb{C}\setminus\{0\}$ such that $|z^{-1}| < |z^{-1} \pm k\pi|$ for $k = 1, 2, \cdots, m$. Then, as $n \longrightarrow \infty$,*

$$\begin{aligned} (TB)_n(nz+1) = &-\pi \sum_{k=1}^{m} \frac{k}{1+e^{2k\pi}} \left[\frac{e^{k\pi(1+nz)}}{(k\pi)^{n+1}} \Gamma(n+1, k\pi nz) \right. \\ &\left. + \frac{e^{k\pi(1-nz)}}{(-k\pi)^{n+1}} \Gamma(n+1, -k\pi nz) \right] + (nz)^n \\ &\cdot \left\{ \frac{\mathrm{csch}\,(1/z)}{2z} + \sum_{k=1}^{m} \frac{2k^2\pi^2 z^2}{\left(e^{-k\pi}+e^{k\pi}\right)\left(z^2k^2\pi^2-1\right)} \right. \\ &- \frac{n}{2(nz)^2} \left[\frac{\mathrm{csch}^3(1/z)(3+\cosh(2/z)-2z\sinh(2/z))}{4z} \right. \\ &\left.\left. + \sum_{k=1}^{m} \frac{4e^{k\pi}k^2\pi^2 z^4\left(z^2k^2\pi^2+3\right)}{\left(1+e^{2k\pi}\right)\left(z^2k^2\pi^2-1\right)^3} \right] + O\left(\frac{1}{n^2}\right) \right\} \end{aligned} \tag{44}$$

Proof. To obtain the corresponding residues, we first let

$$f(w) = \frac{w}{2\sinh w} = \frac{p(w)}{q(w)}. \tag{45}$$

Then,

$$r_k = \frac{p(w_k)}{q'(w_k)} = \frac{w_k}{2\cosh(w_k)}, k = 1, 2 \cdots ., m. \tag{46}$$

Now, for $k = 1, 2 \cdots ., m-1$,

$$\cosh w_k = \cosh k\pi = \frac{e^{k\pi}+e^{-k\pi}}{2}\cdot \tag{47}$$

Hence,

$$\mathrm{r_k} = \frac{k\pi}{e^{k\pi}+\mathrm{e}^{-k\pi}}\cdot \tag{48}$$

On the other hand, for $k = 1, 2 \cdots ., m$,

$$\cosh w_{-(k)} = \cosh(-k\pi), k = 1, 2, \cdots m = \cosh k\pi = \frac{e^{k\pi}+e^{-k\pi}}{2}\cdot \tag{49}$$

Thus, for $k = 1, 2 \cdots ., m$,

$$\mathrm{r_{-k}} = \frac{-k\pi}{\mathrm{e}^{k\pi}+\mathrm{e}^{-k\pi}}\cdot \tag{50}$$

Hence, for $k = 1, 2 \cdots ., m$ we have

$$\begin{aligned} r_k &= \frac{k\pi}{e^{k\pi}+e^{-k\pi}} \text{ for } w_k = k\pi, \\ r_{-k} &= \frac{-k\pi}{e^{k\pi}+e^{-k\pi}} \text{ for } w_{-k} = -k\pi\cdot \end{aligned} \tag{51}$$

We then evaluate some derivatives of the function $h_m(w)$

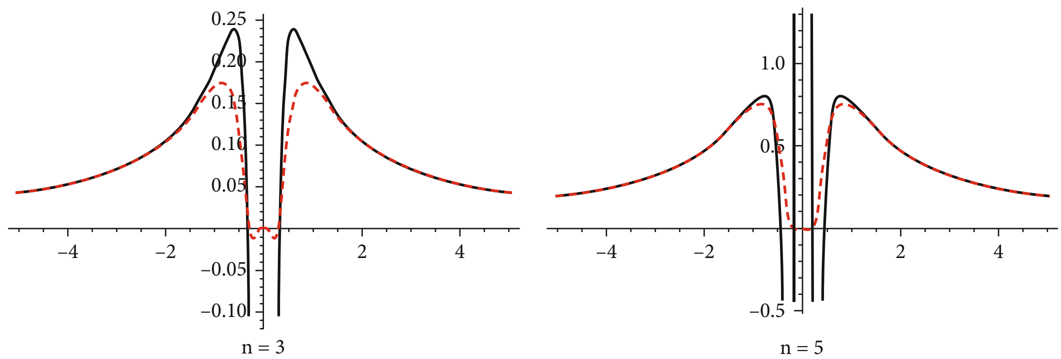

Figure 3: Solid lines represent $T_n(nz+1)$ for several values of n, whereas dashed lines represent the right-hand side of (34) with $z \equiv x$, both normalized by the factor $(1+|x/\alpha|^n)^{-1}$ where $\alpha = 0.2$.

