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Research Article

Two-Scale Network Dynamic Model for Energy Commodity Processes

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In this work, we examine the relationship between different energy commodity spot prices. To do this, multivariate stochastic models with and without external random interventions describing the price of energy commodities are developed. Random intervention process is described by a continuous jump process. The developed mathematical model is utilized to examine the relationship between energy commodity prices. The time-varying parameters in the stochastic model are estimated using the recently developed parameter identification technique called local lagged adapted generalized method of moment (LLGMM). The LLGMM method provides an iterative scheme for updating statistic coefficients in a system of generalized method of moment/observation equations. The usefulness of the LLGMM approach is illustrated by applying to energy commodity data sets for state and parameter estimation problems. Moreover, the forecasting and confidence interval problems are also investigated (U.S. Patent Pending for the LLGMM method described in this manuscript).

1. Introduction

Understanding the economy evolution in response to structural changes in the energy commodity network system is important to professional economists. The relationship between the different energy sources and their uses provide insights into many important energy issues. The quantitative behavior of energy commodities in which the trend in price of one commodity coincides with the trend in prices of other commodities has always raised the question of whether there is any relationship between prices of energy commodities. If there is any relationship, then what comes to mind is the extent to which one commodity influences the other. Petroleum, natural gas, coal, nuclear fuel, and renewable energy are termed as primary energy components of the energy goods network system because other sources of energy depend on them. Natural gas is usually found near petroleum. Hence, natural gas and crude oil are rivals in production and substitutes in consumption, and energy theory suggests that the two prices should be related. The electric power sector uses primary energy such as coal to generate electricity, which makes electricity a secondary rather than a primary energy source. According to the US Energy Information Administration (EIA), the major energy goods consumed in the United States are petroleum (oil), natural gas, coal, nuclear, and renewable energy. The majority of users are residential and commercial buildings, industry, transportation, and electric power generators. The pattern of fuel usage varies widely by sector [1]. For example, 71% of total petroleum oil provides 93% of the energy used for transportation; 23% of total petroleum oil provides 17% of energy used for residential and commercial use; 5% of total petroleum oil provides 40% of energy used for industrial use; but only 1% of total petroleum oil provides about 1% of the energy used to generate electric power, whereas coal provides 46% of the energy used to generate electric power and natural gas provides 20% of the energy used to generate electric power. This analysis suggests that the strength of interactions between coal and electricity will be stronger than that of crude oil and electricity, or natural gas and electricity.

Energy price forecasts are highly uncertain. We might expect the price of the natural gas and crude oil to follow

TABLE 1: Estimates $\widehat{m}_k$, $u_1(\widehat{m}_k, k)$, $\kappa_{1,1}(\widehat{m}_k, k)$, $\kappa_{1,2}(\widehat{m}_k, k)$, $\kappa_{1,3}(\widehat{m}_k, k)$, $u_2(\widehat{m}_k, k)$, $\kappa_{2,1}(\widehat{m}_k, k)$, $\kappa_{2,2}(\widehat{m}_k, k)$, $\kappa_{2,3}(\widehat{m}_k, k)$, $u_3(\widehat{m}_k, k)$, $\kappa_{3,1}(\widehat{m}_k, k)$, $\kappa_{3,2}(\widehat{m}_k, k)$, and $\kappa_{3,3}(\widehat{m}_k, k)$.

		Natural gas				Crude oil				Coal			
t_k	$\widehat{m}_k$	$u_1(\widehat{m}_k, k)$	$\kappa_{1,1}(\widehat{m}_k, k)$	$\kappa_{1,2}(\widehat{m}_k, k)$ $\times 10^{-16}$	$\kappa_{1,3}(\widehat{m}_k, k)$ $\times 10^{-18}$	$u_2(\widehat{m}_k, k)$	$\kappa_{2,1}(\widehat{m}_k, k)$ $\times 10^{-18}$	$\kappa_{2,2}(\widehat{m}_k, k)$	$\kappa_{2,3}(\widehat{m}_k, k)$ $\times 10^{-18}$	$u_3(\widehat{m}_k, k)$	$\kappa_{3,1}(\widehat{m}_k, k)$ $\times 10^{-18}$	$\kappa_{3,2}(\widehat{m}_k, k)$ $\times 10^{-18}$	$\kappa_{3,3}(\widehat{m}_k, k)$
11	1	4.1593	0.0211	0	0	57.7000	0	0	0	16.7407	0	0	0
12	3	4.2000	0.0111	0	0	58.6313	0.0011	0.0310	-0.0012	16.2395	0	0	-0.0376
13	5	4.0616	0.0679	-0.0054	-0.0035	58.5378	-0.0035	0.0205	0.0032	16.2680	0	0	0.1069
14	5	4.0616	-0.0242	-0.0179	0	61.4809	0.0020	0.0098	0	15.5249	0	0	-0.0294
15	8	4.0910	0.6416	-0.2898	0	58.9282	-0.0036	0.0128	0.0071	16.8286	0	0	0.0513
16	8	4.0160	0.2101	0	0	59.6867	-0.0051	0.0080	0.0071	17.0888	0	0	0.0415
17	8	4.9575	0.1876	0	0	60.6244	0.0024	0.0052	0	17.4120	-0.0003	0.0001	0.0555
18	8	4.9575	-0.1947	0	0	61.0700	0	0	0	17.2374	0	-0	0
19	6	4.7336	-1.4476	5.8820	0	61.9414	0	0.0043	-0.0086	16.8438	0.0001	0.0001	0.0768
20	6	2.5646	0.3319	0.7261	0	62.7899	0	0.0053	0.0082	18.3022	-0.0083	0.0027	0.0558
…	…	…	…	…	…	…	…	…	…	…	…	…	…
495	8	3.9654	0.0591	-0	0	108.2457	0.0038	0.0049	-0.0023	33.1313	0.0027	0.0009	0.0363
496	5	4.0421	0.0616	0.0001	0.0017	107.5186	0	0	0	33.4224	-0.0005	0.0003	0.0214
497	6	4.0514	0.0127	-0.0002	0.0020	109.8836	0	0	-0.0001	33.3388	0.0002	0	0.0443
498	7	4.1646	0.0442	-0.0012	-0.0053	107.8013	-0.0021	0.0033	0.0038	33.2862	-0.0002	0.0006	0.0343
499	6	4.1226	0.0352	-0.0020	0	108.1554	-0.0005	0.0032	0.0039	33.2862	0.0010	0.0001	0.0068
500	6	4.2625	0.0733	-0.0002	0	110.5101	-0.0032	0.0033	0.0016	36.1647	0.0003	0.0003	0.0079
501	8	3.1551	0	0	-0.0009	110.3071	0.0014	0.0025	0	34.7467	0	0	0
502	4	4.1564	0.0914	-0.0002	0	111.1186	0	0.0013	-0.0031	49.4050	0.0026	-0.0002	0.0211
503	5	4.5799	0.0467	0.0004	0	112.0057	0	0.0027	-0.0043	34.7207	-0.0001	-0.0001	0.0216
504	4	4.3061	0.0236	0.0002	0.0007	112.3186	0	0.0021	0.0015	34.4483	0.0019	0.0003	0.0170
505	9	4.4325	-0.0015	-0.0018	0.0030	106.3345	0	0.0043	0.0001	33.7160	0	-0.0006	0.0265
…	…	…	…	…	…	…	…	…	…	…	…	…	…
1102	7	3.5429	-0.0286	-0.0006	-0.0028	110.3777	0.0006	0.0045	0	5.2399	0	0.0013	0.0008
1103	4	3.5601	0.1028	0.0001	0.0001	111.1585	-0.0003	0.0083	0	5.4824	0	0.0077	0.0485
1104	4	3.5314	0.0809	0.0018	0.0090	109.0996	-0.0007	0.0095	0.0013	11.0949	-0.0018	0.0005	0.1175
1105	4	3.4439	0.1551	-0.0008	-0.0015	106.5667	0.0033	0.0073	-0.0020	4.8300	-0.0012	-0.0003	0.1283
1106	6	3.8206	0.2258	0.0004	0	104.7497	0	0	0.0027	4.8300	0	0.0008	0
1107	4	3.6917	0.2132	-0.0001	-0.0008	105.1229	0.0011	0.0039	0	4.3586	-0.0005	0.0004	0.1418
1108	5	3.7871	0	0	0	105.3595	0.0006	0.0027	-0.0009	4.8000	0.0006	-0.0001	0.1265
1109	4	3.8445	-0.0405	-0.0011	0.0011	102.9022	-0.0044	0.0037	0.0039	5.0279	0	0	0
1110	5	3.8399	0.0212	0.0004	0	102.8313	-0.0020	0.0045	0.0018	4.6817	0.0021	0.0041	0.0536

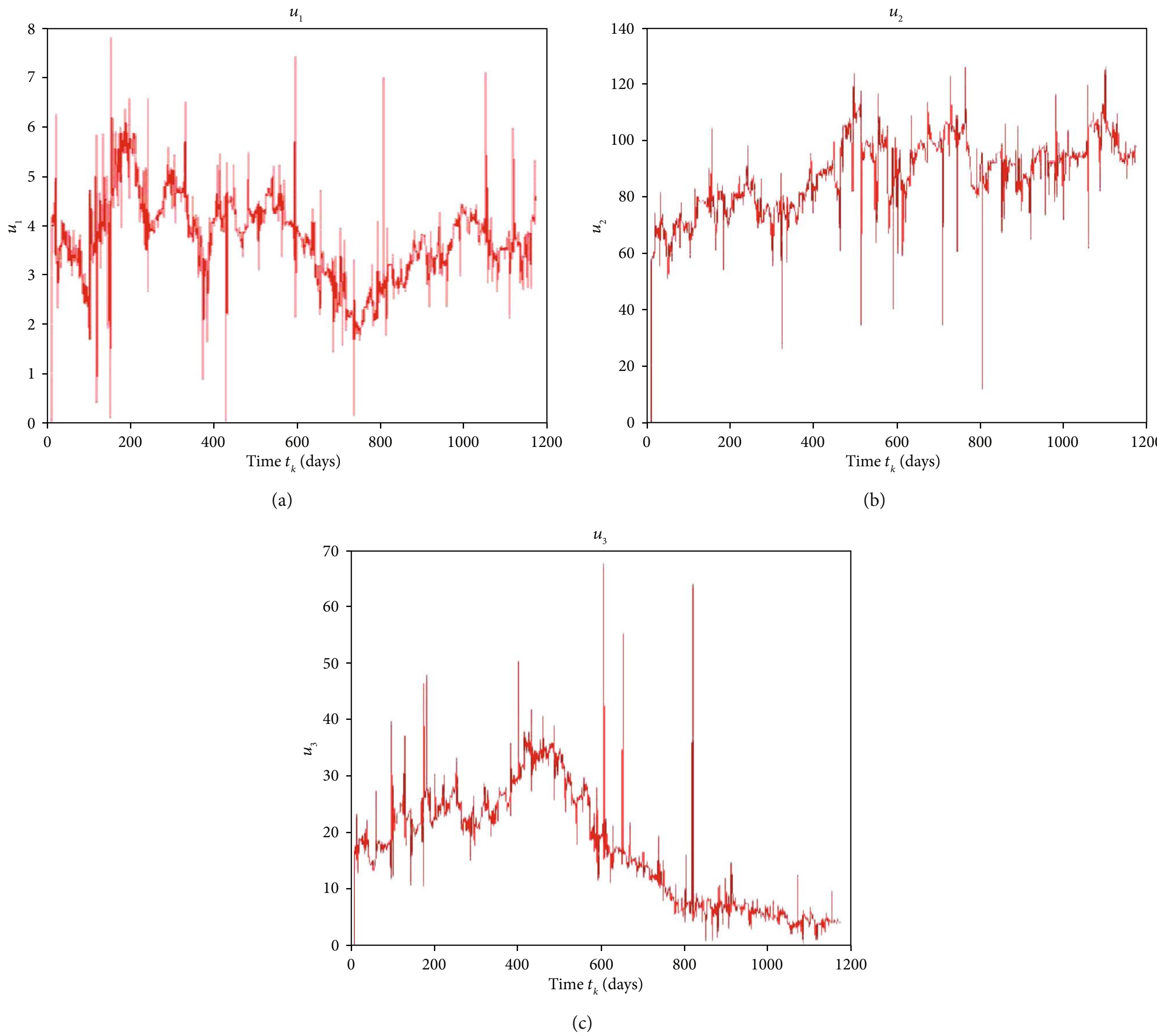


Figure 1: The graph of mean level $u_1(\widehat{m}_k, k)$, $u_2(\widehat{m}_k, k)$, and $u_3(\widehat{m}_k, k)$ for natural gas, crude oil, and coal, respectively.

the same trend because they are often found mixed with oil in oil wells and also of the fact that natural gas is often used in petroleum refining and exploration. Recently, Ramberg and Parsons [2] showed that the cointegration relationship between natural gas and crude oil does not appear to be stable through time. They claimed that though there is cointegration between the two energy prices, there are also recurrent exogeneous factors such as seasonality, episodic heat waves, cold waves, and supply interruption from the hurricane affecting the trends in the prices. Brown and Yücel [3] also observed that the price of natural gas pulled away from oil prices in 2000, 2002, 2003, and late 2005. Oil prices are influenced by several factors, including some that have mainly short-term impacts and other factors, such as expectations about the future world demand for petroleum, other liquids, and production decisions of the Organization of the Petroleum Exporting Countries (OPEC) [1]. Supply and demand in the world oil market are balanced through responses to price movement with considerable complexity in the evolution of underlying supply and demand expectation process. For petroleum and other liquids, the key determinants of long-term supply and prices can be summarized in four broad categories [1]: the economics of non-OPEC supply, OPEC investment and production decisions, the economics of other liquids supply, and world demand for petroleum and other liquids.

An understanding of how changes in the price of one energy commodity are expressed in terms of other energy commodity is needed. This would prove to be useful in predicting price behavior over the long run and further facilitates profit-maximizing processes. To check if there is really indeed a relationship between energy commodities, the need to be able to create a model which explains such commodity price relationship over short- and long-time intervals arises. The relationships between energy commodities have been addressed in [2–15]. The error correction model [2–5, 7, 8]

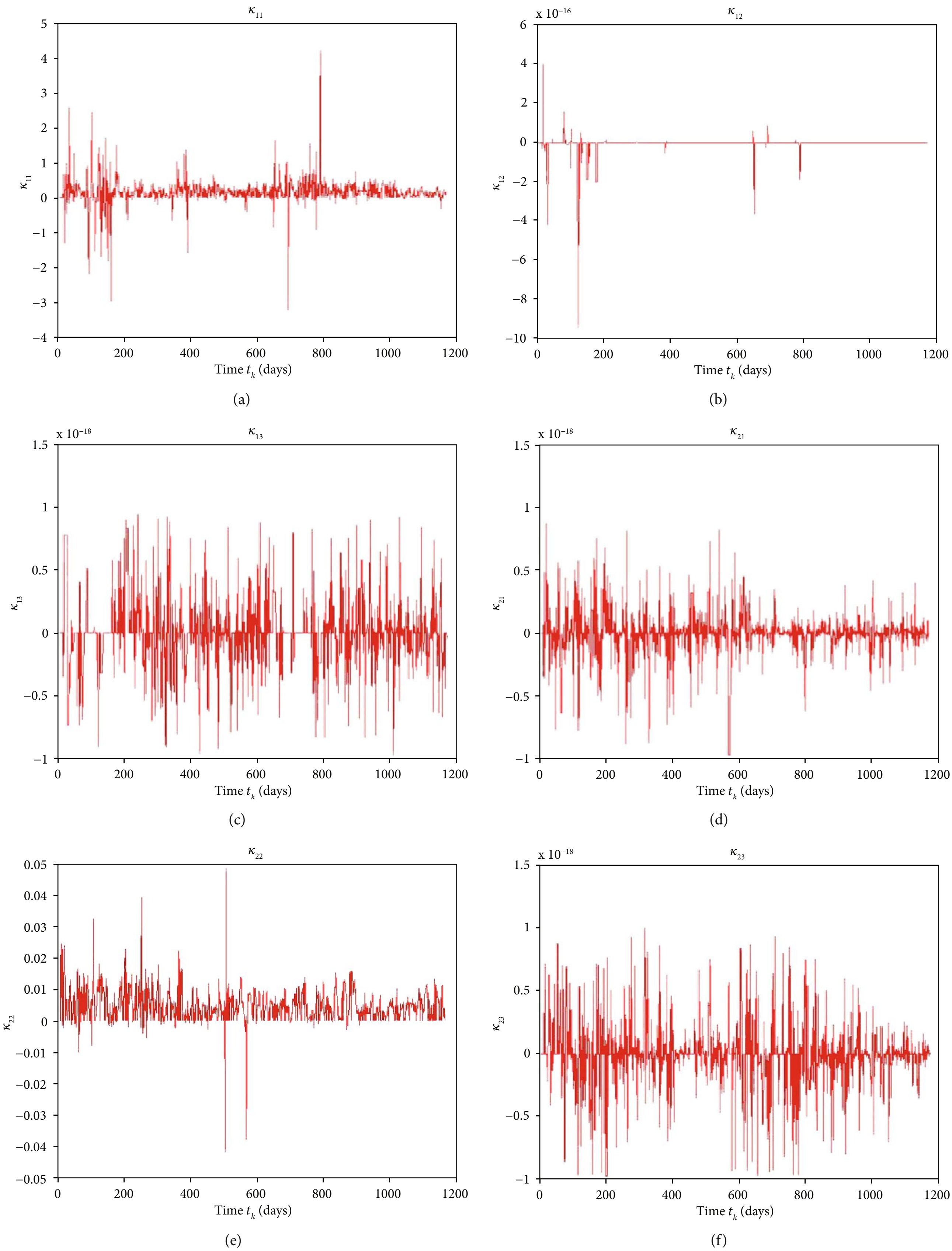


Figure 2: Continued.

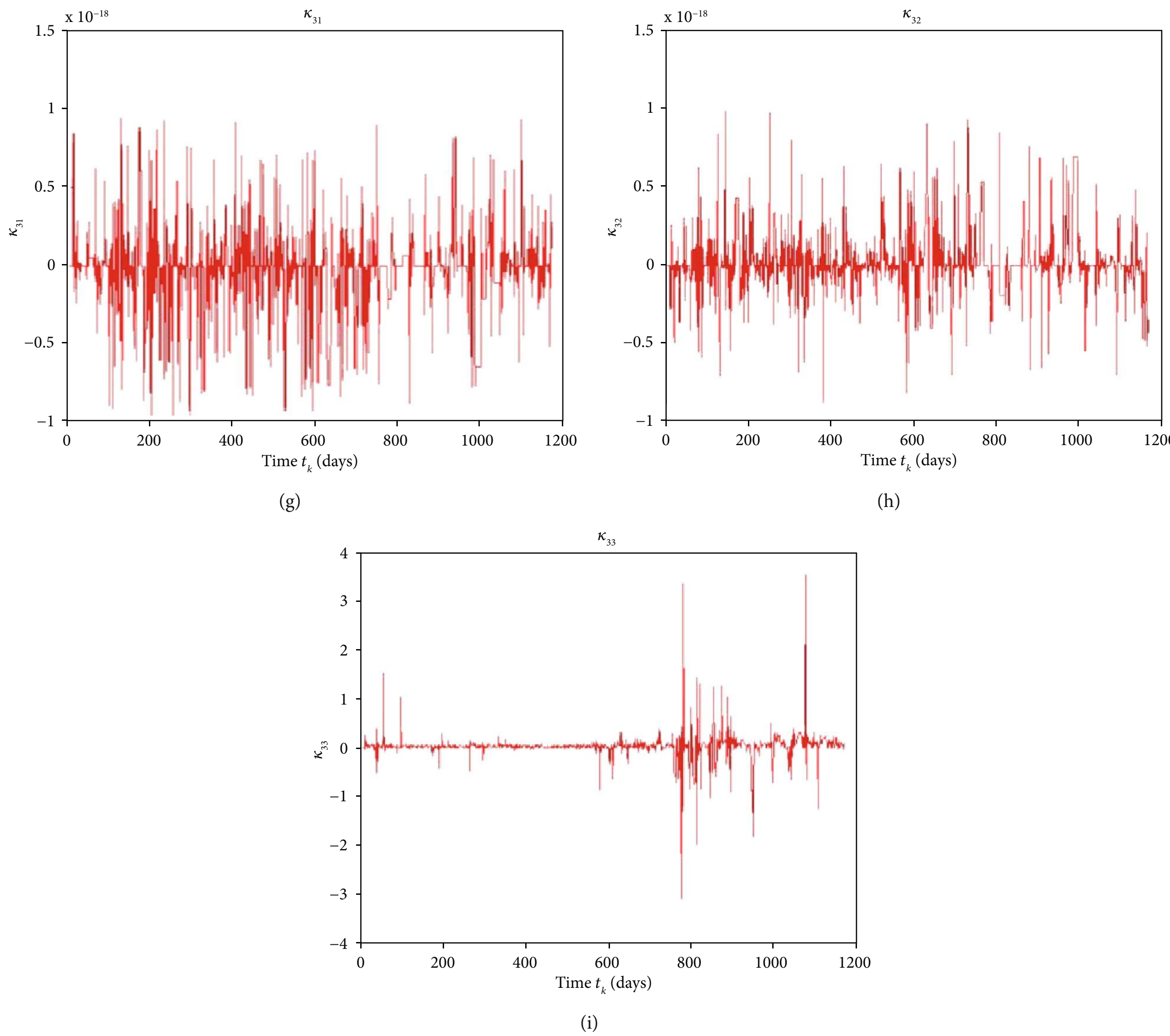


Figure 2: The graph of interaction coefficients $\kappa_{1,1}(\widehat{m}_k, k)$, $\kappa_{1,2}(\widehat{m}_k, k)$, $\kappa_{1,3}(\widehat{m}_k, k)$, $\kappa_{2,1}(\widehat{m}_k, k)$, $\kappa_{2,2}(\widehat{m}_k, k)$, $\kappa_{2,3}(\widehat{m}_k, k)$, $\kappa_{3,1}(\widehat{m}_k, k)$, $\kappa_{3,2}(\widehat{m}_k, k)$, and $\kappa_{3,3}(\widehat{m}_k, k)$.

is the most commonly used model by authors to describe the relationship between energy commodities. Moreover, Hartley et al. [7] have concluded that the U. S. natural gas and crude oil remain linked in their long-term movements. In addition, it is exhibited that there is strong evidence of a stable relationship between these two energy commodities which are affected by short-run seasonal fluctuations and other factors. The rule of thumb [7] has long been used in the energy industry to relate the natural gas prices to crude oil prices. The rule denoted by the 10-to-1 rule states that the price of natural gas is one-tenth of the price of crude oil prices. Similarly, the 6-to-1 rule states that the price of natural gas is one-sixth of the price of crude oil. It has been examined by Brown and Yücel [3] that these two rules do not perform well when used to assess the relationship between US natural gas price and West Texas Intermediate (WTI) crude oil price for the past 20^+ years. Moreover, their error-correcting model specifies the relationship between the two commodities. Using this model, they show that when certain factors are taken into account, movements in crude oil prices can shape the price of natural gas. Vezzoli [9] in his work applies an ordinary least squares (OLS) regression on the log of natural gas and crude oil prices. Using this model, he was able to conclude that there is a relationship between natural gas and crude oil prices. Bachmeir and Griffin [4] showed that crude oil, coal, and natural gas in the United States have weak cross-cointegration using the error correction model. Ramberg and Parsons [2] show that any simple formula between natural gas and crude oil prices will leave a portion of the natural gas price unexplained. Furthermore, the relationship between natural gas and crude oil using a vector error correction model [2, 3] under the

Table 2: Estimates $\delta_{1,1}(\widehat{m}_k, k)$, $\delta_{1,2}(\widehat{m}_k, k)$, $\delta_{1,3}(\widehat{m}_k, k)$, $\delta_{2,1}(\widehat{m}_k, k)$,$\delta_{2,2}(\widehat{m}_k, k)$, $\delta_{2,3}(\widehat{m}_k, k)$, $\delta_{3,1}(\widehat{m}_k, k)$, $\delta_{3,2}(\widehat{m}_k, k)$, and $\delta_{3,3}(\widehat{m}_k, k)$.

t_k		Natural gas			Crude oil			Coal	
	$\delta_{1,1}(\widehat{m}_k, k)$	$\delta_{1,2}(\widehat{m}_k, k)$	$\delta_{1,3}(\widehat{m}_k, k)$	$\delta_{2,1}(\widehat{m}_k, k)$	$\delta_{2,2}(\widehat{m}_k, k)$	$\delta_{2,3}(\widehat{m}_k, k)$	$\delta_{3,1}(\widehat{m}_k, k)$	$\delta_{3,2}(\widehat{m}_k, k)$	$\delta_{3,3}(\widehat{m}_k, k)$
11	0.0123	0.0012	0.0001	0	0.0223	0	0.0412	0	0.0022
12	0.0024	0.0011	0.0121	0.0234	0.0245	0	0	0	0.0112
13	0.0001	1.3425	1.7280	1.9811	0.9899	0.9731	0.6374	0.6374	0.0123
14	0	1.1267	0.6027	2.3258	0.1213	3.9128	1.6564	1.6564	0.0004
15	1.15260	0.4287	0.6210	2.3252	0.0006	0.5083	1.6650	1.6650	0.4565
16	4.9354	0	0	2.3217	0.0120	1.1124	1.6724	1.6724	0.8762
17	4.1360	0.0989	3.6877	1.6425	0	0	1.7719	1.7719	0
18	3.0410	0.1527	0	1.3105	0.9167	1.3451	1.7630	1.7630	0
19	2.7713	0	0	1.1052	0	3.3241	1.7400	1.7400	0
20	2.8461	0.0012	0.2221	0.1196	5.1929	0	0.6532	0.9876	0.0082
⋯	⋯	⋯	⋯	⋯	⋯	⋯	⋯	⋯	⋯
⋯	⋯	⋯	⋯	⋯	⋯	⋯	⋯	⋯	⋯
494	2.9961	0.0586	0	0.5529	0	0.42339	0	0	0.5187
495	5.9059	0	0.0584	0.5488	0.8947	0	0.0017	0.0021	0.0001
496	0.1121	0	0.6613	0.5767	0.9899	0	0.8763	0	0.9827
497	1.1229	0.0095	0.0988	0.6499	5.8547	0	1.1317	1.1317	0.0012
498	0.6946	0.0101	0	0	5.8298	0.0320	1.0294	1.0294	0.0321
499	0.7353	0.0066	0.0384	0	5.7180	0.0330	0.7317	0.7317	0.0431
500	1.7509	0.0069	0.0283	0.4307	5.6133	0.0413	0.4826	0.4826	0.0783
501	2.1299	0.0077	0.0282	0.5043	5.6282	0.0308	0.4272	0.4272	0.0002
502	0.9778	0.0077	0.0255	0.2878	4.6543	0.0322	0.5239	0.5239	0.0098
503	0.9872	0	0	0.2909	4.5544	0.0411	1.4523	1.4523	0.0087
504	1.1329	0	0	0.3707	0	0.1128	2.4181	2.4181	0
505	1.9178	0	0	0.3812	1.3243	0.1724	4.9207	4.9207	0
⋯	⋯	⋯	⋯	⋯	⋯	⋯	⋯	⋯	⋯
⋯	⋯	⋯	⋯	⋯	⋯	⋯	⋯	⋯	⋯
1102	0	0.0331	0.0056	0.9297	3.9502	0	0.2853	1.8033	1.1355
1103	1.5077	0.0626	0.0332	1.1017	2.8221	0	0	0	1.4133
1104	0	0.0435	0.5821	0.1939	4.5585	0	0	0	1.1672
1105	0	0	1.52970	0.1922	3.2418	0.7273	0.2726	0.2726	0
1106	4.4476	0.323	0.5112	3.5487	3.8113	1.0179	0.3296	0.3296	0
1107	2.4312	0.0011	0.0435	0.2001	2.6026	0.9354	0	0	1.7245
1108	2.5079	0.1232	0.4542	0.3781	0	0.8825	0.1878	0.1878	0
1109	1.7828	0.0431	0.3210	0.4024	0	0.8812	0	0	1.3191
1110	1.2706	0.0056	1.1123	0.3252	0	0.8078	0	0	1.0233

cointegration between the two energy commodities and other factors such as recurrent exogeneous factors are presented. Villar and Joutz [10] list some economic factors linking natural gas and crude oil prices, while testing for market integration in the United Kingdom during the time when natural gas was deregulated. Asche et al. [6] have integrated the prices of the energy commodities: natural gas, electricity, and crude oil.

Most of this work is centered around the relationship between prices of energy commodities. We are interested in an interdependence of certain energy commodities. Moreover, a nonlinear multivariate interconnected stochastic model of energy commodities and sources with and without external random intervention processes is developed. Also, parameter and state estimation problems of continuous time nonlinear stochastic dynamic process are motivated to initiate an alternative innovative approach. This led to the introduction of the concept of statistic processes, namely, local sample mean and sample variance which further led to the development of an interconnected discrete-time dynamic system of local statistic processes and its mathematical model discussed in Otunuga et al. [16, 17]. This paved the way for extending the LLGMM [16] to a multivariate local lagged adapted generalized method of moments case. The parameters in the multivariate model are estimated using the LLGMM method. These estimated parameters help in

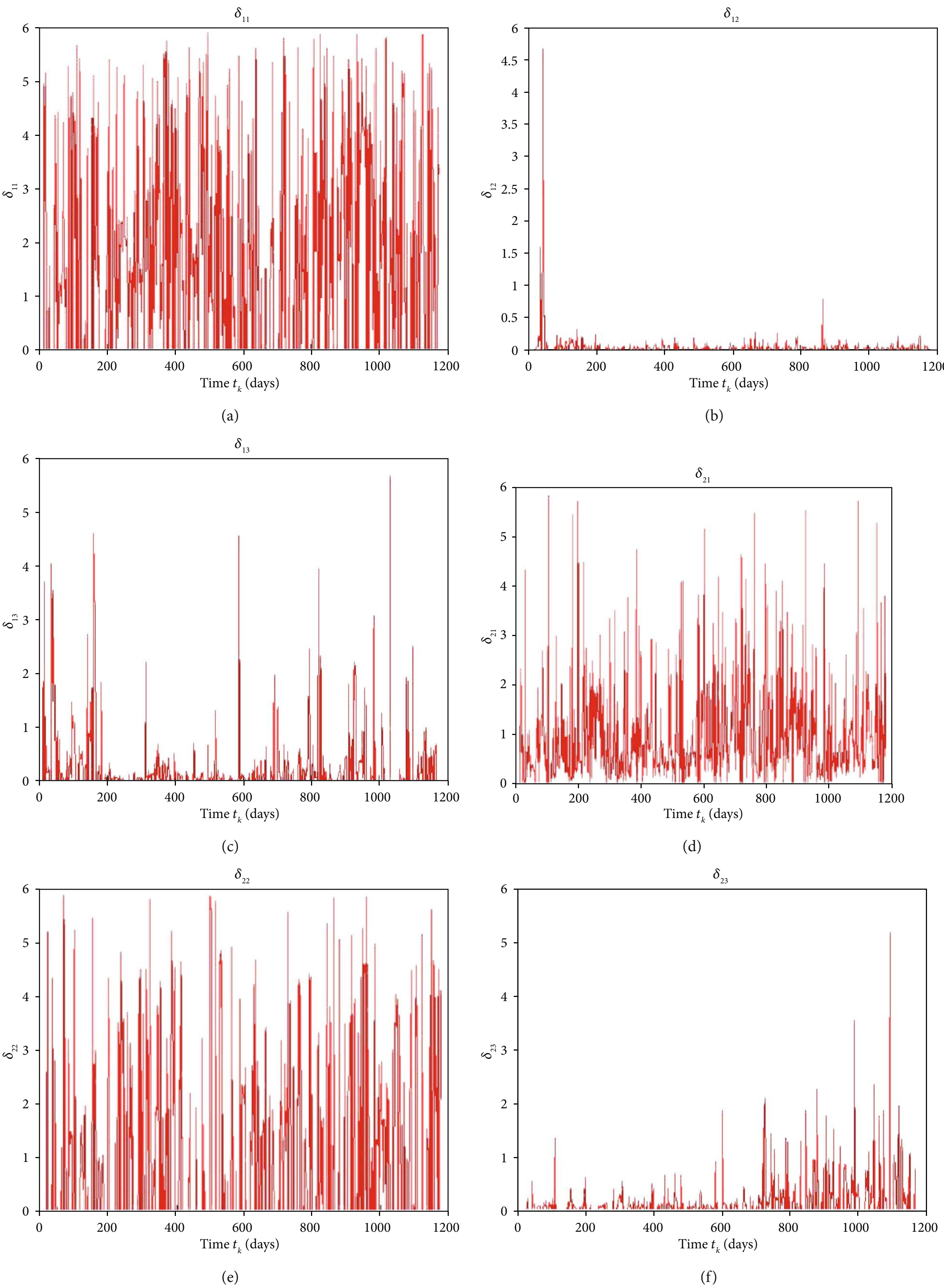
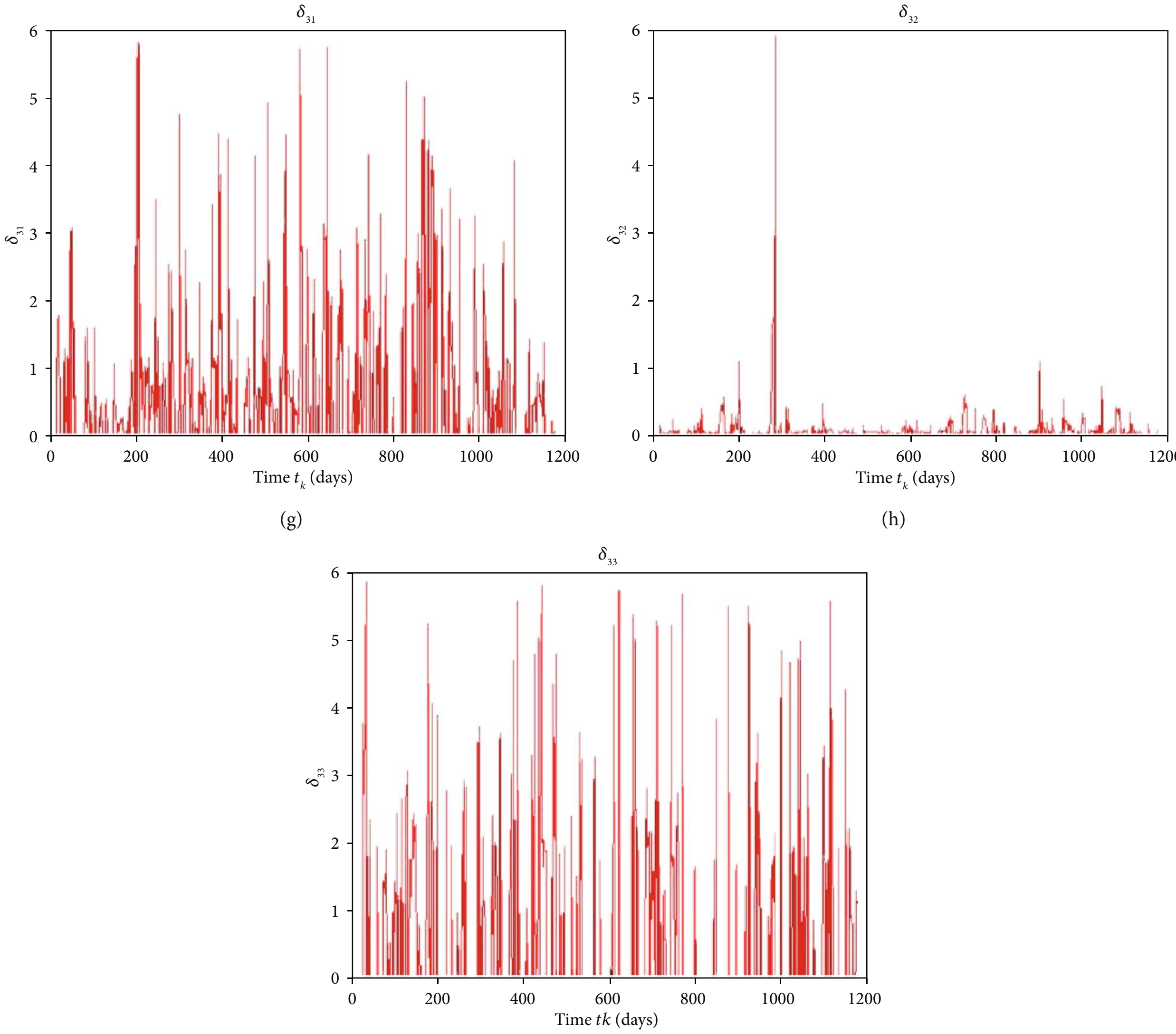


Figure 3: Continued.



(g) (h) (i)

FIGURE 3: The graph of interaction coefficients $\delta_{1,1}(\widehat{m}_k, k)$, $\delta_{1,2}(\widehat{m}_k, k)$, $\delta_{1,3}(\widehat{m}_k, k)$, $\delta_{2,1}(\widehat{m}_k, k)$, $\delta_{2,2}(\widehat{m}_k, k)$, $\delta_{2,3}(\widehat{m}_k, k)$, $\delta_{3,1}(\widehat{m}_k, k)$, $\delta_{3,2}(\widehat{m}_k, k)$, and $\delta_{3,3}(\widehat{m}_k, k)$.

analyzing the short- and long-term relationships between the energy commodities. It has been shown in Appendices A, B, C, and D of the work of Otunuga et al. [16] that the LLGMM parameter estimation scheme performs better than existing nonparametric statistical methods. A performance comparison (with appropriate statistical indices) of the LLGMM method with existing orthogonality condition based on generalized method of moments (OCBGMM) and aggregated generalized method of moments (AGMM) methods using the energy commodity stochastic model is discussed in their work. The method is applied to estimate time-varying parameter estimates in a stochastic differential equation governing energy commodities, stock price processes, and transmission of infectious diseases in the work of Otunuga et al. [16], Otunuga [18], and Mummert and Otunuga [19], respectively. Algorithm for implementing the LLGMM approach is presented in Otunuga et al. [17]. In this work, the usefulness of this approach is further illustrated by applying the technique to study the relationship between three energy commodity data sets: Henry Hub natural gas, crude oil, and coal data sets for state and parameter estimation problems. Interested readers are directed to the work of Otunuga et al. [16–18, 20] to read more about the LLGMM parameter estimation procedure. Moreover, the forecasting and confidence interval problems are also investigated.

The organization of this paper is as follows.

In Section 2, we derive a multivariate stochastic dynamical model for energy commodities. In Section 3, the multivariate model derived in Section 2 is validated. Due to random intervention, we introduce interventions described by a continuous jump process in our model in Section 4. In Section 5, we discuss the discrete-time dynamic model for sample mean and covariance processes with jump using the work of Otunuga et al. in [16, 17]. In Section 6, we discuss about

Table 3: Estimates $\widehat{m}_k$, $\beta_1(\widehat{m}_k,k)$, $\gamma_{1,1}(\widehat{m}_k,k)$, $\gamma_{1,2}(\widehat{m}_k,k)$, $\gamma_{1,3}(\widehat{m}_k,k)$, $\widehat{m}_k$, $\beta_2(\widehat{m}_k,k)$, $\gamma_{2,1}(\widehat{m}_k,k)$, $\gamma_{2,2}(\widehat{m}_k,k)$, $\gamma_{2,3}(\widehat{m}_k,k)$, $\widehat{m}_k$,$\beta_3(\widehat{m}_k,k)$, $\gamma_{3,1}(\widehat{m}_k,k)$, $\gamma_{3,2}(\widehat{m}_k,k)$, and $\gamma_{3,3}(\widehat{m}_k,k)$.

t_k	Natural gas				Crude oil				Coal			
	$\beta_1(\widehat{m}_k,k)$	$\gamma_{1,1}(\widehat{m}_k,k)$	$\gamma_{1,2}(\widehat{m}_k,k)$	$\gamma_{1,3}(\widehat{m}_k,k)$	$\beta_2(\widehat{m}_k,k)$	$\gamma_{2,1}(\widehat{m}_k,k)$	$\gamma_{2,2}(\widehat{m}_k,k)$	$\gamma_{2,3}(\widehat{m}_k,k)$	$\beta_3(\widehat{m}_k,k)$	$\gamma_{3,1}(\widehat{m}_k,k)$	$\gamma_{3,2}(\widehat{m}_k,k)$	$\gamma_{3,3}(\widehat{m}_k,k)$
11	0.3462	0.2579	-0.0039	0.0218	0.0029	0.0023	0.0056	0.0124	0	0	0.0432	0.0012
12	0.1681	0.3497	-0.0109	0.0248	0	0	0	0	0.6812	-0.3513	0.0248	0
13	0.1592	0.3755	-0.0102	0.0228	-0.0490	0.0228	0	-0.0027	1.0795	-0.2904	0.0135	0
14	-0.0889	0.5001	-0.0069	0.0257	-0.3700	-0.1689	0	0.0633	-0.6689	-0.1969	0.0166	0.0177
15	0.7025	0.6513	-0.0242	0.0376	-0.0903	-0.1737	0	0.0477	-0.8220	-0.0944	0.0061	0.0291
16	0.6727	0.6513	0.0188	-0.1048	0.3425	-0.1847	0	0.0252	-0.0480	0.0694	-0.0112	0.0165
17	0.3253	0.3674	-0.0103	0.0140	-0.4531	-0.1457	0	0.0638	-3.6240	0.1086	0.0266	0.0569
18	0.1523	0.3433	0.0014	-0.0163	4.0859	0.2320	0.3114	-0.2889	-1.3277	-0.0566	0.0088	0.0353
19	-7.9573	0.3433	-0.2058	1.1496	0.1389	-0.0677	0.0004	0.0065	-0.9232	0.0368	-0.0277	0.0883
20	0.0514	0.3028	0.0041	-0.0192	-0.1464	-0.0409	0.0004	0.0178	-1.9138	0.0530	-0.0104	0.0836
…	…	…	…	…	…	…	…	…	…	…	…	…
495	0.0552	0.2994	0.0015	-0.0063	1.8435	-0.1608	0.1040	-0.0368	0.5815	-0.3597	-0.0050	0.0350
496	0.1799	0.1860	0.0004	-0.0067	1.5682	-0.0268	0.4723	-0.0502	0.8586	-0.2029	-0.0094	0.0241
497	0.8047	0.1923	0.0023	-0.0314	6.6699	-0.2655	0.4723	-0.1649	0.7191	-0.2061	-0.0053	0.0172
498	0.2742	0.2651	0.0020	-0.0145	1.0042	0.0088	0.0226	-0.0315	0.3978	-0.1680	-0.0034	0.0161
499	0.4915	0.2295	-0.0006	-0.0125	1.3074	0.3761	0.0073	-0.0872	0.1425	-0.1899	-0.0026	0.0224
500	0.5659	0.1618	0.0008	-0.0194	0.4040	-0.0889	0.0392	-0.0011	0.3674	-0.2313	-0.0073	0.0331
501	0.4498	0.1679	0.0010	-0.0167	0.4230	-0.1297	0.0434	0.0035	-0.9002	1.1703	-0.0775	0.1011
502	0.4836	0.1850	-0.0001	-0.0139	0.5570	-0.1502	0.0384	0.0022	-0.2313	0.6524	-0.0496	0.0663
503	0.4696	0.1850	0.1224	-0.0919	-0.0441	0.0299	0.0384	-0.0023	3.7804	0.0120	-0.0498	0.0389
504	-0.0456	0.1850	0.0088	-0.0270	0.6112	-0.0820	0.0425	-0.0080	6.4696	0.4005	-0.0950	0.0543
505	0.0464	1.7125	-0.0423	0.1339	0.7135	-0.1115	0.1135	-0.0082	2.2295	0.0897	-0.0357	0.0306
…	…	…	…	…	…	…	…	…	…	…	…	…
1102	0.6765	0.0455	-0.0020	-0.0908	0.2863	-0.2183	0.1891	0.1028	4.3927	3.8144	0.1072	0.0250
1103	1.1804	0.4214	-0.0149	0.0837	-2.1858	0.4491	0.1891	0.1135	-6.1960	0.7446	0.0261	0.0144
1104	0.1069	0.2489	-0.0009	-0.0014	-2.1178	0.3406	0.1959	0.1826	-6.4415	0.0339	-0.0037	0.1429
1105	0.2367	0.0128	0.0065	-0.0019	0.2633	0.0565	0.0742	-0.0997	0.6510	2.4930	0.0714	0.1429
1106	0.1178	0	0	-0.0014	0.1384	0.0784	0.2014	-0.0904	0.6510	2.4930	0.6341	0.1429
1107	0.1466	0.4648	-0.0002	-0.0271	0.5787	-0.0297	0.1305	-0.0979	-4.5897	-0.0971	0.0240	0.0509
1108	0.3240	0.2478	0.1212	-0.0074	0.4293	-0.0678	0.0721	-0.0389	-4.5961	-0.0734	0.0233	0.0507
1109	0.121	0	0	-0.0021	0.2282	-0.0468	0.0721	-0.0133	-4.5961	-2.6776	0.5625	0.4702
1110	0.002	0	0	-0.0056	0.1523	0.0121	0.0011	-0.0129	9.5959	0.9045	-0.1478	0.0499

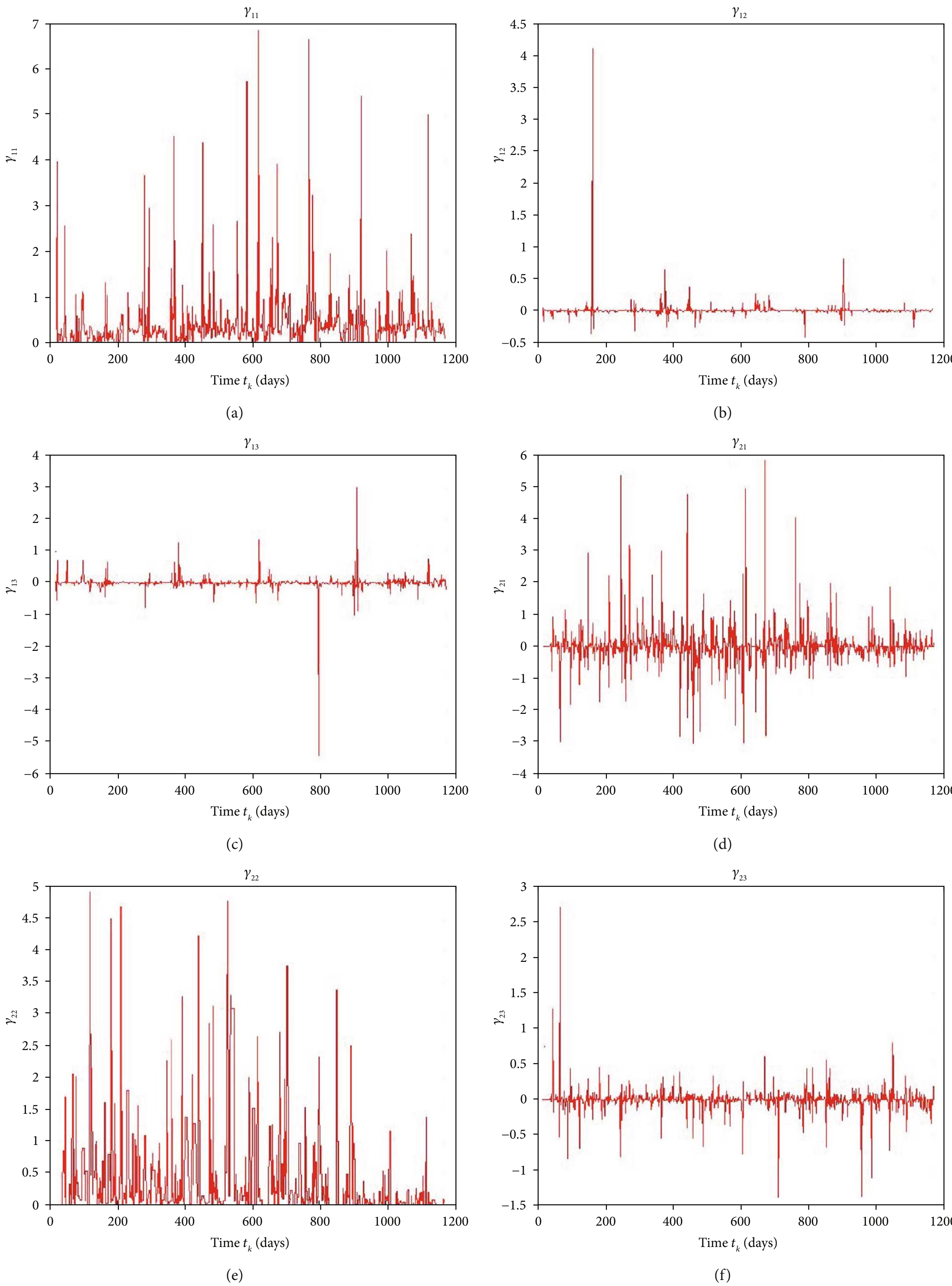
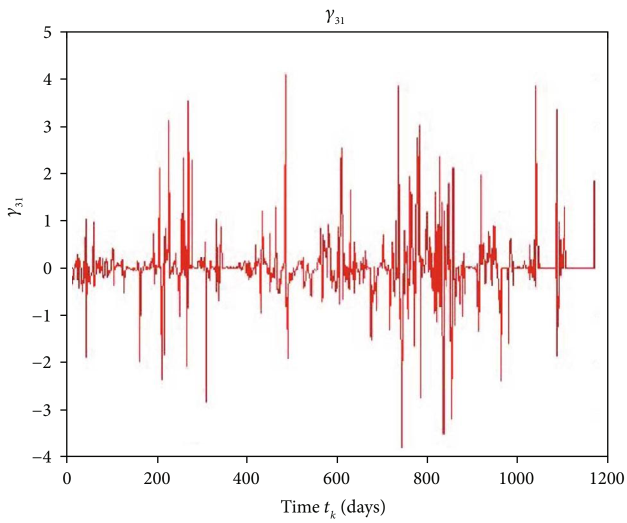
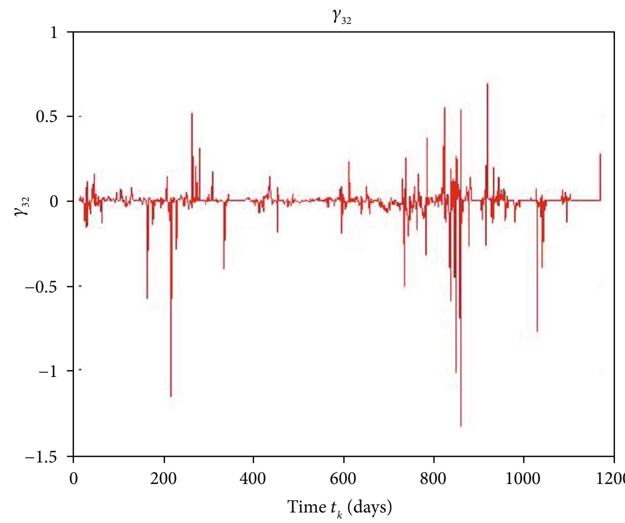


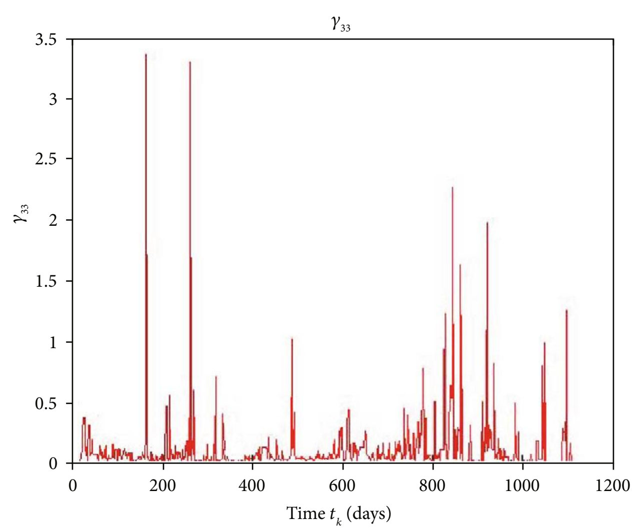
Figure 4: Continued.



(g)



(h)



(i)

Figure 4: The graph of interaction coefficients $\gamma_{1,1}(\widehat{m}_k, k)$, $\gamma_{1,2}(\widehat{m}_k, k)$, $\gamma_{1,3}(\widehat{m}_k, k)$, $\gamma_{2,1}(\widehat{m}_k, k)$, $\gamma_{2,2}(\widehat{m}_k, k)$, $\gamma_{2,3}(\widehat{m}_k, k)$, $\gamma_{3,1}(\widehat{m}_k, k)$, $\gamma_{3,2}(\widehat{m}_k, k)$, and $\gamma_{3,3}(\widehat{m}_k, k)$.

parametric estimation techniques. We construct an observation system from a nonlinear stochastic functional differential equation. In Section 7, using the method of moments [21], in the context of lagged adaptive expectation process [22], we briefly outline a procedure to estimate the state parameters for the dynamical model with jump and the model without jump locally. Moreover, the usefulness of computational algorithm is illustrated by applying the procedure to test for the relationship between Henry Hub natural gas, crude oil, and coal for the state and parameter estimation problems. In Section 8, the forecasting and confidence interval problems are also addressed.

2. Model Derivation

Let $p = [p_1, p_2, \cdots, p_n]^T$ be a vector of n energy commodity prices considered to have long-run or short-run relationship with each other, with $p_j(t)$ being the price of the jth energy commodity at time t. The economic principles of demand and supply processes suggest that the price of an energy commodity will remain within a given finite expected lower and upper bounds. We define $u_j \in \mathfrak{R}_+ = (0,\infty)$ and $l_j \geq 0$ as the expected upper and lower limits of the jth energy commodity spot prices, respectively. In the absence of interactions between the energy commodities p_j, $j \in I(1, n)$, where

$$I(a, b) = \{z \in \mathbb{Z} : a \leq z \leq b\}, \tag{1}$$

the market potential for the jth commodity per unit of time at time t can be characterized by $(u_j - p_j)(l_j + p_j)$. This simple idea leads to the following economic principle regarding the

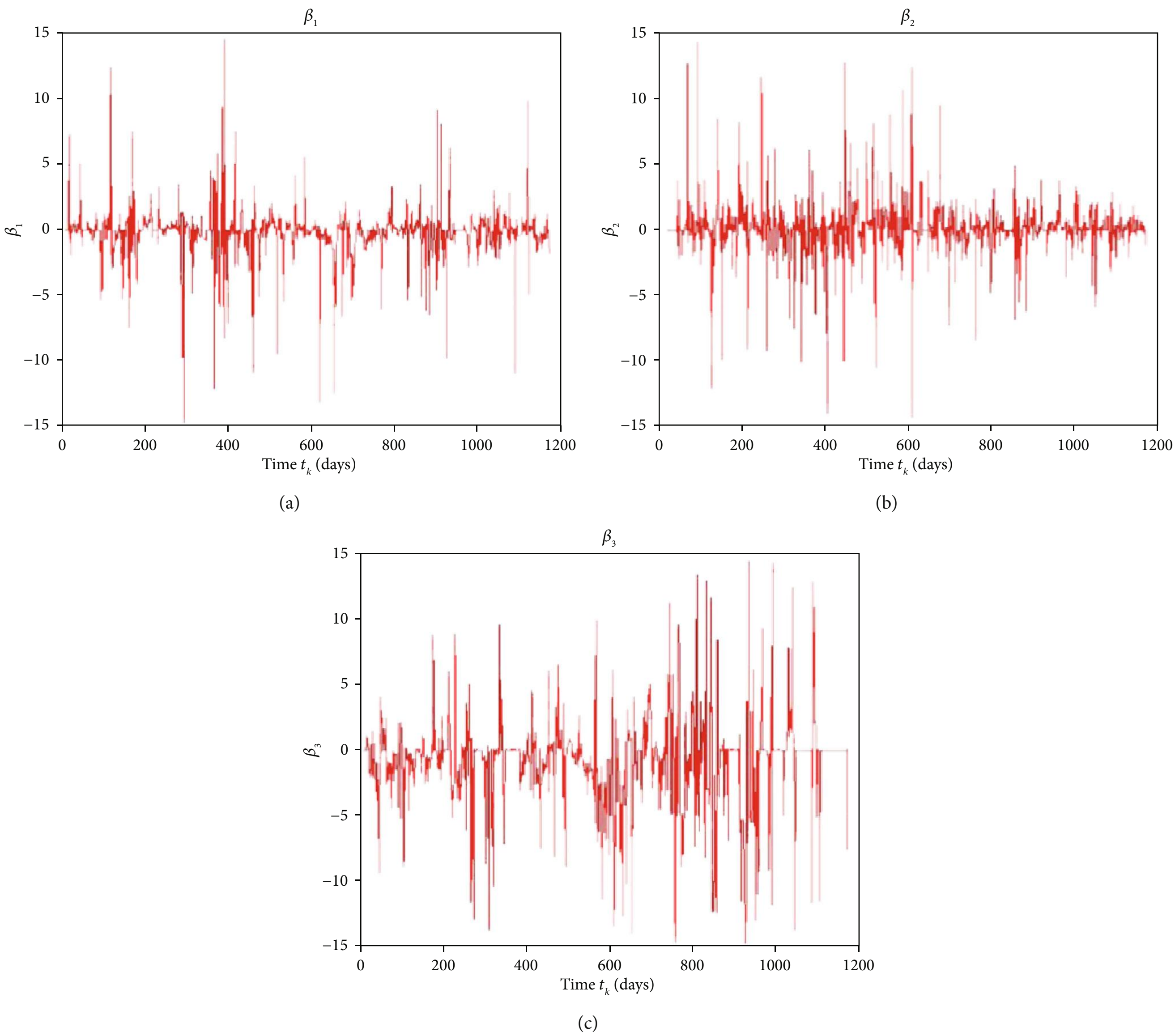


FIGURE 5: The graph of $\beta_1(\widehat{m}_k, k)$, $\beta_2(\widehat{m}_k, k)$, and $\beta_3(\widehat{m}_k, k)$ for natural gas, crude oil, and coal, respectively.

dynamic of the price p_j of the jth energy commodity. The change in spot price of the jth energy commodity $\Delta p_j(t) = p_j(t+\Delta t) - p_j(t)$ over the interval of length $|\Delta t|$ is directly proportional to the market potential price, that is,

$$\Delta p_j(t) \propto \left(u_j - p_j\right)\left(l_j + p_j\right)\Delta t. \tag{2}$$

This implies that

$$dp_j = \alpha_j\left(u_j - p_j\right)\left(l_j + p_j\right)dt, \tag{3}$$

for some constant α_j. From this deterministic mathematical model, if $\alpha_j > 0$, we note that as the price falls below the expected price u_j, the price of the jth commodity rises, and as the price lies above u_j, there is a tendency for the price to fall. Similar argument follows if $\alpha_j < 0$. Hence u_j is the equilibrium state of (3).

In a real world situation, the expected upper price limit u_j of the jth commodity is not a constant parameter. It varies with time, and moreover it is subjected to random environmental perturbations. Therefore, we consider

$$u_j = y_j + \xi_j, \tag{4}$$

where ξ_j is a white noise process that characterizes the measure of random fluctuation of the upper price limit of the jth commodity; here y_j stands for the mean of the energy spot

Table 4: Estimates $\sigma_{1,1}(\widehat{m}_k, k)$, $\sigma_{1,2}(\widehat{m}_k, k)$, $\sigma_{1,3}(\widehat{m}_k, k)$, $\sigma_{2,1}(\widehat{m}_k, k)$, $\sigma_{2,2}(\widehat{m}_k, k)$, $\sigma_{2,3}(\widehat{m}_k, k)$, $\sigma_{3,1}(\widehat{m}_k, k)$, $\sigma_{3,2}(\widehat{m}_k, k)$, and $\sigma_{3,3}(\widehat{m}_k, k)$.

t_k		Natural gas			Crude oil			Coal	
	$\sigma_{1,1}(\widehat{m}_k, k)$	$\sigma_{1,2}(\widehat{m}_k, k)$	$\sigma_{1,3}(\widehat{m}_k, k)$	$\sigma_{2,1}(\widehat{m}_k, k)$	$\sigma_{2,2}(\widehat{m}_k, k)$	$\sigma_{2,3}(\widehat{m}_k, k)$	$\sigma_{3,1}(\widehat{m}_k, k)$	$\sigma_{3,2}(\widehat{m}_k, k)$	$\sigma_{3,3}(\widehat{m}_k, k)$
11	0	0	0	0	0	0	0	0	0
12	0.0485	0.0004	0.0032	0.2734	0.0166	0	0	0	0.0000
13	0	0	0	0	0	0	0	0	0
14	0.2120	0.1386	0.0133	1.2573	0.4773	0.1195	0	0.0665	0.0086
15	0.4246	0.1318	0.0021	2.1081	0.4894	0.1211	0	0.6107	0.0696
16	0.5538	0.0778	0.1501	0	0.2524	0.0811	0.0651	0.4251	0.0635
17	1.1121	0.0469	0.2230	0	0.1848	0.2463	0	0.4458	0.0478
18	1.5347	0.0180	0.2178	0	0.1877	0.1602	0.5681	0.0592	0.0115
19	1.1315	0.0619	0.2221	0	0.2673	0.2465	0.4999	0.0569	0.0127
20	2.0845	0.0536	0.1866	0	0.1700	0.0781	0.3789	0.3174	0.0046
…	…	…	…	…	…	…	…	…	…
495	0	0.0036	0.0406	0.2286	0.0600	0.0172	0	0.9387	0.0182
496	0.1588	0.0035	0.0107	1.4847	0.3163	0.0102	0	0	0.0016
497	0.1551	0.0009	0.0065	0	0.1453	0	0.7777	0	0.0033
498	0.1576	0.0011	0.0073	0	0.1679	0	0.5334	0	0.0060
499	0.1197	0.0006	0.0059	1.9414	0.2391	0.0172	0.4405	0.1432	0.0097
500	0.3600	0.0001	0.0049	1.9554	0.3960	0.0079	0.6331	0.1410	0.0093
501	0.0514	0.0033	0.0049	2.0436	0.3499	0.0111	0.7690	0.1376	0.0089
502	0.2503	0.0034	0.0042	2.0837	0.1744	0.0132	0.6198	0.1274	0.0066
503	0.1195	0.0147	0.0165	0	0.4283	0.0060	1.1613	0.1530	0.0049
504	0.0974	0.0144	0.0027	0	0.2241	0.0048	0.4778	0.0574	0.0043
505	0.1422	0.0060	0.0131	0	0.2023	0.0054	0.5604	0.0669	0.0004
…	…	…	…	…	…	…	…	…	…
1102	0.1898	0.0016	0.0413	0.8313	0.0767	0.0381	0.6875	0	0.1451
1103	0.2094	0.0015	0.0352	0.8262	0.0673	0.0451	0.7298	0.2808	0.0147
1104	0.1711	0.0011	0.0040	0.6648	0.0915	0.0462	0.5563	0.1831	0.0105
1105	0.1816	0.0012	0.0116	0.6658	0.1049	0.0371	0.6591	0.2874	0.0057
1106	0.1191	0.0011	0.0116	0.6260	0.1155	0.0393	0	0.0196	0.0060
1107	0.0417	0.0012	0.0041	0.4992	0.0781	0.0382	0	0	0.0065
1108	0.1058	0.0033	0.0045	0.0019	0.0589	0.0421	0	0	0.0018
1109	0.1740	0.0021	0	0	0.0446	0.0316	2.1187	0	0.4511
1110	0.2912	0.0021	0.0163	0.0385	0.0342	0.0037	0	1.1563	0.0257

price process of the jth commodity at time t. It is further assumed that y_j is governed by a similar dynamic forces described in (3), that is,

$$dy_j = \mu_j \left(u_j - y_j\right)\left(v_j + y_j\right) dt, \qquad (5)$$

where $\mu_j \in \mathfrak{R}_+$ is defined as the mean reversion rate of the mean of the jth commodity. By following the argument used in (4), we incorporate the effects of random environmental perturbations into the lower limit v_j of the mean of the jth commodity:

$$v_j = \nu_j + e_j, \qquad (6)$$

where $\nu_j \geq 0$, and e_j is a white noise process describing the measure of random influence on the mean price of the jth commodity.

Substituting expressions in (4) and (6) into (3) and (5), respectively, we obtain

$$\begin{cases} dy_j = \mu_j \left(u_j - y_j\right)\left(\nu_j + y_j\right) dt + \mu_j \left(u_j - y_j\right) e_j(t) dt \\ dp_j = \alpha_j \left(y_j - p_j\right)\left(l_j + p_j\right) dt + \alpha_j \left(l_j + p_j\right) \xi_j(t) dt. \end{cases} \qquad (7)$$

In the absence of interactions and using (7), the system of stochastic model for isolated expected spot and spot prices

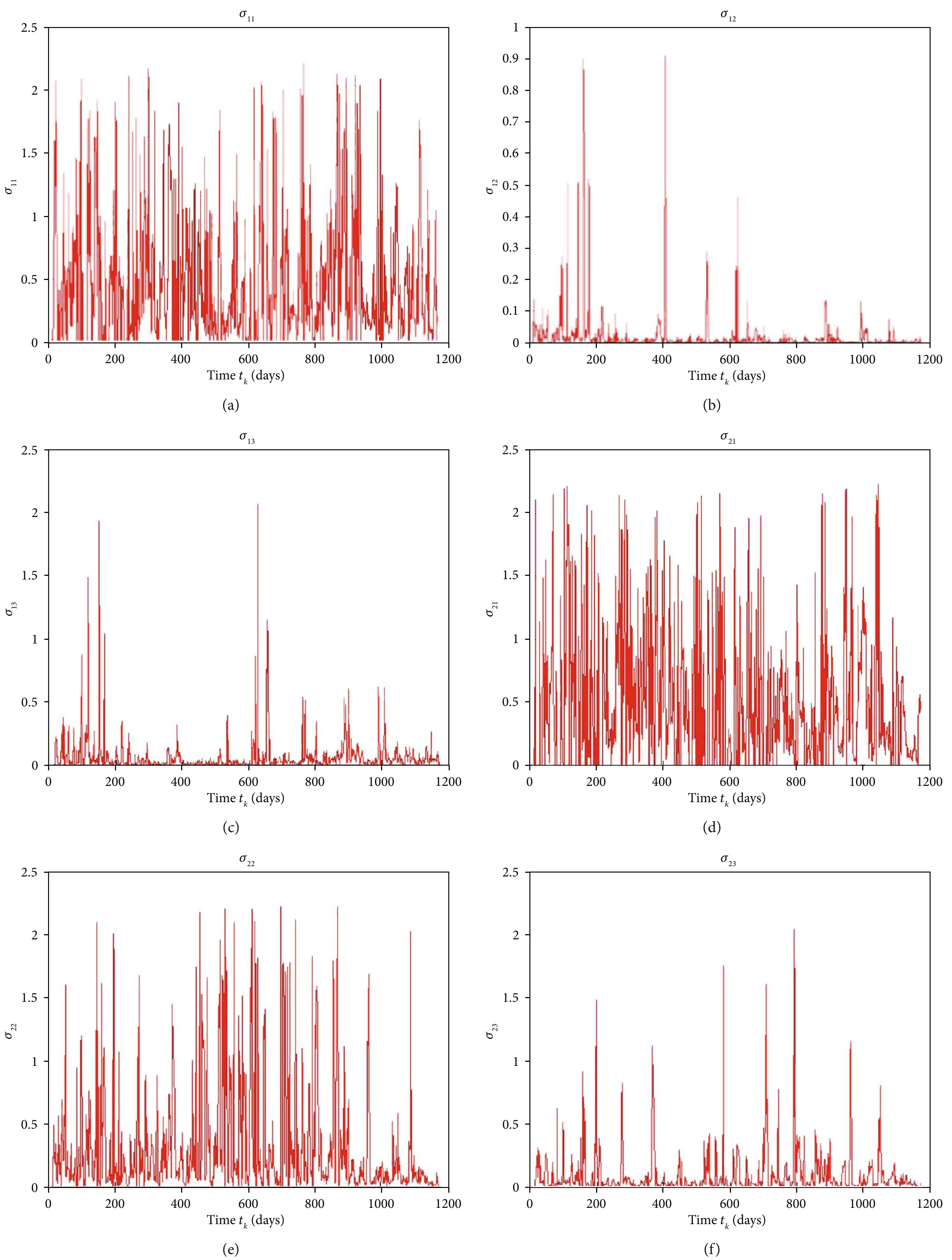
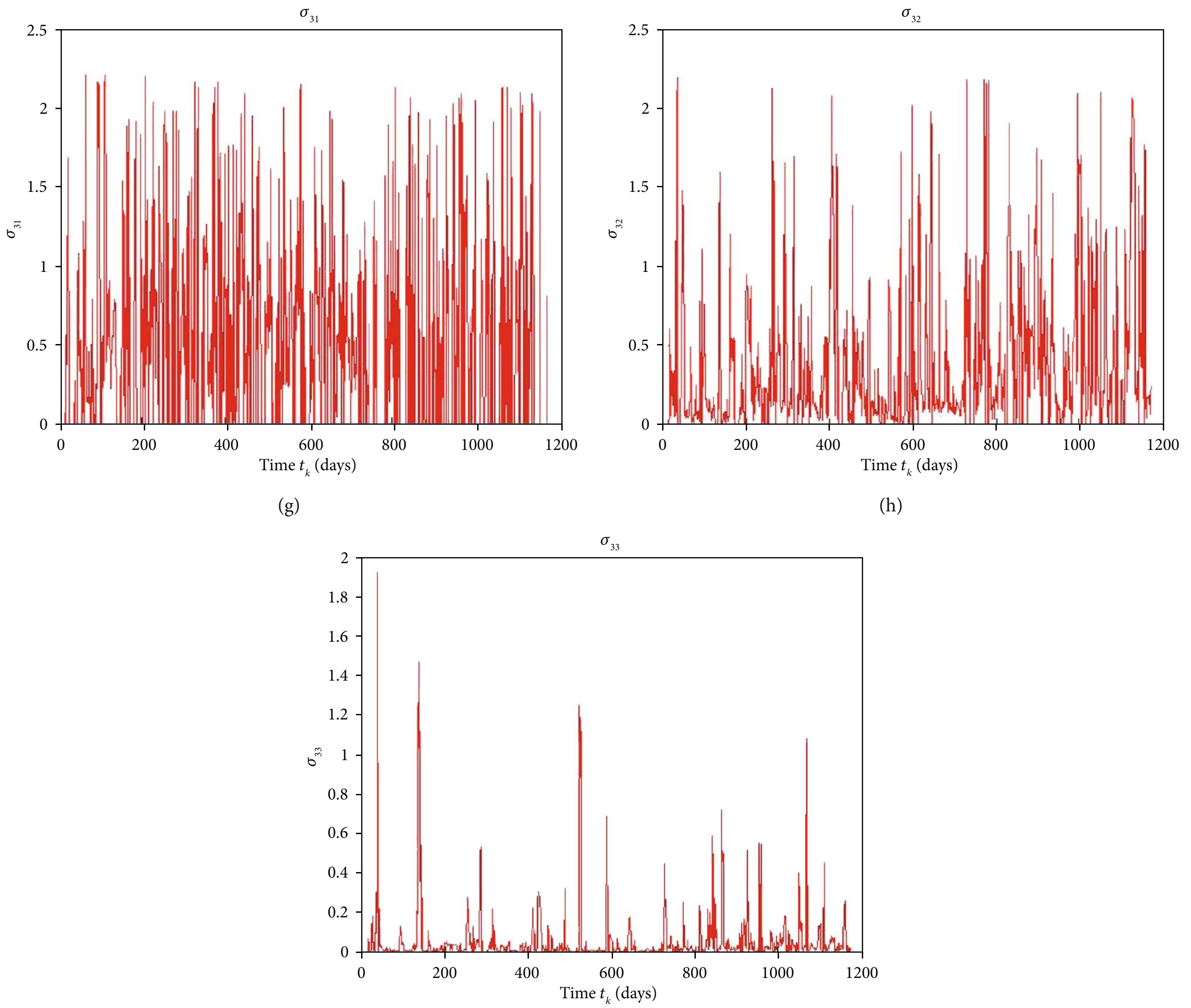


Figure 6: Continued.



(g) (h) (i)

FIGURE 6: The graph of $\sigma_{1,1}(\widehat{m}_k, k)$, $\sigma_{1,2}(\widehat{m}_k, k)$, $\sigma_{1,3}(\widehat{m}_k, k)$, $\sigma_{2,1}(\widehat{m}_k, k)$, $\sigma_{2,2}(\widehat{m}_k, k)$, $\sigma_{2,3}(\widehat{m}_k, k)$, $\sigma_{3,1}(\widehat{m}_k, k)$, $\sigma_{3,2}(\widehat{m}_k, k)$, and $\sigma_{1,1}(\widehat{m}_k, k)$ for natural gas, crude oil, and coal, respectively.

processes are described by the following nonlinear system of stochastic differential equations:

$$\begin{cases} dy_j = \mu_j\left(u_j - y_j\right)\left(v_j + y_j\right)dt + \delta_{j,j}\left(u_j - y_j\right)dW_{j,j}(t), & y_j(t_0) = y_{j0}, \\ dp_j = \alpha_j\left(y_j - p_j\right)\left(l_j + p_j\right)dt + \sigma_{j,j}\left(l_j + p_j\right)dZ_{j,j}(t), & p_j(t_0) = p_{j0}, j \in I(1, n), \end{cases} \tag{8}$$

where

$$\begin{cases} \mu_j e_j(t)dt = \delta_{j,j}dW_{j,j}(t), \\ \alpha_j \xi_j(t)dt = \sigma_{j,j}dZ_{j,j}(t), \end{cases} \tag{9}$$

and $\delta_{j,j}, \sigma_{i,j}$ are nonnegative for $j \in I(1, n)$.

In the presence of interactions, for each $j \in I(1, n)$, we consider both deterministic and stochastic interaction functions. For each $j \in I(1, n)$, we define the jth aggregate interaction functions $\mathbf{g}_j : [t_0, \infty) \times \mathfrak{R}^n \longrightarrow \mathfrak{R}$ and $\mathbf{h}_j : [t_0, \infty) \times \mathfrak{R}^n \longrightarrow \mathfrak{R}$ for the jth mean energy spot price process $y_j(t)$ and the energy spot price process $\mathbf{p}_j(t)$ in energy commodity market network system, respectively. Moreover,

Table 5: Real and simulated estimates (without jump) for natural gas, crude oil, and coal.

t_k	Real	Natural gas Simulated $p_1^s(\widehat{m}_k, k)$	Real	Crude oil Simulated $p_2^s(\widehat{m}_k, k)$	Real	Coal Simulated $p_3^s(\widehat{m}_k, k)$
11	4.0200	4.0500	58.9900	56.5200	16.5900	16.8000
12	3.9900	4.0500	59.5200	59.3099	17.4600	16.8635
13	3.7500	3.6690	61.4500	59.3377	17.8900	17.8086
14	3.7700	3.6341	60.4900	59.4191	17.5500	17.0859
15	3.4100	3.3967	61.1500	59.6974	17.4100	17.0859
16	3.3500	3.3967	62.4800	59.6974	16.7500	17.0859
17	3.4900	3.4537	63.4100	61.2177	17.6600	19.0677
18	3.5500	3.4537	65.0900	61.4561	17.5200	16.0578
19	3.9200	3.8618	66.3100	61.6529	18.5000	19.0677
20	3.8600	3.8618	68.5900	60.9364	19.0600	19.0677
⋯	⋯	⋯	⋯	⋯	⋯	⋯
495	4.1900	4.0368	107.1800	104.1295	32.7600	31.3108
496	4.3300	4.1868	110.8400	111.1245	33.6500	32.7737
497	4.3300	4.1025	111.7200	112.4675	33.7100	33.4888
498	4.3700	4.0964	111.6800	110.8795	34.7500	35.5907
499	4.3200	4.1042	111.7200	104.2465	34.5400	32.9391
500	4.3500	4.0548	112.3100	109.9535	34.0400	36.2674
501	4.3800	4.0548	112.3800	109.9995	33.1000	36.2674
502	4.5100	4.3249	113.3900	104.3254	33.6700	34.8915
503	4.6000	4.3555	113.0300	113.2356	33.9400	35.0472
504	4.6000	4.3491	110.6000	103.9435	33.8300	32.8992
505	4.5900	4.3609	108.7900	104.9995	32.0200	32.8992
⋯	⋯	⋯	⋯	⋯	⋯	⋯
1102	3.7200	3.5963	108.2300	110.5149	4.7700	2.8861
1103	3.7300	3.5963	106.2600	105.8076	5.0100	5.6871
1104	3.6800	3.4099	104.7000	105.8076	4.9800	5.3821
1105	3.6600	3.4356	103.6200	105.8076	4.7300	4.9221
1106	3.5900	3.4636	103.2200	106.9547	4.6800	4.2352
1107	3.5200	3.2573	102.6800	105.4047	4.6300	5.8172
1108	3.4900	2.8981	103.1000	102.4928	4.7400	6.0376
1109	3.5100	2.8981	102.8600	102.4928	4.3300	5.1121
1110	3.4800	3.0267	102.3600	102.4928	4.1800	4.8978

we assume that these functions have the following structural forms:

$$\begin{cases} \mathbf{g}_j(t,\mathbf{y}) = \mathbf{g}_j\left(t, k_{j,1}y_1, k_{j,2}y_2, \cdots, k_{j,n}y_n\right), \\ \boldsymbol{h}_j(t,\mathbf{p}) = \boldsymbol{h}_j\left(t, \gamma_{j,1}p_1, \gamma_{j,2}p_2, \cdots, \gamma_{j,n}p_n\right), \end{cases} \tag{10}$$

where $k_{j,i}$ and $\gamma_{j,i}$ are elements of $n\times n$ interconnection matrices denoted by $\mathbf{E}_g$ and $\mathbf{E}_h$, respectively. In (10), $k_{j,i}$ and $\gamma_{j,i}$ represent a degree of interaction of the jth commodity with the ith commodity in the energy commodity market network system.

For the matrix $\mathbf{E}_g$, $k_{j,i}=0$ with fixed $i \in I(1,n)$ if the ith commodity in the energy market network system does not influence the jth commodity. Likewise, for the matrix $\mathbf{E}_h$, $\gamma_{j,i}=0$ with fixed $i\in I(1,n)$, if the jth commodity in the energy market network system subcomponent of $\mathbf{p}$ is totally unaffected by the influence of the ith commodity. Finally, we introduce interactions in the diffusion coefficients with respect to the jth commodity of the energy market network system under random environmental perturbations as $\boldsymbol{\psi}_j : [t_0,\infty)\times\mathfrak{R}^n \longrightarrow \mathfrak{R}^n$ and $\boldsymbol{\Lambda}_j : [t_0,\infty)\times\mathfrak{R}^n \longrightarrow \mathfrak{R}^n$ for each $j\in I(1,n)$. The diffusion part is of the form

$$\begin{cases} \boldsymbol{\psi}_j(t,\mathbf{y})\cdot \mathbf{e}_j(t)dt = \sum_{l=1}^{n} \boldsymbol{\psi}_{j,l}(t,y_l)dW_{j,l}(t), \\ \boldsymbol{\Lambda}_j(t,\mathbf{p})\cdot \boldsymbol{\xi}_j(t)dt = \sum_{l=1}^{n} \boldsymbol{\Lambda}_{j,l}(t,p_l)dZ_{j,l}(t), \end{cases} \tag{11}$$

where $\mathbf{e}_j$ and $\boldsymbol{\xi}_j$ are n-dimensional white noise processes; $\cdot$ stands for dot product.

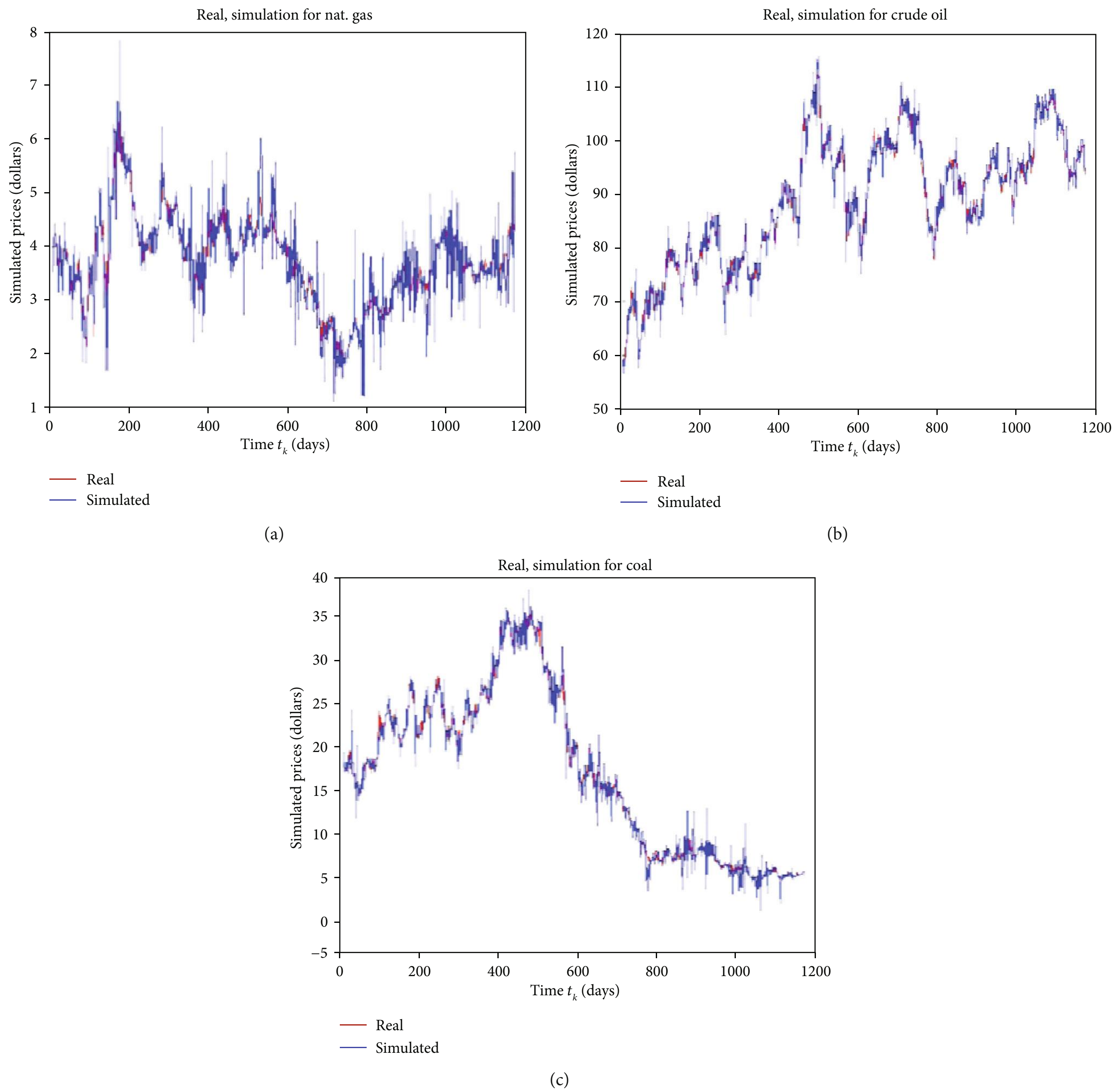


Figure 7: Real and simulated prices (without jump) for natural gas, crude oil, and coal.

Table 6: Some result for the jump times of the system $(\mathbf{y}, \mathbf{p})$.

T	17	44	61	87	157	200	422	464	483	502	722	754	870	930	1113

We assume that the interaction functions (10) and (11) have the following forms:

$$\begin{cases} \mathbf{g}(t, \mathbf{y}) = \boldsymbol{\gamma}(t, \mathbf{y}) G(t, \mathbf{y}), \\ \mathbf{h}(t, \mathbf{p}) = \boldsymbol{\lambda}(t, \mathbf{p}) \mathbf{H}(t, \mathbf{p}), \\ \boldsymbol{\psi}(t, y) = \boldsymbol{\gamma}(t, \mathbf{y}) \boldsymbol{\Psi}(t, \mathbf{y}), \\ \boldsymbol{\Lambda}(t, \mathbf{p}) = \boldsymbol{\lambda}(t, \mathbf{p}) \boldsymbol{\Phi}(t, \mathbf{p}), \end{cases} \tag{12}$$

where $\mathbf{g}(t, \mathbf{y}) = [\mathbf{g}_1(t, \mathbf{y}), \cdots, \mathbf{g}_j(t, \mathbf{y}), \cdots, \mathbf{g}_n(t, \mathbf{y})]^\top$, $\mathbf{h}(t, \mathbf{p}) = [\mathbf{h}_1(t, \mathbf{p}), \cdots, \mathbf{h}_j(t, \mathbf{p}), \cdots, \mathbf{h}_n(t, \mathbf{p})]^\top$ are defined in (10), $\boldsymbol{\psi}(t, \mathbf{y}) = (\boldsymbol{\psi}_{j,l}(t, \mathbf{y}))_{n \times n}$, and $\boldsymbol{\Lambda}(t, \mathbf{p}) = (\boldsymbol{\Lambda}_{j,l}(t, \mathbf{p}))_{n \times n}$, $\boldsymbol{\gamma}(t, y) = \text{diag}\,(u_1 - y_1, \cdots, u_j - y_j, \cdots, u_n - y_n)$ and $\lambda(t, p) = \text{diag}\,(l_1 + p_1, \cdots, l_j + p_j, \cdots, l_n + p_n)$; $\mathbf{G}$, and $\mathbf{H}$ are $n \times 1$ column vectors; $\boldsymbol{\Psi} = \text{diag}\,(\boldsymbol{\psi}_1, \cdots, \boldsymbol{\psi}_j, \cdots, \boldsymbol{\psi}_n)$ and $\boldsymbol{\Phi} = \text{diag}\,(\boldsymbol{\Lambda}_1, \cdots, \boldsymbol{\Lambda}_j, \cdots, \boldsymbol{\Lambda}_n)$ are block diagonal matrices; $\boldsymbol{\psi}_j = [\boldsymbol{\psi}_{j,1}, \cdots, \boldsymbol{\psi}_{j,l}, \cdots, \boldsymbol{\psi}_{j,n}]$, $\boldsymbol{\Lambda}_j = [\boldsymbol{\Lambda}_{j,1}, \cdots, \boldsymbol{\Lambda}_{j,l}, \cdots, \boldsymbol{\Lambda}_{j,n}]$. We also assume that $\mathbf{G}$, $\mathbf{H}$, $\boldsymbol{\Psi}$, and $\boldsymbol{\Phi}$ satisfy the local Lipschitz condition. This assumption implies that $\mathbf{g}$, $\mathbf{h}$, $\boldsymbol{\psi}$, and $\boldsymbol{\Lambda}$ also satisfy local Lipschitz condition.

Thus, the interconnected energy commodity network system is described by

$$\begin{cases} dy_j = \left(u_j - y_j\right)\left[\left(\mu_j\left(v_j + y_j\right) + \mathbf{G}_j(t,\mathbf{y})\right)dt + \delta_{j,j}dW_{j,j}(t) + \sum_{l=1}^{n}\mathbf{\Psi}_{j,l}(t,\mathbf{y})dW_{j,l}(t)\right], y_j(t_0) = y_{j0}, \\ dp_j = \left(l_j + p_j\right)\left[\left(\alpha_j\left(y_j - p_j\right) + \mathbf{H}_j(t,\mathbf{p})\right)dt + \sigma_{j,j}dZ_{j,j}(t) + \sum_{l=1}^{n}\mathbf{\Phi}_{j,l}(t,\mathbf{p})dZ_{j,l}(t)\right], p_j(t_0) = p_{j0}, j \in I(1,n), \end{cases} \tag{13}$$

where the parameters $\mu_j > 0$; $\alpha_j > 0$; $u_j > 0$; $v_j \geq 0$; $l_j \geq 0$; $\delta_{j,j} > 0$; $\sigma_{j,j} > 0$; and for $j \neq l$, $\delta_{j,l} \geq 0$; $\sigma_{j,l} \geq 0$; $j, l \in I(1,n)$; for $j \in I(1,n)$, $\mathbf{W}_j$ and $\mathbf{Z}_j$ are n-dimensional independent Wiener processes defined on a filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathscr{P})$; for $l \neq i$, $\mathbb{E}[dW_{j,l}dW_{k,i}] = 0$, and for $l = i$, $\mathbb{E}[dW_{j,l}dW_{k,i}] = dt$; the filtration function $(\mathscr{F}_t)_{t\geq 0}$ is right continuous; for each $t \geq 0$, each $\mathscr{F}_t$ contains all $\mathscr{P}$-null sets in $\mathscr{F}$; the n-dimensional random vectors $\mathbf{y}(t_0)$ and $\mathbf{p}(t_0)$ are $\mathscr{F}_{t_0}$ measurable.