defined in (33) at the saddle point z^{-1}. Then for $k = 1, 2 \cdots., m$, we write

$$\begin{aligned} h_m(w) &= -\sum_{k=1}^{m} \frac{r_k}{w-w_k} - \sum_{k=1}^{m} \frac{r_{-k}}{w-w_{-k}} \\ &= -\sum_{k=1}^{m} \frac{k\pi/e^{k\pi} + e^{-k\pi}}{w-k\pi} - \sum_{k=1}^{m} \frac{-k\pi/e^{k\pi} + e^{-k\pi}}{w+k\pi} \\ &= \sum_{k=1}^{m} \frac{k\pi}{e^{k\pi}+e^{-k\pi}} \left(\frac{1}{w+k\pi} - \frac{1}{w-k\pi} \right) \\ &= \sum_{k=1}^{m} \frac{2(k\pi)^2}{\left(e^{k\pi}+e^{-k\pi}\right)\left(k^2\pi^2 - w^2\right)}. \end{aligned} \tag{52}$$

Using (51) we can compute some derivatives of $h_m(w)$ and evaluate it at the saddle point z^{-1}. We obtain the results below.

$$h_m^{(1)}(w) = \sum_{k=1}^{m} \frac{4k^2\pi^2 w}{\left(e^{-k\pi}+e^{k\pi}\right)\left(k^2\pi^2 - w^2\right)^2}, \tag{53}$$

$$h_m^{(2)}(w) = \sum_{k=1}^{m} \frac{4e^{k\pi} k^2\pi^2 \left(k^2\pi^2 + 3w^2\right)}{\left(1+e^{2k\pi}\right)\left(k^2\pi^2 - w^2\right)^3}. \tag{54}$$

Equations (51) and (53) yield

$$h_m^{(0)}\left(z^{-1}\right) = \sum_{k=1}^{m} \frac{2k^2\pi^2 z^2}{\left(e^{-k\pi}+e^{k\pi}\right)\left(z^2k^2\pi^2 - 1\right)} \tag{55}$$

$$h_m^{(2)}\left(z^{-1}\right) = \sum_{k=1}^{m} \frac{4e^{k\pi} k^2\pi^2 z^4 \left(z^2k^2\pi^2 + 3\right)}{\left(1+e^{2k\pi}\right)\left(z^2k^2\pi^2 - 1\right)^3}. \tag{56}$$

Using equations (26) and (32) for $k = 1, 2 \cdots., m$, we have

$$\begin{aligned} (TB)_n(nz+1) = &-\sum_{k=1}^{m} \frac{\left(k\pi/e^{k\pi}+e^{-k\pi}\right) e^{k\pi nz}}{(k\pi)^{n+1}} \Gamma(n+1, k\pi nz) \\ &- \sum_{k=1}^{m} \frac{\left(-k\pi/e^{k\pi}+e^{-k\pi}\right) e^{-k\pi nz}}{(-k\pi)^{n+1}} \Gamma(n+1, -k\pi nz) \\ &+ (nz)^n \left\{ \frac{f^{(0)}\left(z^{-1}\right) + h_m^{(0)}\left(z^{-1}\right) p_0(n)}{0!} \right. \\ &+ \frac{f^{(1)}\left(z^{-1}\right) + h_m^{(1)}\left(z^{-1}\right) p_1(n)}{1!(nz)} \\ &\left. + \frac{f^{(2)}\left(z^{-1}\right) + h_m^{(2)}\left(z^{-1}\right) p_2(n)}{2!(nz)^2} + O\left(\frac{1}{n^2}\right) \right\}. \end{aligned} \tag{57}$$

In Theorem 2, we have $p_0(n) = 1, p_1(n) = 0, p_2(n) = -n$, $f^{(0)}(z^{-1}) = (\operatorname{csch}(1/z)/2z)$, $f^{(2)}(z^{-1}) = \operatorname{csch}^3(1/z)(3 + \cosh(2/z) - 2z \sinh(2/z))/4z$ and using (54) and (55), we obtain the desired result. □

The figure below depicts the graphs of $(TB)_n(nz+1)$ (in solid lines) generated using relation (8) and the graphs of the approximate values of $(TB)_n(nz+1)$ (in dashed lines) generated using the expansion at the right-hand side of (43). Using Mathematica, we obtain the graph below.