The network system of stochastic differential equations in (13) can be written as follows:

$$\begin{cases} d\mathbf{y} = \mathbf{a}(t, \mathbf{y})dt + \mathbf{Y}(t, \mathbf{y})d\mathbf{W}(t), \mathbf{y}(t_0) = \mathbf{y}_0, \\ d\mathbf{p} = \mathbf{b}(t, \mathbf{y}, \mathbf{p})dt + \boldsymbol{\sigma}(t, \mathbf{p})d\mathbf{Z}(t), \mathbf{p}(t_0) = \mathbf{p}_0, \end{cases} \tag{14}$$

where

$$\begin{cases} \mathbf{a}(t,\mathbf{y}) = \begin{pmatrix} (u_1 - y_1)[\mu_1(v_1 + y_1) + \mathbf{G}_1(t,\mathbf{y})] \\ (u_2 - y_2)[\mu_2(v_2 + y_2) + \mathbf{G}_2(t,\mathbf{y})] \\ \vdots \\ (u_n - y_n)[\mu_n(v_n + y_n) + \mathbf{G}_n(t,\mathbf{y})] \end{pmatrix}, \mathbf{b}(t,\mathbf{y},\mathbf{p}) = \begin{pmatrix} (l_1 + p_1)[\alpha_1(y_1 - p_1) + \mathbf{H}_1(t,\mathbf{p})] \\ (l_2 + p_2)[\alpha_2(y_2 - p_2) + \mathbf{H}_2(t,\mathbf{p})] \\ \vdots \\ (l_n + p_n)[\alpha_n(y_n - p_n) + \mathbf{H}_n(t,\mathbf{p})] \end{pmatrix}, \\ \mathbf{Y}(t,y) = \operatorname{diag}\left(A_1(\mathbf{y}), \cdots, A_j(\mathbf{y}), .., A_n(\mathbf{y})\right), \boldsymbol{\sigma}(t,\mathbf{p}) = \operatorname{diag}\left(B_1(\mathbf{p}), \cdots, B_j(\mathbf{p}), \cdots, B_n(\mathbf{p})\right), \end{cases} \tag{15}$$

and

$$\begin{cases} \mathbf{A}_j(y) = \left(u_j - y_j\right)\left(\boldsymbol{\Psi}_{j,1} \quad \boldsymbol{\Psi}_{j,2} \quad \cdots \quad \boldsymbol{\Psi}_{j,j-1} \quad \delta_{j,j} + \boldsymbol{\Psi}_{j,j} \quad \boldsymbol{\Psi}_{j,j+1} \quad \cdots \quad \boldsymbol{\Psi}_{j,n}\right), \\ \mathbf{B}_j(p) = \left(l_j + p_j\right)\left(\boldsymbol{\Phi}_{j,1} \quad \boldsymbol{\Phi}_{j,2} \quad \cdots \quad \boldsymbol{\Phi}_{j,j-1} \quad \sigma_{j,j} + \boldsymbol{\Phi}_{j,j} \quad \boldsymbol{\Phi}_{j,j+1} \quad \cdots \quad \boldsymbol{\Phi}_{j,n}\right); \end{cases} \tag{16}$$

$\mathbf{W} = [\mathbf{W}_1, \cdots \mathbf{W}_j, \cdots, \mathbf{W}_n]^T$ and $\mathbf{Z} = [\mathbf{Z}_1, \cdots \mathbf{Z}_j, \cdots, \mathbf{Z}_n]^T$ are block matrices; $\mathbf{W}_j = [W_{j,1}, \cdots W_{j,2}, \cdots, W_{j,n}]^T$, $\mathbf{Z}_j = [Z_{j,1}, \cdots Z_{j,2}, \cdots, Z_{j,n}]^T$; and $\mathbf{Y}(t, \mathbf{y})$, $\boldsymbol{\sigma}(t, \mathbf{p})$ are a $n \times n$ block matrix with each entries having order $1 \times n$.

In the next section, we outline the model validation problems of (14), namely, the existence and uniqueness of solution process.

3. Mathematical Model Validation

Here, we validate the mathematical model derived in Section 2. We note that the classical existence and uniqueness theorem [23–25] is not directly applicable to (14). We need to modify the existence and uniqueness results. The modification is based on Theorem 3.4 [23]. We show the global existence of the solution process of a system of differential equations (14). We note that the rate functions $\mathbf{a}$, $\mathbf{b}$, $\mathbf{Y}$, and $\boldsymbol{\sigma}$ in stochastic system of differential equations (14) do not satisfy the classical existence and uniqueness conditions [23]. However, these rate functions do satisfy the local Lipschitz condition. Therefore, we construct sequences of functions for the drift and diffusion coefficients of interconnected dynamic system (14) so that the classical conditions for existence and uniqueness theorem are applicable. The construction of modification scheme is as follows: First, we define a cylindrical subset $[t_0, \infty) \times U_m$ of $[0, \infty) \times \mathfrak{R}^n$ for $t_0 \in [0, \infty)$, $m \geq 1$, where U_m is an n- dimensional sphere with radius m defined by

$$U_m = \mathbb{B}(\mathbf{x}_0, m) = \{\mathbf{x} \in \mathfrak{R}^n : \|\mathbf{x} - \mathbf{x}_0\| < m\}, \tag{17}$$

for any $m \geq 1$. We note that U_m is inscribed in n-dimensional parallelepiped $\mathbb{R}(\mathbf{x} - \mathbf{x}_0, m) = [-m, m] \times \cdots \times [-m, m]$ in $\mathfrak{R}^n$.

The developed stochastic network model (14) can be written as:

$$\begin{cases} d\mathbf{y} = \mathbf{a}^m(t, \mathbf{y})dt + \mathbf{Y}^m(t, \mathbf{y})d\mathrm{W}(t), \mathbf{y}(t_0) = \mathbf{y}_0, \\ d\mathbf{p} = \mathbf{b}^m(t, y, \mathbf{p})dt + \boldsymbol{\sigma}^m(t, \mathbf{p})d\mathbf{Z}(t), \mathbf{p}(t_0) = \mathbf{p}_0, \end{cases} \tag{18}$$

where

$$\begin{cases} \mathbf{a}^m(t, \mathbf{y}) = \mathbf{a}(t, \mathbf{q}(\mathbf{y}, m)), \\ \mathbf{Y}^m(t, \mathbf{y}) = \mathbf{Y}(t, \mathbf{q}(\mathbf{y}, m)), \\ \mathbf{b}^m(t, \mathbf{y}, \mathbf{p}) = \mathbf{b}(t, \mathbf{q}(\mathbf{y}, m), \mathbf{q}(\mathbf{p}, m)), \\ \boldsymbol{\sigma}^m(t, \mathbf{p}) = \boldsymbol{\sigma}(t, \mathbf{q}(\mathbf{p}, m)). \end{cases} \tag{19}$$

Here, for each $j \in I(1, n)$ and $\mathbf{x} \in \mathfrak{R}^n$, we define $\mathbf{q}_j(\mathbf{x}, m) = \max\{-m, \min\{x_j - x_{0j}, m\}\}$. Hence, $\mathbf{q}(\mathbf{x}, m) = [\mathbf{q}_1(\mathbf{x}, m), \cdots, \mathbf{q}_j(\mathbf{x}, m), \cdots, \mathbf{q}_n(\mathbf{x}, m)]^T$.

Remark 1. We observe that $\mathbf{q}(\mathbf{x}, m)$ satisfies the global Lipschitz condition on $\mathfrak{R}^n$ with a Lipschitz constant 1. Using this, together with the local Lipschitz condition assumption on the drift and diffusion coefficients of stochastic differential equations (14), the modified rate coefficient functions in (18) satisfy the classical existence and uniqueness conditions [26, 27]. Thus, its solution is denoted by $(\mathbf{y}_m, \mathbf{p}_m)$, for $m \geq 1$. Moreover, $(\mathbf{y}, \mathbf{p})$ is nonnegative whenever $\mathbf{y}_0$, $\mathbf{p}_0 \in \mathfrak{R}^n_+$.

Now, we apply Theorems 3.4 and 3.5 of [26] in the context of the modified system of stochastic differential equations (18) and Remark 1 to establish the global existence of solution of stochastic differential equations in (13). For this purpose, we outline the argument used in the proof of these theorems. In addition to the local Lipschitz conditions on the drift and diffusion coefficients, we further impose the following hypothesis on the coefficients:

$(\mathbf{H}_1)$

$$\begin{cases} \left|\mathbf{g}_j(t, \mathbf{y})\right| \leq a_{1,j} + \kappa_j\|\mathbf{y}\|, \\ \left|\boldsymbol{h}_j(t, \mathbf{p})\right| \leq d'_{1,j} + \gamma_j\|\mathbf{p}\|, \\ |\boldsymbol{\psi}_{j,l}(t, \mathbf{y})| \leq a_{2,j} + \tilde{\delta}_{j,l}\|\mathbf{y}\|, \\ \left|\boldsymbol{\Lambda}_{j,l}(t, \mathbf{p})\right| \leq d'_{2,j} + \tilde{\sigma}_{j,l}\|\mathbf{p}\|, \end{cases} \tag{20}$$

where for $i \in I(1, 2)$, $a_{i,j}$, $d'_{i,j}$ are nonnegative; κ_j, γ_j, $\tilde{\delta}_{j,l}$, $\tilde{\sigma}_{j,l} \in \mathfrak{R}_+$. From (18), we further remark that the dynamic of the mean spot price vector $\mathbf{y}$ is decoupled with the dynamic of spot price p. Now, we first apply Theorems 3.4 and 3.5 of [23] in the context of modified system of stochastic differential equations (18) and hypothesis $(\mathbf{H}_1)$ to establish the global existence of solution of the completely decoupled subsystem of stochastic differential equations in (18). For this purpose, we outline the argument used in the proof of these theorems.

Definition 2. Let $\boldsymbol{\tau}_m$ be the first exit time of the solution process $\mathbf{y}_m$ from the set $\mathbb{B}(\mathbf{y}_0, m)$. Define $\boldsymbol{\tau}$ to be the (finite or infinite) limit of the monotone increasing sequence $\boldsymbol{\tau}_m$ as $m \longrightarrow \infty$.

$$\boldsymbol{\tau} = \lim_{m\to\infty} \boldsymbol{\tau}_m. \tag{21}$$

We wish to show that

$$\text{Prob}\,(\boldsymbol{\tau} = \infty) = 1. \tag{22}$$

In the following, we present a result that is parallel to Theorem 3.5 [26] in the context of the completely decoupled subsystem of stochastic differential equation (14). For this purpose, it is enough to exhibit the global existence result for the transformed system (18).

Lemma 3. *For $m \geq 1$, and $y_0 \in \mathfrak{R}^n_+$, let $\mathbf{y}_m(t) = y_m(t, t_0, y_0)$ be the solution of the completely decoupled subsystem of (18), and let the hypothesis $(\boldsymbol{H}_1)$ be satisfied. Let V_1 be a function defined on $[t_0,\infty) \times \mathfrak{R}^n_+$ into $\mathfrak{R}_+$ defined by*

$$V_1(t, y) = \ln\left(\|\mathbf{y}\|^2 + e\right), \tag{23}$$

Then, there exists some constant $\mathbf{c}_1 > 0$ such that

$$\begin{cases} \mathbf{L}V_1 \leq \mathbf{c}_1 V_1, \\ V_{1,m} = \inf_{\|y\|>m} V_1(t, \mathbf{y}) \longrightarrow \infty as\, m \longrightarrow \infty, \end{cases} \tag{24}$$

where $\mathbf{L}$ is the differential operator with respect to (14); $e = \exp(1)$.

Moreover, the global existence of solution of the completely decoupled subsystem of (14) is follows.

Proof. It is obvious that $V_1 \in \mathscr{C}_{1,2}$ on $[t_0,\infty) \times \mathfrak{R}^n_+ \to \mathfrak{R}_+$. In fact, $(\partial V_1(t, \mathbf{y}))/\partial y_j = 2y_j/(\mathbf{y} + e)$, $\partial^2 V_1(t, \mathbf{y})/\partial y_j^2 = (2/(\mathbf{y} + e)) - (4y_j^2/(\|\mathbf{y}\|^2 + e)^2)$, and $\partial^2 V_1(t, \mathbf{y})/\partial y_i y_j = -4y_i y_j/(\|\mathbf{y}\|^2 + e)^2$ exist and are continuous functions defined on

$[t_0,\infty) \times \mathfrak{R}_+^n \longrightarrow \mathfrak{R}$. Moreover, the $\mathbf{L}$ operator with respect to the completely decoupled component is as follows:

$$\begin{aligned}
\mathbf{L}V_1(t,\mathbf{y}) &= \sum_{j=1}^{n}\left[\mu_j\left(u_j-y_j\right)\left(v_j+y_j\right)+\mathbf{g}_j(t,\mathbf{y})\right]\frac{\partial V_1(t,y)}{\partial y_j} \\
&+\frac{1}{2}\sum_{j=1}^{n}\left[\left[\delta_{j,j}\left(u_j-y_j\right)+\boldsymbol{\psi}_{jj}(t,\mathbf{y})\right]^2\right. \\
&\left.+\sum_{l\neq j}^{n}\boldsymbol{\psi}_{j,l}^2(t,\mathbf{y})\right]\frac{\partial^2 V_1(t,y)}{\partial y_j^2}+\frac{1}{2}\sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n} \\
&\cdot\left[\sum_{l\neq i,j}^{n}\psi_{i,l}(t,\mathbf{y})\psi_{j,l}(t,\mathbf{y})+2\left[\delta_{i,i}(u_i-y_i)+\psi_{i,i}\right]\psi_{j,i}\right] \\
&\cdot\frac{\partial^2 V_1(t,y)}{\partial y_i y_j} \\
&=\sum_{j=1}^{n}\mu_j\left(-\left[y_j-\left(\frac{u_j-v_j}{2}\right)\right]^2+\left(\frac{u_j+v_j}{2}\right)^2\right)\frac{2y_j}{\left(\|\mathbf{y}\|^2+e\right)} \\
&+\sum_{j=1}^{n}\frac{2\mathbf{g}_j(t,\mathbf{y})y_j}{\left(\|\mathbf{y}\|^2+e\right)}+\frac{1}{2}\sum_{j=1}^{n}\left[\delta_{j,j}\left(u_j-y_j\right)+\psi_{j,j}\right]^2 \\
&\cdot\left(\frac{2}{\left(\|\mathbf{y}\|^2+e\right)}-\frac{4y_j^2}{\left(\|\mathbf{y}\|^2+e\right)^2}\right)+\frac{1}{2}\sum_{j=1}^{n}\sum_{l\neq j}^{n}\psi_{j,l}^2(t,\mathbf{y}) \\
&\cdot\left(\frac{2}{\left(\|\mathbf{y}\|^2+e\right)}-\frac{4y_j^2}{\left(\|\mathbf{y}\|^2+e\right)^2}\right)-\frac{1}{2}\sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n} \\
&\cdot\left[\sum_{l\neq i,j}^{n}\psi_{i,l}(t,y_l)\psi_{j,l}(t,\mathbf{y})+2\left[\delta_{i,i}(u_i-y_i)+\psi_{i,i}\right]\psi_{j,i}\right] \\
&\cdot\frac{4y_iy_j}{\left(\|\mathbf{y}\|^2+e\right)^2}\le 2\sum_{j=1}^{n}\mu_j\left(\frac{u_j+v_j}{2}\right)^2\frac{y_j}{\left(\|\mathbf{y}\|^2+e\right)} \\
&+\sum_{j=1}^{n}\frac{2\mathbf{g}_j(t,\mathbf{y})y_j}{\left(\|\mathbf{y}\|^2+e\right)}+\sum_{j=1}^{n}\frac{\left[\delta_{j,j}\left(u_j-y_j\right)+\psi_{jj}(t,\mathbf{y})\right]^2}{\left(\|\mathbf{y}\|^2+e\right)} \\
&+\sum_{j=1}^{n}\sum_{l\neq j}^{n}\frac{\psi_{j,l}^2(t,y)}{\left(\|\mathbf{y}\|^2+e\right)}-\frac{1}{2}\sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n} \\
&\cdot\left[\sum_{l\neq i,j}^{n}\psi_{i,l}(t,y_l)\psi_{j,l}(t,\mathbf{y})+2\left[\delta_{i,i}(u_i-y_i)+\psi_{i,i}\right]\psi_{j,i}\right] \\
&\cdot\frac{4y_iy_j}{\left(\|\mathbf{y}\|^2+e\right)^2}\le 2\sum_{j=1}^{n}\mu_j\left(\frac{u_j+v_j}{2}\right)^2 \\
&+\sum_{j=1}^{n}\frac{\left(a_{1,j}+\kappa_j\|\mathbf{y}\|^2\right)^2+y_j^2}{\left(\|\mathbf{y}\|^2+e\right)} \\
&+2\sum_{j=1}^{n}\left[\delta_{j,j}^2(u_j+1)^2+\left(a_{2,j}+\tilde{\delta}_{j,j}\right)^2\right] \\
&+\sum_{j=1}^{n}\sum_{l\neq j}^{n}\left(a_{2,j}+\tilde{\delta}_{j,l}\right)^2+2\sum_{i=1}^{n}\sum_{j\neq i}^{n}\sum_{l\neq i,j}^{n}\left(a_{2,i}+\tilde{\delta}_{i,l}\right)\left(a_{2,j}+\tilde{\delta}_{j,l}\right) \\
&+4\sum_{i=1}^{n}\sum_{j\neq i}^{n}\left(a_{2,j}+\tilde{\delta}_{j,i}\right)\left[\delta_{i,i}(u_i+1)+\left(a_{2,i}+\tilde{\delta}_{i,i}\right)\right] \\
&\le c_1V_1(t,\mathbf{y}),
\end{aligned} \tag{25}$$

where

$$\begin{aligned}
c_1 &= 2\sum_{j=1}^{n}\mu_j\left(\frac{u_j+v_j}{2}\right)^2+1+\sum_{j=1}^{n}\left(a_{1,j}+\kappa_j\right)^2 \\
&+2\sum_{j=1}^{n}\left[\delta_{j,j}^2\left(u_j+1\right)^2+\left(a_{2,j}+\tilde{\delta}_{j,j}\right)^2\right] \\
&+\sum_{j=1}^{n}\sum_{l\neq j}^{n}\left(a_{2,j}+\tilde{\delta}_{j,l}\right)^2+2\sum_{i=1}^{n}\sum_{j\neq i}^{n}\sum_{l\neq i,j}^{n}\left(a_{2,i}+\tilde{\delta}_{i,l}\right)\left(a_{2,j}+\tilde{\delta}_{j,l}\right) \\
&+4\sum_{i=1}^{n}\sum_{j\neq i}^{n}\left(a_{2,j}+\tilde{\delta}_{j,i}\right)\left[\delta_{i,i}(u_i+1)+\left(a_{2,i}+\tilde{\delta}_{i,i}\right)\right].
\end{aligned} \tag{26}$$

Furthermore, $\inf_{\|\mathbf{y}\|>m} V_1(t,\mathbf{y}) \longrightarrow \infty$ as $m \longrightarrow \infty$.

To show that $\mathrm{Prob}(\boldsymbol{\tau}=\infty)=1$, we define a function

$$V(t,\mathbf{y}) = V_1(t,\mathbf{y})\exp\left\{-\mathbf{c}_1(t-t_0)\right\}. \tag{27}$$

It is obvious that $\mathbf{L}V \le 0$. By defining $\tau_m(t)=\min(\tau_m,t)$; $\mathscr{Y}(t)=\mathbf{y}_m(t)$ for $t<\tau_m$; and imitating the argument of Lemma 4 [23], we have

$$\mathbb{E}\{V_1(\tau_m(t),\mathscr{Y}(\tau_m(t))\}\le e^{c_1(t-t_0)}\mathbb{E}V_1(t_0,\mathbf{y}(t_0)). \tag{28}$$

Hence,

$$\begin{aligned}
\mathrm{Prob}\{\tau_m\le t\} &\le \frac{e^{\mathbf{c}_1(t-t_0)}\mathbb{E}V_1(t_0,\mathscr{Y}(t_0))}{\inf_{\|y\|>m,u>t_0}V_1(u,y)}\longrightarrow 0 \text{ as } m \\
&\longrightarrow \infty \text{ by } (24).
\end{aligned} \tag{29}$$

The global existence and uniqueness of the solution of the first component of (18) follows by letting $m\longrightarrow\infty$. Hence, from this and the fact that the solution process of transformed system (18) is almost surely identical with the solution process of the original system (14), we conclude the global existence and uniqueness of (14).

Following the idea of Lemma 3, we present a global existence and uniqueness of solution of the system of stochastic differential equations governing the subsystem p in (14).

Lemma 4. *For $m\ge 1$, and $\boldsymbol{y}_0$, $\mathbf{p}_0\in\mathfrak{R}_+^n$, let $\mathbf{p}_m(t)=\mathbf{p}_m(t,t_0,\mathbf{p}_0)$ be the solution of the system of stochastic differential equations governing the subsystem $\mathbf{p}$ described in (18), and let the hypothesis ($\boldsymbol{H}_1$) be satisfied. Let V_2 be a nonnegative function on $[t_0,\infty)\times\mathfrak{R}_+^n$ into $\mathfrak{R}_+$ defined by*

$$V_2(t,\mathbf{p})=\ln\left(\|\mathbf{p}\|^2+e\right)+\sum_{j=1}^{n}\frac{\alpha_j}{2}\int_t^T\left(y_j(s)+l_j\right)^2ds, \tag{30}$$

Then, there exist a constant $c > 0$ such that

$$\begin{cases} \mathbf{L}V_2 \le cV_2 \\ V_{2,m} = \inf_{\|\mathbf{p}\|>m} V_2(t,\mathbf{p}) \longrightarrow \infty \text{ as } m \longrightarrow \infty. \end{cases} \tag{31}$$

where $\mathbf{L}$ is the differential operator with respect to (14); $e = \exp(1)$.

Moreover, the global existence of the solution of the system of stochastic differential equations governing the subsystem p described in (14) is as follows.

Proof. It is obvious that $V_2 \in \mathscr{C}_{1,2}$ on $[t_0,\infty)\times\mathfrak{R}_+^n \longrightarrow \mathfrak{R}_+$. In fact, $\partial V_2(t,\mathbf{p})/\partial t = -\sum_{j=1}^n (\alpha_j/2)(y_j(t)+l_j)^2$, $\partial V_2(t,\mathbf{p})/\partial p_j = 2p_j/(\|\mathbf{p}\|^2+e)$, $\partial^2 V_2(t,\mathbf{p})/\partial p_j^2 = (2/(\|\mathbf{p}\|^2+e)) - (4p_j^2/(\|\mathbf{p}\|^2+e)^2)$, and $\partial^2 V_2(t,\mathbf{p})/\partial p_i p_j = -4p_ip_j/(\|\mathbf{p}\|^2+e)^2$ exist and are continuous functions defined on $[t_0,\infty)\times\mathfrak{R}_+^n \longrightarrow \mathfrak{R}$. Moreover, the $\mathbf{L}$ operator with respect to the system of stochastic differential equations governing the subsystem $\mathbf{p}$ described in (14) is obtained as follows:

$$\begin{aligned}
\mathbf{L}V_2(t,\mathbf{p}) &= -\sum_{j=1}^n \frac{\alpha_j}{2}\left(y_j(t)+l_j\right)^2 + \sum_{j=1}^n \Big[\alpha_j\left(y_j-p_j\right)\left(l_j+p_j\right) \\
&\quad + \mathbf{h}_j(t,\mathbf{p})\Big]\frac{2p_j}{(\|\mathbf{p}\|^2+e)}, \\
&\quad + \frac{1}{2}\sum_{j=1}^n\Bigg[\left[\sigma_{j,j}\left(l_j+p_j\right)+\mathbf{\Lambda}_{j,j}(t,\mathbf{p})\right]^2 \\
&\quad + \sum_{l\neq j}^n \mathbf{\Lambda}_{j,l}(t,\mathbf{p})^2\Bigg]\left[\frac{2}{(\|\mathbf{p}\|^2+e)} - \frac{4p_j^2}{(\|\mathbf{p}\|^2+e)^2}\right] \\
&\quad + \frac{1}{2}\sum_{i=1}^n\sum_{\substack{j=1\\ j\neq i}}^n\left[\sum_{l=1}^n \mathbf{\Lambda}_{i,l}\mathbf{\Lambda}_{j,l} + 2[\sigma_{i,i}(l_i+p_i)+\mathbf{\Lambda}_{i,i}]\mathbf{\Lambda}_{j,i}\right] \\
&\quad \cdot\left[-\frac{4p_ip_j}{(\|\mathbf{p}\|^2+e)^2}\right] \\
&= -\sum_{j=1}^n\frac{\alpha_j}{2}\left(y_j+l_j\right)^2 + \sum_{j=1}^n\alpha_j\left(-\left[p_j-\frac{y_j-l_j}{2}\right]^2\right. \\
&\quad \left. + \left(\frac{y_j+l_j}{2}\right)^2\right)\frac{2p_j}{(\|\mathbf{p}\|^2+e)} \\
&\quad + \sum_{j=1}^n\frac{2\mathbf{h}_j(t,\mathbf{p})p_j}{(\|\mathbf{p}\|^2+e)} + \frac{1}{2}\sum_{j=1}^n\left[\sigma_{j,j}\left(l_j+p_j\right)+\mathbf{\Lambda}_{j,j}(t,\mathbf{p})\right]^2 \\
&\quad \cdot\left(\frac{2}{(\|\mathbf{p}\|^2+e)} - \frac{4p_j^2}{(\|\mathbf{p}\|^2+e)^2}\right) \\
&\quad + \frac{1}{2}\sum_{j=1}^n\sum_{l\neq j}^n\mathbf{\Lambda}_{j,l}(t,\mathbf{p})^2\left(\frac{2}{(\|\mathbf{p}\|^2+e)} - \frac{4p_j^2}{(\|\mathbf{p}\|^2+e)^2}\right) \\
&\quad - \frac{1}{2}\sum_{i=1}^n\sum_{\substack{j=1\\ j\neq i}}^n\left[\sum_{l=1}^n \mathbf{\Lambda}_{i,l}\mathbf{\Lambda}_{j,l} + 2[\sigma_{i,i}(l_i+p_i)+\mathbf{\Lambda}_{i,i}]\mathbf{\Lambda}_{j,i}\right] \\
&\quad \cdot\frac{4p_ip_j}{(\|\mathbf{p}\|^2+e)^2} \le -\sum_{j=1}^n\frac{\alpha_j}{2}\left(y_j+l_j\right)^2 \\
&\quad + \sum_{j=1}^n\frac{\alpha_j}{2}\left(y_j+l_j\right)^2\frac{p_j}{(\|\mathbf{p}\|^2+e)} \\
&\quad + \sum_{j=1}^n\frac{\left[d'_{1,j}+\gamma_j\|\mathbf{p}\|\right]^2+p_j^2}{(\|\mathbf{p}\|^2+e)} \\
&\quad + \sum_{j=1}^n\frac{\left[\sigma_{j,j}\left(l_j+p_j\right)+\mathbf{\Lambda}_{j,j}(t,\mathbf{p})\right]^2}{(\|\mathbf{p}\|^2+e)} \\
&\quad + \sum_{j=1}^n\sum_{l\neq j}^n\left(d'_{2,j}+\tilde{\sigma}_{j,l}\right)^2 + 2\sum_{i=1}^n\sum_{j\neq i}^n\sum_{l\neq i,j}^n\left(d'_{2,i}+\tilde{\sigma}_{i,l}\right) \\
&\quad \cdot\left(d'_{2,j}+\tilde{\sigma}_{j,l}\right) + 4\sum_{i=1}^n\sum_{j\neq i}^n\left(d'_{2,j}+\tilde{\sigma}_{j,i}\right) \\
&\quad \cdot\left[\sigma_{i,i}(l_i+1)+\left(d'_{2,i}+\tilde{\sigma}_{i,i}\right)\right] \le cV_2(t,\mathbf{p}),
\end{aligned} \tag{32}$$

where

$$\begin{aligned}
c &= 1 + \sum_{j=1}^n\left[d'_{1,j}+\gamma_j\right]^2 + 2\sum_{j=1}^n\left[\sigma_{j,j}^2\left(l_j+1\right)^2+\left(d'_{2,j}+\tilde{\sigma}_{j,j}\right)^2\right] \\
&\quad + \sum_{j=1}^n\sum_{l\neq j}^n\left(d'_{2,j}+\tilde{\sigma}_{j,l}\right)^2 + 2\sum_{i=1}^n\sum_{j\neq i}^n\sum_{l\neq i,j}^n\left(d'_{2,i}+\tilde{\sigma}_{i,l}\right)\left(d'_{2,j}+\tilde{\sigma}_{j,l}\right) \\
&\quad + 4\sum_{i=1}^n\sum_{j\neq i}^n\left(d'_{2,j}+\tilde{\sigma}_{j,i}\right)\left[\sigma_{i,i}(l_i+1)+\left(d'_{2,i}+\tilde{\sigma}_{i,i}\right)\right].
\end{aligned} \tag{33}$$

Furthermore, $\inf_{\|p\|>m} V_2(t,\mathbf{p}) \longrightarrow \infty$ as $m \longrightarrow \infty$. Following the final argument used in proving the global existence of y in Lemma 3, we conclude that there exists a unique global solution to the interconnected system of stochastic differential (14).

In the next section, we discuss the case where we incorporate jump process into the system $(\mathbf{y},\mathbf{p})$.

4. Energy Commodity Model with and without Jumps

Due to random interventions that affects the price of energy commodities, we introduce random perturbations described by a continuous jump in our model. We follow the approach discussed in [28, 29] where a class of stochastic hybrid dynamic processes is investigated.

Let $K \geq 0$ be the number of jumps on the interval $[t_0, T]$, for $T > 0$. For $K \neq 0$, let $T_1, \cdots, T_K$ be the jump times over a time interval $[t_0, T]$ such that $t_0 = T_0 \leq T_1 < \cdots < T_K \leq T$, where T_i denotes the time at which the ith jump occurred in the system $(\mathbf{y}, \mathbf{p})$. For $K = 0$, no jump has occurred on the interval $[t_0, T]$. We denote the ith subinterval by $T_{i-1} \leq t < T_i$. Knowing the global existence and uniqueness solution process of the system (14) on the interval $[t_0, T]$, $T > 0$ in Section 3, for $i \in I(1, K^*)$, where $K^* = K$ if $T_K = T$, and $K^* = K + 1$ if $T_K < T$ and $K \neq 0$, we consider the solution process on each subinterval $[T_{i-1}, T_i)$, between jumps. For $i \in I(1, K^*)$ and $K = 0$, we have $I(1, K) = \varnothing$ and $I(1, K^*) = \{1\}$. In this case, we consider the solution process on $[t_0, T]$. The interconnected system is governed by the following system of continuous time stochastic process:

$$\begin{cases} d\mathbf{y}^{i-1} = \mathbf{a}^{i-1}(t, \mathbf{y})dt + \mathbf{Y}^{i-1}(t, \mathbf{y})d\mathbf{W}(t), \mathbf{y}(T_{i-1}) = \mathbf{y}^{i-1}, t \in [T_{i-1}, T_i) \\ d\mathbf{p}^{i-1} = \mathbf{b}^{i-1}(t, \mathbf{y}, \mathbf{p})dt + \boldsymbol{\sigma}^{i-1}(t, \mathbf{p})d\mathbf{Z}(t), \mathbf{p}(T_{i-1}) = \mathbf{p}^{i-1}, t \in [T_{i-1}, T_i), i \in I(1, K^*) \\ \mathbf{y}^i = \mathbf{\Pi}^i \mathbf{y}^{i-1}\left(T_i^-, T_{i-1}, \mathbf{y}^{i-1}\right), \\ \mathbf{p}^i = \boldsymbol{\Theta}^i \mathbf{p}^{i-1}\left(T_i^-, T_{i-1}, \mathbf{y}^{i-1}, \mathbf{p}^{i-1}\right), \end{cases} \tag{34}$$

where

$$\begin{aligned} \Pi^i &= \operatorname{diag}\left(\pi_1^i, \pi_2^i, \cdots, \pi_n^i\right), \\ \Theta^i &= \operatorname{diag}\left(\theta_1^i, \theta_2^i, \cdots, \theta_n^i\right), \end{aligned} \tag{35}$$

$(\mathbf{y}^{i-1}(t, T_{i-1}, \mathbf{y}^{i-1}), \mathbf{p}^{i-1}(t, T_{i-1}, \mathbf{y}^{i-1}, \mathbf{p}^{i-1}))$ is the solution of the system of equation (34); for $K \neq 0$ and $i \in I(1, K^*)$, Π^i and Θ^i are jump coefficient matrices. These jump coefficients are estimated by $\widehat{\pi}_j^i = y_j(T_i)/\lim_{t\to T_i^-} y_j^{i-1}(t, T_{i-1}, \mathbf{y}^{i-1})$; $\widehat{\theta}_j^i = p_j(T_i)/\lim_{t\to T_i^-} p_j^{i-1}(t, T_{i-1}, \mathbf{y}^{i-1}, \mathbf{p}^{i-1})$ for $j \in I(1, n)$. Following the approach in [28, 29], the solution of (34) takes the following representation:

$$\mathbf{y}(t, t_0, \mathbf{y}_0) = \begin{cases} \mathbf{y}^0(t, t_0, \mathbf{y}_0), \mathbf{y}(t_0) = \mathbf{y}_0, t_0 \leq t < T_1, \\ \mathbf{y}^1\left(t, T_1, \mathbf{y}^1\right), \mathbf{y}^1 = \mathbf{y}(T_1), T_1 \leq t < T_2, \\ \cdots \\ \mathbf{y}^{i-1}\left(t, T_{i-1}, \mathbf{y}^{i-1}\right), \mathbf{y}^{i-1} = \mathbf{y}(T_{i-1}), T_{i-1} \leq t < T_i, \\ \cdots \\ \mathbf{y}^K\left(t, T_K, \mathbf{y}^K\right), \mathbf{y}^K = \mathbf{y}(T_K), T_K \leq t \leq T, \end{cases}$$

$$p(t, t_0, y_0, p_0) = \begin{cases} \mathbf{p}^0(t, t_0, \mathbf{y}_0, \mathbf{p}_0), \mathbf{p}(t_0) = \mathbf{p}_0, t_0 \leq t < T_1, \\ \mathbf{p}^1\left(t, T_1, \mathbf{y}^1, \mathbf{p}^1\right), \mathbf{p}^1 = \mathbf{p}(T_1), T_1 \leq t < T_2, \\ \cdots \\ \mathbf{p}^{i-1}\left(t, T_{i-1}, \mathbf{y}^{i-1}, \mathbf{p}^{i-1}\right), \mathbf{p}^{i-1} = \mathbf{p}(T_{i-1}), T_{i-1} \leq t < T_i, \\ \cdots \\ \mathbf{p}^K\left(t, T_K, \mathbf{y}^K, \mathbf{p}^K\right), \mathbf{p}^K = \mathbf{p}(T_K), T_K \leq t \leq T, \end{cases} \tag{36}$$

Remark 5. In case of no jump, that is $K = 0$, (34) and (36) reduce to

$$\begin{cases} d\mathbf{y} = \mathbf{a}(t, \mathbf{y})dt + \mathbf{Y}(t, \mathbf{y})d\mathbf{W}(t), \mathbf{y}(t_0) = \mathbf{y}_0, \\ d\mathbf{p} = \mathbf{b}(t, \mathbf{y}, \mathbf{p})dt + \boldsymbol{\sigma}(t, \mathbf{p})d\mathbf{Z}(t), \mathbf{p}(t_0) = \mathbf{p}_0, t_0 \leq t \leq T, \end{cases} \tag{37}$$

$$\begin{cases} \mathbf{y}(t, t_0, \mathbf{y}_0), \quad \mathbf{y}(t_0) = \mathbf{y}_0, \\ \mathbf{p}(t, t_0, \mathbf{y}_0, \mathbf{p}_0), \quad \mathbf{p}(t_0) = \mathbf{p}_0, t_0 \leq t < T, \end{cases} \tag{38}$$

respectively.

5. Discrete-Time Dynamic Model for Local Sample Mean and Covariance Processes

In this section, we extend the discrete-time dynamic model for local sample mean and variance processes given in Otunuga et al. [16, 17] to a multivariate case including the jump process. The development of idea and model of statistic for mean and covariance processes was motivated by the state and parameter estimation problems of the continuous time nonlinear stochastic dynamic model of the energy commodity market network. For this purpose, we need to introduce a few definitions and notations. For more information, we direct the readers to the work of Otunuga et al. [16, 17].

For each $i \in I(1, K^*)$, let τ_{i-1} and $\bar{\gamma}_{i-1}$ be finite constant time delays such that $0 < \bar{\gamma}_{i-1} \leq \tau_{i-1}$. If $K = 0$, then there is no jump and we simply write $\tau_{i-1} = \tau$ and $\bar{\gamma}_{i-1} = \bar{\gamma}$. Here, τ_{i-1} characterizes the influence of the past performance history of state of dynamic process. $\bar{\gamma}_{i-1}$ describes the reaction or response time delays. In general, these time delays are unknown and random variables. These types of delays play an important role in developing mathematical models of continuous time [30] and discrete-time [22, 31] dynamic processes. Based on the practical nature of data collection process, it is essential to either transform these time delays into positive integers or design a suitable data collection schedule or discretization process. For this purpose, for each $i \in I(1, K^*)$, we describe the discrete version of time delays of τ_i and γ_i as follows:

$$\begin{cases} r_{i-1} = \left[\left|\dfrac{\tau_{i-1}}{\Delta t_{i-1}}\right|\right] + 1, \\ \bar{\bar{\gamma}}_{i-1} = \left[\left|\dfrac{\bar{\gamma}_{i-1}}{\Delta t_{i-1}}\right|\right] + 1, \text{for } i \in I(1, K^*). \end{cases} \quad (39)$$

Here, $[|.|]$ is the integer part of a real number, and for the sake of simplicity, we assume that $0 < \bar{\gamma}_{i-1} < 1$, $(\bar{\bar{\gamma}}_{i-1} = 1)$.

Definition 6. Let $\mathbf{x} = [x_1, x_2, \cdots, x_n]^T$ be a continuous time multivariate stochastic process defined on an interval $[t_0 - \tau, T]$ into $\mathfrak{R}^n$, for some $T > 0$. For $t \in [t_0 - \tau, T]$, let $\mathscr{F}_t$ be an increasing subsigma algebra of a complete probability space $(\Omega, \mathscr{F}, \mathscr{P})$ for which $\mathbf{x}(t)$ is $\mathscr{F}_t$ measurable. For each $i \in I(1, K^*)$, let $\mathbb{P}$ and $\mathbb{P}^{i-1}$ be a partition of $[t_0 - \tau, T]$ and $[T_{i-1} - \tau_{i-1}, T_i]$, respectively. The partition $\mathbb{P}^{i-1}$ is derived by decomposing the partition $\mathbb{P}$. For each $i \in I(1, K^*)$, the partitions $\mathbb{P}$ and $\mathbb{P}^{i-1}$ are defined as follows:

$$\begin{cases} \mathbb{P} = \{t_k : t_k = t_0 + k\Delta t, \quad k \in I(-r, N)\}, \\ \mathbb{P}^{i-1} = \left\{t_k^{i-1} : t_k^{i-1} = T_{i-1} + k\Delta t, k \in I(-r_{i-1}, N_{i-1})\right\}, \end{cases} \quad (40)$$

where t_k^{i-1} represents time t_k in the interval $[T_{i-1} - \tau_{i-1}, T_i]$, $\Delta t = (T - t_0)/N \equiv (T_i - T_{i-1})/N_{i-1}$, and $T_0 = t_0$.

Remark 7. We define $\mathscr{S}_i = \sum_{l=1}^{i} N_{l-1}$ with $\mathscr{S}_0 = 0$. For $K \neq 0$, we note that we can write $\mathbb{P}$ as $\{t_0 < t_1 < \cdots < t_{N_0} < t_{N_0+1} < \cdots . < t_{N_0+N_1} < t_{N_0+N_1+1} < \cdots . < t_{\mathscr{S}_{i-1}} < t_{\mathscr{S}_{i-1}+1} < \cdots < t_{\mathscr{S}_i} < \cdots < T\}$. From this, it follows directly that the jump times T_i are contained in $\mathbb{P}$. For any $t_k^{i-1} \in \mathbb{P}^{i-1}$, $k \in [0, N_{i-1}]$, we have $t_k^{i-1} \in \mathbb{P}$. Hence, there is a relationship between elements of $\mathbb{P}^{i-1}$ with $\mathbb{P}$ that is described by $t_k^{i-1} = t_{\mathscr{S}_{i-1}+k}$, for $k \in I(0, N_i)$. In fact, the relationship between a set of jump times $\{T_1, T_2, \cdots, T_K\}$ and the partition $\mathbb{P}$ defined in (40) is as $T_i \rightarrow t_{\mathscr{S}_i}$, where N_{i-1} is the size of partition $\mathbb{P}^{i-1}$ of the subinterval $[T_{i-1}, T_i]$. It follows that $\mathscr{S}_{K^*} = N$. Using these facts, and noting that if $K = 0$, then $\mathbb{P}^{i-1} = \mathbb{P}$, $N_{i-1} = N$, $\tau_{i-1} = \tau$, $\gamma_{i-1} = \gamma$, $r_{i-1} = r$, $\eta_{i-1} = \eta$, $t_k^{i-1} = t_k$ and (40) reduces to

$$\mathbb{P}^{i-1} = \left\{t_k^{i-1} : t_k^{i-1} = T_{i-1} + k\Delta t, \, k \in I(-r_{i-1}, N_{i-1})\right\}. \quad (41)$$

For each $i \in I(1, K^*)$, let $\{\mathbf{x}^{i-1}(t_k^{i-1})\}_{k=-r_{i-1}}^{N_{i-1}}$ be a finite sequence corresponding to the stochastic process x and partition $\mathbb{P}^{i-1}$ defined in (41). We simply write $\mathbf{x}(t_k^{i-1}) \equiv \mathbf{x}^{i-1}(t_k^{i-1})$. We further recall that $x(t_k^{i-1})$ is $\mathscr{F}_{t_k^{i-1}}$ measurable for $k \in I(-r_{i-1}, N_{i-1})$. We also recall the definition of forward time shift operator F [16, 17, 32]:

$$F^l \mathbf{x}\left(t_k^{i-1}\right) = \mathbf{x}\left(t_{k+l}^{i-1}\right), \; l \geq 0. \quad (42)$$

Definition 8. For $q_{i-1} = 1$ and $r_{i-1} \geq 1$, each $k \in I(0, N_{i-1})$ and each $m_k^{i-1} \in I(2, r_{i-1} + \mathscr{S}_{i-1} + k - 1)$, a partition P_k^{i-1} of closed interval $[t_{k-m_k^{i-1}}^{i-1}, t_{k-1}^{i-1}]$ is called local at time t_k^{i-1}, and it is defined by

$$P_k^{i-1} := t_{k-m_k^{i-1}}^{i-1} < t_{k-m_k^{i-1}+1}^{i-1} < \cdots < t_{k-1}^{i-1}. \quad (43)$$

Moreover, P_k^{i-1} is referred to as the m_k^{i-1} point subpartition of the partition $\mathbb{P}^{i-1}$ in (41) of the closed subinterval $[t_{k-m_k^{i-1}}^{i-1}, t_{k-1}^{i-1}]$ of $[-\tau_{i-1}, T_i]$.

Remark 9. We note that for $K = 0$, that is, no jump, we have $P_k^{i-1} = P_k$, $m_k^{i-1} = m_k$, $t_{k-m_k^{i-1}}^{i-1} = t_{k-m_k}^{i-1}$, and $t_{k-1}^{i-1} = t_{k-1}$, where P_k is referred as the m_k point subpartition of partition $\mathbb{P}$ in (40) of the closed subinterval $[t_{k-m_k}, t_k]$ of $[t_0 - \tau, T]$ for $k \in I(0, N)$.

Definition 10. For each $i \in I(1, K^*)$, $k \in I(0, N_{i-1})$, and $m_k^{i-1} \in I(2, r_{i-1} + \mathscr{S}_{i-1} + k - 1)$, a local finite sequence $\mathbf{S}_{m_k^{i-1},k}$ at t_k^{i-1} of size m_k^{i-1} is the restriction [33] of $\{\mathbf{x}(t_k^{i-1})\}_{k=-r_{i-1}}^{Ni-1}$ to P_k^{i-1} in (43). This restriction sequence is defined by

$$\mathbf{S}_{m_k^{i-1},k} := \left\{F^l \mathbf{x}\left(t_{k-1}^{i-1}\right)\right\}_{l=-m_k^{i-1}+1}^{0}. \quad (44)$$

As m_k^{i-1} varies from 2 to $r_{i-1} + \mathscr{S}_{i-1} + k - 1$, the corresponding respective local sequence $\mathbf{S}_{m_k^{i-1},k}$ at t_k^{i-1} varies from $\{\mathbf{x}(t_l^{i-1})\}_{l=k-2}^{k-1}$ to $\{\mathbf{x}(t_l^{i-1})\}_{l=-r_{i-1}+\mathscr{S}_{i-1}+1}^{k-1}$. As a result of this, the sequence defined in (44) is also called a m_k^{i-1} local moving sequence. Furthermore, the average corresponding to the local sequence $S_{m_k^{i-1},k}$ in (44) is defined by

$$\bar{\mathbf{S}}_{m_k^{i-1},k} = \frac{1}{m_k^{i-1}} \sum_{l=-m_k^{i-1}+1}^{0} F^l \mathbf{x}\left(t_{k-1}^{i-1}\right). \quad (45)$$

The average/mean defined in (45) is also called the m_k^{i-1} local average/mean. For $i \in I(1, K^*)$ and $k \in I(0, N_{i-1})$, the m_k^{i-1} local covariance matrix corresponding to the local sequence $\mathbf{S}_{m_k^{i-1},k}$ in (44) is defined by

$$\sum_{m_k^{i-1},k} = \begin{pmatrix} s_{m_k^{i-1},k}^{1,1} & s_{m_k^{i-1},k}^{1,2} & s_{m_k^{i-1},k}^{1,3} & \cdots & s_{m_k^{i-1},k}^{1,n} \\ s_{m_k^{i-1},k}^{2,1} & s_{m_k^{i-1},k}^{2,2} & s_{m_k^{i-1},k}^{2,3} & \cdots & s_{m_k^{i-1},k}^{2,n} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ s_{m_k^{i-1},k}^{n,1} & s_{m_k^{i-1},k}^{n,2} & s_{m_k^{i-1},k}^{n,3} & \cdots & s_{m_k^{i-1},k}^{n,n} \end{pmatrix}, \quad (46)$$

where $s^{j,l}_{m^{i-1}_k,k} \equiv s^{j,l}_{m^{i-1}_k,k}(x)$, $j, l \in I(1,n)$ is the local sample covariance statistic between x_j and x_l at t^{i-1}_k described by

$$s^{j,l}_{m^{i-1}_k,k} := \begin{cases} \dfrac{1}{m^{i-1}_k} \displaystyle\sum_{a=-m^{i-1}_k+1}^{0} \left(F^a x_j\left(t^{i-1}_{k-1}\right) - \dfrac{1}{m^{i-1}_k} \sum_{b=-m^{i-1}_k+1}^{0} F^b x_j\left(t^{i-1}_{k-1}\right) \right) \left(F^a x_l\left(t^{i-1}_{k-1}\right) - \dfrac{1}{m^{i-1}_k} \sum_{b=-m^{i-1}_k+1}^{0} F^b x_l\left(t^{i-1}_{k-1}\right) \right), & \text{for small } m^{i-1}_k, \\ \dfrac{1}{m^{i-1}_k - 1} \displaystyle\sum_{a=-m^{i-1}_k+1}^{0} \left(F^a x_j\left(t^{i-1}_{k-1}\right) - \dfrac{1}{m^{i-1}_k} \sum_{b=-m^{i-1}_k+1}^{0} F^b x_j\left(t^{i-1}_{k-1}\right) \right) \left(F^a x_l\left(t^{i-1}_{k-1}\right) - \dfrac{1}{m^{i-1}_k} \sum_{b=-m^{i-1}_k+1}^{0} F^b x_l\left(t^{i-1}_{k-1}\right) \right), & \text{for large } m^{i-1}_k. \end{cases} \tag{47}$$

Denoting $\mathbf{x}(k) \equiv \mathbf{x}(t^{i-1}_k)$ for $i \in I(1, K^*)$ and $k \in I(1, N_{i-1})$, the discrete-time interconnected dynamic model of local sample mean $\bar{S}_{m_k,k}$ and variance $s^2_{m_k,k}$ processes (DTIDMLSMVSP) [16, 17] is given by

$$\begin{cases} \bar{\mathbf{S}}_{m^{i-1}_{k-d_{i-1}+1},k-d_{i-1}+1} = \dfrac{m^{i-1}_{k-d_{i-1}}}{m^{i-1}_{k-d_{i-1}+1}} \bar{\mathbf{S}}_{m^{i-1}_{k-d_{i-1}},k-d_{i-1}} + \boldsymbol{\eta}_{m^{i-1}_{k-d_{i-1}},k-d_{i-1}}, \quad \bar{\mathbf{S}}_{m^{i-1}_{T_{i-1}},T_{i-1}} = \bar{\mathbf{S}}_{T_{i-1}} \\ \displaystyle\sum_{m^{i-1}_k,k} = \begin{cases} \dfrac{m^{i-1}_k - 1}{m^{i-1}_k} \left[\displaystyle\sum_{j=1}^{d_{i-1}} \left[\dfrac{m^{i-1}_{k-j}}{\prod_{l=0}^{j-1} m^{i-1}_{k-l}} \right] \sum_{m^{i-1}_{k-j},k-j} + \dfrac{m^{i-1}_{k-d_{i-1}}}{\prod_{l=0}^{d_{i-1}-1} m^{i-1}_{k-l}} \bar{\mathbf{S}}_{m^{i-1}_{k-d_{i-1}},k-d_{i-1}} \bar{\mathbf{S}}^T_{m^{i-1}_{k-d_{i-1}},k-d_{i-1}} \right] + \boldsymbol{\varepsilon}_{m^{i-1}_{k-1},k-1}, \\ \text{for small } m^{i-1}_k,\ m^{i-1}_{k-1} \le m^{i-1}_k, \\ \displaystyle\sum_{j=1}^{d_{i-1}} \left[\dfrac{m^{i-1}_{k-j} - 1}{\prod_{l=0}^{j-1} m^{i-1}_{k-l}} \right] \sum_{m^{i-1}_{k-j},k-j} + \dfrac{m^{i-1}_{k-d_{i-1}}}{\prod_{l=0}^{d_{i-1}-1} m^{i-1}_{k-l}} \bar{\mathbf{S}}_{m^{i-1}_{k-d_{i-1}},k-d_{i-1}} \bar{\mathbf{S}}^T_{m^{i-1}_{k-d_{i-1}},k-d_{i-1}} + \boldsymbol{\varepsilon}_{m^{i-1}_{k-1},k-1} \\ \text{for large } m^{i-1}_k,\ m^{i-1}_{k-1} \le m^{i-1}_k \\ \displaystyle\sum_{m^{i-1}_j,j} = \sum_{j,} i \in I(1, K^*),\ j \in I(-d_{i-1}, 0), \text{ initial conditions} \end{cases} \end{cases} \tag{48}$$

where d_{i-1} is the order of the system and

$$
\boldsymbol{\eta}=\begin{pmatrix}\eta^1\\ \eta^2\\ \vdots\\ \eta^n\end{pmatrix},
$$

$$
\varepsilon_{m_k^{i-1},k}=\begin{pmatrix}
\varepsilon^{1,1}_{m_k^{i-1},k} & \varepsilon^{1,2}_{m_k^{i-1},k} & \varepsilon^{1,3}_{m_k^{i-1},k} & \cdots & \varepsilon^{1,n}_{m_k^{i-1},k}\\
\varepsilon^{2,1}_{m_k^{i-1},k} & \varepsilon^{2,2}_{m_k^{i-1},k} & \varepsilon^{2,3}_{m_k^{i-1},k} & \cdots & \varepsilon^{2,n}_{m_k^{i-1},k}\\
\vdots & \ddots & \ddots & \ddots & \vdots\\
\varepsilon^{n,1}_{m_k^{i-1},k} & \varepsilon^{n,2}_{m_k^{i-1},k} & \varepsilon^{n,3}_{m_k^{i-1},k} & \cdots & \varepsilon^{n,n}_{m_k^{i-1},k}
\end{pmatrix},
$$

$$
\boldsymbol{\epsilon}_{m_k^{i-1},k}=\begin{pmatrix}
\varepsilon^{1,1}_{m_k^{i-1},k} & \varepsilon^{1,2}_{m_k^{i-1},k} & \varepsilon^{1,3}_{m_k^{i-1},k} & \cdots & \varepsilon^{1,n}_{m_k^{i-1},k}\\
\varepsilon^{2,1}_{m_k^{i-1},k} & \varepsilon^{2,2}_{m_k^{i-1},k} & \varepsilon^{2,3}_{m_k^{i-1},k} & \cdots & \varepsilon^{2,n}_{m_k^{i-1},k}\\
\vdots & \ddots & \ddots & \ddots & \vdots\\
\varepsilon^{n,1}_{m_k^{i-1},k} & \varepsilon^{n,2}_{m_k^{i-1},k} & \varepsilon^{n,3}_{m_k^{i-1},k} & \cdots & \varepsilon^{n,n}_{m_k^{i-1},k}
\end{pmatrix},
$$

$$
\begin{cases}
\eta^j_{m^{i-1}_{k-d_{i-1}},k-d_{i-1}}=\dfrac{1}{m^{i-1}_{k-d_{i-1}+1}}\left[\displaystyle\sum_{\iota=-m^{i-1}_{k-d_{i-1}+1}+1}^{-m^{i-1}_{k-d_{i-1}}+1}F^{\iota}x_j(k-d_{i-1})-F^{-m^{i-1}_{k-d_{i-1}}+1}x_j(k-d_{i-1})-F^{-m^{i-1}_{k-d_{i-1}}}x_j(k-d_{i-1})+F^0x_j(k-d_{i-1})\right],\\
\varepsilon^{j,l}_{m^{i-1}_{k-1},k-1}=\dfrac{m_k^{i-1}-1}{m_k^{i-1}}\left[\displaystyle\sum_{\iota=1}^{d_{i-1}}\frac{F^{-\iota+1}x_j(k-1)F^{-\iota+1}x_l(k-1)}{\prod_{a=0}^{\iota-1}m^{i-1}_{k-a}}-\sum_{\iota=1}^{d_{i-1}}\frac{F^{-\iota+1-m^{i-1}_{k-\iota}}x_j(k-1)F^{-\iota+1-m^{i-1}_{k-\iota}}x_l(k-1)}{\prod_{a=0}^{\iota-1}m^{i-1}_{k-a}}-\sum_{\iota=1}^{d_{i-1}}\frac{F^{-\iota+2-m^{i-1}_{k-\iota}}x_j(k-1)F^{-\iota+2-m^{i-1}_{k-\iota}}x_l(k-1)}{\prod_{a=0}^{\iota-1}m^{i-1}_{k-a}}\right]\\
\qquad+\dfrac{m_k^{i-1}-1}{m_k^{i-1}}\left[\displaystyle\sum_{\iota=1}^{d_{i-1}}\left[\frac{\sum_{v=-\iota+2-m^{i-1}_{k-\iota+1}}^{-\iota+2-m^{i-1}_{k-\iota}}F^vx_j(k-1)F^vx_l(k-1)}{\prod_{a=0}^{\iota-1}m^{i-1}_{k-a}}\right]+\sum_{\iota=1}^{d_{i-1}}\left[\frac{\sum_{v,s=-\iota+2-m^{i-1}_{k-\iota+1}\,v\neq s}^{-\iota+1}F^vx_j(k-1)F^sx_l(k-1)}{\prod_{a=0}^{\iota-1}m^{i-1}_{k-a}}\right]\right]\\
\qquad-\dfrac{1}{m_k^{i-1}}\displaystyle\sum_{v,s=-m_k^{i-1}+1\,v\neq s}^{0}F^vx_j(k-1)F^sx_l(k-1),\\
\epsilon^{j,l}_{m^{i-1}_{k-1},k-1}=\displaystyle\sum_{\iota=1}^{d_{i-1}}\frac{F^{-\iota+1}x_j(k-1)F^{-\iota+1}x_l(k-1)}{\prod_{a=0}^{\iota-1}m^{i-1}_{k-a}}-\sum_{\iota=1}^{d_{i-1}}\frac{F^{-\iota+1-m^{i-1}_{k-\iota}}x_j(k-1)F^{-\iota+1-m^{i-1}_{k-\iota}}x_l(k-1)}{\prod_{a=0}^{\iota-1}m^{i-1}_{k-a}}-\sum_{\iota=1}^{d_{i-1}}\frac{F^{-\iota+2-m^{i-1}_{k-\iota}}x_j(k-1)F^{-\iota+2-m^{i-1}_{k-\iota}}x_l(k-1)}{\prod_{a=0}^{\iota-1}m^{i-1}_{k-a}}\\
\qquad+\displaystyle\sum_{\iota=1}^{d_{i-1}}\left[\frac{\sum_{v=-\iota+2-m^{i-1}_{k-\iota+1}}^{-\iota+2-m^{i-1}_{k-\iota}}F^vx_j(k-1)F^vx_l(k-1)}{\prod_{a=0}^{\iota-1}m^{i-1}_{k-a}}\right]+\sum_{\iota=1}^{d_{i-1}}\left[\frac{\sum_{v,s=-\iota+2-m^{i-1}_{k-\iota+1}\,v\neq s}^{-\iota+1}F^vx_j(k-1)F^sx_l(k-1)}{\prod_{a=0}^{\iota-1}m^{i-1}_{k-a}}\right]\\
\qquad-\dfrac{1}{m_k^{i-1}-1}\displaystyle\sum_{v,s=-m_k^{i-1}+1\,v\neq s}^{0}F^vx_j(k-1)F^sx_l(k-1).
\end{cases}
\tag{49}
$$

The proof of this is given in [16].

6. Parametric Estimation

In this section, we consider a parameter estimation problem in drift and diffusion coefficients of (34). This is achieved by utilizing the lagged adaptive process [22] and the interconnected discrete-time dynamics of local sample mean and variances statistic processes model (48). For each $i\in I(1,K^*)$, we consider a general interconnected hybrid system described by the system of stochastic differential equations:

$$
\begin{cases}
d\mathbf{x}^{i-1}=\mathbf{f}^{i-1}(t,\mathbf{x})dt+\boldsymbol{\sigma}^{i-1}(t,\mathbf{x})d\mathbf{W}(t),\ \mathbf{x}(T_{i-1})=\mathbf{x}^{i-1},\ t\in[T_{i-1},T_i),\\
\mathbf{x}^i=\Gamma^i\mathbf{x}^{i-1}\left(T_i^-,T_{i-1},\mathbf{x}^{i-1}\right),
\end{cases}
\tag{50}
$$

where $\Gamma^i = \operatorname{diag}(\Gamma_1^i, \Gamma_2^i, \cdots, \Gamma_j^i, \cdots, \Gamma_n^i)$ is the jump coefficient matrix; the jump times T_i's are defined in (34). For each $j \in I(1, n)$, the estimate of the jump coefficient Γ_j^i is given by $\Gamma_j^i = x_j(T_i) / \lim_{t \to T_i^-} x_j^{i-1}(t, T_{i-1}, x^{i-1})$.

Let $V \in C[[-\tau, \infty] \times \mathfrak{R}^n, \mathfrak{R}^m]$, and its partial derivatives V_t, $\partial V/\partial x$, and $\partial^2 V/\partial x^2$ exist and are continuous on each interval $[T_{i-1}, T_i]$. We apply Itô-Doob stochastic differential formula [34] to V, and we obtain

$$\begin{cases} dV(t, \mathbf{x}^{i-1}) = \mathbf{L}V(t, \mathbf{x}^{i-1})dt + V_{\mathrm{x}}(t, \mathbf{x}^{i-1})\sigma(t, \mathbf{x}^{i-1})dW(t), \; \mathbf{x}(T_{i-1}) = \mathbf{x}^{i-1}, \; t \in [T_{i-1}, T_i), \\ V(T_i, \mathbf{x}^i) = V(T_i, \Gamma^i \mathbf{x}^{i-1}(T_i^-, T_{i-1}, \mathbf{x}^{i-1})). \end{cases} \tag{51}$$

where the L operator is defined by

$$\begin{cases} \mathbf{L}V(t, x^{i-1}) = V_t(t, \mathbf{x}^{i-1}) + V_x(t, \mathbf{x}^{i-1})f(t, \mathbf{x}^{i-1}) + \dfrac{1}{2} tr(V_{xx}(t, \mathbf{x}^{i-1}))c(t, \mathbf{x}^{i-1}), \\ c(t, \mathbf{x}^{i-1}) = \boldsymbol{\sigma}^{i-1}(t, \mathbf{x}^{i-1})\boldsymbol{\sigma}^{i-1}(t, \mathbf{x}^{i-1})^{\top}. \end{cases} \tag{52}$$

For (50) and (51), we present the Euler-type discretization scheme [23]:

$$\begin{cases} \Delta\mathbf{x}^{i-1}(t_k^{i-1}) = \mathbf{f}(t_{k-1}^{i-1}, \mathbf{x}(t_{k-1}^{i-1}))\Delta t_k^{i-1} + \boldsymbol{\sigma}^{i-1}(t_{k-1}^{i-1}, \mathbf{x}(t_{k-1}^{i-1}))\Delta\mathbf{W}(t_k^{i-1}), \; k \in I(1, N_{i-1}), \\ \mathbf{x}^i = \Gamma^i \mathbf{x}^{i-1}(T_i^-, T_{i-1}, \mathbf{x}^{i-1}), \\ \Delta V(t_k^{i-1}, \mathbf{x}^{i-1}(t_k^{i-1})) = LV(t_{k-1}^{i-1}, \mathbf{x}^{i-1}(t_{k-1}^{i-1}))\Delta t_k^{i-1} + V_x(t_{k-1}^{i-1}, \mathbf{x}^{i-1}(t_{k-1}^{i-1}))\boldsymbol{\sigma}^{i-1}(t_{k-1}^{i-1}, \mathbf{x}(t_{k-1}^{i-1}))\Delta W(t_k^{i-1}), \\ V(T_i, \mathbf{x}^i) = V(T_i, \Gamma^i \mathbf{x}^{i-1}(T_i^-, T_{i-1}, \mathbf{x}^{i-1})). \end{cases} \tag{53}$$

Define $\mathscr{F}_{t_{k-1}}^{i-1} \equiv \mathscr{F}_{k-1}^{i-1}$ as the filtration process up to time t_{k-1}^{i-1}. With regard to the continuous time dynamic system (50) and its transformed system (51), the more general moments of $\Delta x(t_k^{i-1})$ are as follows:

$$\begin{cases} \mathbb{E}[\Delta\mathbf{x}(t_k^{i-1}) | \mathscr{F}_{k-1}^{i-1}] = \mathbf{f}^{i-1}(t_{k-1}^{i-1}, \mathbf{x}^{i-1}(t_{k-1}^{i-1}))\Delta t_k^{i-1}, \\ \mathbb{E}\left[(\Delta x^{i-1}(t_k^{i-1}) - \mathbb{E}[\Delta\mathbf{x}^{i-1}(t_k^{i-1}) | \mathscr{F}_{k-1}^{i-1}])(\Delta\mathbf{x}^{i-1}(t_k^{i-1}) - \mathbb{E}[\Delta\mathbf{x}^{i-1}(t_k^{i-1}) | \mathscr{F}_{k-1}^{i-1}])^T | \mathscr{F}_{k-1}^{i-1}\right] = \boldsymbol{\sigma}^{i-1}(t_{k-1}^{i-1}, \mathbf{x}^{i-1}(t_{k-1}^{i-1}))\boldsymbol{\sigma}^{i-1}(t_{k-1}^{i-1}, \mathbf{x}^{i-1}(t_{k-1}^{i-1}))^T \Delta t_k^{i-1}, \\ \mathbb{E}[\Delta V(t_k^{i-1}, \mathbf{x}^{i-1}(t_k^{i-1})) | \mathscr{F}_{k-1}^{i-1}] = LV(t_{k-1}^{i-1}, \mathbf{x}^{i-1}(t_k^{i-1}),)\Delta t_k^{i-1}, \\ \mathbb{E}[(\Delta V(t_k^{i-1}, \mathbf{x}^{i-1}(t_k^{i-1})) - \mathbb{E}[\Delta V(t_k^{i-1}, \mathbf{x}^{i-1}(t_k^{i-1})) | \mathscr{F}_{k-1}^{i-1}])(\Delta V(t_k^{i-1}, \mathbf{x}^{i-1}(t_k^{i-1})) - \mathbb{E}[\Delta V(t_k^{i-1}, \mathbf{x}^{i-1}(t_k^{i-1})) | \mathscr{F}_{k-1}^{i-1}])^T | \mathscr{F}_{k-1}^{i-1}\right] = B(t_{k-1}^{i-1}, \mathbf{x}^{i-1}(t_{k-1}^{i-1})), \end{cases} \tag{54}$$

where $B(t_{k-1}^{i-1}, \mathbf{x}(t_{k-1}^{i-1})) = V_x(t_{k-1}^{i-1}, \mathbf{x}^{i-1}(t_{k-1}^{i-1}))c(t_{k-1}^{i-1}, \mathbf{x}(t_{k-1}^{i-1}))V_x(t_{k-1}^{i-1}, \mathbf{x}(t_{k-1}^{i-1}))^{\top}\Delta t_k^{i-1}$, and T stands for the transpose of the matrix. From (53) and (54), we have

$$\begin{cases} \Delta \mathbf{x}^{i-1}(t_k^{i-1}) = \mathbb{E}\left[\Delta \mathbf{x}^{i-1}(t_k^{i-1}) \middle| \mathcal{F}_{k-1}^{i-1}\right] + \boldsymbol{\sigma}^{i-1}(t_{k-1}^{i-1}, \mathbf{x}^{i-1}(t_{k-1}^{i-1}))\Delta \mathbf{W}(t_k^{i-1}), \ k \in I(1, N_{i-1}), \\ \Delta V(t_k^{i-1}, \mathbf{x}^{i-1}(t_k^{i-1})) = \mathbb{E}\left[\Delta V(t_k^{i-1}, \mathbf{x}^{i-1}(t_k^{i-1})) \middle| \mathcal{F}_{k-1}^{i-1}\right] + V_{\mathbf{x}}^{i-1}(t_{k-1}^{i-1}, \mathbf{x}^{i-1}(t_{k-1}^{i-1}))\boldsymbol{\sigma}^{i-1}(t_{k-1}^{i-1}, \mathbf{x}^{i-1}(t_{k-1}^{i-1}))\Delta W(t_k). \end{cases} \tag{55}$$

This provides the basis for the development of the concept of lagged adaptive expectation process [22] with respect to continuous time stochastic dynamic system (51). This indeed leads to a formulation of m_k^{i-1} local generalized method of moments at t_k^{i-1}.

In the following, we state a result that exhibits the existence of the solution of the system of nonlinear algebraic equations. For the sake of easy reference, we shall state the implicit function theorem without proof.

Theorem 11. *Implicit Function Theorem [33]. Let $\boldsymbol{F} = \{F_1, F_2, \cdots, F_q\}$ be a vector-valued function defined on an open set $S \in \mathfrak{R}^{q+k}$ with values in $\mathfrak{R}^q$. Suppose $\boldsymbol{F} \in C'$ on S. Let $(\boldsymbol{u}_0; \boldsymbol{v}_0)$ be a point in S for which $\boldsymbol{F}(\boldsymbol{u}_0; \boldsymbol{v}_0) = 0$ and for which the $q \times q$ determinant of the Jacobian matrix det $[\boldsymbol{J}_{\boldsymbol{F}}(\boldsymbol{v}_0)] \neq 0$. Then, there exists a k-dimensional open set $\boldsymbol{T}_0$ containing $\boldsymbol{v}_0$ and unique vector-valued function $\boldsymbol{g}$, defined on $\boldsymbol{T}_0$ and having values in $\mathfrak{R}^q$, such that $\boldsymbol{g} \in C'$ on $\boldsymbol{T}_0$, $\boldsymbol{g}(\boldsymbol{v}_0) = \boldsymbol{u}_0$, and $\boldsymbol{F}(\boldsymbol{g}(\boldsymbol{v}); \boldsymbol{v}) = 0$ for every $\boldsymbol{v} \in \boldsymbol{T}_0$.*

6.1. Special Case: Illustration. For each $j, l \in \boldsymbol{I}(1, n)$ and each $i \in \boldsymbol{I}(1, K^*)$, we consider a special case of (14).

$$\begin{cases} dy_j = \left(u_j^{i-1} - y_j\right)\left[\kappa_{j,j}^{i-1} y_j + \sum_{l \neq j}^{n} \kappa_{j,l}^{i-1} y_l\right] dt + \delta_{j,j}^{i-1}\left(u_j^{i-1} - y_j\right) dW_{j,j}(t) + \left(u_j^{i-1} - y_j\right)\sum_{l \neq j}^{n} \delta_{j,l}^{i-1} y_l dW_{j,l}(t), y_j(T_{i-1}) = y_j^{i-1}, \ t \in [T_{i-1}, T_i), \\ y_j^i = \pi_j^i y_j^{i-1}(T_i^-, T_{i-1}, \boldsymbol{y}^{i-1}), \\ dp_j(t) = p_j\left[\gamma_{j,j}^{i-1}\left(y_j - p_j\right) + \beta_j^{i-1} + \sum_{l \neq j}^{n} \gamma_{j,l}^{i-1} p_l(t)\right] dt + \sigma_{j,j}^{i-1} p_j dZ_{j,j}(t) + p_j \sum_{l \neq j}^{n} \sigma_{j,l}^{i-1} p_l dZ_{j,l}(t), p_j(T_{i-1}) = p_j^{i-1}, \ t \in [T_{i-1}, T_i), \\ p_j^i = \theta_j^i p_j^{i-1}(T_i^-, T_{i-1}, \boldsymbol{y}^{i-1}, \boldsymbol{p}^{i-1}). \end{cases} \tag{56}$$

Here, $\kappa_{j,l}^{i-1}$, u_j^{i-1}, β_j^{i-1}, $\gamma_{j,l}^{i-1}$, $\delta_{j,l}^{i-1}$, and $\sigma_{j,l}^{i-1}$ are the system parameters on the jump subinterval $[T_{i-1}, T_i)$; u_j^{i-1}, $\kappa_{j,j}^{i-1}$, $\gamma_{j,j}^{i-1}$, $\delta_{j,j}^{i-1}$, and $\sigma_{j,j}^{i-1}$ are positive; and for $l \neq j$, $\kappa_{j,l}^{i-1}$, $\gamma_{j,l}^{i-1} \in \mathfrak{R}$; $\delta_{j,l}^{i-1}$, $\sigma_{j,l}^{i-1}$ are nonnegative. W and Z are independent standard Wiener process on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathcal{P})$ with the properties described in (14). It follows that the interconnected system of stochastic differential equations (56) has $4n^2 + 2n$ parameters. Also,

$$\{\kappa_{j,l}\}_{l \neq j}^{i-1} \begin{cases} > 0 \text{ if } y_l \text{ is cooperating with } y_j, \\ < 0 \text{ if } y_l \text{ is competing with } y_j, \\ = 0 \text{ if there is no interaction between } y_l \text{ and } y_j, \ j, l \in I(1, n), \end{cases} \tag{57}$$

and likewise,

$$\left\{\gamma_{j,l}\right\}_{l\neq j}^{i-1}\begin{cases} >0 \text{ if } p_l \text{ is cooperating with } p_j, \\ <0 \text{ if } p_l \text{ is competing with } p_j, \\ =0 \text{ if there is no interaction between } p_l \text{ and } p_j,\ j,l\in I(1,n). \end{cases} \tag{58}$$

Remark 12. For the case $K=0$, (56) is reduced to

$$\begin{cases} dy_j = \left(u_j - y_j\right)\left[\kappa_{j,j}y_j + \sum_{l\neq j}^{n}\kappa_{j,l}y_l\right]dt + \delta_{j,j}\left(u_j-y_j\right)dW_{j,j}(t) + \left(u_j-y_j\right)\sum_{l\neq j}^{n}\delta_{j,l}y_l dW_{j,l}(t), y_j(t_0)=y_{j0},\ t\in[t_0,T], \\ dp_j(t) = p_j\left[\gamma_{j,j}\left(y_j-p_j\right)+\beta_j+\sum_{l\neq j}^{n}\gamma_{j,l}p_l(t)\right]dt + \sigma_{j,j}p_j dZ_{j,j}(t) + p_j\sum_{l\neq j}^{n}\sigma_{j,l}p_l dZ_{j,l}(t), p_j(t_0)=p_{j0},\ t\in[t_0,T], \end{cases} \tag{59}$$

where for $j,l\in I(1,n)$, the parameters $\kappa_{j,l}$, u_j, β_j, $\gamma_{j,l}$, $\delta_{j,l}$, and $\sigma_{j,l}$ are the system parameters on the interval $[t_0,T]$; u_j, $\kappa_{j,j}$, $\gamma_{j,j}$, $\delta_{j,j}$, and $\sigma_{j,j}$ are positive; and for $l\neq j$, $\kappa_{j,l}, \gamma_{j,l}\in\mathfrak{R}$; $\delta_{j,l}$, $\sigma_{j,l}$ are nonnegative. For each $j\in I(1,n)$, we pick a Lyapunov function

$$\begin{cases} V_{1j}\left(t,y_j\right) = \left(y_j\right)^q, \\ V_{2j}\left(t,p_j\right) = \left(p_j\right)^q,\ q\in\mathbb{Z}_+, \end{cases} \tag{60}$$

in (51) for (56). Using the Itô differential formula [24], we have

$$\begin{cases} dV_{1j} = \left[q\left(y_j\right)^{q-1}\left(u_j^{i-1}-y_j\right)\left(\kappa_{j,j}^{i-1}y_j+\sum_{l\neq j}^{n}\kappa_{j,l}^{i-1}y_l\right)+\frac{1}{2}q(q-1)\left(y_j\right)^{q-2}\left(u_j^{i-1}-y_j\right)^2\left(\left(\delta_{j,j}^{i-1}\right)^2+\sum_{l\neq j}^{n}\left(\delta_{j,l}^{i-1}\right)^2 y_l^2\right)\right]dt \\ \quad + q\left(y_j\right)^{q-1}\left(u_j^{i-1}-y_j\right)\left[\delta_{j,j}^{i-1}dW_{j,j}(t)+\sum_{l\neq j}^{n}\delta_{j,l}^{i-1}y_l dW_{j,l}(t)\right], y_j(T_{i-1})=y_j^{i-1}, t\in[T_{i-1},T_i) \\ V_{1j}^i = \left(\pi_j^i\right)^q y_j\left(T_i^-,T_{i-1},\boldsymbol{y}^{i-1}\right)^q, \text{if } t=T_i, \\ dV_{2j} = \left(p_j\right)^q\left[q\left(\gamma_j^{i-1}\left(y_j-p_j\right)+\beta_j^{i-1}+\sum_{l\neq j}^{n}\gamma_{j,l}^{i-1}p_l\right)+\frac{1}{2}q(q-1)\left(\left(\sigma_{j,j}^{i-1}\right)^2+\sum_{l\neq j}^{n}\left(\sigma_{j,l}^{i-1}\right)^2 p_l^2\right)\right]dt \\ \quad + q\left(p_j\right)^q\left[\sigma_{j,j}^{i-1}dZ_{j,j}(t)+\sum_{l\neq j}^{n}\sigma_{j,l}^{i-1}p_l dZ_{j,l}(t)\right], p_j(T_{i-1})=p_j^{i-1}, t\in[T_{i-1},T_i), \\ V_{2j}^i = \left(\theta_j^i\right)^q p_j\left(T_i^-,T_{i-1},\boldsymbol{y}^{i-1},\mathbf{p}^{i-1}\right)^q, \text{if } t=T_i, \end{cases} \tag{61}$$

By setting $\Delta t_k^{i-1} = t_k^{i-1} - t_{k-1}^{i-1} = \Delta t$, $\Delta \boldsymbol{y}(t_k^{i-1}) = \boldsymbol{y}(t_k^{i-1}) - \boldsymbol{y}(t_{k-1}^{i-1})$, and $\Delta \mathbf{p}(t_k^{i-1}) = \mathbf{p}(t_k^{i-1}) - \mathbf{p}(t_{k-1}^{i-1})$, the combined Euler discretized scheme for (61) is

$$\begin{cases} \Delta\left(y_j\right)^q(t_k^{i-1}) = \left[q\left(y_j\right)^{q-1}(t_{k-1}^{i-1})\left(u_j^{i-1} - y_j(t_{k-1}^{i-1})\right)\left(\kappa_{j,j}^{i-1} y_j(t_{k-1}^{i-1}) + \sum_{l\neq j}^{n} \kappa_{j,l}^{i-1} y_l(t_{k-1}^{i-1})\right) + \frac{1}{2}q(q-1)\left(y_j\right)^{q-2}(t_{k-1}^{i-1})\left(u_j^{i-1} - y_j(t_{k-1}^{i-1})\right)^2\left(\left(\delta_{j,j}^{i-1}\right)^2 + \sum_{l\neq j}^{n}\left(\delta_{j,l}^{i-1}\right)^2 y_l^2(t_{k-1}^{i-1})\right)\right]\Delta t \\ \quad + q\left(y_j\right)^{q-1}(t_{k-1}^{i-1})\left(u_j^{i-1} - y_j(t_{k-1}^{i-1})\right)\left[\delta_{j,j}^{i-1}\Delta W_{j,j}(t_k^{i-1}) + \sum_{l\neq j}^{n}\delta_{j,l}^{i-1} y_l \Delta W_{j,l}(t_k^{i-1})\right], y_j(T_{i-1}) = y_j^{i-1}, t_k^{i-1} \in [T_{i-1}, T_i), \\ \left(y_j^i\right)^q = \left(\pi_j^i\right)^q y_j(T_i^-, T_{i-1}, \mathbf{y}^{i-1})^q, \text{ if } t = T_i, \\ \Delta\left(p_j\right)^q(t_k^{i-1}) = \left(p_j\right)^q(t_{k-1}^{i-1})\left[q\left(\gamma_{j,j}^{i-1}\left(y_j(t_{k-1}^{i-1}) - p_j(t_{k-1}^{i-1})\right) + \beta_j^{i-1} + \sum_{l\neq j}^{n}\gamma_{j,l}^{i-1} p_l(t_{k-1}^{i-1})\right) + \frac{1}{2}q(q-1)\left(\left(\sigma_{j,j}^{i-1}\right)^2 + \sum_{l\neq j}^{n}\left(\sigma_{j,l}^{i-1}\right)^2 p_l^2(t_{k-1}^{i-1})\right)\right]\Delta t \\ \quad + q\left(p_j\right)^q(t_{k-1}^{i-1})\left[\sigma_{j,j}^{i-1}\Delta Z_{j,j}(t_k^{i-1}) + \sum_{l\neq j}^{n}\sigma_{j,l}^{i-1} p_l(t_{k-1}^{i-1})\Delta Z_{j,l}(t_k^{i-1})\right], p_j(T_{i-1}) = p_j^{i-1}\ t_k^{i-1} \in [T_{i-1}, T_i), q \in I(1, n+1), \\ \left(p_j^i\right)^q = \left(\theta_j^i\right)^q p_j(T_i^-, T_{i-1}, \mathbf{y}^{i-1}, \mathbf{p}^{i-1})^q, \text{ if } t = T_i. \end{cases} \tag{62}$$

where $\{\mathbf{y}(t_k^{i-1})\}_{k=-r_{i-1}}^0$, $\{\mathbf{p}(t_k^{i-1})\}_{k=-r_{i-1}}^0$ are given finite sequence of $\mathscr{F}_{T_{i-1}}^{i-1}$ measurable random vectors and are independent of $\{\Delta W(t_k^{i-1})\}_{k=0}^{N_{i-1}}$, $\{\Delta Z(t_k^{i-1})\}_{k=0}^{N_{i-1}}$, respectively. We define $\Delta(y_j)^q(t_k^{i-1}) = (y_j)^q(t_k^{i-1}) - (y_j)^q(t_{k-1}^{i-1})$ and $\Delta(p_j)^q(t_k^{i-1}) = (p_j)^q(t_k^{i-1}) - (p_j)^q(t_{k-1}^{i-1})$.

Applying conditional expectation to (62) with respect to $\mathscr{F}_{t_{k-1}}^{i-1} \equiv \mathscr{F}_{k-1}^{i-1}$, we obtain

$$\begin{cases} \begin{cases} \mathbb{E}\left[\Delta\left(y_j\right)^q(t_k^{i-1})|\mathscr{F}_{k-1}^{i-1}\right] = \left[q\left(y_j\right)^{q-1}(t_{k-1}^{i-1})\left(u_j^{i-1} - y_j(t_{k-1}^{i-1})\right)\left(\kappa_{j,j}^{i-1} y_j(t_{k-1}^{i-1}) + \sum_{l\neq j}^{n}\kappa_{j,l}^{i-1} y_l(t_{k-1}^{i-1})\right) + \frac{q(q-1)}{2\Delta t_i}\left(y_j\right)^{q-2}(t_{k-1}^{i-1})\mathbb{E}\left[\left(\Delta y_j(t_k^{i-1}) - \mathbb{E}\left[\Delta y_j(t_k^{i-1})|\mathscr{F}_{k-1}^{i-1}\right]\right)^2|\mathscr{F}_{k-1}^{i-1}\right]\right] \\ \quad \cdot \Delta t \text{ for } t_k^{i-1} \in [T_{i-1}, T_i), \\ \mathbb{E}\left[\left(y_j^i\right)^q(t_k^{i-1})|\mathscr{F}_{k-1}^{i-1}\right] = \left(\pi_j^i\right)^q\left(y_j\right)^q(T_i^-, T_{i-1}, \mathbf{y}^{i-1}), \text{ if } t_k^{i-1} = T_i, \end{cases} \\ \begin{cases} \mathbb{E}\left[\Delta\left(p_j\right)^q(t_k^{i-1})|\mathscr{F}_{k-1}^{i-1}\right] = \left[q\left(p_j\right)^q(t_{k-1}^{i-1})\left(\gamma_{j,j}^{i-1}\left(y_j(t_{k-1}^{i-1}) - p_j(t_{k-1}^{i-1})\right) + \beta_j^{i-1} + \sum_{l\neq j}^{n}\gamma_{j,l}^{i-1} p_l(t_{k-1}^{i-1})\right) + \frac{q(q-1)}{2\Delta t_i} p_j^{q-2}(t_{k-1}^{i-1})\mathbb{E}\left[\left(\Delta p_j(t_k^{i-1}) - \mathbb{E}\left[\Delta p_j(t_k^{i-1})|\mathscr{F}_{k-1}^{i-1}\right]\right)^2|\mathscr{F}_{k-1}^{i-1}\right]\right] \\ \quad \cdot \Delta t, \text{ for } t_k^{i-1} \in [T_{i-1}, T_i), \\ \mathbb{E}\left[\left(p_j^i\right)^q|\mathscr{F}_{i-1}\right] = \left(\theta_j^i\right)^q\left(p_j\right)^q(T_i^-, T_{i-1}, \mathbf{y}^{i-1}, \mathbf{p}^{i-1}), \text{ if } t_k^{i-1} = T_i, q \in I(1, n+1), \end{cases} \end{cases} \tag{63}$$

$$\begin{cases} \mathbb{E}\left[\left(\Delta\left(y_j\right)^q(t_k^{i-1}) - \mathbb{E}\left[\Delta\left(y_j\right)^q(t_k^{i-1})|\mathscr{F}_{k-1}^{i-1}\right]\right)(\Delta(y_l)^q(t_k^{i-1}) - \mathbb{E}[\Delta(y_l)^q(t_k^{i-1})|\mathscr{F}_{k-1}^{i-1}])|\mathscr{F}_{k-1}^{i-1}\right] = \\ q^2\left(y_j y_l\right)^{q-1}(t_{k-1}^{i-1})\left(u_j^{i-1} - y_j(t_{k-1}^{i-1})\right)(u_l^{i-1} - y_l(t_{k-1}^{i-1}))\left[\delta_{j,j}^{i-1}\delta_{l,j}^{i-1} y_j(t_{k-1}^{i-1}) + \delta_{l,l}^{i-1}\delta_{j,l}^{i-1} y_l(t_{k-1}^{i-1}) + \sum_{r=1,j,l\neq r}^{n}\delta_{j,r}^{i-1}\delta_{l,r}^{i-1} y_r^2(t_{k-1}^{i-1})\right]\Delta t, \\ t_k^{i-1} \in [T_{i-1} - \tau_{i-1}, T_i), \\ \mathbb{E}\left[\left(\Delta\left(p_j\right)^q(t_k^{i-1}) - \mathbb{E}\left[\Delta\left(p_j\right)^q(t_k^{i-1})|\mathscr{F}_{k-1}^{i-1}\right]\right)(\Delta(p_l)^q(t_k^{i-1}) - \mathbb{E}[\Delta(p_l)^q(t_k^{i-1})|\mathscr{F}_{k-1}^{i-1}])|\mathscr{F}_{k-1}^{i-1}\right] = \\ q^2\left(p_j p_l\right)^q(t_{k-1}^{i-1})\left[2\sigma_{j,j}^{i-1}\sigma_{l,j}^{i-1} y_j(t_{k-1}^{i-1}) + \sum_{r=1,j,l\neq r}^{n}\sigma_{j,r}^{i-1}\sigma_{l,r}^{i-1} y_r^2(t_{k-1}^{i-1})\right], q, j \neq l \in I(1, n), \end{cases} \tag{64}$$

where $\mathscr{F}_{k-1}^{i-1}$ is the filtration up to time t_{k-1}^{i-1}. From (62), (63) and (64) are reduced to

$$\begin{cases} \Delta\left(y_j\right)^q\left(t_k^{i-1}\right) = \mathbb{E}\left[\Delta\left(y_j\right)^q\left(t_k^{i-1}\right)|\mathscr{F}_{k-1}^{i-1}\right] + q\left(y_j\right)^{q-1}\left(t_{k-1}^{i-1}\right)\left(u_j^{i-1} - y_j\left(t_{k-1}^{i-1}\right)\right)\left[\delta_{j,j}{}^{i-1}\Delta W_{j,j}\left(t_k^{i-1}\right) + \sum_{l\neq j}^{n}\delta_{j,l}{}^{i-1}y_l\Delta W_{j,l}\left(t_k^{i-1}\right)\right], \\ \quad y_j(T_{i-1}) = y_j^{i-1},\ t_k^{i-1}\in[T_{i-1},T_i), \\ \left(y_j^i\right)^q = \left(\pi_j^i\right)^q\left(y_j\right)^q\left(T_i^-, T_{i-1}, \mathbf{y}^{i-1}\right), \text{if } t_k^{i-1} = T_i, \\ \Delta\left(p_j\right)^q\left(t_k^{i-1}\right) = \mathbb{E}\left[\Delta\left(p_j\right)^q\left(t_k^{i-1}\right)|\mathscr{F}_{k-1}^{i-1}\right] + q\left(p_j\right)^q\left(t_{k-1}^{i-1}\right)\left[\sigma_{j,j}^{i-1}\Delta Z_{j,j}\left(t_k^{i-1}\right) + \sum_{l\neq j}^{n}\sigma_{j,l}^{i-1}p_l\left(t_{k-1}^{i-1}\right)\Delta Z_{j,l}\left(t_k^{i-1}\right)\right], \\ \quad p_j(T_{i-1}) = p_j^{i-1},\ t_k^{i-1}\in[T_{i-1},T_i),\ q\in I(1,n+1), j\in I(1,n), \\ \left(p_j^i\right)^q = \left(\theta_j^i\right)^q\left(p_j\right)^q\left(T_i^-, T_{i-1}, \mathbf{y}^{i-1}, \mathbf{p}^{i-1}\right), \text{if } t_k^{i-1} = T_i. \end{cases} \tag{65}$$

Equation (65) provides the basis for the development of the concept of lagged adaptive expectation process [22] with respect to continuous time stochastic dynamic systems (56) and (61).