The graphs show the accuracy of approximation (43) for several values of n for real values of the uniform parameter z. For a real argument the approximation of $(TB)_n(nz+1)$ with enlarged validity in the monotonic region is better than the approximation in (23). It can also be observed that in the oscillatory region the accuracy is better compare to the approximation in (23).

The next theorem contains the asymptotic expansion of Tangent-Genocchi polynomials with enlarged region of validity.

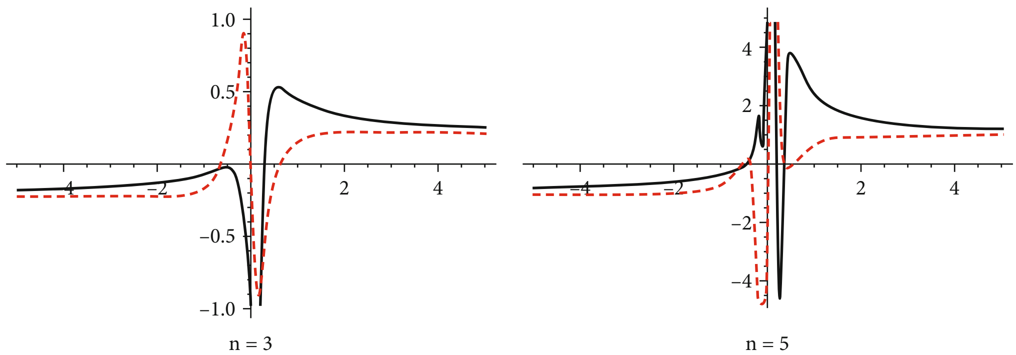

Figure 4: Solid lines represent $(TB)_n(nz+1)$ for several values of n, whereas dashed lines represent the right-hand side of (23) with $z \equiv x$, both normalized by the factor $(1+|x/\alpha|^n)^{-1}$ where $\alpha = 0.2$.

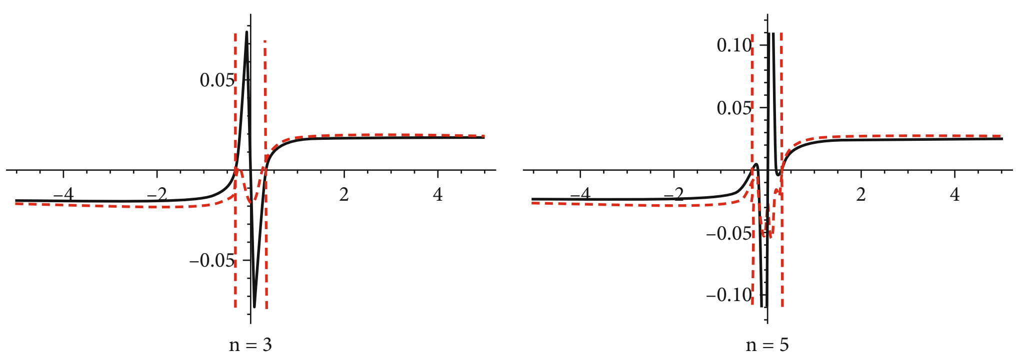

Figure 5: Solid lines represent $(TB)_n(nz+1)$ for several values of n, whereas dashed lines represent the right-hand side of (43) with $z \equiv x$, both normalized by the factor $(1+|x/\alpha|^n)^{-1}$ where $\alpha = 0.2$.

Theorem 6. *For $z \in \mathbb{C}\setminus\{0\}$ such that $|z^{-1}| < |z^{-1} \pm (2k+1)(\pi/2)i|$ for $k = 0, 1, \cdots, m-1$. Then, as $n \longrightarrow \infty$,*

$$\begin{aligned}(TG)_n(TG)_n(nz+1) = {} & \frac{2^n}{\pi^n i^{n+1}} \sum_{k=0}^{m-1} \frac{(-1)^{k+1}}{(2k+1)^n} \left[e^{2k+1/2\pi inz}\,\Gamma\left(n+1, \frac{2k+1}{2}\pi inz\right) \right. \\ & \left. + (-1)^{n+1} e^{-2k+1/2\pi inz}\,\Gamma\left(n+1, -\frac{2k+1}{2}\pi inz\right)\right] \\ & + (nz)^n \left\{ \frac{\operatorname{sech}(1/z)}{z} + \sum_{k=0}^{m-1} \frac{4(-1)^{k+1}(1+2k)\pi z}{4+\pi^2(z+2kz)^2} \right. \\ & - \frac{n}{2(nz)^2}\left[\frac{\operatorname{sech}^3(1/z)(-3+\cosh(2/z)-2z\sinh(2/z))}{2z} \right. \\ & \left. + \sum_{k=0}^{m-1} \frac{32(-1)^k(1+2k)\pi z^3\left[-4+3\pi^2(z+2kz)^2\right]}{\left[4+\pi^2(z+2kz)^2\right]^3}\right] \\ & \left. + O\left(\frac{1}{n^2}\right)\right\}.\end{aligned} \tag{58}$$