For $k \in I(0, N_{i-1})$, applying the lagged adaptive expectation process [22] and Definitions 6, 8, and 10 and using (63), (64), and (65), we formulate a local observation/measurement process at t_k^{i-1} as an algebraic functions of m_k^{i-1} local functions of restriction of the finite sample sequence $\{\mathbf{y}(t_l^{i-1})\}_{l=-r_{i-1}}^{N_{i-1}}$ and $\{\mathbf{p}(t_l^{i-1})\}_{l=-r_{i-1}}^{N_{i-1}}$ to subpartition P_k^{i-1} in Definition 8:

$$\begin{aligned} \frac{1}{m_k^{i-1}}\sum_{\iota=k-m_k^{i-1}}^{k-1}\mathbb{E}\left[\Delta\left(y_j\right)^q\left(t_\iota^{i-1}\right)|\mathscr{F}_{\iota-1}^{i-1}\right] &= \left\{\frac{1}{m_k^{i-1}}\sum_{\iota=k-m_k^{i-1}}^{k-1}\left[q\left(y_j\right)^{q-1}\left(t_{\iota-1}^{i-1}\right)\left(u_j^{i-1} - y_j\left(t_{\iota-1}^{i-1}\right)\right)\left(\kappa_{j,j}^{i-1}y_j\left(t_{\iota-1}^{i-1}\right) + \sum_{l\neq j}^{n}\kappa_{j,l}^{i-1}y_l\left(t_{\iota-1}^{i-1}\right)\right)\right.\right. \\ &\quad \left.\left. + \frac{q(q-1)}{2\Delta t}\left(y_j\right)^{q-2}\left(t_{\iota-1}^{i-1}\right)s_{m_k^{i-1},k,\Delta y_j}^{j,j}\right]\Delta t, \right. \\ \frac{1}{m_k^{i-1}}\sum_{\iota=k-m_k^{i-1}}^{k-1}\mathbb{E}\left[\Delta\left(p_j\right)^q\left(t_\iota^{i-1}\right) | \mathscr{F}_{\iota-1}^{i-1}\right] &= \left\{\frac{1}{m_k^{i-1}}\sum_{\iota=k-m_k^{i-1}}^{k-1}\left[q\left(p_j\right)^q\left(t_{\iota-1}^{i-1}\right)\left(\gamma_{j,j}^{i-1}\left(y_j\left(t_{\iota-1}^{i-1}\right) - p_j\left(t_{\iota-1}^{i-1}\right)\right) + \beta_j^{i-1} + \sum_{l\neq j}^{n}\gamma_{j,l}p_l\left(t_{\iota-1}^{i-1}\right)\right)\right.\right. \\ &\quad \left.\left. + \frac{q(q-1)}{2\Delta t}p_j^{q-2}\left(t_{\iota-1}^{i-1}\right)s_{m_k^{i-1},k,\Delta p_j}^{j,j}\right]\Delta t, q\in I(1,n+1), \right. \end{aligned} \tag{66}$$

$$\begin{cases} \widehat{s}_{m_k^{i-1},k}^{j,l}\left(\Delta(y)^q\right) = \frac{1}{m_k^{i-1}}\sum_{\iota=k-m_k^{i-1}}^{k-1}q^2\left(y_jy_l\right)^{q-1}\left(t_{\iota-1}^{i-1}\right)\left(u_j^{i-1} - y_j\left(t_{\iota-1}^{i-1}\right)\right)\left(u_l^{i-1} - y_l\left(t_{\iota-1}^{i-1}\right)\right) \\ \qquad \left[\delta_{j,j}^{i-1}\delta_{l,j}^{i-1}y_j\left(t_{\iota-1}^{i-1}\right) + \delta_{l,l}^{i-1}\delta_{j,l}^{i-1}y_l\left(t_{k-1}^{i-1}\right) + \sum_{r=1 j\neq l\neq r}^{n}\delta_{j,r}^{i-1}\delta_{l,r}^{i-1}y_r^2\left(t_{\iota-1}^{i-1}\right)\right], \\ \widehat{s}_{m_k^{i-1},k}^{j,l}\left(\Delta(p)^q\right) = \frac{1}{m_k^{i-1}}\sum_{\iota=k-m_k^{i-1}}^{k-1}q^2\left(p_jp_l\right)^q\left(t_{\iota-1}^{i-1}\right)\left[\sigma_{j,j}^{i-1}\sigma_{l,j}^{i-1}p_j\left(t_{\iota-1}^{i-1}\right) + \sigma_{l,l}^{i-1}\sigma_{j,l}^{i-1}p_l\left(t_{\iota-1}^{i-1}\right) + \sum_{r=1 j\neq l\neq r}^{n}\sigma_{j,r}^{i-1}\sigma_{l,r}^{i-1}p_r^2\left(t_{\iota-1}^{i-1}\right)\right], \\ \quad j\neq l\in I(1,n),\ q\in I(1,2n). \end{cases} \tag{67}$$

For each $i \in I(1, K^*)$ and each $j \neq l \in I(1, n)$, we define

$$\begin{aligned}
&F_{1q}\left(u_j^{i-1}, \left\{\kappa_{j,r}^{i-1}\right\}_{r=1}^{n}\right) \equiv F_{1q}\left(\mathbb{E}\left[\Delta\left(y_j^q\right)\left(t_\iota^{i-1}\right)\middle|\mathscr{F}_{\iota-1}^{i-1}\right]; u_j^{i-1}, \left\{\kappa_{j,r}^{i-1}\right\}_{r=1}^{n}\right),\\
&F_{2q}\left(\left\{\delta_{j,r}^{i-1}, \delta_{l,r}^{i-1}\right\}_{r=1}^{n}\right) \equiv F_{2q}\left(\mathbb{E}\left[\Delta\left(y_j^q\right)\left(t_\iota^{i-1}\right)\middle|\mathscr{F}_{\iota-1}^{i-1}\right], \mathbb{E}\left[\Delta\left(y_l^q\right)\left(t_\iota^{i-1}\right)\middle|\mathscr{F}_{\iota-1}^{i-1}\right]; \left\{\delta_{j,r}^{i-1}, \delta_{l,r}^{i-1}\right\}_{r=1}^{n}\right),\\
&G_{1q}\left(\beta_j^{i-1}, \left\{\gamma_{j,r}^{i-1}\right\}_{r=1}^{n}\right) \equiv G_{1q}\left(\mathbb{E}\left[\Delta\left(p_j^q\right)\left(t_\iota^{i-1}\right) \mid \mathscr{F}_{\iota-1}^{i-1}\right]; \beta_j^{i-1}, \left\{\gamma_{j,r}^{i-1}\right\}_{r=1}^{n}\right),\\
&G_{2q}\left(\left\{\sigma_{j,r}^{i-1}, \sigma_{l,r}^{i-1}\right\}_{r=1}^{n}\right) \equiv G_{2q}\left(\mathbb{E}\left[\Delta\left(p_j^q\right)\left(t_\iota^{i-1}\right) \mid \mathscr{F}_{\iota-1}^{i-1}\right], \mathbb{E}\left[\Delta\left(p_l^q\right)\left(t_\iota^{i-1}\right) \mid \mathscr{F}_{\iota-1}^{i-1}\right]; \left\{\sigma_{j,r}, \sigma_{l,r}^{i-1}\right\}_{r=1}^{n}\right),
\end{aligned} \tag{68}$$

by

$$\begin{aligned}
F_{1q}\left(u_j^{i-1}, \left\{\kappa_{j,r}^{i-1}\right\}_{r=1}^{n}\right) &= \frac{1}{m_k^{i-1}} \sum_{\iota=k-m_k^{i-1}}^{k-1} \left\{\left[q\left(y_j\right)^{q-1}\left(t_{\iota-1}^{i-1}\right)\left(u_j^{i-1} - y_j\left(t_{\iota-1}^{i-1}\right)\right)\left(\sum_{r=1}^{n} \kappa_{j,r} y_r\left(t_{\iota-1}^{i-1}\right)\right)\right.\right.\\
&\quad \left.\left. + \frac{q(q-1)}{2\Delta t}\left(y_j\right)^{q-2}\left(t_{\iota-1}^{i-1}\right)\widehat{s}_{m_k^{i-1},k}^{j,j}\left(\Delta y_j\right)\right]\Delta t\right\} - \frac{1}{m_k^{i-1}} \sum_{\iota=k-m_k^{i-1}}^{k-1} \mathbb{E}\left[\Delta\left(y_j\right)^q\left(t_\iota^{i-1}\right)\middle|\mathscr{F}_{\iota-1}^{i-1}\right],\ q \in I(1, n+1),\\
F_{2q}\left(\left\{\delta_{j,r}^{i-1}, \delta_{l,r}^{i-1}\right\}_{r=1}^{n}\right) &= \frac{1}{m_k^{i-1}} \sum_{\iota=k-m_k^{i-1}}^{k-1} q^2\left(y_j y_l\right)^{q-1}\left(t_{\iota-1}^{i-1}\right)\left(u_j^{i-1} - y_j\left(t_{\iota-1}^{i-1}\right)\right) \times \left(u_l^{i-1} - y_l\left(t_{\iota-1}^{i-1}\right)\right)\left[\delta_{j,j}^{i-1}\delta_{l,j}^{i-1} y_j\left(t_{\iota-1}^{i-1}\right) + \delta_{l,l}^{i-1}\delta_{j,l}^{i-1} y_l\left(t_{\iota-1}^{i-1}\right)\right.\\
&\quad \left. + \sum_{r=1 j\neq l\neq r}^{n} \delta_{j,r}^{i-1}\delta_{l,r}^{i-1} y_r^2\left(t_{\iota-1}^{i-1}\right)\right] - \widehat{s}_{m_k^{i-1},k,}^{j,l}\left(\Delta(y)^q\right),\ j \neq l \in I(1, n), q \in I(1, 2n),\\
G_{1q}\left(\beta_j^{i-1}, \left\{\gamma_{j,r}^{i-1}\right\}_{r=1}^{n}\right) &= \frac{1}{m_k^{i-1}} \sum_{\iota=k-m_k^{i-1}}^{k-1} \left\{\left[q\left(p_j\right)^q\left(t_{\iota-1}^{i-1}\right)\left(\gamma_{j,j}^{i-1}\left(y_j\left(t_{\iota-1}^{i-1}\right) - p_j\left(t_{\iota-1}^{i-1}\right)\right) + \beta_j^{i-1} + \sum_{r\neq j}^{n} \gamma_{j,r}^{i-1} p_r\left(t_{\iota-1}^{i-1}\right)\right)\right.\right.\\
&\quad \left.\left. + \frac{q(q-1)}{2\Delta t} p_j^{q-2}\left(t_{\iota-1}^{i-1}\right)\widehat{s}_{m_k^{i-1},k}^{j,j}\left(\Delta p_j\right)\right]\Delta t\right\} - \frac{1}{m_k^{i-1}} \sum_{\iota=k-m_k^{i-1}}^{k-1} \mathbb{E}\left[\Delta\left(p_j\right)^q\left(t_\iota^{i-1}\right) \mid \mathscr{F}_{\iota-1}^{i-1}\right], q \in I(1, n+1),\\
G_{2q}\left(\left\{\sigma_{j,r}^{i-1}, \sigma_{l,r}^{i-1}\right\}_{r=1}^{n}\right) &= \frac{1}{m_{\mathrm{k}}^{i-1}} \sum_{\iota=k-m_k^{i-1}}^{k-1} q^2\left(p_j p_l\right)^q\left(t_{\iota-1}^{i-1}\right)\left[\sigma_{j,j}^{i-1}\sigma_{l,j}^{i-1} p_j\left(t_{\iota-1}^{i-1}\right)\right] + \sigma_{l,l}^{i-1}\sigma_{j,l}^{i-1} p_l\left(t_{\iota-1}^{i-1}\right)\\
&\quad + \sum_{r=1 j\neq l\neq r}^{n} \sigma_{j,r}^{i-1}\sigma_{l,r}^{i-1} p_r^2\left(t_{\iota-1}^{i-1}\right)] - \widehat{s}_{m_k^{i-1},k,\Delta(p)^q}^{j,l}, j \neq l \in I(1, n), q \in I(1, 2n).
\end{aligned} \tag{69}$$

Table 7: Estimates π_1^i, π_2^i, π_3^i, θ_1^i, θ_2^i, and θ_3^i.

i	T_i	Natural gas	Crude oil	Coal	T_i	Natural gas	Crude oil	Coal
	T_i	Π_1^i	Π_2^i	Π_3^i	T_i	θ_1^i	θ_2^i	θ_3^i
1	17	1.0031	1.1219	1.0256	17	1.0049	1.1219	1.0493
2	44	0.9213	0.9727	1.0410	44	0.9352	1.0084	0.9249
3	61	0.9482	0.9671	0.9661	61	0.9997	0.9427	0.9404
4	87	0.8859	0.9974	0.9653	87	0.7389	1.0452	0.9905
5	157	1.0435	0.9350	1.0432	157	1.0933	1.0019	1.0049
6	200	1.0309	1.0199	1.0382	200	0.9826	1.0210	0.9794
7	422	1.0270	0.9775	0.9669	422	0.9706	0.9939	0.9917
8	464	0.9581	1.0462	1.0523	464	1.0128	1.0508	1.0324
9	483	0.9765	0.9787	1.0291	483	1.0382	1.0328	1.0246
10	502	1.0532	1.0737	1.0136	502	1.0359	1.0073	1.0162
11	722	0.9812	0.9959	0.9919	722	0.9700	0.9695	1.0011
12	754	1.0003	1.0009	0.9189	754	1.0137	0.9987	1.3481
13	870	1.0579	0.9921	1.1378	870	1.0328	1.0033	1.1420
14	930	1.0275	0.9907	0.9978	930	0.9995	0.9812	1.1848
15	1113	1.0009	0.9960	1.0706	1113	0.9304	0.9801	0.9897

For every $j \in I(1, n)$, we have

$$\begin{cases} F_{1q}\left(u_j^{i-1}, \left\{\kappa_{j,r}^{i-1}\right\}_{r=1}^{n}\right) = 0, q \in I(1, n+1), \\ F_{2q}\left(\left\{\delta_{j,r}^{i-1}, \delta_{l,r}^{i-1}\right\}_{r=1}^{n}\right) = 0, q \in I(1, 2n), \\ G_{1q}\left(\beta_j^{i-1}, \left\{\gamma_{j,r}^{i-1}\right\}_{r=1}^{n}\right) = 0, q \in I(1, n+1), \\ G_{2q}\left(\left\{\delta_{j,r}^{i-1}, \delta_{l,r}^{i-1}\right\}_{r=1}^{n}\right) = 0, q \in I(1, 2n). \end{cases} \tag{70}$$

We define $F_1 = \{F_{1q}\}_{q \in I(1,n+1)}$, $F_2 = \{F_{2q}\}_{q \in I(1,n)}$, $G_1 = \{G_{1q}\}_{q \in I(1,n+1)}$, and $G_2 = \{G_{2q}\}_{q \in I(1,n)}$. Thus, provided that the determinant of each of the Jacobian matrices $JF_1(u_j^{i-1}, \{\kappa_{j,r}^{i-1}\}_{r=1}^{n})$, $JF_2(u_j^{i-1}, \{\delta_{j,r}^{i-1}\}_{r=1}^{n})$, $JG_1(\beta_j^{i-1}, \{\gamma_{j,r}^{i-1}\}_{r=1}^{n})$ and $JG_2(\{\sigma_{j,r}^{i-1}, \sigma_{l,r}^{i-1}\}_{r=1}^{n})$ are not zero, by the application of Theorem 11 (Implicit Function Theorem), we conclude that for every nonconstant m_k^{i-1}-local sequence $\{y_j(t_l^{i-1})\}_{l=k-m_k^{i-1}}^{k-1}$ and $\{p_j(t_l^{i-1})\}_{l=k-m_k^{i-1}}^{k-1}$, there exist a unique solution $(\widehat{u}_j^{i-1}(m_k^{i-1}, t_k^{i-1})$, $\widehat{\beta}_j^{i-1}(m_k^{i-1}, t_k^{i-1})$, $\{\widehat{\kappa}_{j,r}^{i-1}\}_{r=1}^{n}(m_k^{i-1}, t_k^{i-1})$, $\{\widehat{\gamma}_{j,r}^{i-1}\}_{r=1}^{n}(m_k^{i-1}, t_k^{i-1})$, $\{\widehat{\delta}_{j,r}^{i-1}\}_{r=1}^{n}(m_k^{i-1}, t_k^{i-1})$, $\{\widehat{\sigma}_{j,r}^{i-1}\}_{r=1}^{n}(m_k^{i-1}, t_k^{i-1}))$ of system of algebraic equations (70) as a point estimates of u_j^{i-1}, $\kappa_{j,r}^{i-1}{}_{r=1}^{n}$, $\gamma_{j,r}^{i-1}{}_{r=1}^{n}$, $\delta_{j,r}^{i-1}{}_{r=1}^{n}$, $\sigma_{j,r}^{i-1}{}_{r=1}^{n}$, $j \in I(1, n)$, respectively. We illustrate this approach using energy commodities natural gas, crude oil and coal [35–37] in the next section.

6.2. Illustration: Application to Energy Commodity. In this subsection, we present an illustration of the derived model on the natural gas, crude oil and coal [35–37]. Here, $j \in I(1, 3)$ and $i \in I(1, K^*)$. Moreover, (56) reduces to

$$\begin{cases} dy_j = \left(u_j^{i-1} - y_j\right)\left[\kappa_{j,j}^{i-1} y_j + \sum_{l \neq j}^{3} \kappa_{j,l}^{i-1} y_l\right] dt + \delta_{j,j}^{i-1}\left(u_j^{i-1} - y_j\right) dW_{j,j}(t) + \left(u_j^{i-1} - y_j\right) \sum_{l \neq j}^{3} \delta_{j,l}^{i-1} y_l dW_{j,l}(t), y_j(T_{i-1}) = y_j^{i-1},\ t \in [T_{i-1}, T_i), \\ y_j^i = \pi_j^i y_j\left(T_i^-, T_{i-1}, y^{i-1}\right), \\ dp_j(t) = p_j\left[\gamma_{j,j}^{i-1}\left(y_j - p_j\right) + \beta_j^{i-1} + \sum_{l \neq j}^{3} \gamma_{j,l}^{i-1} p_l(t)\right] dt + \sigma_{j,j}^{i-1} p_j dZ_{j,j}(t) + p_j \sum_{l \neq j}^{3} \sigma_{j,l}^{i-1} p_l dZ_{j,l}(t), p_j(T_{i-1}) = p_j^{i-1},\ t \in [T_{i-1}, T_i), \\ p_j^i = \theta_j^i p_j\left(T_i^-, T_{i-1}, \mathbf{y}^{i-1}, \mathbf{p}^{i-1}\right). \end{cases} \tag{71}$$

Table 8: Estimates $\widehat{m}_k$, $u_1(\widehat{m}_k,k)$, $\kappa_{1,1}(\widehat{m}_k,k)$, $\kappa_{1,2}(\widehat{m}_k,k)$, $\kappa_{1,3}(\widehat{m}_k,k)$, $\widehat{m}_k$, $u_2(\widehat{m}_k,k)$, $\kappa_{2,1}(\widehat{m}_k,k)$, $\kappa_{2,2}(\widehat{m}_k,k)$, $\kappa_{2,3}(\widehat{m}_k,k)$, $\widehat{m}_k$, $u_3(\widehat{m}_k,k)$, $\kappa_{3,1}(\widehat{m}_k,k)$, $\kappa_{3,2}(\widehat{m}_k,k)$, and $\kappa_{3,3}(\widehat{m}_k,k)$.

		Natural gas				Crude oil				Coal			
t_k	$\widehat{m}_k$	$u_1(\widehat{m}_k,k)$	$\kappa_{1,1}(\widehat{m}_k,k)$	$\kappa_{1,2}(\widehat{m}_k,k)$ $\times 10^{-16}$	$\kappa_{1,3}(\widehat{m}_k,k)$ $\times 10^{-16}$	$u_2(\widehat{m}_k,k)$	$\kappa_{2,1}(\widehat{m}_k,k)$ $\times 10^{-16}$	$\kappa_{2,2}(\widehat{m}_k,k)$	$\kappa_{2,3}(\widehat{m}_k,k)$ $\times 10^{-16}$	$u_3(\widehat{m}_k,k)$	$\kappa_{3,1}(\widehat{m}_k,k)$ $\times 10^{-16}$	$\kappa_{3,2}(\widehat{m}_k,k)$ $\times 10^{-16}$	$\kappa_{3,3}(\widehat{m}_k,k)$
11	1	4.1593	0	0	0	57.7000	0	0	0	16.7407	0	0	0
12	4	4.2000	0	0	0	58.6313	0.0011	0.0310	-0.0012	16.2395	0	0	-0.0376
13	6	4.0616	0.0679	-0.0054	-0.0035	58.5378	-0.0035	0.0205	0.0032	16.2680	0	0	0.1069
14	2	4.0616	-0.0242	-0.0179	-0.0035	61.4809	0.0020	0.0098	0	15.5249	0	0	-0.0294
15	7	4.0910	0.6416	-0.2898	0.0078	22.5758	-0.0057	0.0085	0.0012	18.7073	0.0009	-0.0021	0.0318
16	4	4.0160	0	0	0.0078	59.6867	-0.0051	0.0080	0	17.0060	0	-0.0021	0
17	2	4.9575	0	0	0.0078	60.3710	-0.0005	0.0207	0	12.8918	-0.0005	-0.0002	0.0318
18	8	4.9575	-0.1947	0	0.0078	62.3437	0.0005	-0.0008	0	16.5954	0.0002	0.0008	0.0662
19	4	3.3190	-0.4472	0.6760	0.0078	74.6911	-0.0008	-0.0019	0	17.9932	0	-0.0002	0
20	1	3.4762	-0.2540	-0.0048	0.0078	65.9190	0.0026	-0.0006	1	17.6485	0	0	0.0499
…	…	…	…	…	…	…	…	…	…	…	…	…	…
494	1	4.1457	0	0	-0.0001	115.1875	0.0002	0.0053	0	33.3359	0.0003	-0.0003	0.0326
495	1	4.2877	0.1184	-0.0014	0	124.5218	0.0008	0.0056	0	30.1732	0.0002	-0	0.0412
496	1	4.2238	0.2582	0.0011	0.0003	106.8349	0.0003	0.0113	-0.0006	34.9907	0.0034	0	0.0097
497	5	4.0998	0.0477	-0.0006	-0.0002	108.4725	-0.0003	0.0162	-0.0033	33.3388	0.0002	0	0.0443
498	8	4.0592	0.0201	0.0010	0	104.8926	0	0	0.0003	35.1174	0	0.0001	0.0207
499	1	4.3433	0.2118	-0.0014	0	109.2551	-0.0002	0.0048	0.0003	33.2862	0.0010	0.0001	0.0068
500	4	2.4519	0	0	0	111.7067	0	0	0	36.1647	0.0003	0.0003	0.0079
501	1	4.2415	0.4108	0	-0.0015	110.7517	-0.0006	0.0026	0.0009	34.9145	-0.0001	-0.0004	0.0407
502	2	4.3633	0.3210	0.0002	-0.0001	103.6326	-0.0019	0.0023	-0.0002	34.8337	-0.0001	-0.0001	0.0140
503	2	4.2911	0.1276	0.0003	0.0043	112.1547	-0.0033	0.0030	0.0027	35.8389	0.0005	0.0001	0.0211
504	7	4.5942	-0.0125	-0.0002	-0.0031	111.1278	0.0010	0.0072	0.0006	33.6875	-0.0021	0	0.0268
505	2	3.1882	0.0666	0.0009	0	106.1919	-0.0009	0.0110	0.0011	33.6640	-0.0011	-0.0002	0.0231
…	…	…	…	…	…	…	…	…	…	…	…	…	…
1102	1	3.5909	0	0	0.0008	110.3777	0.0006	0.0045	0	5.1761	0.0067	-0.0029	-0.0044
1103	6	3.5303	0.1166	0.0002	-0.0003	111.1585	-0.0003	0.0083	0	5.4558	-0.0019	0.0014	0.0600
1104	4	3.5314	0.0809	0.0018	0	109.0996	-0.0007	0.0095	0.0013	4.8000	0.0005	0.0006	0.1742
1105	1	3.7100	0.2234	-0.0013	-0.0015	106.5667	0.0033	0.0073	-0.0020	5.4226	-0.0082	0.0020	0.0932
1106	8	3.4084	0.1098	0.0001	0	106.5989	0.0003	0.0030	0.0023	5.3360	-0.0023	0.0005	0.0956
1107	7	3.5520	0.1086	0.0001	-0.0070	103.4473	-0.0020	0.0037	-0.0045	4.3586	-0.0005	0.0004	0.1418
1108	5	3.9233	0.0601	0.0007	0	102.8550	0	0.0040	0	4.6582	-0.0010	0	0.1388
1109	8	3.5328	0.0417	0	0	103	-0.0002	0.0089	-0.0005	4.9663	-0.0019	0.0008	0.1279
1110	5	3.8399	0.0212	0.0004	0	102.8800	0	0	0	4.7286	-0.0037	-0.0030	0.0740

For each $j \in I(1,3)$, following the argument used in Section 6.1, we have

$$\begin{aligned} F_{1q}\left(u_j^{i-1}, \left\{\kappa_{j,r}^{i-1}\right\}_{r=1}^{3}\right) &= \frac{1}{m_k^{i-1}} \sum_{\iota=k-m_k^{i-1}}^{k-1} \left\{\left[q\left(y_j\right)^{q-1}\left(t_{\iota-1}^{i-1}\right)\left(u_j^{i-1} - y_j\left(t_{\iota-1}^{i-1}\right)\right)\left(\sum_{r=1}^{3}\kappa_{j,r}y_r\left(t_{\iota-1}^{i-1}\right)\right) + \frac{q(q-1)}{2\Delta t}\left(y_j\right)^{q-2}\left(t_{\iota-1}^{i-1}\right)\widehat{s}^{j,j}_{m_k^{i-1},k}\left(\Delta y_j\right)\right]\Delta t\right\} \\ &- \frac{1}{m_k^{i-1}} \sum_{\iota=k-m_k^{i-1}}^{k-1} \mathbb{E}\left[\Delta\left(y_j\right)^{q}\left(t_\iota^{i-1}\right)\middle|\mathscr{F}_{\iota-1}^{i-1}\right],\ q \in I(1,4), \\ F_{2q}\left(\left\{\delta_{j,r}^{i-1}, \delta_{l,r}^{i-1}\right\}_{r=1}^{3}\right) &= \frac{1}{m_k^{i-1}} \sum_{\iota=k-m_k^{i-1}}^{k-1} q^2\left(y_j y_l\right)^{q-1}\left(t_{\iota-1}^{i-1}\right)\left(u_j^{i-1} - y_j\left(t_{\iota-1}^{i-1}\right)\right) \times \left(u_l^{i-1} - y_l\left(t_{\iota-1}^{i-1}\right)\right)\left[\delta_{j,j}^{i-1}\delta_{l,j}^{i-1}y_j\left(t_{\iota-1}^{i-1}\right) + \delta_{l,l}^{i-1}\delta_{j,l}^{i-1}y_l\left(t_{\iota-1}^{i-1}\right)\right. \\ &\left. + \sum_{\substack{r=1 \\ j\neq l\neq r}}^{3} \delta_{j,r}^{i-1}\delta_{l,r}^{i-1}y_r^2\left(t_{\iota-1}^{i-1}\right)\right] - \widehat{s}^{j,l}_{m_k^{i-1},k,}\left(\Delta(y)^q\right),\ j \neq l \in I(1,3),\ q \in I(1,6), \end{aligned} \tag{72}$$

$$\begin{aligned} G_{1q}\left(\beta_j^{i-1}, \left\{\gamma_{j,r}^{i-1}\right\}_{r=1}^{3}\right) &= \frac{1}{m_k^{i-1}} \sum_{\iota=k-m_k^{i-1}}^{k-1} \left\{\left[q\left(p_j\right)^{q}\left(t_{\iota-1}^{i-1}\right)\left(\gamma_{j,j}^{i-1}\left(y_j\left(t_{\iota-1}^{i-1}\right) - p_j\left(t_{\iota-1}^{i-1}\right)\right) + \beta_j^{i-1} + \sum_{r\neq j}^{3}\gamma_{j,r}^{i-1}p_r\left(t_{\iota-1}^{i-1}\right)\right)\right.\right. \\ &\left.\left. + \frac{q(q-1)}{2\Delta t}p_j^{q-2}\left(t_{\iota-1}^{i-1}\right)\widehat{s}^{j,j}_{m_k^{i-1},k}\left(\Delta p_j\right)\right]\Delta t\right\} - \frac{1}{m_k^{i-1}} \sum_{\iota=k-m_k^{i-1}}^{k-1} \mathbb{E}\left[\Delta\left(p_j\right)^{q}\left(t_\iota^{i-1}\right) \mid \mathscr{F}_{\iota-1}^{i-1}\right],\quad q \in I(1,4), \\ G_{2q}\left(\left\{\sigma_{j,r}^{i-1}, \sigma_{l,r}^{i-1}\right\}_{r=1}^{3}\right) &= \frac{1}{m_k^{i-1}} \sum_{\iota=k-m_k^{i-1}}^{k-1} q^2\left(p_j p_l\right)^{q}\left(t_{\iota-1}^{i-1}\right)\left[\sigma_{j,j}^{i-1}\sigma_{l,j}^{i-1}p_j\left(t_{\iota-1}^{i-1}\right) + \sigma_{l,l}^{i-1}\sigma_{j,l}^{i-1}p_l\left(t_{\iota-1}^{i-1}\right) + \sum_{r=1 j\neq l\neq r}^{3}\sigma_{j,r}^{i-1}\sigma_{l,r}^{i-1}p_r^2\left(t_{\iota-1}^{i-1}\right)\right] \\ &- \widehat{s}^{j,l}_{m_k^{i-1},\mathrm{k},\Delta(p)^q},\ j \neq l \in I(1,3),\ q \in I(1,6), \end{aligned}$$

$$\begin{cases} F_{1q}\left(u_j^{i-1}, \left\{\kappa_{j,r}^{i-1}\right\}_{r=1}^{3}\right) = 0,\ q \in I(1,4), \\ F_{2q}\left(\left\{\delta_{j,r}^{i-1}, \delta_{l,r}^{i-1}\right\}_{r=1}^{3}\right) = 0,\ q \in I(1,6), \\ G_{1q}\left(\beta_j^{i-1}, \left\{\gamma_{j,r}^{i-1}\right\}_{r=1}^{3}\right) = 0,\ q \in I(1,4), \\ G_{2q}\left(\left\{\delta_{j,r}^{i-1}, \delta_{l,r}^{i-1}\right\}_{r=1}^{3}\right) = 0,\ q \in I(1,6). \end{cases} \tag{73}$$

We also have $F_1 = \{F_{1q}\}_{q\in I(1,4)}$, $F_2 = \{F_{2q}\}_{q\in I(1,3)}$, $G_1 = \{G_{1q}\}_{q\in I(1,4)}$, and $G_2 = \{G_{2q}\}_{q\in I(1,3)}$. For each $j \in I(1,3)$, it follows that the determinant of the Jacobian of $F_1(u_j^{i-1}, \{\kappa_{j,r}^{i-1}\}_{r\in I(1,3)})$, that is,

$JF_1(u_j^{i-1}, \{\kappa_{j,r}^{i-1}\}_{r\in I(1,3)})$, is not zero provided that all parameters $\{\kappa_{j,r}\}_{r\in I(1,3)}$ are not zero or provided the sequence $\{y_j^{i-1}(t_r^{i-1})\}_{r=k-m_k^{i-1}-1}^{k-1}$ is neither zero nor constant. Likewise, determinants of the Jacobians $JF_2(u_j^{i-1}, \{\delta_{j,r}^{i-1}\}_{r\in I(1,n)})$, $JG_1(\beta_j^{i-1}, \{\gamma_{j,r}^{i-1}\}_{r\in I(1,n)})$ and $JG_2(\{\sigma_{j,r}^{i-1}\}_{r\in I(1,n)})$ are nonzero if $\{\delta_{j,r}\}_{j,l\in I(1,3)}$, $\{\gamma_{j,r}\}_{j,l\in I(1,3)}$, and $\{\sigma_{j,r}\}_{j,l\in I(1,3)}$ are not zero or provided the sequence $\{y_j^{i-1}(t_r^{i-1})\}_{r=k-m_k^{i-1}-1}^{k-1}$ and $\{p_j^{i-1}(t_r^{i-1})\}_{r=k-m_k^{i-1}-1}^{k-1}$ are neither zero nor constant for $j \in I(1,3)$.

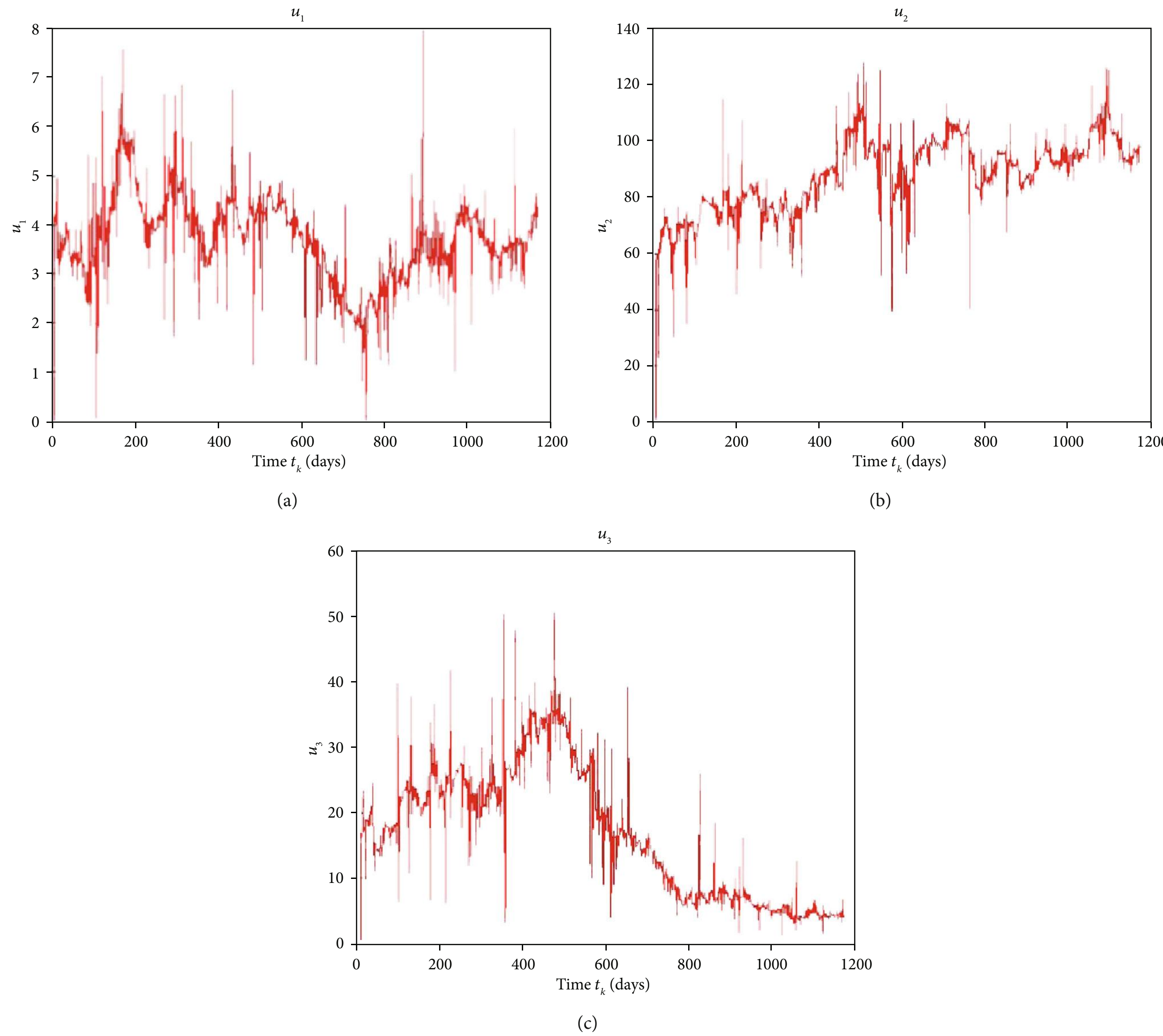


FIGURE 8: The graph of mean level $u_1(\widehat{m}_k, k)$, $u_2(\widehat{m}_k, k)$, and $u_3(\widehat{m}_k, k)$ for natural gas, crude oil, and coal, respectively.

Remark 13. If the sample $\{y_j^{i-1}(t_r^{i-1})\}_{r=k-m_k^{i-1}-1}^{k-1}$ is a constant sequence, it follows from (62) ($q=1$) and the fact that $\Delta(y_j^{i-1}(t_k^{i-1}))=0$ and $s^{j,j}_{m_k^{i-1},k}(\Delta y_j)=0$ that we can set $u_j^{i-1}(m_k^{i-1}, t_k^{i-1}) = 1/m_k^{i-1}\sum_{r=k-m_k^{i-1}}^{k-1} y_j^{i-1}(m_k^{i-1}, t_k^{i-1})$. It also follows from (66) that $\kappa_{j,r}^{i-1}(m_k^{i-1}, t_k^{i-1}) = 0$.

7. Computational Algorithm

In this section, we outline computational, data organizational, and simulation schemes. We introduce the ideas of iterative data process and data simulation time schedules in relation to the real-time data observation/collection schedule. For the computational estimation of continuous time stochastic dynamic system state and parameters, it is essential to identify an admissible set of local conditional sample average and sample covariance parameters, namely, the size of local conditional sample in the context of a partition of time interval $[T_{i-1} - \tau_{i-1}, T_i]$. Moreover, the discrete-time dynamic model of conditional sample mean and sample covariance statistic processes in Section 5 and the theoretical parameter estimation scheme in Section 6 motivates to outline a computational scheme in a systematic and coherent manner. A brief conceptual computational scheme and simulation process summary is described below.

7.1. Coordination of Data Observation, Iterative Process, and Simulation Schedules. Without loss of generality, we assume that the real data observation/collection partition schedules

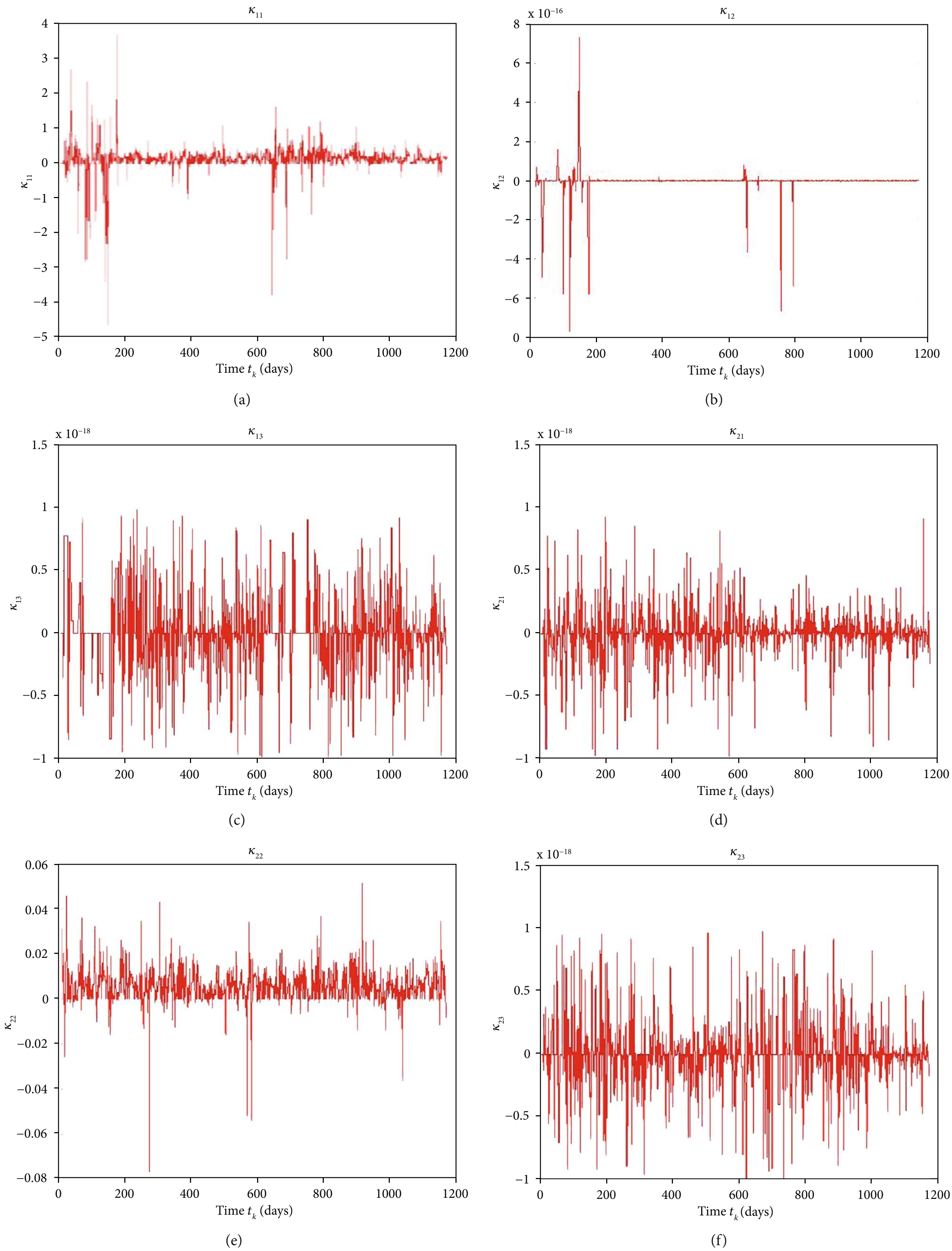


Figure 9: Continued.

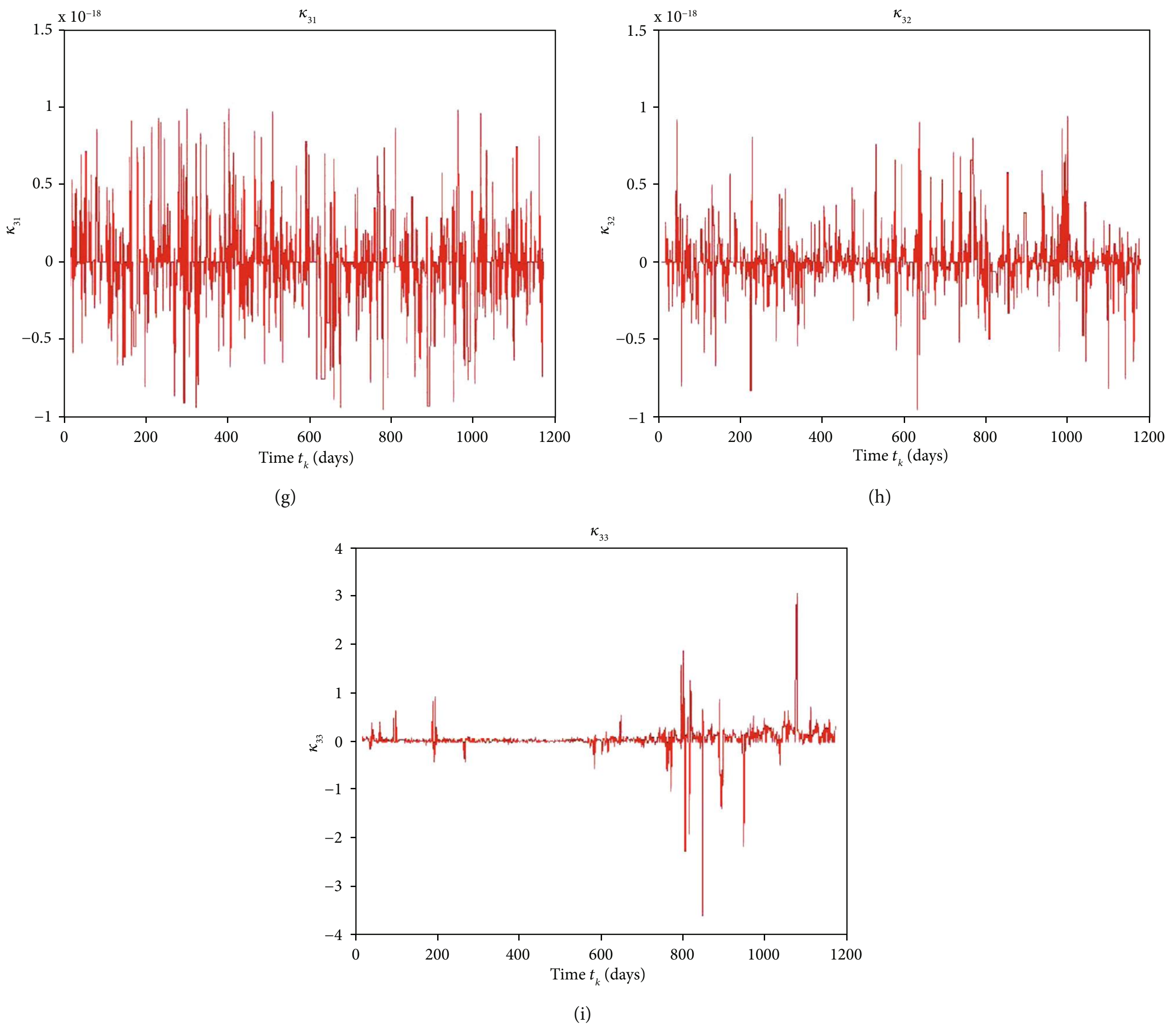


FIGURE 9: The graph of interaction coefficients $\kappa_{1,1}(\widehat{m}_k, k)$, $\kappa_{1,2}(\widehat{m}_k, k)$, $\kappa_{1,3}(\widehat{m}_k, k)$, $\kappa_{2,1}(\widehat{m}_k, k)$, $\kappa_{2,2}(\widehat{m}_k, k)$, $\kappa_{2,3}(\widehat{m}_k, k)$, $\kappa_{3,1}(\widehat{m}_k, k)$, $\kappa_{3,2}(\widehat{m}_k, k)$, and $\kappa_{3,3}(\widehat{m}_k, k)$.

$\mathbb{P}^{i-1}$, $i \in I(1, K^*)$ are defined in (41). We present definitions of iterative process and simulation time schedule following the definitions in Otunuga et al. [16].

Definition 14. The iterative process time schedule in relation with the real data collection schedule is defined by

$$\left\{ I\mathbb{P}^{i-1} = \left\{ F^{-r_{i-1}} t_k^{i-1} : \text{for } t_k^{i-1} \in \mathbb{P}^{i-1} \right\}, \text{for } i \in I(1, K^*), k \in I(-r_{i-1}, N_{i-1}), \tag{74}$$

where $F^{-r_{i-1}} t_k^{i-1} = t_{k-r_{i-1}}^{i-1}$ is a forward shift operator [32].

The simulation time is based on the order d_{i-1} of the time series model of m_k^{i-1} local conditional sample mean and covariance processes in (48).

Remark 15. For the case where $K = 0$, we have $I\mathbb{P}_{i-1} = I\mathbb{P}$, where $\mathbb{P}^{i-1} = \mathbb{P}$ is defined in (40). This is the iterative time schedule in the absence of jumps.

Definition 16. The simulation process time schedule in relation with the real data observation schedule is defined by

$$S\mathbb{P}^{i-1} = \begin{cases} \left\{ F^{r_{i-1}} t_k^{i-1} : \text{for } t_k^{i-1} \in \mathbb{P}^{i-1} \right\}, & \text{if } d_{i-1} \le r_{i-1}, \\ \left\{ F^{d_{i-1}} t_k^{i-1} : \text{for } t_k^{i-1} \in \mathbb{P}^{i-1} \right\}, & \text{if } d_{i-1} > r_{i-1}, k \in I(-r_{i-1}, N_{i-1}). \end{cases} \tag{75}$$

Remark 17. For each $i \in I(1, K^*)$, the initial times of iterative and simulation processes are equal to the real data

Table 9: Estimates $\delta_{1,1}(\widehat{m}_k,k)$, $\delta_{1,2}(\widehat{m}_k,k)$, $\delta_{1,3}(\widehat{m}_k,k)$, $\delta_{2,1}(\widehat{m}_k,k)$, $\delta_{2,2}(\widehat{m}_k,k)$, $\delta_{2,3}(\widehat{m}_k,k)$, $\delta_{3,1}(\widehat{m}_k,k)$, $\delta_{3,2}(\widehat{m}_k,k)$, and $\delta_{3,3}(\widehat{m}_k,k)$.

t_k		Natural gas			Crude oil			Coal	
	$\delta_{1,1}(\widehat{m}_k,k)$	$\delta_{1,2}(\widehat{m}_k,k)$	$\delta_{1,3}(\widehat{m}_k,k)$	$\delta_{2,1}(\widehat{m}_k,k)$	$\delta_{2,2}(\widehat{m}_k,k)$	$\delta_{2,3}(\widehat{m}_k,k)$	$\delta_{3,1}(\widehat{m}_k,k)$	$\delta_{3,2}(\widehat{m}_k,k)$	$\delta_{3,3}(\widehat{m}_k,k)$
11	0.0062	0.0010	0.0001	1.5277	0.0078	0.0011	0	0	0.0218
12	0.0182	0.9002	0	1.6227	0.0010	0	0	0	0.0988
13	0.0239	0.0802	1.7280	0	1.7694	0	0.6374	0.6374	0.0959
14	0	0.0001	0.6027	2.3258	0	0	1.6564	1.6564	0.0847
15	0	0.8001	0.6210	2.3252	0	0	1.6650	1.6650	0.0111
16	0.0455	0.0007	3.6877	2.3217	0	1.2215	1.6724	1.6724	0
17	0	0.9876	0	1.6425	0	0	1.7719	1.7719	0
18	3.0410	0.9351	0	1.3105	0	0.1070	1.7630	1.7630	0.0434
19	2.7713	0.6680	0	1.1052	0	0	1.7400	1.7400	0
20	2.8461	1.7795	0	0.1196	0	0.0983	0	0.4555	0
…	…	…	…	…	…	…	…	…	…
495	1.1229	0	0.0584	0.5488	0.1104	0.0761	0	0	1.3987
496	0.6946	0	0.6613	0.5767	0.0715	0.0610	0	0	1.3017
497	1.1229	0.0095	0.0988	0.6499	0.0870	0.0633	1.1317	1.1317	1.3069
498	0.6946	0.0101	0	0	0	0.0320	1.0294	1.0294	1.5410
499	0.7353	0.0066	0.0384	0	0.0922	0.0330	0.7317	0.7317	1.2225
500	1.7509	0.0069	0.0283	0.4307	0.4545	0.0413	0.4826	0.4826	1.2254
501	2.1299	0.0077	0.0282	0.5043	0.7873	0.0308	0.4272	0.4272	1.5587
502	0.9778	0.0077	0	0.2878	0	0	0.5239	0.5239	1.8713
503	0.9872	0	0	0.2909	0	0	1.4523	1.4523	1.8874
504	1.1329	0	0	0.3707	0.4261	0	0	0	0
505	1.9178	0	0	0.3812	0.7292	0.1724	0	0	0
…	…	…	…	…	…	…	…	…	…
1102	0	0.0331	0.7183	0.9297	0.0434	0.0680	0	0	1.1355
1103	1.5077	0.0626	0.2048	1.1017	0.0421	0.1510	0	0	1.4133
1104	0.4444	0.0435	0.4622	0.1939	0.1078	0	0.0814	0	1.1672
1105	3.5933	0	0.3646	0.1922	0	0.7273	0.2726	0.2726	1.3023
1106	2.4964	0	0.3919	0	0.0684	1.0179	0.3296	0.3296	1.4111
1107	2.4600	0	0.8995	0.2001	0.1510	0.9354	0	0	1.7245
1108	2.0262	0	0.6325	0.3781	0.0814	0.8825	0.1878	0.1878	1.0915
1109	1.7828	0	0.6116	0.4024	0.0332	0.8812	0	0	1.3191
1110	1.2706	0	0.1001	0.3252	0.0155	0.8078	0	0	1.0233

times $t^{i-1}_{r_{i-1}}$ and $t^{i-1}_{d_{i-1}}$, whenever $d_{i-1} \le r_{i-1}$ and $d_{i-1} > r_{i-1}$, respectively. The iterative process and simulation process times with jump are $t^{i-1}_{k+r_{i-1}}$ and $t^{i-1}_{k+d_{i-1}}$, $i \in I(1, K^*)$, respectively.

7.2. Conceptual Computational Parameter Estimation Scheme. For the conceptual computational dynamic system parameter estimation, we need to introduce a few concepts of local admissible sample/data observation size m^{i-1}_k local admissible conditional finite sequence at $t^{i-1}_k \in S\mathbb{P}^{i-1}$ and local finite sequence of parameter estimates at t^{i-1}_k.

Definition 18. For each $i \in I(1, K^*)$, and $t^{i-1}_k \in I(T_{i-1} - \tau_{i-1}, T_i)$, we define local admissible sample/data observation size m^{i-1}_k at t^{i-1}_k as $m^{i-1}_k \in OS^{i-1}_k$, where

$$OS^{i-1}_k = \begin{cases} I(2, r_{i-1} + \mathcal{S}_{i-1} + k - 1), & \text{if } d_{i-1} \le r_{i-1}, \\ I(2, d_{i-1} + \mathcal{S}_{i-1} + k - 1), & \text{if } d_{i-1} > r_{i-1}, k \in I(0, N_{i-1}). \end{cases} \tag{76}$$

Moreover, OS^{i-1}_k is referred as the local admissible set of lagged sample/data observation size at t^{i-1}_k.

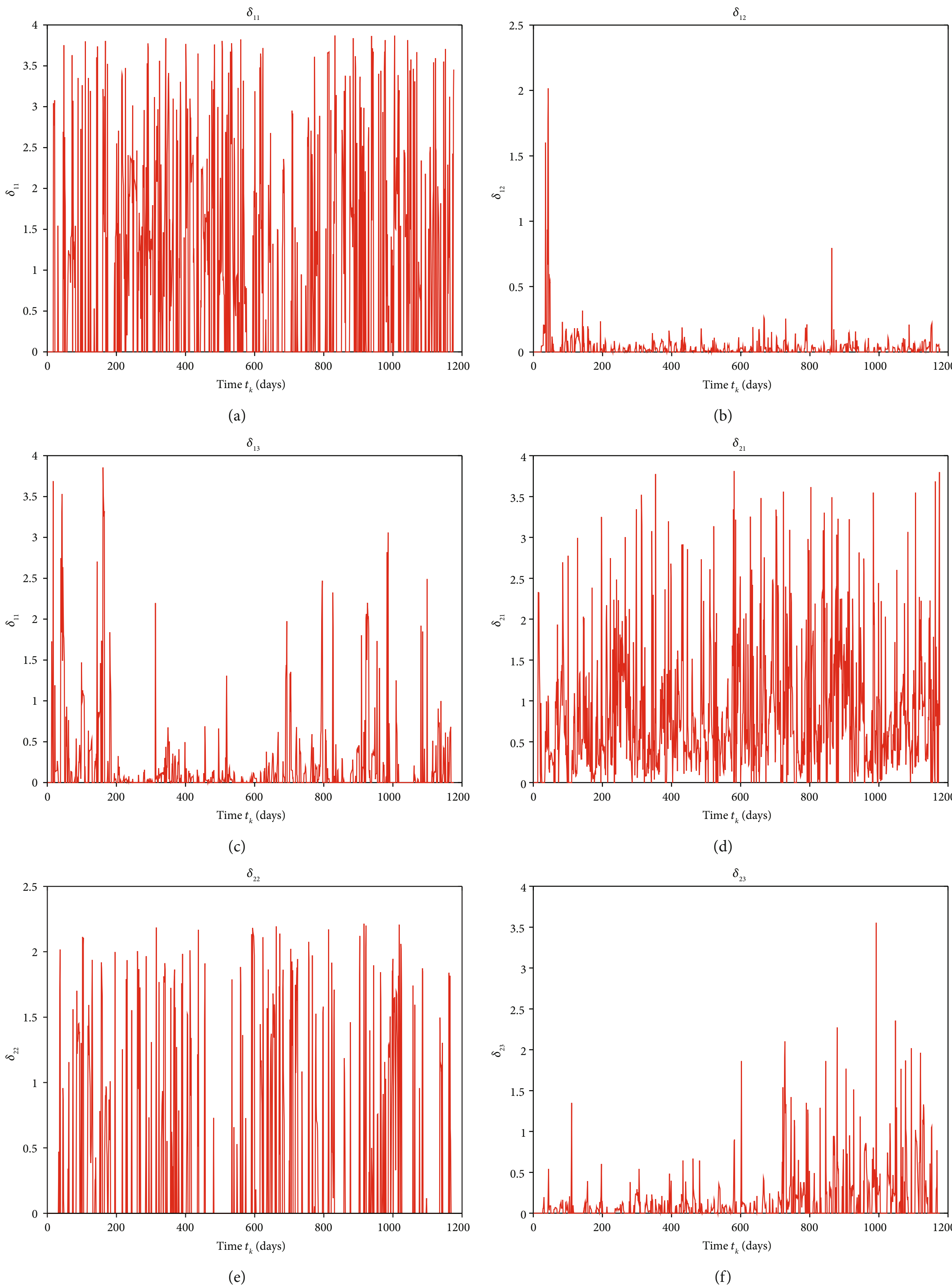
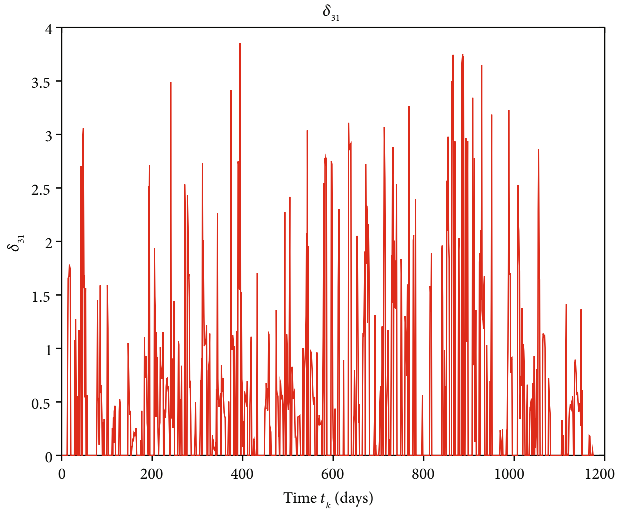
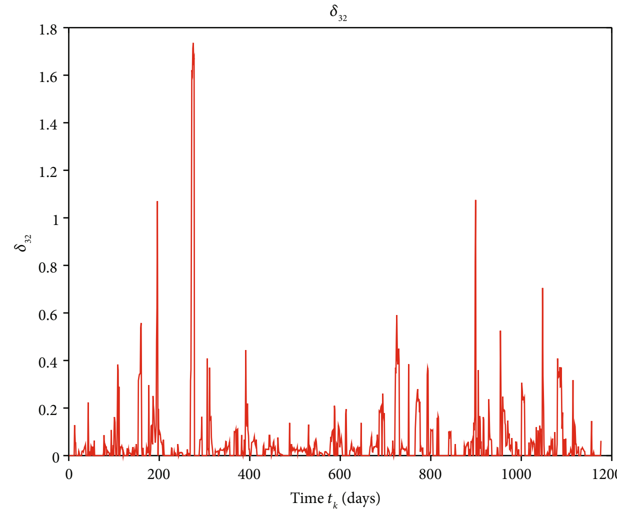


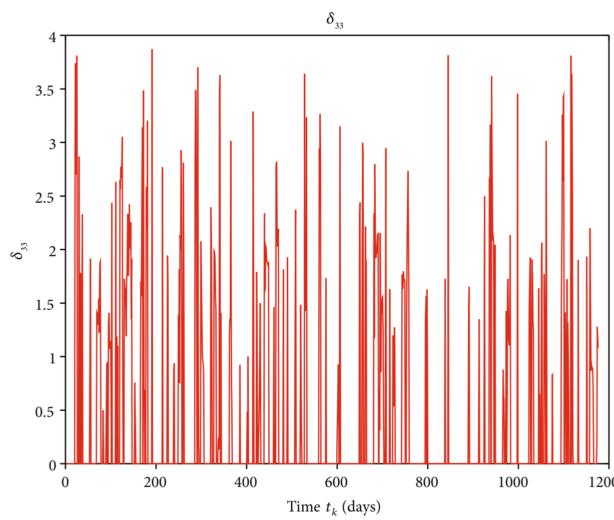
Figure 10: Continued.



(g)



(h)



(i)

FIGURE 10: The graph of interaction coefficients $\delta_{1,1}(\widehat{m}_k, k)$, $\delta_{1,2}(\widehat{m}_k, k)$, $\delta_{1,3}(\widehat{m}_k, k)$, $\delta_{2,1}(\widehat{m}_k, k)$, $\delta_{2,2}(\widehat{m}_k, k)$, $\delta_{2,3}(\widehat{m}_k, k)$, $\delta_{3,1}(\widehat{m}_k, k)$, $\delta_{3,2}(\widehat{m}_k, k)$, and $\delta_{3,3}(\widehat{m}_k, k)$.

Remark 19. We note that if $K = 0$, $\mathcal{S}_{i-1} = 0$, the point $t_k^{i-1} = t_k \in [t_0, T]$. Thus, (76) is reduced to

$$OS_k^{i-1} = OS_k = \begin{cases} I(2, r + k - 1), & \text{if } d \le r, \\ I(2, d + k - 1), & \text{if } d > r, k \in I(0, N). \end{cases} \tag{77}$$

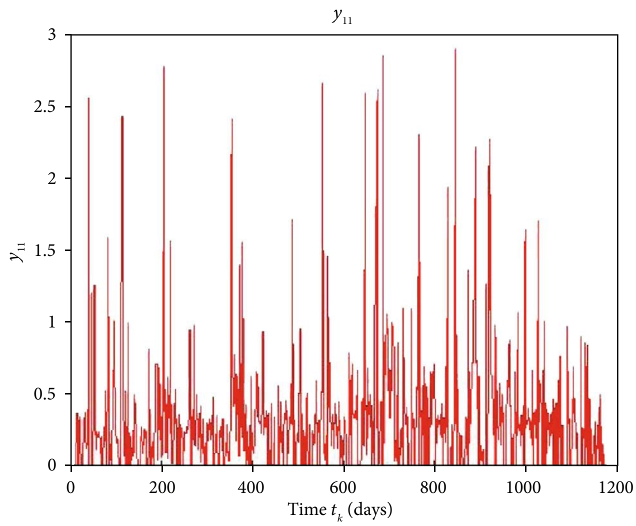
Definition 20. For each $i \in I(1, K^*)$, $m_k^{i-1} \in OS_k^{i-1}$ in Definition 18 and $k \in I(0, N_{i-1})$, a m_k^{i-1} local admissible lagged adapted finite restriction sequence of conditional sample/data observation at time t_k^{i-1} to subpartition P_k^{i-1} of $\mathbb{P}^{i-1}$ in Definition 8 is defined by $(\{\mathbb{E}[\mathbf{y}^{i-1}(t_l^{i-1}) | \mathcal{F}_{l-1}^{i-1}]\}_{l=k-m_k^{i-1}}^{k-1}$, $\{\mathbb{E}[\mathbf{p}^{i-1}(t_l^{i-1}) | \mathcal{F}_{l-1}^{i-1}]\}_{l=k-m_k^{i-1}}^{k-1})$. Moreover, a m_k^{i-1} class of admissible lagged adapted finite sequences of conditional sample/-data observation of size m_k^{i-1} at t_k^{i-1} is defined by

$$\mathcal{A}\mathcal{S}_k^{i-1} = \begin{cases} \left\{ \{\mathbb{E}[\mathbf{y}^{i-1}(t_l^{i-1}) | \mathcal{F}_{l-1}^{i-1}]\}_{l=k-m_k^{i-1}}^{k-1} \right\}_{m_k^{i-1} \in OS_k^{i-1}}, \\ \left\{ \{\mathbb{E}[\mathbf{p}^{i-1}(t_l^{i-1}) | \mathcal{F}_{l-1}^{i-1}]\}_{l=k-m_k^{i-1}}^{k-1} \right\}_{m_k^{i-1} \in OS_k^{i-1}}. \end{cases} \tag{78}$$

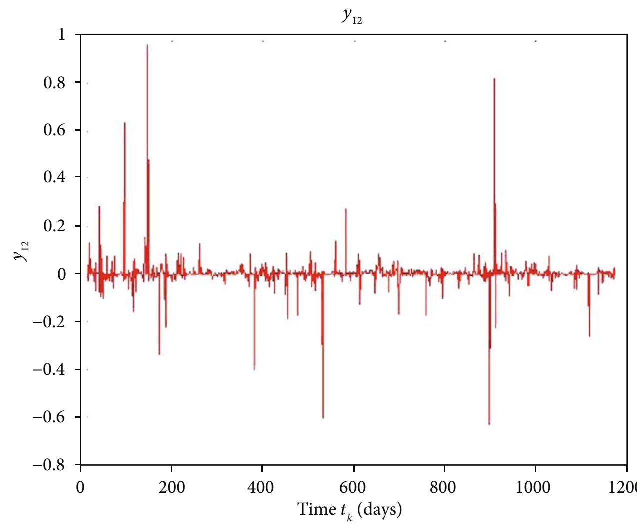
In the case of energy commodity model, for each $i \in I(1, K^*)$, $m_k^{i-1} \in OS_k^{i-1}$, we find corresponding m_k^{i-1} local

TABLE 10: Estimates $\beta_1(\hat{m}_k,k)$, $\gamma_{1,1}(\hat{m}_k,k)$, $\gamma_{1,2}(\hat{m}_k,k)$, $\gamma_{1,3}(\hat{m}_k,k)$, $\beta_2(\hat{m}_k,k)$, $\gamma_{2,1}(\hat{m}_k,k)$, $\gamma_{2,2}(\hat{m}_k,k)$, $\gamma_{2,3}(\hat{m}_k,k)$, $\beta_3(\hat{m}_k,k)$, $\gamma_{3,1}(\hat{m}_k,k)$, $\gamma_{3,2}(\hat{m}_k,k)$, and $\gamma_{3,3}(\hat{m}_k,k)$.

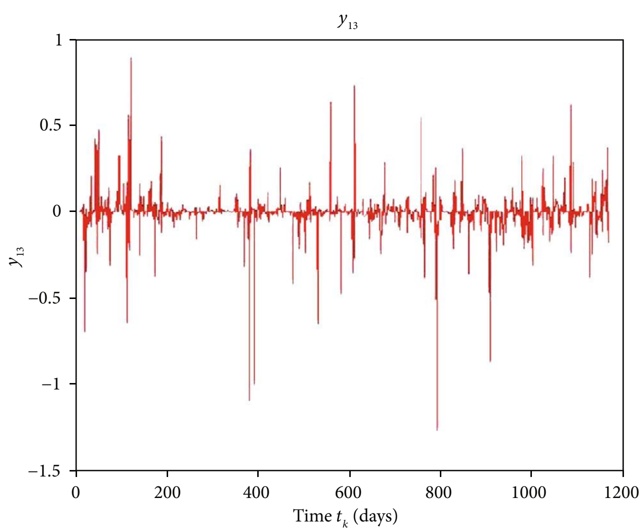
t_K	Natural gas				Crude oil				Coal			
	$\beta_1(\hat{m}_k,k)$	$\gamma_{1,1}(\hat{m}_k,k)$	$\gamma_{1,2}(\hat{m}_k,k)$	$\gamma_{1,3}(\hat{m}_k,k)$	$\beta_2(\hat{m}_k,k)$	$\gamma_{2,1}(\hat{m}_k,k)$	$\gamma_{2,2}(\hat{m}_k,k)$	$\gamma_{2,3}(\hat{m}_k,k)$	$\beta_3(\hat{m}_k,k)$	$\gamma_{3,1}(\hat{m}_k,k)$	$\gamma_{3,2}(\hat{m}_k,k)$	$\gamma_{3,3}(\hat{m}_k,k)$
11	0	0	0	0	0	0	0	0	0	0	0	0
12	0.1681	0.3497	-0.0109	0.0248	-0.4815	0.1626	-0.4066	-0.0123	0.7665	-0.3259	0.0205	-0.0198
13	0.1592	0.3755	-0.0102	0.0228	-0.7778	-0.5752	-0.0578	0.1870	1.0795	-0.2904	0.0135	-0.0217
14	7.3439	0.3755	0.0488	-0.5478	1.7680	-0.5555	-0.2058	0.0291	0.8543	-0.2056	0.0034	-0.0064
15	0.3336	0.3652	-0.0127	0.0213	-0.8999	0.0601	-0.1110	0.0405	0.5144	-0.1264	-0.0062	0.0127
16	0.4709	0.2780	-0.0116	0.0104	-0.8999	0.0601	-0.1110	-0.0292	0.0017	-0.0002	0	0
17	0.3277	0.2780	0.0768	-0.2633	1.3349	0.0027	-0.0330	-0.0750	-0.6285	-0.0569	0.0016	0.0262
18	0.3277	1.3156	-0.1491	-0.1646	1.5419	-0.0088	0.1205	-0.0892	-1.3275	-0.0703	0.0080	0.0386
19	0	0	0	0	-0.1785	-0.0368	-0.0062	0.0189	-0.8091	-0.0001	-0.0083	0.0466
20	0.6985	0.4990	-0.0069	-0.0187	-0.1513	-0.0778	-0.0096	0.0255	-0.0182	0.0080	0.0001	0.0003
…	…	…	…	…	…	…	…	…	…	…	…	…
495	0.1201	0.2264	0.0007	-0.0056	0.2756	0.1131	-0.9587	-0.0087	-0.0288	-0.0780	-0.0019	0.0136
496	0.1809	0.2085	0.0009	-0.0082	0.2898	-0.0431	0.7329	-0.0133	0.1324	-0.1434	-0.0018	0.0158
497	0.2442	0.1597	0.0007	-0.0093	3.1030	-0.0495	-0.0462	-0.0862	0.9772	-0.2426	-0.0065	0.0177
498	0.2742	0.2651	0.0020	-0.0145	1.3147	0.0148	-0.0009	-0.0411	0.2770	-0.1888	-0.0037	0.0217
499	0.3320	0.3298	-0.0009	-0.0070	0.8430	0.0070	0.0283	-0.0265	0.0931	-0.1551	-0.0041	0.0237
500	0.5035	0.2337	-0.0007	-0.0128	1.3320	0.3949	-0.3308	-0.0838	0.4175	-0.2331	-0.0034	0.0222
501	0.6328	0.2612	-0.0034	-0.0077	0.3251	-0.1148	0.0548	0.0042	0.7896	-0.2546	-0.0093	0.0305
502	0.5403	0.2457	-0.0014	-0.0113	0.4863	-0.1315	0.0343	0.0020	3.7990	0.0420	-0.0524	0.0421
503	0.4794	0.2098	-0.0028	-0.0050	0.1239	0.0013	0.0202	-0.0040	9.6735	0.0736	-0.1152	0.0648
504	-0.3308	-0.5600	0.0258	-0.0737	0.2867	-0.0391	0.0009	-0.0034	4.2547	0.3506	-0.0669	0.0382
505	1.1680	0.8346	-0.0274	0.0542	0.1198	-0.0588	-0.3412	0.0092	2.2295	0.0897	-0.0357	0.0306
…	…	…	…	…	…	…	…	…	…	…	…	…
1102	0.6765	0.0455	-0.0020	-0.0908	0.4026	-0.2544	0.2045	0.1058	-5.9294	0.6777	0.0292	0.0068
1103	1.1804	0.4214	-0.0149	0.0837	-0.6549	0.1780	0.0070	0.0018	-6.3380	0.7440	0.0291	0.0106
1104	0.1069	0.2489	-0.0009	-0.0014	-2.1178	0.3406	0.1959	0.1826	-3.8701	0.5681	0.0157	0.0021
1105	0.0139	0.2777	-0.0001	-0.0008	0.3958	-0.0274	0.0642	-0.0620	-4.0701	0.1880	0.0091	0.0514
1106	-0.2513	0.4043	0.0031	-0.0164	0.4097	0.0060	0.1536	-0.0907	-5.0668	0.3261	0.0178	0.0419
1107	0.0670	0.3163	-0	-0.0145	0.2906	0.0485	0.2310	-0.0989	-5.0668	0.4016	0.0308	0.1474
1108	1.0112	0.6861	-0.0091	-0.0107	0.4281	0.0048	0.1337	-0.0933	-5.0668	0.4650	0.0304	0.1295
1109	0.5020	0.5370	-0.0030	-0.0375	0.3645	-0.0168	0.1078	-0.0641	-5.0668	0.4156	0.0311	0.1396
1110	0.1420	0.3295	0.0009	-0.0484	0.1728	-0.0189	-0.0164	-0.0230	-6.6650	0.4509	0.0099	0.0831



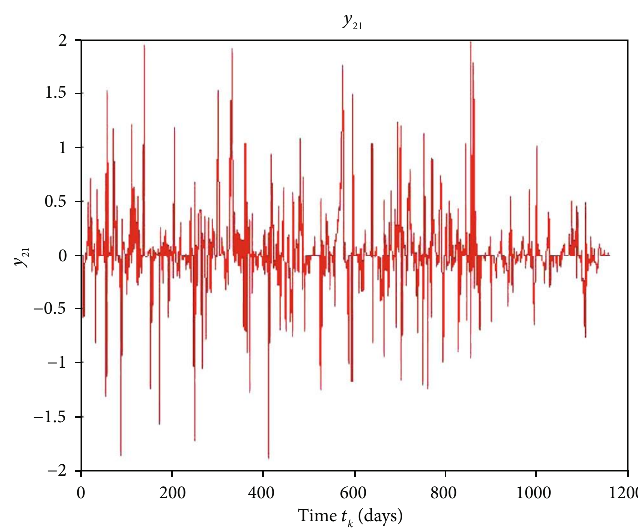
(a)



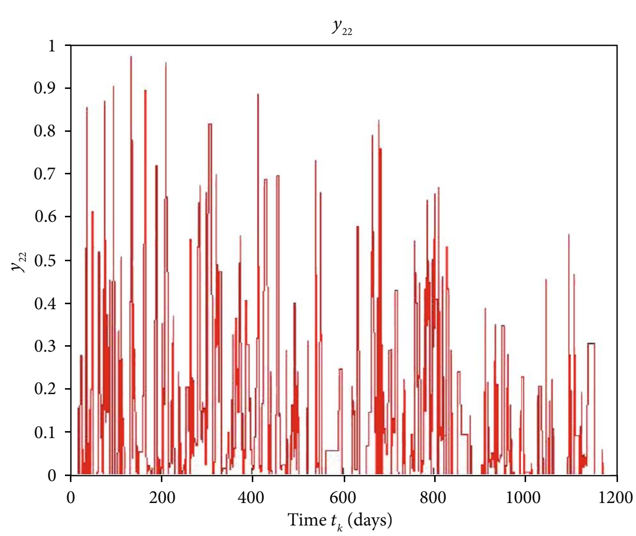
(b)



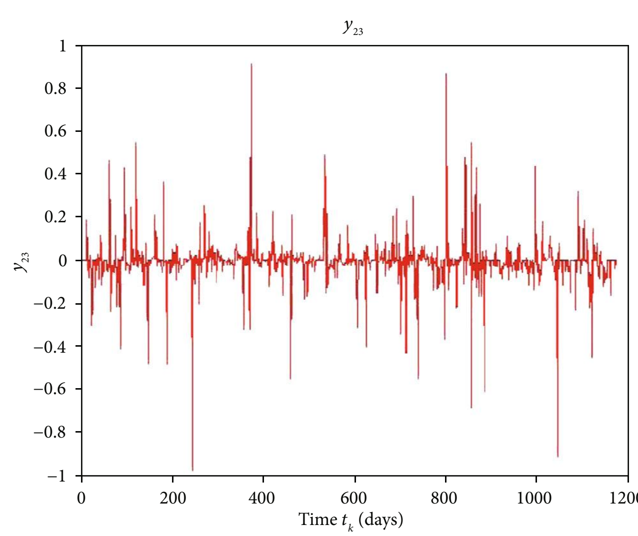
(c)



(d)



(e)



(f)

Figure 11: Continued.

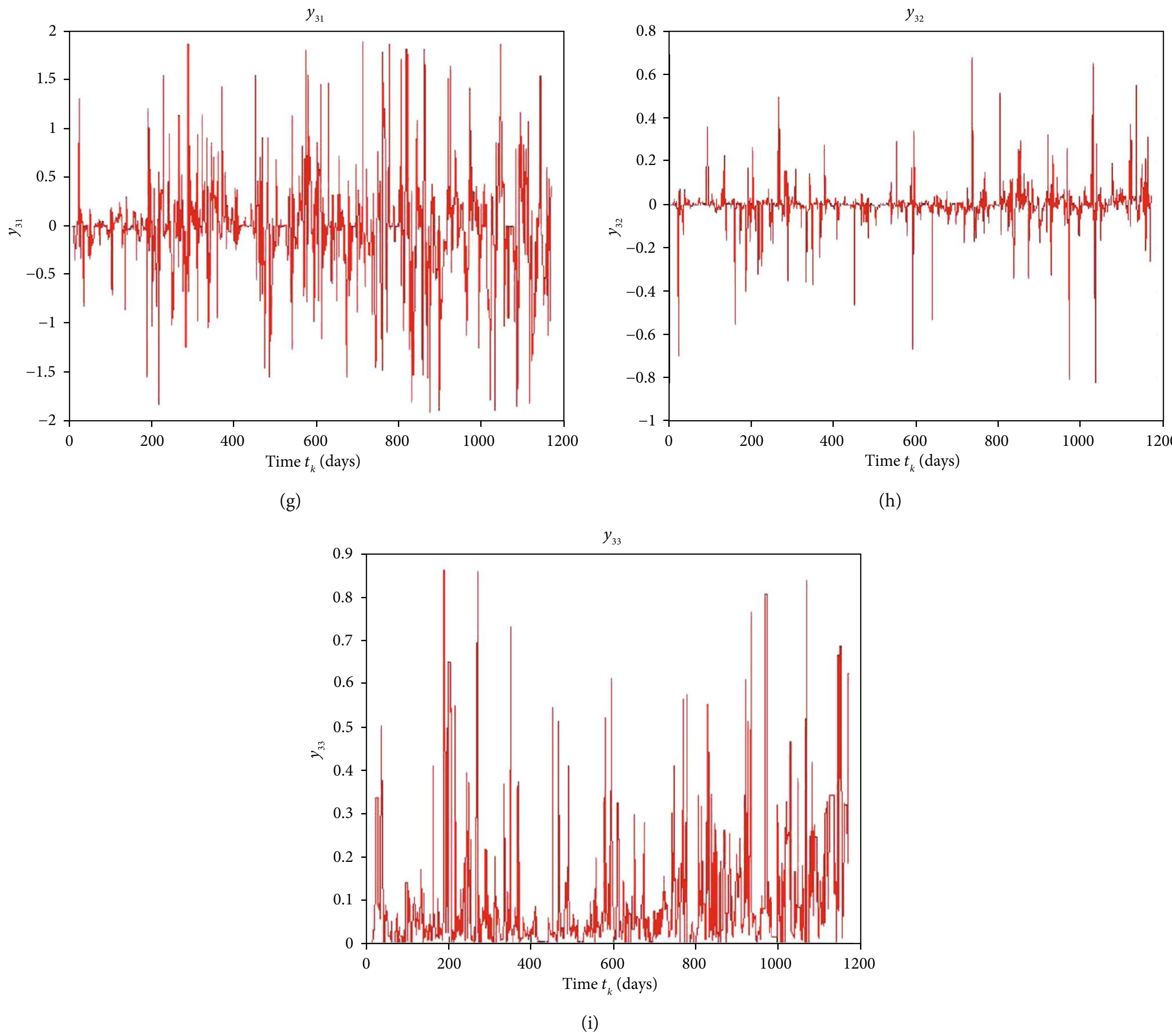
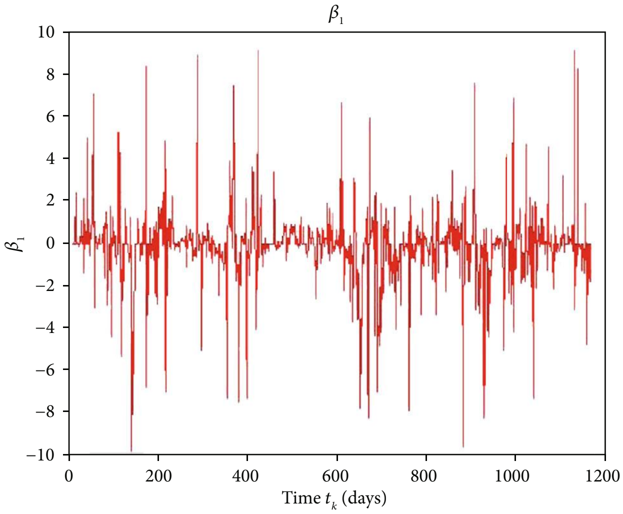


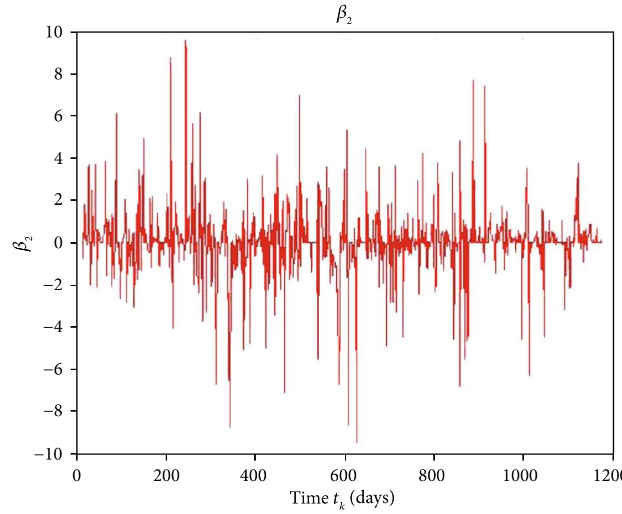
Figure 11: The graph of interaction coefficients $\gamma_{1,1}(\widehat{m}_k, k)$, $\gamma_{1,2}(\widehat{m}_k, k)$, $\gamma_{1,3}(\widehat{m}_k, k)$, $\gamma_{2,1}(\widehat{m}_k, k)$, $\gamma_{2,2}(\widehat{m}_k, k)$, $\gamma_{2,3}(\widehat{m}_k, k)$, $\gamma_{3,1}(\widehat{m}_k, k)$, $\gamma_{3,2}(\widehat{m}_k, k)$, and $\gamma_{3,3}(\widehat{m}_k, k)$.

admissible adapted finite sequence of conditional sample/data observation at t_k^{i-1}, $(\{\mathbb{E}[\mathbf{y}^{i-1}(t_l^{i-1})|\mathscr{F}_{l-1}^{i-1}]\}_{l=k-m_k^{i-1}}^{k-1}$, $\{\mathbb{E}[\mathbf{p}^{i-1}(t_l^{i-1})|\mathscr{F}_{l-1}^{i-1}]\}_{l=k-m_k^{i-1}}^{k-1})$. For $i \in I(1, K^*)$, using this sequence and solutions of (70), we compute

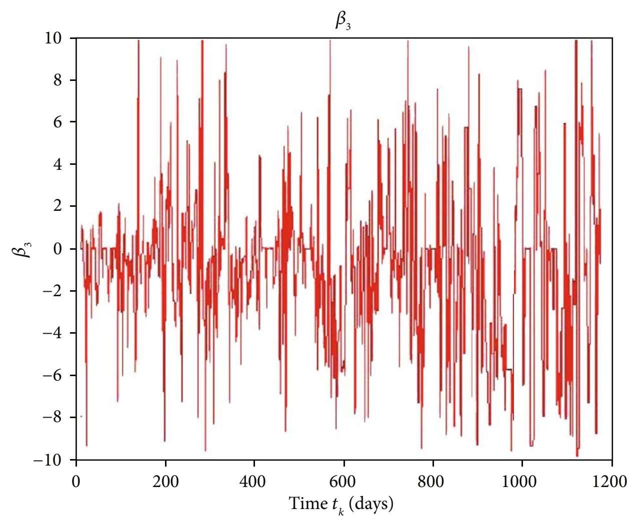
$$\left\{ u_j^{i-1}\left(m_k, t_k^{i-1}\right), \beta_j^{i-1}\left(m_k^{i-1}, t_k^{i-1}\right), \kappa_{j,l}^{i-1}\left(m_k^{i-1}, t_k^{i-1}\right), \gamma_{j,l}^{i-1}\left(m_k^{i-1}, t_k^{i-1}\right), \delta_{j,l}^{i-1}\left(m_k^{i-1}, t_k^{i-1}\right), \sigma_{j,l}^{i-1}\left(m_k^{i-1}, t_k^{i-1}\right), k \in [0, N_{i-1}], \right. \tag{79}$$



(a)



(b)



(c)

FIGURE 12: The graph of $\beta_1(\widehat{m}_k, k)$, $\beta_2(\widehat{m}_k, k)$, and $\beta_3(\widehat{m}_k, k)$ for natural gas, crude oil, and coal, respectively.

for $j, l \in I(1, n)$. This leads to a local finite sequence of parameter estimates at t_k^{i-1} defined on OS_k^{i-1} as follows:

$$\left\{\left(\widehat{u}_j^{i-1}\left(m_k^{i-1}, t_k^{i-1}\right), \widehat{\beta}_j^{i-1}\left(m_k^{i-1}, t_k^{i-1}\right), \widehat{\kappa}_{j,l}^{i-1}\left(m_k^{i-1}, t_k^{i-1}\right), \widehat{\gamma}_{j,l}^{i-1}\left(m_k^{i-1}, t_k^{i-1}\right), \widehat{\delta}_{j,l}^{i-1}\left(m_k^{i-1}, t_k^{i-1}\right), \widehat{\sigma}_{j,l}^{i-1}\left(m_k^{i-1}, t_k^{i-1}\right)\right)\right\}_{m_k^{i-1} \in OS_k^{i-1}}. \tag{80}$$

The above defined collection is denoted by

$$(\mathcal{U}_k, \mathcal{B}_k, \mathcal{K}_k, \boldsymbol{\gamma}_k, \boldsymbol{\delta}_k, \boldsymbol{\sigma}_k) = \begin{cases} \left\{\left(\widehat{u}_j^{i-1}\left(m_k^{i-1}, t_k^{i-1}\right), \widehat{\beta}_j^{i-1}\left(m_k^{i-1}, t_k^{i-1}\right), \widehat{\kappa}_{j,l}^{i-1}\left(m_k^{i-1}, t_k^{i-1}\right), \widehat{\gamma}_{j,l}^{i-1}\left(m_k^{i-1}, t_k^{i-1}\right), \widehat{\delta}_{j,l}^{i-1}\left(m_k^{i-1}, t_k^{i-1}\right), \widehat{\sigma}_{j,l}^{i-1}\left(m_k^{i-1}, t_k^{i-1}\right)\right)\right\}_{m_k^{i-1} \in OS_k^{i-1}}, \\ \text{for } j \in I(1, n), i \in I(1, K^*). \end{cases} \tag{81}$$

7.3. Conceptual Computation of State Simulation Scheme. For the development of a conceptual computational scheme, we need to employ the method of induction. The presented simulation scheme is based on the idea of lagged adaptive expectation process [17]. For $j, l \in I(1,n)$, an autocorrelation function (ACF) analysis [21, 32] performed on $(s^{j,j}_{m^{i-1}_k,k}(\mathbf{y}), s^{j,j}_{m^{i-1}_k,k}(\mathbf{p}))$ suggests that the interconnected discrete-time dynamic model of local conditional sample mean and sample variance statistics in (48) is of order $d_{i-1} = 2$. In view of this, we need to identify the initial data. We begin with a given initial data $(\mathbf{y}^{i-1}(T_{i-1}), \mathbf{p}^{i-1}(T_{i-1}))$, $(\{\sum_{m^{i-1}_0,t^{i-1}_0}(y)\}_{m^{i-1}_0 \in OS^{i-1}_0}, \{\sum_{m^{i-1}_0,t^{i-1}_0}(p)\}_{m^{i-1}_0 \in OS^{i-1}_0}), (\{\sum_{m^{i-1}_{-1},t^{i-1}_{-1}}(y)\}_{m^{i-1}_{-1} \in OS^{i-1}_{-1}}, \{\sum_{m^{i-1}_{-1},t^{i-1}_{-1}}(p)\}_{m^{i-1}_{-1} \in OS^{i-1}_{-1}})$, $(\{\bar{S}_{m^{i-1}_{-1},t^{i-1}_{-1}}(y)\}_{m^{i-1}_{-1} \in OS^{i-1}_{-1}}, \{\bar{S}_{m^{i-1}_{-1},t^{i-1}_{-1}}(p)\}_{m^{i-1}_{-1} \in OS^{i-1}_{-1}})$ as defined in (45), (46), (47), and (48).