Proof. We let

$$f(w) = \frac{w}{\cosh w} = \frac{p(w)}{q(w)}. \tag{59}$$

Then

$$r_k = \frac{p(w_k)}{q'(w_k)} = \frac{w_k}{\sinh(w_k)}, \quad k = 0, 1, 2 \cdots, m-1. \tag{60}$$

Now, for $w_k = (2k+1)(\pi/2)i$, $k = 0, 1, 2 \cdots, m-1$, we see from Theorem 4 that

$$\sinh w_k = (-1)^k i, \tag{61}$$

so that

$$r_k = \frac{(2k+1)(\pi/2)i}{(-1)^k i} = (-1)^k (2k+1)\frac{\pi}{2}. \tag{62}$$

On the other hand, for $w_{-k} = -(2k+1)(\pi/2)i$, $k = 0, 1, 2 \cdots, m-1$, we see from Theorem 4 that

$$\sinh w_{-(k)} = (-1)^{k+1} i. \tag{63}$$

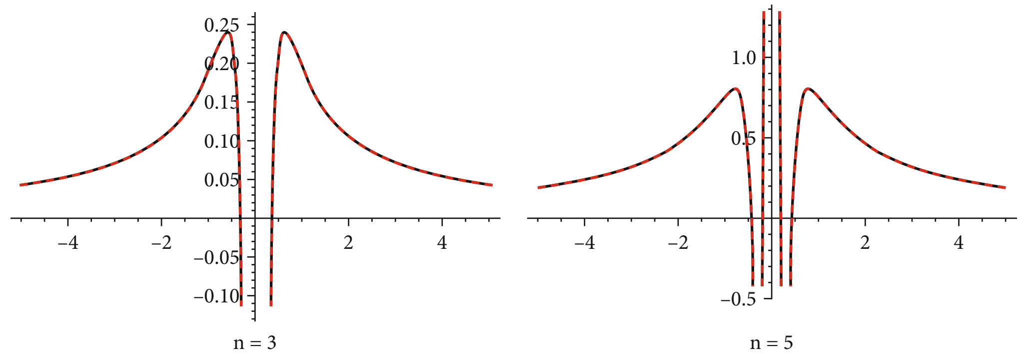

FIGURE 6: Solid lines represent $(TG)_n(nz+1)$ for several values of n, whereas dashed lines represent the right-hand side of (58) with $z \equiv x$, both normalized by the factor $(1+|x/\alpha|^n)^{-1}$ where $\alpha = 0.2$.

This gives

$$\mathrm{r}_{-\mathrm{k}} = \frac{-(2\mathrm{k}+1)(\pi/2)\mathrm{i}}{(-1)^{\mathrm{k}+1}\mathrm{i}} = (-1)^{\mathrm{k}}(2\mathrm{k}+1)\frac{\pi}{2}. \tag{64}$$

Hence, for $k = 0, 1, 2 \cdots., m-1$ we have

$$\mathrm{r}_{\mathrm{k}} = \mathrm{r}_{-\mathrm{k}} = (-1)^{\mathrm{k}}(2\mathrm{k}+1)\frac{\pi}{2}, \tag{65}$$

for both

$$w_k = \frac{2k+1}{2}\pi i \text{ and } w_{-k} = -\frac{2k+1}{2}\pi i \cdot \tag{66}$$