Let $(\mathbf{y}^s(m^{i-1}_k, t^{i-1}_k), \mathbf{p}^s(m^{i-1}_k, t^{i-1}_k)) \equiv (\mathbf{y}^{i-1,s}(m^{i-1}_k, t^{i-1}_k), \mathbf{p}^{i-1,s}(m^{i-1}_k, t^{i-1}_k))$ be a simulated value of $(\mathbb{E}[\mathbf{y}^{i-1}(t^{i-1}_k)|\mathcal{F}^{i-1}_{k-1}], \mathbb{E}[\mathbf{p}^{i-1}(t^{i-1}_k)|\mathcal{F}^{i-1}_{k-1}])$ at time t^{i-1}_k corresponding to an admissible sequence $\{\mathbb{E}[\mathbf{y}^{i-1}(t^{i-1}_l)|\mathcal{F}^{i-1}_{l-1}], \mathbb{E}[\mathbf{p}^{i-1}(t^{i-1}_l)|\mathcal{F}^{i-1}_{l-1}]\}^{k-1}_{l=k-m_k} \in \mathcal{AS}^{i-1}_k$. For $q=1$, and $j \in \boldsymbol{I}(1,n)$, the simulated value $(y^s_j(m^{i-1}_k, t^{i-1}_k), p^s_j(m^{i-1}_k, t^{i-1}_k))$ is generated from the discretized Euler scheme (62) as follows:

$$\begin{cases} y^s_j(m^{i-1}_k, t^{i-1}_k) = y^s_j(m^{i-1}_{k-1}, t^{i-1}_{k-1}) + \left(u^{i-1}_j(m^{i-1}_{k-1}, t^{i-1}_{k-1}) - y^s_j(m^{i-1}_{k-1}, t^{i-1}_{k-1})\right)\left[\kappa^{i-1}_{j,j}(m^{i-1}_{k-1}, t^{i-1}_{k-1}) y^s_j(m^{i-1}_{k-1}, t^{i-1}_{k-1}) + \sum_{l\neq j}^{n} \kappa^{i-1}_{j,l}(m^{i-1}_{k-1}, t^{i-1}_{k-1}) y^s_l(m^{i-1}_{k-1}, t^{i-1}_{k-1})\right]\Delta t \\ \quad + \left(u^{i-1}_j(m^{i-1}_{k-1}, t^{i-1}_{k-1}) - y^s_j(m^{i-1}_{k-1}, t^{i-1}_{k-1})\right)\left[\delta^{i-1}_{j,j}(m^{i-1}_{k-1}, t^{i-1}_{k-1}) \Delta W_{j,j}(m^{i-1}_k, t^{i-1}_k) + \sum_{l\neq j}^{n} \delta^{i-1}_{j,l}(m^{i-1}_{k-1}, t^{i-1}_{k-1}) y^s_l(m^{i-1}_{k-1}, t^{i-1}_{k-1}) \Delta W_{j,l}(m^{i-1}_k, t^{i-1}_k)\right], \\ \quad \cdot\ t^{i-1}_k \in [T_{i-1}, T_i), \\ p^s_j(m^{i-1}_k, t^{i-1}_k) = p^s_j\left[\gamma^{i-1}_{j,j}(m^{i-1}_{k-1}, t^{i-1}_{k-1})\left(y^s_j(m^{i-1}_{k-1}, t^{i-1}_{k-1}) - p^s_j(m^{i-1}_{k-1}, t^{i-1}_{k-1})\right) + \beta^{i-1}_j(m^{i-1}_{k-1}, t^{i-1}_{k-1}) + \sum_{l\neq j}^{n} \gamma^{i-1}_{j,l}(m^{i-1}_{k-1}, t^{i-1}_{k-1}) p^s_l(m^{i-1}_{k-1}, t^{i-1}_{k-1})\right]\Delta t \\ \quad + \sigma^{i-1}_{j,j}(m^{i-1}_{k-1}, t^{i-1}_{k-1}) p^s_j(m^{i-1}_{k-1}, t^{i-1}_{k-1}) \Delta Z_{j,j}(m^{i-1}_k, t^{i-1}_k) + p^s_j(m^{i-1}_{k-1}, t^{i-1}_{k-1}) \sum_{l\neq j}^{n} \sigma^{i-1}_{j,l}(m^{i-1}_{k-1}, t^{i-1}_{k-1}) p^s_l(m^{i-1}_{k-1}, t^{i-1}_{k-1}) \Delta Z_{j,l}(m^{i-1}_k, t^{i-1}_k), \\ \quad \cdot\ t^{i-1}_k \in [T_{i-1}, T_i). \end{cases} \tag{82}$$

To find the simulated value $y^{i,s}_j(T_i)$ and $p^{i,s}_j(T_i)$, we need to estimate $\widehat{\pi}^i_j$ and $\widehat{\theta}^i_j$ by first simulating $\lim_{t\to T^-_i} y^{i-1}_j(t, T_{i-1}, \mathbf{y}^{i-1,s}) \equiv y^s_j(m^{i-1}_{N_{i-1}}, t^{i-1}_{N_{i-1}})$ and $\lim_{t\to T^-_i} p^{i-1}_j(t, T_{i-1}, \mathbf{y}^{i-1,s}, \mathbf{p}^{i-1,s}) \equiv p^s_j(m^{i-1}_{N_{i-1}}, t^{i-1}_{N_{i-1}})$ as follows:

$$\begin{aligned} y^s_j\left(m^{i-1}_{N_{i-1}}, t^{i-1}_{N_{i-1}}\right) &= y^s_j\left(m^{i-1}_{N_{i-1}-1}, t^{i-1}_{N_{i-1}-1}\right) + \left(u^{i-1}_j\left(m^{i-1}_{N_{i-1}-1}, t^{i-1}_{N_{i-1}-1}\right) - y^s_j\left(m^{i-1}_{N_{i-1}-1}, t^{i-1}_{N_{i-1}-1}\right)\right)\left[\sum_{l=1}^{n} \kappa^{i-1}_{j,l}\left(m^{i-1}_{N_{i-1}-1}, t^{i-1}_{N_{i-1}-1}\right) y^s_l\left(m^{i-1}_{N_{i-1}-1}, t^{i-1}_{N_{i-1}-1}\right)\right]\Delta t \\ &\quad + \left(u^{i-1}_j\left(m^{i-1}_{N_{i-1}-1}, t^{i-1}_{N_{i-1}-1}\right) - y^s_j\left(m^{i-1}_{N_{i-1}-1}, t^{i-1}_{N_{i-1}-1}\right)\right)\left[\delta^{i-1}_{j,j}\left(m^{i-1}_{N_{i-1}-1}, t^{i-1}_{N_{i-1}-1}\right) \Delta W_{j,j}\left(m^{i-1}_{N_{i-1}}, t^{i-1}_{N_{i-1}}\right)\right. \\ &\quad \left. + \sum_{l\neq j}^{n} \delta^{i-1}_{j,l}\left(m^{i-1}_{N_{i-1}-1}, t^{i-1}_{N_{i-1}-1}\right) y^s_l\left(m^{i-1}_{N_{i-1}-1}, t^{i-1}_{N_{i-1}-1}\right) \Delta W_{j,l}\left(m^{i-1}_{N_{i-1}}, t^{i-1}_{N_{i-1}}\right)\right], \\ p^s_j\left(m^{i-1}_{N_{i-1}}, t^{i-1}_{N_{i-1}}\right) &= p^s_j\left[\gamma^{i-1}_{j,j}\left(m^{i-1}_{N_{i-1}-1}, t^{i-1}_{N_{i-1}-1}\right)\left(y^s_j\left(m^{i-1}_{N_{i-1}-1}, t^{i-1}_{N_{i-1}-1}\right) - p^s_j\left(m^{i-1}_{N_{i-1}-1}, t^{i-1}_{N_{i-1}-1}\right)\right) + \beta^{i-1}_j\left(m^{i-1}_{N_{i-1}-1}, t^{i-1}_{N_{i-1}-1}\right)\right. \\ &\quad \left. + \sum_{l\neq j}^{n} \gamma^{i-1}_{j,l}\left(m^{i-1}_{N_{i-1}-1}, t^{i-1}_{N_{i-1}-1}\right) p^s_l\left(m^{i-1}_{N_{i-1}-1}, t^{i-1}_{N_{i-1}-1}\right)\right]\Delta t + \sigma^{i-1}_{j,j}\left(m^{i-1}_{N_{i-1}-1}, t^{i-1}_{N_{i-1}-1}\right) p^s_j\left(m^{i-1}_{N_{i-1}-1}, t^{i-1}_{N_{i-1}-1}\right) \Delta Z_{j,j}\left(m^{i-1}_{N_{i-1}}, t^{i-1}_{N_{i-1}}\right) \\ &\quad + p^s_j\left(m^{i-1}_{N_{i-1}-1}, t^{i-1}_{N_{i-1}-1}\right) \sum_{l\neq j}^{n} \sigma^{i-1}_{j,l}\left(m^{i-1}_{N_{i-1}-1}, t^{i-1}_{N_{i-1}-1}\right) p^s_l\left(m^{i-1}_{N_{i-1}-1}, t^{i-1}_{N_{i-1}-1}\right) \Delta Z_{j,l}\left(m^{i-1}_{N_{i-1}}, t^{i-1}_{N_{i-1}}\right). \end{aligned} \tag{83}$$

From this, we calculate $\widehat{\pi}_j^i$ and $\widehat{\theta}_j^i$ as:

$$\widehat{\pi}_j^i = \frac{\mathbb{E}\left[y_j^{i-1}(T_i)\middle|\mathscr{F}_{T_i-1}^{i-1}\right]}{y_j^s\left(m_{N_{i-1}}^{i-1}, t_{N_{i-1}}^{i-1}\right)}$$
$$\widehat{\theta}_j^i = \frac{\mathbb{E}\left[p_j^{i-1}(T_i)\middle|\mathscr{F}_{T_i-1}^{i-1}\right]}{p_j^s\left(m_{N_{i-1}}^{i-1}, t_{N_{i-1}}^{i-1}\right)}. \quad (84)$$

Thus, $y_j^{i,s}(T_i) = \widehat{\pi}_j^i y_j^{i-1,s}(T_i^-, T_{i-1}, \mathbf{y}^{i-1,s})$ and $p_j^{i,s}(T_i) = \widehat{\theta}_j^i p_j^{i-1,s}(T_i^-, T_{i-1}, \mathbf{y}^{i-1,s}, \mathbf{p}^{i-1,s})$. Let $(\{\mathbf{y}^s(m_k^{i-1}, t_k^{i-1})\}_{m_k^{i-1}\in OS_k^{i-1}}, \{\mathbf{p}^s(m_k^{i-1}, t_k^{i-1})\}_{m_k^{i-1}\in OS_k^{i-1}})$ be a m_k^{i-1}- local sequence of simulated values corresponding to m_k^{i-1}-admissible lagged adapted finite sequence of conditional observation belonging to $\mathscr{AS}_k^{i-1}$, and corresponding term of sequence. $(\mathscr{U}_k, \mathscr{B}_k, \mathscr{K}_k, \boldsymbol{\gamma}_k, \boldsymbol{\delta}_k, \boldsymbol{\sigma}_k)$. Thus, for each $i \in I(1, K^*)$, $(\{\mathbf{y}^s(m_k^{i-1}, t_k^{i-1})\}_{m_k^{i-1}\in OS_k^{i-1}}, \{\mathbf{p}^s(m_k^{i-1}, t_k^{i-1})\}_{m_k^{i-1}\in OS_k^{i-1}})$ are the finite sequence correspondence of simulated values of $(\mathbb{E}[\mathbf{y}^{i-1}(t_k^{i-1})|\mathscr{F}_{k-1}^{i-1}], \mathbb{E}[\mathbf{p}^{i-1}(t_k^{i-1})|\mathscr{F}_{k-1}^{i-1}])$ at t_k^{i-1}.

7.4. Mean-Square Suboptimal Procedure. To find the best estimate of $(\mathbb{E}[y(t_k^{i-1})|\mathscr{F}_{k-1}^{i-1}], \mathbb{E}[\mathbf{p}(t_k^{i-1})|\mathscr{F}_{k-1}^{i-1}])$ using a local admissible finite sequence $(\{\mathbf{y}^s(m_k^{i-1}, t_k^{i-1})\}_{m_k^{i-1}\in OS_k^{i-1}}, \{\mathbf{p}^s(m_k^{i-1}, t_k^{i-1})\}_{m_k^{i-1}\in OS_k^{i-1}})$, we need to compute a finite sequence of quadratic mean square error corresponding to $(\{\mathbf{y}^s(m_k^{i-1}, t_k^{i-1})\}_{m_k^{i-1}\in OS_k^{i-1}}, \{\mathbf{p}^s(\mathrm{m}_k^{i-1}, t_k^{i-1})\}_{m_k^{i-1}\in OS_k^{i-1}})$. The quadratic mean square error is defined below.

Definition 21. For each $i \in I(1, K^*)$, the quadratic mean square error of $(\mathbb{E}[\mathbf{y}(t_k^{i-1})|\mathscr{F}_{k-1}^{i-1}], \mathbb{E}[\mathbf{p}(t_k^{i-1})|\mathscr{F}_{k-1}^{i-1}])$ relative to each member of the term of local admissible sequence $(\{\mathbf{y}^s(m_k^{i-1}, t_k^{i-1})\}_{m_k^{i-1}\in OS_k^{i-1}}, \{\mathbf{p}^s(m_k^{i-1}, t_k^{i-1})\}_{m_k^{i-1}\in OS_k^{i-1}})$ of simulated values is defined by

$$\left\{ \Xi_{m_k^{i-1}, t_k^{i-1}} = \left\|\mathbf{y}^s\left(m_k^{i-1}, t_k^{i-1}\right) - \mathbb{E}\left[\mathbf{y}^{i-1}\left(t_k^{i-1}\right)\middle|\mathscr{F}_{k-1}^{i-1}\right]\right\|^2 + \left\|\mathbf{p}^s\left(m_k^{i-1}, t_k^{i-1}\right) - \mathbb{E}\left[\mathbf{p}^{i-1}\left(t_k^{i-1}\right)\middle|\mathscr{F}_{k-1}^{i-1}\right]\right\|^2. \right. \quad (85)$$

For any arbitrary small positive number ε and for each time t_k^{i-1}, to find the the best estimate from the admissible simulated values of simulated sequence of $(\{\mathbf{y}^s(m_k^{i-1}, t_k^{i-1})\}_{m_k^{i-1}\in OS_k^{i-1}}, \{\mathbf{p}^s(\mathrm{m}_k^{i-1}, t_k^{i-1})\}_{m_k^{i-1}\in OS_k^{i-1}})$ for $(\mathbb{E}[\mathbf{y}(t_k^{i-1})|\mathscr{F}_{k-1}^{i-1}], \mathbb{E}[\mathbf{p}(t_k^{i-1})|\mathscr{F}_{k-1}^{i-1}])$, we determine the following sub-optimal admissible set of m_k^{i-1}-size local conditional sample

$$\left\{ \mathscr{M}_{t_k^{i-1}} = \left\{ m_k^{i-1} \in OS_k^{i-1} : \Xi_{m_k^{i-1}, t_k^{i-1}} < \varepsilon \right\}, \quad \text{for } i \in I(1, K^*). \right. \quad (86)$$

Among these collected values, the value that gives the minimum $\Xi_{m_k^{i-1}, t_k^{i-1}}$ for $k \in [0, N_{i-1}]$ are recorded as $\widehat{m}_k^{i-1}$. If more than one value exist, then the largest of such m_k^{i-1}'s is recorded as $\widehat{m}_k^{i-1}$. If condition (86) is not met at time t_k^{i-1}, the value of m_k^{i-1} where the minimum $\min_{m_k^{i-1}} \Xi_{m_k^{i-1}, t_k^{i-1}}$ is attained is recorded as $\widehat{m}_k^{i-1}$. The $\varepsilon-$ level sub-optimal estimates of the parameters $\{\widehat{u}_j^{i-1}(m_k^{i-1}, t_k^{i-1}), \widehat{\kappa}_{j,l}^{i-1}(m_k^{i-1}, t_k^{i-1}), \widehat{\beta}_j^{i-1}(m_k^{i-1}, t_k^{i-1}), \widehat{\delta}_{j,l}^{i-1}(m_k^{i-1}, t_k^{i-1}), \widehat{\gamma}_{j,l}^{i-1}(m_k^{i-1}, t_k^{i-1}), \widehat{\sigma}_{j,l}^{i-1}(m_k^{i-1}, t_k^{i-1})\}$ are recorded as $\{u_j^{i-1}(\widehat{m}_{k,y_j}^{i-1}, k), \kappa_{j,l}^{i-1}(\widehat{m}_{k,y_j}^{i-1}, k), \beta_j^{i-1}(\widehat{m}_{k,p_j}^{i-1}, k), \delta_{j,l}^{i-1}(\widehat{m}_{k,y_j}^{i-1}, k), \gamma_{j,l}^{i-1}(\widehat{m}_{k,p_j}^{i-1}, k), \sigma_{j,l}^{i-1}(\widehat{m}_{k,p_j}^{i-1}, k)\}$. Finally, the simulated value $\mathbf{y}^s(m_k^{i-1}, t_k^{i-1})$, $\mathbf{p}^s(m_k^{i-1}, t_k^{i-1})$ at time t_k^{i-1} with $\widehat{m}_k^{i-1}$ is now recorded as the best estimate for $\mathbb{E}[\mathbf{y}^{i-1}(t_k^{i-1}) \mid \mathscr{F}_{k-1}^{i-1}]$ and $\mathbb{E}[\mathbf{p}^{i-1}(t_k^{i-1}) \mid \mathscr{F}_{k-1}^{i-1}]$. The value $\mathbf{y}^s(\widehat{m}_k^{i-1}, k)$, $\mathbf{p}^s(\widehat{m}_k^{i-1}, k)$ is called the $\varepsilon-$ sub-optimal simulated value of $\mathbf{y}^s(m_k^{i-1}, t_k^{i-1})$ and $\mathbf{p}^s(m_k^{i-1}, t_k^{i-1})$ of $\mathbb{E}[\mathbf{y}^{i-1}(t_k^{i-1}) \mid \mathscr{F}_{k-1}^{i-1}]$ and $\mathbb{E}[\mathbf{p}^{i-1}(t_k^{i-1}) \mid \mathscr{F}_{k-1}^{i-1}]$ at t_k^{i-1}.

7.5. Illustration: Application of Conceptual Computational Algorithm to Energy Commodity Data Set. In this subsection, we apply the above conceptual computational algorithm to study the relationship between three energy commodities by setting $n = 3$ in (56). The three energy commodities are daily Henry Hub Natural gas data set, daily crude oil data set, and daily coal data set for the period of 05/04/2009-01/03/2014, [35–37]. Thus, for each pair (y_1, p_1), (y_2, p_2), and (y_3, p_3), the drift and diffusion coefficient function of the stochastic dynamic equation governing (y_j, p_j), for $j \in I(1, 3)$ has 4 and 3 parameters each to be estimated, respectively. Thus, there are 42 parameters to be estimated in total. Using $\Delta t = 1$, $\varepsilon = 0.001$, for each $j \in I(1, 3)$, the $\varepsilon-$ level sub-optimal estimates of parameters $u_j^{i-1}(\widehat{m}_k^{i-1}, k)$, $\beta_j^{i-1}(\widehat{m}_k^{i-1}, k)$, $\kappa_{j,l}^{i-1}(\widehat{m}_k^{i-1}, k)$, $\gamma_{j,l}^{i-1}(\widehat{m}_k^{i-1}, k)$, $\delta_{j,l}^{i-1}(\widehat{m}_k^{i-1}, k)$, $\sigma_{j,l}^{i-1}(\widehat{m}_k^{i-1}, k)$, $l \in I(1, 3)$, at each real data times are exhibited below.

7.5.1. Illustration: Relationship between Natural Gas, Crude Oil, and Coal: Without Incorporating Jump Process. In this subsubsection, we analyze the relationship between natural gas, crude oil, and coal without the jump process. For $j, l \in I(1, 3)$, the stochastic dynamic system governing the three energy commodities is described in (59) of Remark 12. Here, (y_1, p_1) denotes the mean spot and the spot price process of natural gas, (y_2, p_2) denotes the mean spot and the spot price process of crude oil, and (y_3, p_3) denotes the mean spot and the spot price process of coal.

Using the discretized scheme (82), we apply the above conceptual computational algorithm for the real-time data sets namely daily Henry Hub natural gas data set, daily crude oil data set, and daily coal data set for the period of 05/04/2009-01/03/2014, [35–37]. Using $\Delta t = 1$, $\varepsilon = 0.001$, $r = 10$, and $d = 2$, the ε-level suboptimal estimates of the parameters at each real data times are described below for each commodity data sets.

The parameters corresponding to the natural gas data set are $u_1(\widehat{m}_k, k)$, $\beta_1(\widehat{m}_k, k)$, $\kappa_{1,1}(\widehat{m}_k, k)$, $\kappa_{1,2}(\widehat{m}_k, k)$, $\kappa_{1,3}(\widehat{m}_k, k)$, $\gamma_{1,1}(\widehat{m}_k, k)$, $\gamma_{1,2}(\widehat{m}_k, k)$, $\gamma_{1,3}(\widehat{m}_k, k)$, $\delta_{1,1}(\widehat{m}_k, k)$, $\delta_{1,2}(\widehat{m}_k, k)$, $\delta_{1,3}(\widehat{m}_k, k)$, $\sigma_{1,1}(\widehat{m}_k, k)$, $\sigma_{1,2}(\widehat{m}_k, k)$, and $\sigma_{1,3}(\widehat{m}_k, k)$. The parameters corresponding to the crude oil data set are $u_2(\widehat{m}_k, k)$, $\beta_2(\widehat{m}_k, k)$, $\kappa_{2,1}(\widehat{m}_k, k)$, $\kappa_{2,2}(\widehat{m}_k, k)$, $\kappa_{2,3}(\widehat{m}_k, k)$, $\gamma_{2,1}(\widehat{m}_k, k)$, $\gamma_{2,2}(\widehat{m}_k, k)$, $\gamma_{2,3}(\widehat{m}_k, k)$, $\delta_{2,1}(\widehat{m}_k, k)$, $\delta_{2,2}(\widehat{m}_k, k)$, $\delta_{2,3}(\widehat{m}_k, k)$, $\sigma_{2,1}(\widehat{m}_k, k)$, $\sigma_{2,2}(\widehat{m}_k, k)$, and $\sigma_{2,3}(\widehat{m}_k, k)$. The parameters corresponding to coal data set are $u_3(\widehat{m}_k, k)$, $\beta_3(\widehat{m}_{k,p_3}, k)$, $\kappa_{3,1}(\widehat{m}_k, k)$, $\kappa_{3,2}(\widehat{m}_k, k)$, $\kappa_{3,3}(\widehat{m}_k, k)$, $\gamma_{3,1}(\widehat{m}_k, k)$, $\gamma_{3,2}(\widehat{m}_k, k)$, $\gamma_{3,3}(\widehat{m}_k, k)$, $\delta_{3,1}(\widehat{m}_k, k)$, $\delta_{3,2}(\widehat{m}_k, k)$, $\delta_{3,3}(\widehat{m}_k, k)$, $\sigma_{3,1}(\widehat{m}_k, k)$, $\sigma_{3,2}(\widehat{m}_k, k)$, and $\sigma_{3,3}(\widehat{m}_k, k)$.

The following table gives the drift coefficient's parameter estimates $u_1(\widehat{m}_k, k)$, $\kappa_{1,1}(\widehat{m}_k, k)$, $\kappa_{1,2}(\widehat{m}_k, k)$, $\kappa_{1,3}(\widehat{m}_k, k)$, $u_2(\widehat{m}_k, k)$, $\kappa_{2,1}(\widehat{m}_k, k)$, $\kappa_{2,2}(\widehat{m}_k, k)$, $\kappa_{2,3}(\widehat{m}_k, k)$, $u_3(\widehat{m}_k, k)$, $\kappa_{3,1}(\widehat{m}_k, k)$, $\kappa_{3,2}(\widehat{m}_k, k)$, and $\kappa_{3,3}(\widehat{m}_k, k)$ for the decoupled dynamical system for y in the case where jump is not incorporated into the dynamical system.

Table 1 shows the estimates of the ε suboptimal size $\widehat{m}_k$, $j \in I(1,3)$, the parameters $u_1(\widehat{m}_k, k)$, $\kappa_{1,1}(\widehat{m}_k, k)$, $\kappa_{1,2}(\widehat{m}_k, k)$, $\kappa_{1,3}(\widehat{m}_k, k)$, $u_2(\widehat{m}_k, k)$, $\kappa_{2,1}(\widehat{m}_k, k)$, $\kappa_{2,2}(\widehat{m}_k, k)$, $\kappa_{2,3}(\widehat{m}_k, k)$, $u_3(\widehat{m}_k, k)$, $\kappa_{3,1}(\widehat{m}_k, k)$, $\kappa_{3,2}(\widehat{m}_k, k)$, and $\kappa_{3,3}(\widehat{m}_k, k)$ for each of the energy commodity data sets. Moreover, $d \le r$ and the initial real data time is $t_r = t_{10}$.

Figure 1 shows the drift coefficient's parameter estimates $u_1(\widehat{m}_k, k)$, $u_2(\widehat{m}_k, k)$, and $u_3(\widehat{m}_k, k)$ for the decoupled dynamical system for $\mathbf{y}$ in the case where jump is not incorporated into the dynamical system.

Figures 1(a)–1(c) are the graphs of $u_1(\widehat{m}_k, k)$, $u_2(\widehat{m}_k, k)$, and $u_3(\widehat{m}_k, k)$ against time t_k for the daily Henry Hub natural gas price [36], daily crude oil price [37], and daily coal price [35] data set, respectively. By plotting the real data sets, it is easily seen that the graphs of $u_1(\widehat{m}_k, k)$, $u_2(\widehat{m}_k, k)$, and $u_3(\widehat{m}_k, k)$ are similar to the graph of the Henry Hub natural gas, crude oil, and coal data set, respectively. We expect this to happen because u_j, $j \in I(1,3)$, are the equilibrium spot price processes of (5). The sample means of the real spot price data sets for natural gas, crude oil, and coal are given by 3.7218, 88.5620, and 16.5543, respectively. It can be seen from Figures 1(a)–1(c) that the graphs of $u_1(\widehat{m}_k, k)$, $u_2(\widehat{m}_k, k)$, and $u_3(\widehat{m}_k, k)$ move around the mean values 3.7218, 88.5620, and 16.5543, respectively. This analysis shows that the parameters u_j, $j \in I(1,3)$, are statistic processes for the respective mean of the data sets at time t_k.

The graph of the interaction parameters $\kappa_{1,1}(\widehat{m}_k, k)$, $\kappa_{1,2}(\widehat{m}_k, k)$, $\kappa_{1,3}(\widehat{m}_k, k)$, $\kappa_{2,1}(\widehat{m}_k, k)$, $\kappa_{2,2}(\widehat{m}_k, k)$, $\kappa_{2,3}(\widehat{m}_k, k)$, $\kappa_{3,1}(\widehat{m}_k, k)$, $\kappa_{3,2}(\widehat{m}_k, k)$, and $\kappa_{3,3}(\widehat{m}_k, k)$ for the decoupled dynamical system for y (with no jump incorporated into the dynamical system) are given in Figure 2 below.

Figures 2(a)–2(i) show the graph of the ε suboptimal interaction coefficient parameters $\kappa_{1,1}(\widehat{m}_k, k)$, $\kappa_{1,2}(\widehat{m}_k, k)$, $\kappa_{1,3}(\widehat{m}_k, k)$, $\kappa_{2,1}(\widehat{m}_k, k)$, $\kappa_{2,2}(\widehat{m}_k, k)$, $\kappa_{2,3}(\widehat{m}_k, k)$, $\kappa_{3,1}(\widehat{m}_k, k)$, $\kappa_{3,2}(\widehat{m}_k, k)$, and $\kappa_{3,3}(\widehat{m}_k, k)$. The interaction coefficients $\kappa_{j,l}$, $j \ne l$ are negligible, because each estimate is $<<10^{-15}$. Thus, this shows that the model describing the mean spot price, y_j, is mainly characterized by the market potential $\kappa_{j,j}(u_j - y_j)y_j$, $j \in I(1,n)$.

Table 2 below shows the estimates of the diffusion coefficient's parameters $\delta_{1,1}(\widehat{m}_k, k)$, $\delta_{1,2}(\widehat{m}_k, k)$, $\delta_{1,3}(\widehat{m}_k, k)$, $\delta_{2,1}(\widehat{m}_k, k)$, $\delta_{2,2}(\widehat{m}_k, k)$, $\delta_{2,3}(\widehat{m}_k, k)$, $\delta_{3,1}(\widehat{m}_k, k)$, $\delta_{3,2}(\widehat{m}_k, k)$, and $\delta_{3,3}(\widehat{m}_k, k)$ for the decoupled dynamical system for $\mathbf{y}$ in the case where jump is not incorporated into the dynamical system.

The graph of the diffusion coefficient's parameter for the decoupled dynamical system for y without jump incorporated into the dynamical system are given in Figure 3 below.

Figures 3(a)–3(i) show the graphs of the ε suboptimal interaction measure of fluctuation coefficient parameters $\delta_{1,1}(\widehat{m}_k, k)$, $\delta_{1,2}(\widehat{m}_k, k)$, $\delta_{1,3}(\widehat{m}_k, k)$, $\delta_{2,1}(\widehat{m}_k, k)$, $\delta_{2,2}(\widehat{m}_k, k)$, $\delta_{2,3}(\widehat{m}_k, k)$, $\delta_{3,1}(\widehat{m}_k, k)$, $\delta_{3,2}(\widehat{m}_k, k)$, and $\delta_{3,3}(\widehat{m}_k, k)$.

Table 3 gives the drift coefficient's parameter estimates $\beta_1(\widehat{m}_k, k)$, $\gamma_{1,1}(\widehat{m}_k, k)$, $\gamma_{1,2}(\widehat{m}_k, k)$, $\gamma_{1,3}(\widehat{m}_k, k)$, $\beta_2(\widehat{m}_k, k)$, $\gamma_{2,1}(\widehat{m}_k, k)$, $\gamma_{2,2}(\widehat{m}_k, k)$, $\gamma_{2,3}(\widehat{m}_k, k)$, $\beta_3(\widehat{m}_k, k)$, $\gamma_{3,1}(\widehat{m}_k, k)$, $\gamma_{3,2}(\widehat{m}_k, k)$, and $\gamma_{3,3}(\widehat{m}_k, k)$ for the dynamical system for $\mathbf{p}$ without jump.

Table 3 shows the estimates of the ε suboptimal size $\widehat{m}_k$, $j \in I(1,3)$ and the parameters $\beta_1(\widehat{m}_k, k)$, $\gamma_{1,1}(\widehat{m}_k, k)$, $\gamma_{1,2}(\widehat{m}_k, k)$, $\gamma_{1,3}(\widehat{m}_k, k)$, $\beta_2(\widehat{m}_k, k)$, $\gamma_{2,1}(\widehat{m}_k, k)$, $\gamma_{2,2}(\widehat{m}_k, k)$, $\gamma_{2,3}(\widehat{m}_k, k)$, $\beta_3(\widehat{m}_k, k)$, $\gamma_{3,1}(\widehat{m}_k, k)$, $\gamma_{3,2}(\widehat{m}_k, k)$, and $\gamma_{3,3}(\widehat{m}_k, k)$, for each of the energy commodity data sets. Moreover, $d \le r$, and the initial real data time is $t_r = t_{10}$.

In the following, we give the graph of the drift coefficient's parameters with estimates in Table 3 in Figure 4 below.

Figures 4(a)–4(i) show the graphs of the ε suboptimal interaction coefficient parameters $\gamma_{1,1}(\widehat{m}_k, k)$, $\gamma_{1,2}(\widehat{m}_k, k)$, $\gamma_{1,3}(\widehat{m}_k, k)$, $\gamma_{2,1}(\widehat{m}_k, k)$, $\gamma_{2,2}(\widehat{m}_k, k)$, $\gamma_{2,3}(\widehat{m}_k, k)$, $\gamma_{3,1}(\widehat{m}_k, k)$, $\gamma_{3,2}(\widehat{m}_k, k)$, and $\gamma_{3,3}(\widehat{m}_k, k)$. According to (58), the estimate $\gamma_{j,l}(\widehat{m}_k, k)$, $j \ne l$, is positive if commodity p_l is cooperating with commodity p_j, and negative if commodity p_l is competing with commodity p_j. There is no interaction between the two commodities if $\gamma_{j,l}(\widehat{m}_k, k) = 0$. It is apparent from the graph of $\gamma_{1,3}(\widehat{m}_k, k)$ that coal is competing with natural gas because the estimates of $\gamma_{1,3}(\widehat{m}_k, k)$ are mostly negative. Also, it is apparent that natural gas and crude oil are either cooperating or competing, depending on the time period.

Figures 5(a)–5(c) are the graphs of $\beta_1(\widehat{m}_k, k)$, $\beta_2(\widehat{m}_k, k)$, and $\beta_3(\widehat{m}_k, k)$ against time t_k for the daily Henry Hub natural gas price data set [36], daily crude oil price data set [37], and daily coal price data set, respectively.

Table 11: Estimates $\sigma_{1,1}(\widehat{m}_k, k)$, $\sigma_{1,2}(\widehat{m}_k, k)$, $\sigma_{1,3}(\widehat{m}_k, k)$, $\sigma_{2,1}(\widehat{m}_k, k)$, $\sigma_{2,2}(\widehat{m}_k, k)$, $\sigma_{2,3}(\widehat{m}_k, k)$, $\sigma_{3,1}(\widehat{m}_k, k)$, $\sigma_{3,2}(\widehat{m}_k, k)$, and $\sigma_{3,3}(\widehat{m}_k, k)$.

t_k		Natural gas			Crude oil			Coal	
	$\sigma_{1,1}(\widehat{m}_k, k)$	$\sigma_{1,2}(\widehat{m}_k, k)$	$\sigma_{1,3}(\widehat{m}_k, k)$	$\sigma_{2,1}(\widehat{m}_k, k)$	$\sigma_{2,2}(\widehat{m}_k, k)$	$\sigma_{2,3}(\widehat{m}_k, k)$	$\sigma_{3,1}(\widehat{m}_k, k)$	$\sigma_{3,2}(\widehat{m}_k, k)$	$\sigma_{3,3}(\widehat{m}_k, k)$
11	0	0	0	0	0	0	0	0.1303	0
12	0.0485	0.0004	0.0032	0.2734	0.0166	0	0.0513	0	0.0000
13	0.7333	0	0	0.9445	0	0	0.2489	0	0
14	0.2120	0.1386	0.0133	0.3877	0.4773	0.1195	0.1365	0.0665	0.0086
15	0.4246	0.1318	0.0021	0.03341	0.4894	0.1211	0.0112	0.6107	0.0696
16	0.5538	0.0778	0.1501	0.07751	0.2524	0.0811	0.0651	0.4251	0.0635
17	0.3907	0.0469	0.2230	0.08746	0.1848	0.2463	0	0.4458	0.0478
18	0.3523	0.0180	0.2178	0.04291	0.1877	0.1602	0.5681	0.0592	0.0115
19	0.5116	0.0619	0.2221	0.03266	0.2673	0.2465	0.4999	0.0569	0.0127
20	0.6431	0.0536	0.1866	0.0939	0.1700	0.0781	0.3789	0.3174	0.0046
…	…	…	…	…	…	…	…	…	…
495	0	0.0036	0.0406	0.2286	0.0600	0.0172	0.0110	0.9387	0.0182
496	0.1588	0.0035	0.0107	0.08183	0.3163	0.0102	0	0	0.0016
497	0.1551	0.0009	0.0065	0.07869	0.1453	0.4821	0.7777	0	0.0033
498	0.1576	0.0011	0.0073	0.0120	0.1679	0.3786	0.5334	0	0.0060
499	0.1197	0.0006	0.0059	0.0721	0.2391	0.0172	0.4405	0.1432	0.0097
500	0.3600	0.0001	0.0049	0.0273	0.3960	0.0079	0.6331	0.1410	0.0093
0.5010	0.0514	0.0033	0.0049	0.0182	0.3499	0.0111	0.7690	0.1376	0.0089
0.5020	0.2503	0.0034	0.0042	0.0222	0.1744	0.0132	0.6198	0.1274	0.0066
0.5030	0.1195	0.0147	0.0165	0	0.4283	0.0060	0	0.1530	0.0049
0.5040	0.0974	0.0144	0.0027	0	0.2241	0.0048	0.4778	0.0574	0.0043
0.5050	0.1422	0.0060	0.0131	0.0085	0.2023	0.0054	0.5604	0.0669	0.0004
…	…	…	…	…	…	…	…	…	…
1102	0.1898	0.0016	0.0413	0.8313	0.0767	0.0381	0.6875	0	0.1451
1103	0.2094	0.0015	0.0352	0.8262	0.0673	0.0451	0.7298	0.2808	0.0147
1104	0.1711	0.0011	0.0040	0.6648	0.0915	0.0462	0.5563	0.1831	0.0105
1105	0.1816	0.0012	0.0116	0.6658	0.1049	0.0371	0.6591	0.2874	0.0057
1106	0.1191	0.0011	0.0116	0.6260	0.1155	0.0393	0	0.0196	0.0060
1107	0.0417	0.0012	0.0041	0.4992	0.0781	0.0382	0.0271	0.2559	0.0065
1108	0.1058	0.0033	0.0045	0.0019	0.0589	0.0421	0.6209	0.8289	0.0018
1109	0.1740	0.0021	0	0.0305	0.0446	0.0316	0.8431	0.2366	0.4511
1110	0.2912	0.0021	0.0163	0.0385	0.0342	0.0037	0.2910	0.0489	0.0257

Table 4 gives the ε suboptimal estimates of the parameters $\sigma_{1,1}(\widehat{m}_k, k)$, $\sigma_{1,2}(\widehat{m}_k, k)$, $\sigma_{1,3}(\widehat{m}_k, k)$, $\sigma_{2,1}(\widehat{m}_k, k)$, $\sigma_{2,2}(\widehat{m}_k, k)$, $\sigma_{2,3}(\widehat{m}_k, k)$, $\sigma_{3,1}(\widehat{m}_k, k)$, $\sigma_{3,2}(\widehat{m}_k, k)$, and $\sigma_{3,3}(\widehat{m}_k, k)$ for each of the energy commodity data sets.

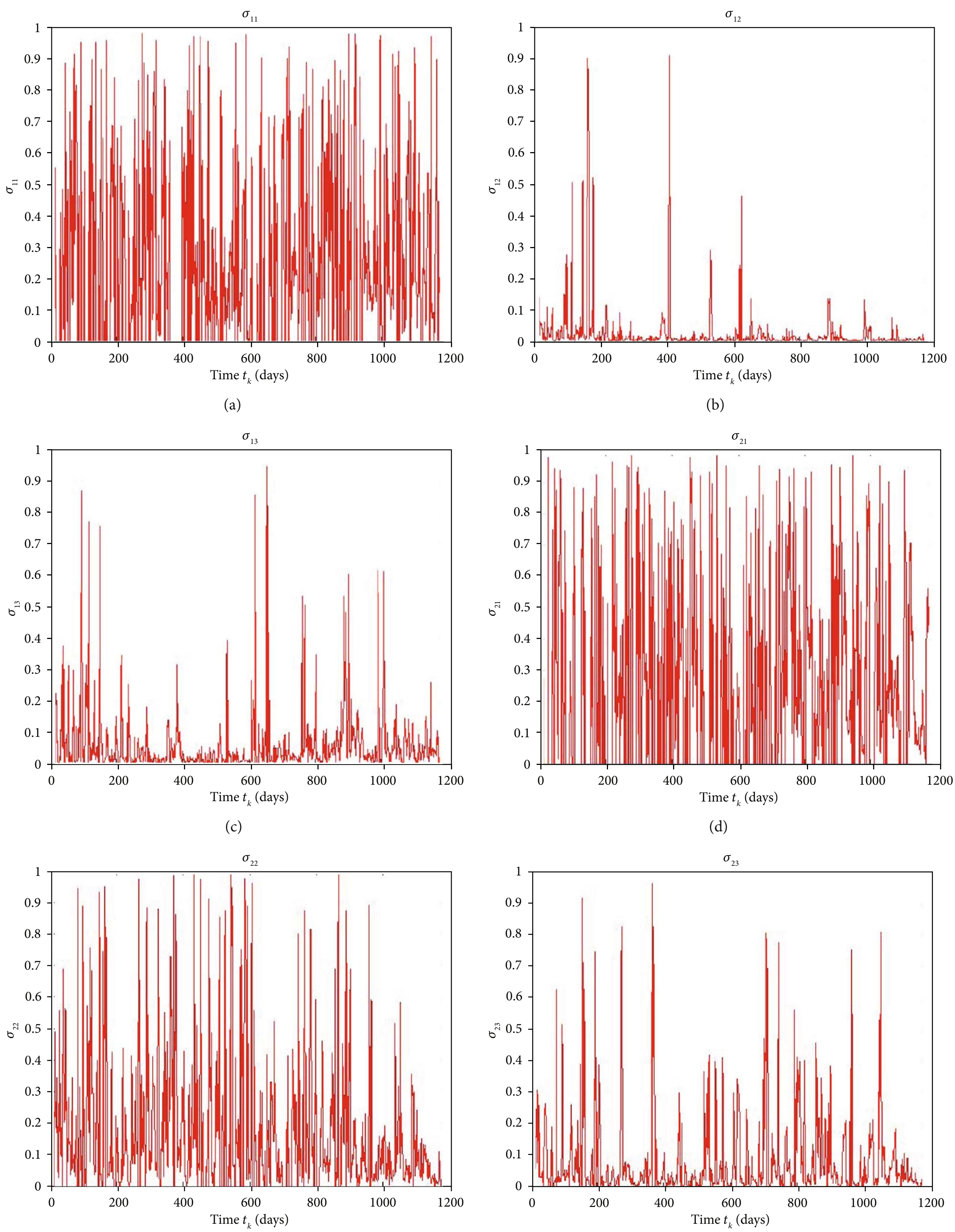
Figures 6(a)–6(i) are the graphs of $\sigma_{1,1}(\widehat{m}_k, k)$, $\sigma_{1,2}(\widehat{m}_k, k)$, $\sigma_{1,3}(\widehat{m}_k, k)$, $\sigma_{2,1}(\widehat{m}_k, k)$, $\sigma_{2,2}(\widehat{m}_k, k)$, $\sigma_{2,3}(\widehat{m}_k, k)$, $\sigma_{3,1}(\widehat{m}_k, k)$, $\sigma_{3,2}(\widehat{m}_k, k)$, and $\sigma_{1,1}(\widehat{m}_k, k)$ against time t_k for the daily Henry Hub natural gas price data set [36], daily crude oil price data set [37], and daily coal price data set, respectively.

Table 5 shows the real and simulated estimates for the spot price processes $p_j(t)$, $j \in I(1, n)$.

Figure 7 below shows the graph of the real and simulated prices for natural gas, crude oil, and coal data set.

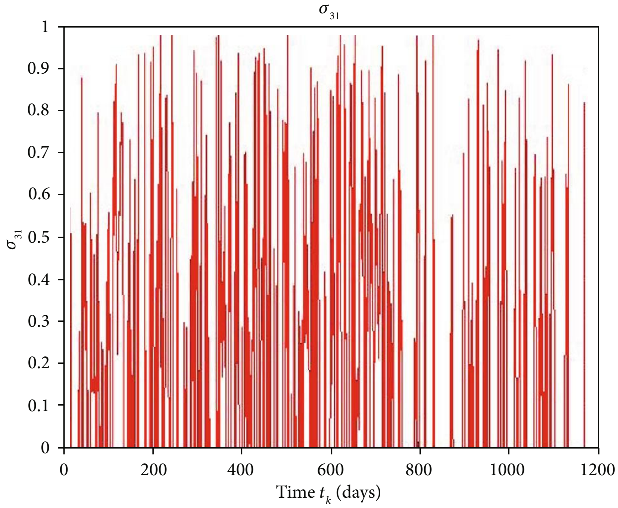
Figures 7(a)–7(c) show the graph of the real and simulated spot prices for the daily Henry Hub natural gas data set [36], daily crude oil data set [37], and daily coal data set [35], respectively. The red line represents the real data set $p(t_k)$, while the blue line represent the simulated data set $p^s(\widehat{m}_k, k)$. Here, we begin by using a starting delay of $r = 10$. The simulation starts from $t_r = t_{10}$. It is clear that the graph fits well. To reduce magnitude of error, we increase the magnitude of time delay. We later compare this result with the case where jump is incorporated into the system.

7.5.2. Relationship between Natural Gas, Crude Oil, and Coal: With Jump Incorporated. In this subsubsection, we analyze the relationship between natural gas, crude oil, and coal with the jump process. Here, we apply the above conceptual computational algorithm described in this section for the real-time data sets, namely, daily Henry Hub natural gas data set, daily crude oil data set, and daily coal data set for the

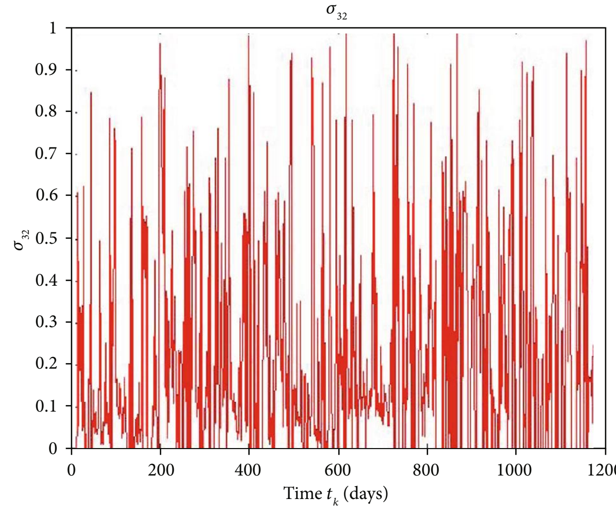


(a) (b) (c) (d) (e) (f)

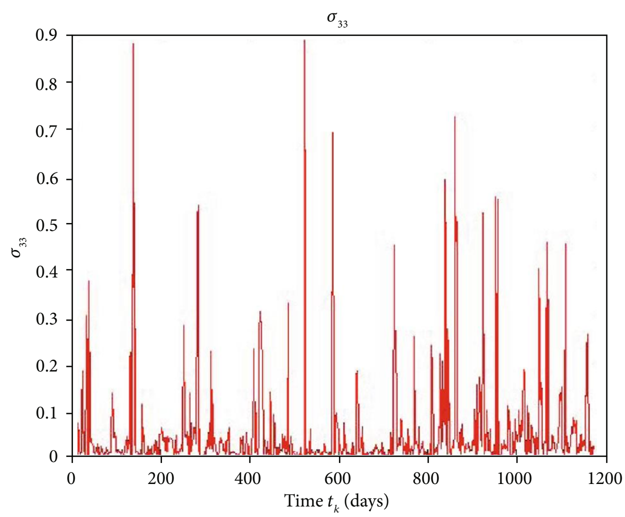
Figure 13: Continued.



(g)



(h)



(i)

Figure 13: The graph of $\sigma_{1,1}(\widehat{m}_k, k)$, $\sigma_{1,2}(\widehat{m}_k, k)$, $\sigma_{1,3}(\widehat{m}_k, k)$, $\sigma_{2,1}(\widehat{m}_k, k)$, $\sigma_{2,2}(\widehat{m}_k, k)$, $\sigma_{2,3}(\widehat{m}_k, k)$, $\sigma_{3,1}(\widehat{m}_k, k)$, $\sigma_{3,2}(\widehat{m}_k, k)$, and $\sigma_{3,3}(\widehat{m}_k, k)$ for natural gas, crude oil, and coal, respectively.

period of 05/04/2009-01/03/2014, [35–37]. For $i \in I(1, K^*)$, $K \neq 0$, we use $\Delta t_{i-1} = 1$, $\varepsilon = 0.001$, $r_{i-1} = 10$, and $d_{i-1} = 2$. The ε-level suboptimal estimates of the parameters at each real data times are described below for each commodity data sets.