Evaluating some derivatives of the function $h_m(w)$ at the saddle point z^{-1} for $k = 0, 1 \cdots., m-1$, we have

$$\begin{aligned} h_m(w) &= -\sum_{k=0}^{m-1}\frac{(-1)^k(2k+1)(\pi/2)}{w-(2k+1/2)\pi i} - \sum_{k=0}^{m-1}\frac{(-1)^k(2k+1)(\pi/2)}{w+(2k+1/2)\pi i} \\ &= \sum_{k=0}^{m-1}\frac{4(-1)^{k+1}(1+2k)\pi w}{(\pi+2k\pi)^2+4w^2}, \end{aligned} \tag{67}$$

which yields,

$$h_m^{(1)}(w) = \sum_{k=0}^{m-1}\frac{4(-1)^{k+1}(1+2k)\pi\left[(\pi+2k\pi)^2-4w^2\right]}{\left[(\pi+2k\pi)^2+4w^2\right]^2}, \tag{68}$$

$$h_m^{(2)}(w) = \sum_{k=0}^{m-1}\frac{32(-1)^k(1+2k)\pi w\left[3(\pi+2k\pi)^2-4w^2\right]}{\left[(\pi+2k\pi)^2+4w^2\right]^3}, \tag{69}$$

$$h_m^{(0)}\left(z^{-1}\right) = \sum_{k=0}^{m-1}\frac{4(-1)^{k+1}(1+2k)\pi z}{4+\pi^2(z+2kz)^2} \tag{70}$$

$$h_m^{(2)}\left(z^{-1}\right) = \sum_{k=0}^{m-1}\frac{32(-1)^k(1+2k)\pi z^3\left[-4+3\pi^2(z+2kz)^2\right]}{\left[4+\pi^2(z+2kz)^2\right]^3}. \tag{71}$$

Using equations (2.18) and (32) for $k = 0, 1, 2 \cdots., m-1$, we have

$$\begin{aligned} (TG)_n(nz+1) = &-\sum_{k=0}^{m-1}\frac{(-1)^k(2k+1)(\pi/2)\,e^{(2k+1/2)\pi inz}}{((2k+1/2)\pi i)^{n+1}}\Gamma \\ &\cdot\left(n+1, \frac{2k+1}{2}\pi inz\right) \\ &-\sum_{k=0}^{m-1}\frac{(-1)^k(2k+1)(\pi/2)\,e^{-(2k+1/2)\pi inz}}{(-(2k+1/2)\pi i)^{n+1}}\Gamma \\ &\cdot\left(n+1, -\frac{2k+1}{2}\pi inz\right) + (nz)^n \\ &\cdot\left\{\frac{f^{(0)}\left(z^{-1}\right)+h_m^{(0)}\left(z^{-1}\right)p_0(n)}{0!}\right. \\ &+\frac{f^{(1)}\left(z^{-1}\right)+h_m^{(1)}\left(z^{-1}\right)p_1(n)}{1!(nz)} \\ &+\frac{f^{(2)}\left(z^{-1}\right)+h_m^{(2)}\left(z^{-1}\right)p_2(n)}{2!(nz)^2} \\ &\left.+O\left(\frac{1}{n^2}\right)\right\}. \end{aligned} \tag{72}$$

In Theorem 3, we have $p_0(n) = 1, p_1(n) = 0, p_2(n) = -n$, $f^{(0)}(z^{-1}) = (\operatorname{sech}(1/z)/z)$,

$$f^{(2)}\left(z^{-1}\right) = \frac{\operatorname{sech}^3(1/z)(-3+\cosh(2/z)-2z\sinh(2/z))}{2z}, \tag{73}$$

and using (58) and (73) for $k = 0, 1, 2 \cdots., m-1$, we obtain (59). □

The following figure depicts the graphs of $(TB)_n(nz+1)$ (in solid lines) generated using relation (9) and the graphs of the approximate values of $(TB)_n(nz+1)$ (in dashed lines) generated using the expansion at the right-hand side of (58).

The graphs show the accuracy of approximation (58) for several values of n for real values of the uniform parameter z. In both monotonic and oscillatory regions, the approximation in (58) shows better accuracy than in (30).

4. Conclusion

Two sets of approximation formulas for Tangent polynomials, Tangent-Bernoulli polynomials and Tangent-Genocchi polynomials are obtained using the saddle point method and the integral representation of these polynomials in terms of hyperbolic functions. The last set of formulas which have enlarged validity give more accurate approximations compared to the first set of formulas as shown in Figures 1–6. It is interesting to establish the asymptotic approximation of other variations of Tangent polynomials like Apostol-Tangent polynomials, Apostol-Tangent-Bernoulli polynomials and Apostol-Tangent-Genocchi polynomials as well as the higher-order versions of these polynomials.

Data Availability

The articles used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The author's declare that they have no conflicts of interest.

Acknowledgments

This research is funded by Cebu Normal University through its Research Institute for Computational Mathematics and Physics (RICMP). The authors would like to thank the anonymous referee for reviewing the paper thoroughly.

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