The parameters corresponding to the model governing natural gas price data set are $u_1^{i-1}(\widehat{m}_k, k)$, $\beta_1^{i-1}(\widehat{m}_k^{i-1}, t_k^{i-1})$, $\kappa_{1,1}^{i-1}(\widehat{m}_k^{i-1}, t_k^{i-1})$, $\kappa_{1,2}^{i-1}(\widehat{m}_k^{i-1}, t_k^{i-1})$, $\kappa_{1,3}^{i-1}(\widehat{m}_k^{i-1}, t_k^{i-1})$, $\gamma_{1,1}^{i-1}(\widehat{m}_k^{i-1}, t_k^{i-1})$, $\gamma_{1,2}^{i-1}(\widehat{m}_k^{i-1}, t_k^{i-1})$, $\gamma_{1,3}^{i-1}(\widehat{m}_k^{i-1}, t_k^{i-1})$, $\delta_{1,1}^{i-1}(\widehat{m}_k^{i-1}, t_k^{i-1})$, $\delta_{1,2}^{i-1}(\widehat{m}_k^{i-1}, t_k^{i-1})$, $\delta_{1,3}^{i-1}(\widehat{m}_k^{i-1}, t_k^{i-1})$, $\sigma_{1,1}^{i-1}(\widehat{m}_k^{i-1}, t_k^{i-1})$, $\sigma_{1,2}^{i-1}(\widehat{m}_k^{i-1}, t_k^{i-1})$, and $\sigma_{1,3}^{i-1}(\widehat{m}_k^{i-1}, t_k^{i-1})$. The parameters corresponding to the model governing crude oil price data set are $u_2^{i-1}(\widehat{m}_k^{i-1}, t_k^{i-1})$, $\beta_2^{i-1}(\widehat{m}_k^{i-1}, t_k^{i-1})$, $\kappa_{2,1}^{i-1}(\widehat{m}_k^{i-1}, t_k^{i-1})$, $\kappa_{2,2}^{i-1}(\widehat{m}_k^{i-1}, t_k^{i-1})$, $\kappa_{2,3}^{i-1}(\widehat{m}_k^{i-1}, t_k^{i-1})$, $\gamma_{2,1}^{i-1}(\widehat{m}_k^{i-1}, t_k^{i-1})$, $\gamma_{2,2}^{i-1}(\widehat{m}_k^{i-1}, t_k^{i-1})$, $\gamma_{2,3}^{i-1}(\widehat{m}_k^{i-1}, t_k^{i-1})$, $\delta_{2,1}^{i-1}(\widehat{m}_k^{i-1}, t_k^{i-1})$, $\delta_{2,2}^{i-1}(\widehat{m}_k^{i-1}, t_k^{i-1})$, $\delta_{2,3}^{i-1}(\widehat{m}_k^{i-1}, t_k^{i-1})$, $\sigma_{2,1}^{i-1}(\widehat{m}_k^{i-1}, t_k^{i-1})$, $\sigma_{2,2}^{i-1}(\widehat{m}_k^{i-1}, t_k^{i-1})$, and $\sigma_{2,3}^{i-1}(\widehat{m}_k^{i-1}, t_k^{i-1})$, while the parameters corresponding to the model governing coal price data set are $u_3^{i-1}(\widehat{m}_k^{i-1}, t_k^{i-1})$, $\beta_3^{i-1}(\widehat{m}_k^{i-1}, t_k^{i-1})$, $\kappa_{3,1}^{i-1}(\widehat{m}_k^{i-1}, t_k^{i-1})$, $\kappa_{3,2}^{i-1}(\widehat{m}_k^{i-1}, t_k^{i-1})$, $\kappa_{3,3}^{i-1}(\widehat{m}_k^{i-1}, t_k^{i-1})$, $\gamma_{3,1}^{i-1}(\widehat{m}_k^{i-1}, t_k^{i-1})$, $\gamma_{3,2}^{i-1}(\widehat{m}_k^{i-1}, t_k^{i-1})$, $\gamma_{3,3}^{i-1}(\widehat{m}_k^{i-1}, t_k^{i-1})$, $\delta_{3,1}^{i-1}(\widehat{m}_k^{i-1}, t_k^{i-1})$, $\delta_{3,2}^{i-1}(\widehat{m}_k^{i-1}, t_k^{i-1})$, $\delta_{3,3}^{i-1}(\widehat{m}_k^{i-1}, t_k^{i-1})$, $\sigma_{3,1}^{i-1}(\widehat{m}_k^{i-1}, t_k^{i-1})$, $\sigma_{3,2}^{i-1}(\widehat{m}_k^{i-1}, t_k^{i-1})$, and $\sigma_{3,3}^{i-1}(\widehat{m}_k^{i-1}, t_k^{i-1})$.

For the sake of simplicity and in order to be able to compare our results in this subsection with the results in Subsection 7.5.1, for each $j, l \in I(1, n)$, we rewrite the parameters $u_j^{i-1}(\widehat{m}_k^{i-1}, t_k^{i-1})$, $\beta_j^{i-1}(\widehat{m}_k^{i-1}, t_k^{i-1})$, $\kappa_{j,l}^{i-1}(\widehat{m}_k^{i-1}, t_k^{i-1})$, $\gamma_{j,l}^{i-1}(\widehat{m}_k^{i-1}, t_k^{i-1})$, $\delta_{j,l}^{i-1}(\widehat{m}_k^{i-1}, t_k^{i-1})$, and $\sigma_{j,l}^{i-1}(\widehat{m}_k^{i-1}, t_k^{i-1})$ after they have been estimated as $u_j(\widehat{m}_k, k)$, $\beta_j(\widehat{m}_k, k)$, $\kappa_{j,l}(\widehat{m}_k, k)$, $\gamma_{j,l}(\widehat{m}_k, k)$, $\delta_{j,l}(\widehat{m}_k, k)$, and $\sigma_{j,l}(\widehat{m}_k, k)$.

We give some results for the jump times of the system $\{T_i\}_{i \in I(1,K^*)}$.

Table 6 shows some result for the jump times of the system $(\mathbf{y}, \mathbf{p})$. These results are derived by recording the times at

Table 12: Real and simulated estimates (with jump) for natural gas, crude oil, and coal.

t_k	Natural gas		Crude oil		Coal	
	Real	Simulated $p_1^s(\widehat{m}_k, k)$	Real	Simulated $p_2^s(\widehat{m}_k, k)$	Real	Simulated $p_3^s(\widehat{m}_k, k)$
11	4.0200	4.0500	58.9900	58.5200	16.5900	16.6000
12	3.9900	3.9600	59.5200	59.5099	17.4600	17.4635
13	3.7500	3.6690	61.4500	60.3377	17.8900	17.8886
14	3.7700	3.7341	60.4900	59.4191	17.5500	17.5188
15	3.4100	3.3967	61.1500	60.1580	17.4100	17.4188
16	3.3500	3.3947	62.4800	62.5028	16.7500	16.7789
17	3.4900	3.3900	63.4100	63.5524	17.6600	17.5341
18	3.5500	3.4957	65.0900	64.9224	17.5200	17.8130
19	3.9200	3.8743	66.3100	66.8239	18.5000	18.8453
20	3.8600	3.8241	68.5900	68.1206	19.0600	18.9453
⋯	⋯	⋯	⋯	⋯	⋯	⋯
494	4.2300	4.2295	106.7000	106.6374	33.2200	33.1852
495	4.1900	4.2191	107.1800	106.9973	32.7600	32.6677
496	4.3300	4.3414	110.8400	111.0084	33.6500	33.8070
497	4.3300	4.3261	111.7200	111.7084	33.7100	33.7061
498	4.3700	4.3863	111.6800	111.2084	34.7500	35.7057
499	4.3200	4.2167	111.7200	111.6126	34.5400	33.5457
500	4.3500	4.3577	112.3100	112.4358	34.0400	34.2862
501	4.3800	4.3535	112.3800	112.5682	33.1000	33.1330
502	4.5100	4.4389	113.3900	113.2925	33.6700	33.6216
503	4.6000	4.6139	113.0300	113.3077	33.9400	33.9216
504	4.6000	4.5964	110.6000	110.8350	33.8300	33.9216
505	4.5900	4.5564	108.7900	108.7598	32.0200	31.9867
⋯	⋯	⋯	⋯	⋯	⋯	⋯
1102	3.7200	3.6616	108.2300	108.2354	4.7700	4.5482
1103	3.7300	3.6477	106.2600	106.1451	5.0100	5.1120
1104	3.6800	3.6748	104.7000	104.6723	4.9800	5.1180
1105	3.6600	3.6861	103.6200	104.6723	4.7300	4.7893
1106	3.5900	3.6436	103.2200	102.9765	4.6800	4.7074
1107	3.5200	3.5213	102.6800	102.7652	4.6300	4.6419
1108	3.4900	3.4564	103.1000	102.9765	4.7400	4.4016
1109	3.5100	3.2596	102.8600	102.9652	4.3300	4.1826
1110	3.4800	3.4604	102.3600	102.4345	4.1800	4.4606

which an entry of a commodity differ by twice the standard deviation or more from the mean of that commodity. These times are now combined into a single array and sorted out in an increasing order.

We give the estimate of the jump coefficient matrices Π^i and Θ^i defined in (34) in Table 7 below.

Table 8 gives the estimates $u_1(\widehat{m}_k, k)$, $\kappa_{1,1}(\widehat{m}_k, k)$, $\kappa_{1,2}(\widehat{m}_k, k)$, $\kappa_{1,3}(\widehat{m}_k, k)$, $u_2(\widehat{m}_k, k)$, $\kappa_{2,1}(\widehat{m}_k, k)$, $\kappa_{2,2}(\widehat{m}_k, k)$, $\kappa_{2,3}(\widehat{m}_k, k)$, $u_3(\widehat{m}_k, k)$, $\kappa_{3,1}(\widehat{m}_k, k)$, $\kappa_{3,2}(\widehat{m}_k, k)$, and $\kappa_{3,3}(\widehat{m}_k, k)$ for the decoupled dynamical system for $\mathbf{y}$ with jump.

Figure 8 shows the parameter estimates $u_1(\widehat{m}_k, k)$, $u_2(\widehat{m}_k, k)$, and $u_3(\widehat{m}_k, k)$ for the decoupled dynamical system for $\mathbf{y}$ with jump.

Figures 8(a)–8(c) are the graphs of $u_1(\widehat{m}_k, k)$, $u_2(\widehat{m}_k, k)$, and $u_3(\widehat{m}_k, k)$ against time t_k for the daily Henry Hub natural gas price data set [36], daily crude oil price data set [37], and daily coal price data set, respectively. By plotting the real data sets, it is easily seen that the graphs of $u_1(\widehat{m}_k, k)$, $u_2(\widehat{m}_k, k)$, and $u_3(\widehat{m}_k, k)$ are similar to the graph of the Henry Hub natural gas, crude oil, and coal data set, respectively. We expect this to happen because u_j, $j \in I(1,3)$, are the equilibrium spot price processes of (5).

The graph of the interaction parameters $\kappa_{1,1}(\widehat{m}_k, k)$, $\kappa_{1,2}(\widehat{m}_k, k)$, $\kappa_{1,3}(\widehat{m}_k, k)$, $\kappa_{2,1}(\widehat{m}_k, k)$, $\kappa_{2,2}(\widehat{m}_k, k)$, $\kappa_{2,3}(\widehat{m}_k, k)$, $\kappa_{3,1}(\widehat{m}_k, k)$, $\kappa_{3,2}(\widehat{m}_k, k)$, and $\kappa_{3,3}(\widehat{m}_k, k)$ for the decoupled dynamical system for y with jump and estimates in Table 8 are given in Figure 9 below.

Figures 9(a)–9(i)) show the graph of the ε suboptimal interaction coefficient parameters $\kappa_{1,1}(\widehat{m}_k, k)$, $\kappa_{1,2}(\widehat{m}_k, k)$, $\kappa_{1,3}(\widehat{m}_k, k)$, $\kappa_{2,1}(\widehat{m}_k, k)$, $\kappa_{2,2}(\widehat{m}_k, k)$, $\kappa_{2,3}(\widehat{m}_k, k)$, $\kappa_{3,1}(\widehat{m}_k, k)$, $\kappa_{3,2}(\widehat{m}_k, k)$, and $\kappa_{3,3}(\widehat{m}_k, k)$. The interaction coefficients $\kappa_{j,l}$, $j \neq l$, are negligible, because each estimate is $<<10^{-15}$. Thus,

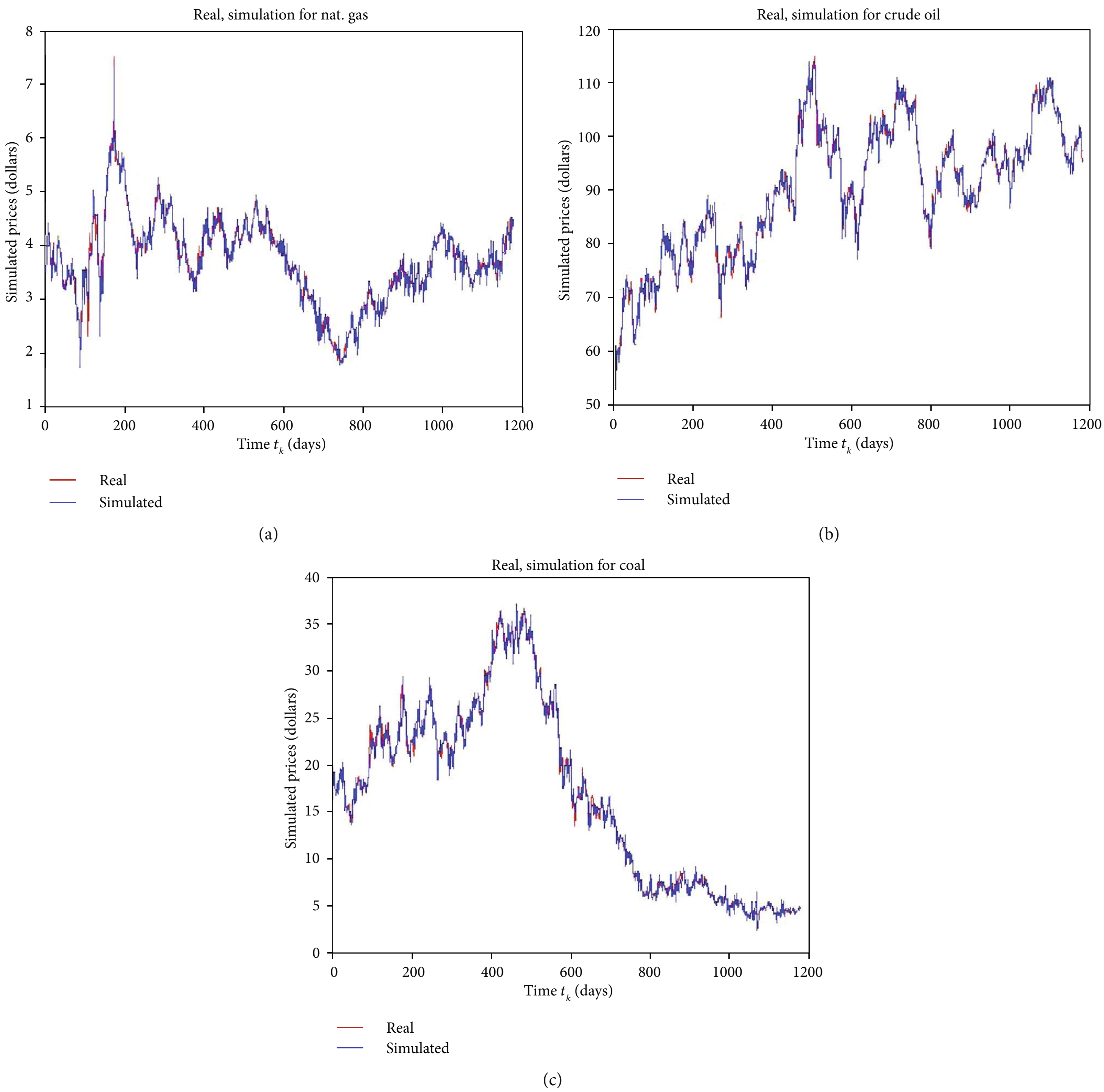


Figure 14: Real and simulated prices (with jump) for natural gas, crude oil, and coal.

this shows that the model describing the mean spot price, y_j, is mainly characterized by the market potential $\kappa_{j,j}^{i-1}(u_j^{i-1} - y_j)y_j$, $j \in I(1, n)$, $i \in I(1, K^*)$.

Table 9 below shows the estimates of the diffussion coefficient's parameters $\delta_{1,1}(\widehat{m}_k, k)$, $\delta_{1,2}(\widehat{m}_k, k)$, $\delta_{1,3}(\widehat{m}_k, k)$, $\delta_{2,1}(\widehat{m}_k, k)$,$\delta_{2,2}(\widehat{m}_k, k)$, $\delta_{2,3}(\widehat{m}_k, k)$, $\delta_{3,1}(\widehat{m}_k, k)$, $\delta_{3,2}(\widehat{m}_k, k)$, and $\delta_{3,3}(\widehat{m}_k, k)$ for the decoupled dynamical system for $\mathbf{y}$ with jump.

The graph of the diffusion coefficient's parameter for the decoupled dynamical system for $\mathbf{y}$ with jump are given in Figure 10 below.

Figures 10(a)–10(i) show the graph of the ε- sub-optimal interaction measure of fluctuation coefficient parameters $\delta_{1,1}(\widehat{m}_k, k)$, $\delta_{1,2}(\widehat{m}_k, k)$, $\delta_{1,3}(\widehat{m}_k, k)$, $\delta_{2,1}(\widehat{m}_k, k)$, $\delta_{2,2}(\widehat{m}_k, k)$, $\delta_{2,3}(\widehat{m}_k, k)$, $\delta_{3,1}(\widehat{m}_k, k)$, $\delta_{3,2}(\widehat{m}_k, k)$, and $\delta_{3,3}(\widehat{m}_k, k)$.

Table 10 gives the drift coefficient's parameter estimates $\beta_1(\widehat{m}_k, k)$, $\gamma_{1,1}(\widehat{\mathrm{m}}_k, k)$, $\gamma_{1,2}(\widehat{m}_k, k)$, $\gamma_{1,3}(\widehat{m}_k, k)$, $\beta_2(\widehat{m}_k, k)$, $\gamma_{2,1}(\widehat{m}_k, k)$, $\gamma_{2,2}(\widehat{m}_k, k)$, $\gamma_{2,3}(\widehat{m}_k, k)$, $\beta_3(\widehat{m}_k, k)$, $\gamma_{3,1}(\widehat{m}_k, k)$, $\gamma_{3,2}(\widehat{m}_k, k)$, and $\gamma_{3,3}(\widehat{m}_k, k)$ for the dynamical system for $\mathbf{p}$ with jump.

Table 10 shows the estimates of the ε suboptimal size $\widehat{m}_k$ and the parameters $\widehat{m}_k$,$\beta_1(\widehat{m}_k, k)$, $\gamma_{1,1}(\widehat{m}_k, k)$, $\gamma_{1,2}(\widehat{m}_k, k)$, $\gamma_{1,3}(\widehat{m}_k, k)$, $\widehat{m}_k$, $\beta_2(\widehat{m}_k, k)$, $\gamma_{2,1}(\widehat{m}_k, k)$, $\gamma_{2,2}(\widehat{m}_k, k)$, $\gamma_{2,3}(\widehat{m}_k, k)$, $\widehat{m}_k$,$\beta_3(\widehat{m}_k, k)$, $\gamma_{3,1}(\widehat{m}_k, k)$, $\gamma_{3,2}(\widehat{m}_k, k)$, and $\gamma_{3,3}(\widehat{m}_k, k)$, for each of the energy commodity data sets. According to (58), the estimate $\gamma_{j,l}(\widehat{m}_k, k)$, $j \neq l$, is positive if

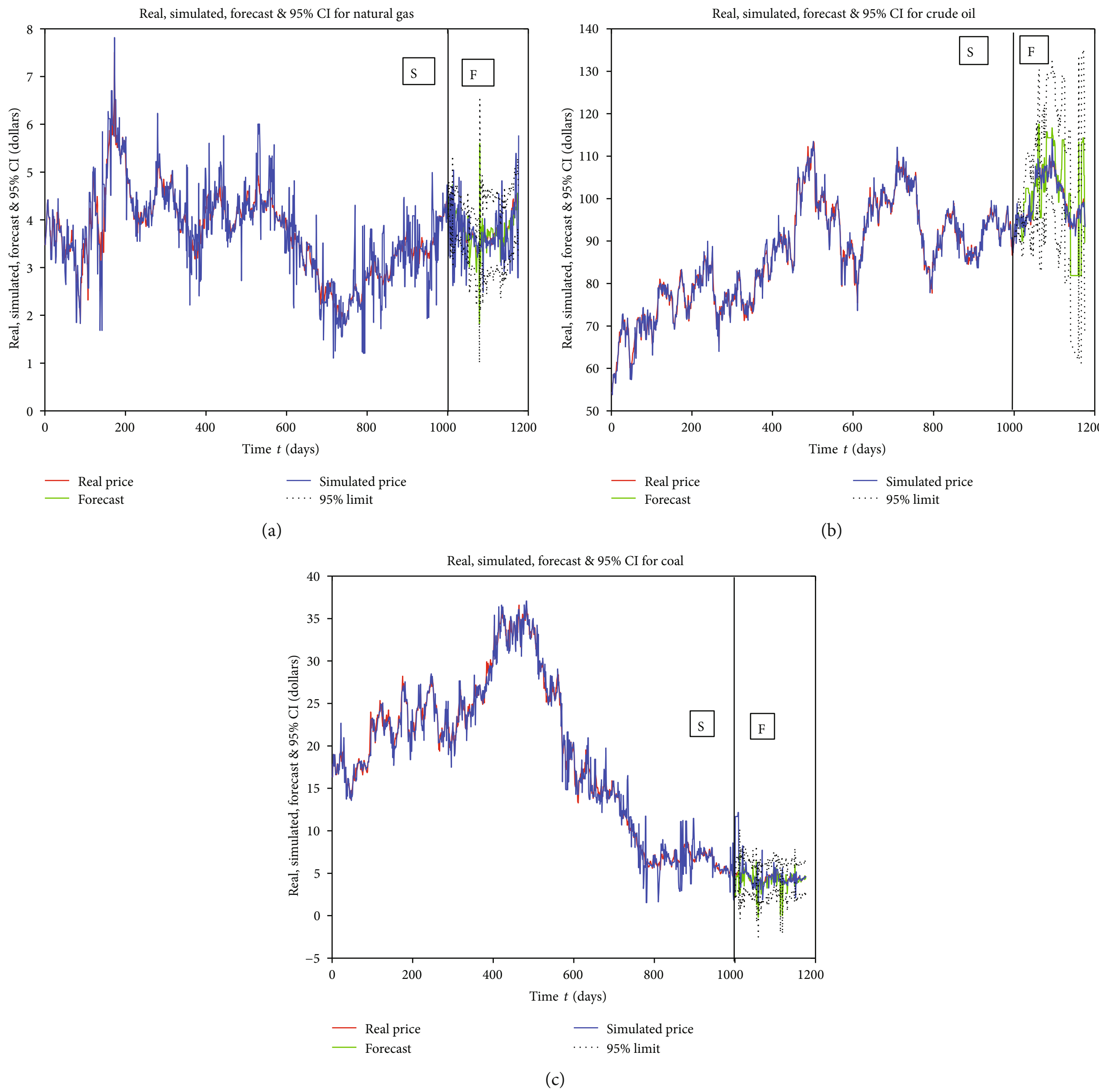


Figure 15: Real, simulated, forecasted prices and 95% C.I. with no jump.

commodity p_l is cooperating with commodity p_j, and negative if commodity p_l is competing with commodity p_j. There is no interaction between the two commodities if $\gamma_{j,l}(\widehat{m}_k, k) = 0$. It is apparent from the graph (from $\gamma_{1,3}(\widehat{m}_k, k)$ in Column 6) that coal is competing with natural gas during this period because the estimates of $\gamma_{1,3}(\widehat{m}_k, k)$ are mostly negative. It is apparent that natural gas and crude oil are either cooperating or competing, depending on the time period.

The graph of the drift coefficient's parameters with estimates in Table 10 are given in Figure 11 below.

Figures 11(a)–11(i) show the graph of the ε suboptimal interaction coefficient parameters $\gamma_{1,1}(\widehat{m}_k, k)$, $\gamma_{1,2}(\widehat{m}_k, k)$, $\gamma_{1,3}(\widehat{m}_k, k)$, $\gamma_{2,1}(\widehat{m}_k, k)$, $\gamma_{2,2}(\widehat{m}_k, k)$, $\gamma_{2,3}(\widehat{m}_k, k)$, $\gamma_{3,1}(\widehat{m}_k, k)$, $\gamma_{3,2}(\widehat{m}_k, k)$, and $\gamma_{3,3}(\widehat{m}_k, k)$. According to (58), the estimate $\gamma_{j,l}(\widehat{m}_k, k)$, $j \neq l$, is positive if commodity p_l is cooperating with commodity p_j, and negative if commodity p_l is competing with commodity p_j. There is no interaction between the two commodities if $\gamma_{j,l}(\widehat{m}_k, k) = 0$. It is apparent from the graph of $\gamma_{1,3}(\widehat{m}_k, k)$ that coal is competing with natural gas because the estimates of $\gamma_{1,3}(\widehat{m}_k, k)$ are mostly negative. This shows that for fixed price p_1 of natural gas, for every \$1 increase in the price of coal, the price of natural gas decreases on the average by 0.0019. Also, it is apparent that natural gas and crude oil are either cooperating or competing, depending on the time period.

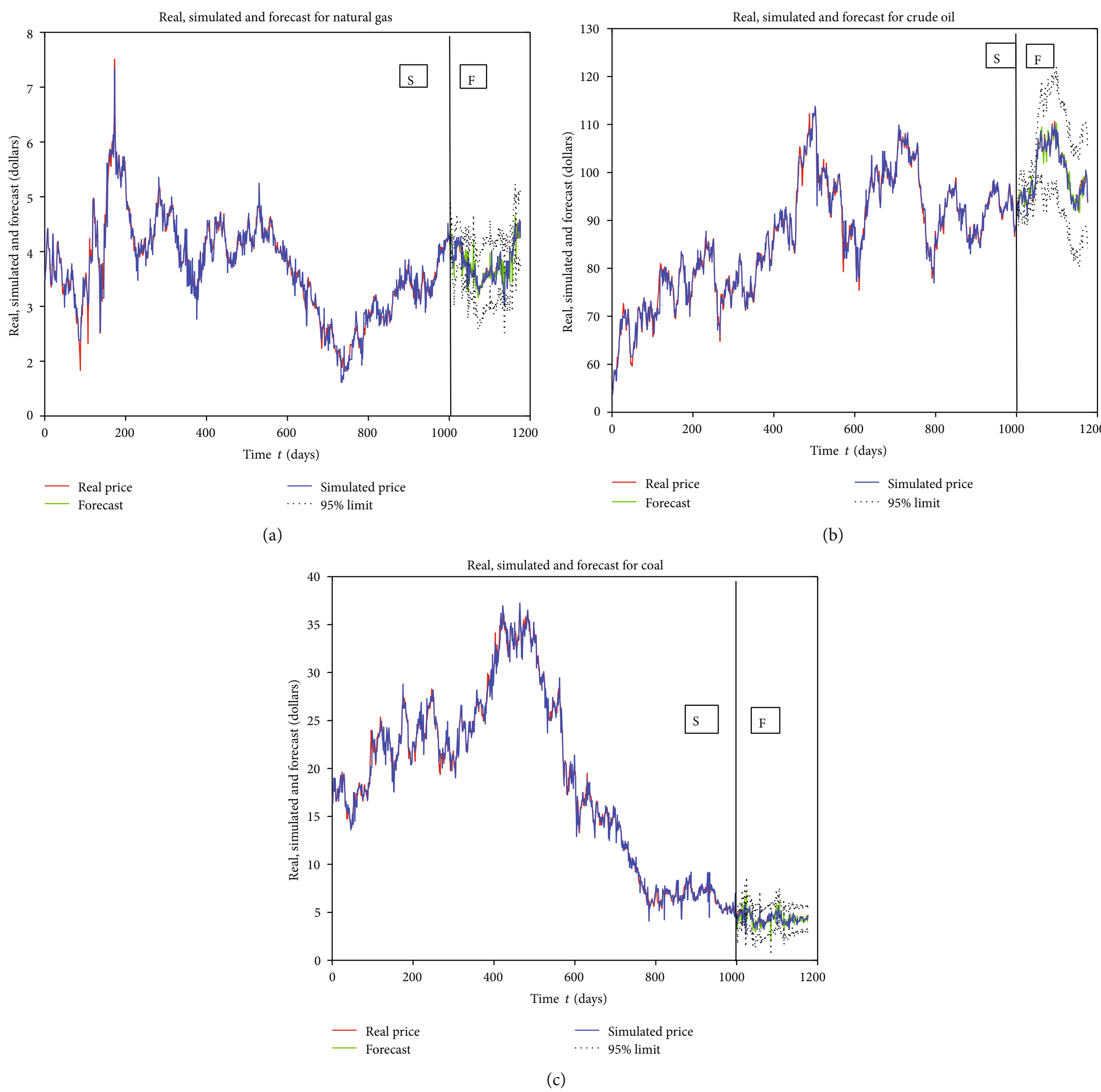


Figure 16: Real, simulated, forecasted prices and 95% CI with jump.

Figures 12(a)–12(c) are the graphs of $\beta_1(\widehat{m}_k, k)$, $\beta_2(\widehat{m}_k, k)$, and $\beta_3(\widehat{m}_k, k)$ against time t_k for the daily Henry Hub natural gas price data set [36], daily crude oil price data set [37], and daily coal price data set, respectively.

Table 11 gives the ε suboptimal estimates of the parameters $u_1(\widehat{m}_k, k)$, $\sigma_{1,1}(\widehat{m}_k, k)$, $\sigma_{1,2}(\widehat{m}_k, k)$, $\sigma_{1,3}(\widehat{m}_k, k)$, $u_2(\widehat{m}_k, k)$, $\sigma_{2,1}(\widehat{m}_k, k)$, $\sigma_{2,2}(\widehat{m}_k, k)$, $\sigma_{2,3}(\widehat{m}_k, k)$, $u_3(\widehat{m}_k, k)$, $\sigma_{3,1}(\widehat{m}_k, k)$, $\sigma_{3,2}(\widehat{m}_k, k)$, and $\sigma_{3,3}(\widehat{\mathrm{m}}_k, k)$ for each of the energy commodity data sets.

Figures 13(a)–13(c) are the graphs of $\sigma_{1,1}(\widehat{m}_k, k)$, $\sigma_{1,2}(\widehat{m}_k, k)$, $\sigma_{1,3}(\widehat{m}_k, k)$, $\sigma_{2,1}(\widehat{m}_k, k)$, $\sigma_{2,2}(\widehat{m}_k, k)$, $\sigma_{2,3}(\widehat{m}_k, k)$, $\sigma_{3,1}(\widehat{m}_k, k)$, $\sigma_{3,2}(\widehat{m}_k, k)$, and $\sigma_{3,3}(\widehat{m}_k, k)$ against time t_k for the daily Henry Hub natural gas price data set [36], daily crude oil price data set [37], and daily coal price data set, respectively.

Table 12 shows the real and simulated estimates for the spot price processes $p_j(t)$, $j \in I(1, n)$.

Figure 14 shows the graph of the real and simulated prices (with jump) for natural gas, crude oil, and coal data set.

Figures 14(a)–14(c) show the graph of the real and simulated spot prices for the daily Henry Hub natural gas data set [36], daily crude oil data set [37], and daily coal data set [35], respectively. The red line represents the real data set $\mathbf{p}$, while the blue line represent the simulated data set $\mathbf{p}^s_{\widehat{m}_k,k}$. It is clear that the graph fits well. To reduce magnitude of error, we increase the magnitude of time delay. It is obvious that these curves fit better than the curves in Figure 7. It follows that the interconnected dynamical system with jump process incorporated into it performs better than the one without jump.

8. Forecasting

In this section, we shall sketch an outline about forecasting problem for the case where there is no jump. The sketch for the case where jump exist is similar. An ε suboptimal simulated value $(\mathbf{y}^s(\widehat{m}_k^{i-1}, t_k^{i-1}), \mathbf{p}^s(\widehat{m}_k^{i-1}, t_k^{i-1}))$ at time t_k^{i-1}, $i \in I(1, K^*)$, is used to define a forecast $(\mathbf{y}^f(\widehat{m}_k^{i-1}, t_k^{i-1}), \mathbf{p}^f(\widehat{m}_k^{i-1}, t_k^{i-1}))$ for $(\mathbf{y}(t_k^{i-1}), \mathbf{p}(t_k^{i-1}))$ at the time t_k^{i-1} for the system of energy commodity model.

8.1. Forecasting for Energy Commodity Model. In the context of Section 6.1, for $i \in I(1, K^*)$, we begin forecasting from time t_k^{i-1}. Using the data set up to time t_{k-1}^{i-1}, we compute $\widehat{m}_a^{i-1}, \widehat{m}_a^{i-1}$, $u_j(\widehat{m}_a^{i-1}, t_a^{i-1})$, $\beta_j(\widehat{m}_a^{i-1}, t_a^{i-1})$, $\kappa_{j,l}(\widehat{m}_a^{i-1}, t_a^{i-1})$, $\gamma_{j,l}(\widehat{m}_a^{i-1}, t_a^{i-1})$, $\delta_{j,l}(\widehat{m}_a^{i-1}, t_a^{i-1})$, $\sigma_{j,l}(\widehat{m}_a^{i-1}, t_a^{i-1})$, and $j, l \in I(1,3)$ for $a \in I(0, k-1)$. We assume that we have no information about the real data set $\{y_j(t_a^{i-1})\}_{a=k}^{N_{i-1}}$. Under these considerations, imitating the computational procedure outlined in Section 7 and using solutions to (66) and (67), we find the estimate of the forecast $\mathbf{y}^f(\widehat{m}_k^{i-1}, t_k^{i-1})$ and $\mathbf{p}^f(\widehat{m}_k^{i-1}, t_k^{i-1})$ at time t_k^{i-1} as follows:

$$
\begin{cases}
y_j^f(\widehat{m}_k^{i-1}, t_k^{i-1}) = y_j^s(\widehat{m}_{k-1}^{i-1}, t_{k-1}^{i-1}) + \left(u_j(\widehat{m}_{k-1}^{i-1}, t_{k-1}^{i-1}) - y_j^s(\widehat{m}_{k-1}^{i-1}, t_{k-1}^{i-1})\right)\left[\kappa_{j,j}(\widehat{m}_{k-1}^{i-1}, t_{k-1}^{i-1}) y_j^s(\widehat{m}_{k-1}^{i-1}, t_{k-1}^{i-1}) + \sum_{l\neq j}^{n} \kappa_{j,l} y_l^s(\widehat{m}_{k-1}^{i-1}, t_{k-1}^{i-1})\right]\Delta t \\
\quad + \delta_{j,j}(\widehat{m}_{k-1}^{i-1}, t_{k-1}^{i-1})\left(u_j(\widehat{m}_{k-1}^{i-1}, t_{k-1}^{i-1}) - y_j^s(\widehat{m}_{k-1}^{i-1}, t_{k-1}^{i-1})\right) W_{j,j}(k) + \left(u_j(\widehat{m}_{k-1}^{i-1}, t_{k-1}^{i-1}) - y_j^s(\widehat{m}_{k-1}^{i-1}, t_{k-1}^{i-1})\right)\sum_{l\neq j}^{n} \delta_{j,l} y_l^s(\widehat{m}_{k-1}^{i-1}, t_{k-1}^{i-1}) W_{j,l}(k), \\
p_j^f(\widehat{m}_k^{i-1}, t_k^{i-1}) = p_j^s(\widehat{m}_{k-1}^{i-1}, t_{k-1}^{i-1}) + p_j^s(\widehat{m}_{k-1}^{i-1}, t_{k-1}^{i-1})\left[\gamma_{j,j}(\widehat{m}_{k-1}^{i-1}, t_{k-1}^{i-1})\left(y_j^s(\widehat{m}_{k-1}^{i-1}, t_{k-1}^{i-1}) - p_j^s(\widehat{m}_{k-1}^{i-1}, t_{k-1}^{i-1})\right) + \beta_j(\widehat{m}_{k-1}^{i-1}, t_{k-1}^{i-1}) + \sum_{l\neq j}^{n} \gamma_{j,l}(\widehat{m}_{k-1}^{i-1}, t_{k-1}^{i-1}) p_l^s(\widehat{m}_{k-1}^{i-1}, t_{k-1}^{i-1})\right]\Delta t \\
\quad + \sigma_{j,j}(\widehat{m}_{k-1}^{i-1}, t_{k-1}^{i-1}) p_j^s(\widehat{m}_{k-1}^{i-1}, t_{k-1}^{i-1}) Z_{j,j}(k) + p_j^s(\widehat{m}_{k-1}^{i-1}, t_{k-1}^{i-1})\sum_{l\neq j}^{n} \sigma_{j,l}(\widehat{m}_{k-1}^{i-1}, t_{k-1}^{i-1}) p_l^s(\widehat{m}_{k-1}^{i-1}, t_{k-1}^{i-1}) Z_{j,l}(k), \; t_k^{i-1} \in [T_{i-1}, T_i).
\end{cases}
\tag{87}
$$

To find the forecasted value $y_j^f(T_i^-)$ and $p_j^f(T_i^-)$, we need to re-estimate $\widehat{\pi}_j^i$ and $\widehat{\theta}_j^i$ by first simulating

$$
\begin{aligned}
\lim_{t\to T_i^-} y_j^f(t, T_{i-1}, \mathbf{y}^{i-1,s}) &\equiv y_j^f\left(m_{N_{i-1}}^{i-1}, t_{N_{i-1}}^{i-1}\right), \\
\lim_{t\to T_i} p_j^f(t, T_{i-1}, \mathbf{y}^{i-1,s}, \mathbf{p}^{i-1,s}) &\equiv p_j^f\left(m_{N_{i-1}}^{i-1}, t_{N_{i-1}}^{i-1}\right),
\end{aligned}
\tag{88}
$$

as follows:

$$
\begin{aligned}
y_j^f\left(m_{N_{i-1}}^{i-1}, t_{N_{i-1}}^{i-1}\right) &= y_j^s\left(m_{N_{i-1}-1}^{i-1}, t_{N_{i-1}-1}^{i-1}\right) + \left(u_j^{i-1}\left(m_{N_{i-1}-1}^{i-1}, t_{N_{i-1}-1}^{i-1}\right)\right. \\
&\quad \left. - y_j^s\left(m_{N_{i-1}-1}^{i-1}, t_{N_{i-1}-1}^{i-1}\right)\right)\left[\sum_{l=1}^{n} \kappa_{j,l}^{i-1}\left(m_{N_{i-1}-1}^{i-1}, t_{N_{i-1}-1}^{i-1}\right)\right. \\
&\quad \left. \cdot y_l^s\left(m_{N_{i-1}-1}^{i-1}, t_{N_{i-1}-1}^{i-1}\right)\right]\Delta t + \left(u_j^{i-1}\left(m_{N_{i-1}-1}^{i-1}, t_{N_{i-1}-1}^{i-1}\right)\right. \\
&\quad \left. - y_j^s\left(m_{N_{i-1}-1}^{i-1}, t_{N_{i-1}-1}^{i-1}\right)\right)\left[\delta_{j,j}^{i-1}\left(m_{N_{i-1}-1}^{i-1}, t_{N_{i-1}-1}^{i-1}\right)\right. \\
&\quad \cdot \Delta W_{j,j}\left(m_{N_{i-1}}^{i-1}, t_{N_{i-1}}^{i-1}\right) + \sum_{l\neq j}^{n} \delta_{j,l}^{i-1}\left(m_{N_{i-1}-1}^{i-1}, t_{N_{i-1}-1}^{i-1}\right) \\
&\quad \left. \cdot y_l^s\left(m_{N_{i-1}-1}^{i-1}, t_{N_{i-1}-1}^{i-1}\right)\Delta W_{j,l}\left(m_{N_{i-1}}^{i-1}, t_{N_{i-1}}^{i-1}\right)\right], \\
p_j^f\left(m_{N_{i-1}}^{i-1}, t_{N_{i-1}}^{i-1}\right) &= p_j^s\left[\gamma_{j,j}^{i-1}\left(m_{N_{i-1}-1}^{i-1}, t_{N_{i-1}-1}^{i-1}\right)\left(y_j^s\left(m_{N_{i-1}-1}^{i-1}, t_{N_{i-1}-1}^{i-1}\right)\right.\right. \\
&\quad \left. - p_j^s\left(m_{N_{i-1}-1}^{i-1}, t_{N_{i-1}-1}^{i-1}\right)\right) + \beta_j^{i-1}\left(m_{N_{i-1}-1}^{i-1}, t_{N_{i-1}-1}^{i-1}\right) \\
&\quad \left. + \sum_{l\neq j}^{n} \gamma_{j,l}^{i-1}\left(m_{N_{i-1}-1}^{i-1}, t_{N_{i-1}-1}^{i-1}\right) p_l^s\left(m_{N_{i-1}-1}^{i-1}, t_{N_{i-1}-1}^{i-1}\right)\right]\Delta t \\
&\quad + \sigma_{j,j}^{i-1}\left(m_{N_{i-1}-1}^{i-1}, t_{N_{i-1}-1}^{i-1}\right) p_j^s\left(m_{N_{i-1}-1}^{i-1}, t_{N_{i-1}-1}^{i-1}\right) \\
&\quad \cdot \Delta Z_{j,j}\left(m_{N_{i-1}}^{i-1}, t_{N_{i-1}}^{i-1}\right) + p_j^s\left(m_{N_{i-1}-1}^{i-1}, t_{N_{i-1}-1}^{i-1}\right) \\
&\quad \cdot \sum_{l\neq j}^{n} \sigma_{j,l}^{i-1}\left(m_{N_{i-1}-1}^{i-1}, t_{N_{i-1}-1}^{i-1}\right) p_l^s\left(m_{N_{i-1}-1}^{i-1}, t_{N_{i-1}-1}^{i-1}\right) \\
&\quad \cdot \Delta Z_{j,l}\left(m_{N_{i-1}}^{i-1}, t_{N_{i-1}}^{i-1}\right), \; t_k^{i-1} = t_{N_{i-1}}^{i-1}.
\end{aligned}
\tag{89}
$$

Imitating argument in (84), we define

$$
\begin{aligned}
\widehat{\pi}_j^i &= \frac{y_j^s\left(m_{N_{i-1}}^{i-1}, t_{N_{i-1}}^{i-1}\right)}{y_j^f\left(m_{N_{i-1}}^{i-1}, t_{N_{i-1}}^{i-1}\right)}, \\
\widehat{\theta}_j^i &= \frac{p_j^s\left(m_{N_{i-1}}^{i-1}, t_{N_{i-1}}^{i-1}\right)}{p_j^f\left(m_{N_{i-1}}^{i-1}, t_{N_{i-1}}^{i-1}\right)}.
\end{aligned}
\tag{90}
$$

Thus, $y_j^f(\boldsymbol{T}_i^-) = \widehat{\pi}_j^i y_j^f(\boldsymbol{T}_i^-, \boldsymbol{T}_{i-1}, \mathbf{y}^{i-1,s})$ and $p_j^f(T_i^-) = \widehat{\theta}_j^i p_j^f(T_i^-, T_{i-1}, \mathbf{y}^{i-1,f}, \mathbf{p}^{i-1,f})$. where the estimates $u_j(\widehat{m}_{k-1}^{i-1}, t_{k-1}^{i-1})$, $\beta_j(\widehat{m}_{k-1}^{i-1}, t_{k-1}^{i-1})$, $\kappa_{j,l}(\widehat{m}_{k-1}^{i-1}, t_{k-1}^{i-1})$, $\gamma_{j,l}(\widehat{m}_{k-1}^{i-1}, t_{k-1}^{i-1})$, $\delta_{j,l}(\widehat{m}_{k-1}^{i-1}, t_{k-1}^{i-1})$, and $\sigma_{j,l}(\widehat{m}_{k-1}^{i-1}, t_{k-1}^{i-1})$, $j, l \in I(1,3)$ are estimated with respect to the known past data set up to the time t_{k-1}^{i-1}. We note that $\mathbf{y}^f_{\widehat{m}_k^{i-1}, t_k^{i-1}}$ is the ε suboptimal estimate for $\mathbf{y}_j(t_k^{i-1})$ at time t_k^{i-1}.

To determine $(\mathbf{y}^f(\widehat{m}_{k+1}^{i-1}, t_{k+1}^{i-1}), \mathbf{p}^f(\widehat{m}_{k+1}^{i-1}, t_{k+1}^{i-1}))$, we need $u_j(\widehat{m}_k^{i-1}, t_k^{i-1})$, $\beta_j(\widehat{m}_k^{i-1}, t_k^{i-1})$, $\kappa_{j,l}(\widehat{m}_k^{i-1}, t_k^{i-1})$, $\gamma_{j,l}(\widehat{m}_k^{i-1}, t_k^{i-1})$, $\delta_{j,l}(\widehat{m}_k^{i-1}, t_k^{i-1})$, and $\sigma_{j,l}(\widehat{m}_k^{i-1}, t_k^{i-1})$, $j, l \in I(1,3)$. Since we only have information of real data up to time t_{k-1}, we use the forecasted estimate $y_j^f(\widehat{m}_k^{i-1}, t_k^{i-1})$ as the estimate of $y_j(t_k^{i-1})$ at time t_k^{i-1} and to estimate $u_j(\widehat{m}_k^{i-1}, t_k^{i-1})$, $\beta_j(\widehat{m}_k^{i-1}, t_k^{i-1})$, $\kappa_{j,l}(\widehat{m}_k^{i-1}, t_k^{i-1})$, $\gamma_{j,l}(\widehat{m}_k^{i-1}, t_k^{i-1})$, $\delta_{j,l}(\widehat{m}_k^{i-1}, t_k^{i-1})$, and $\sigma_{j,l}(\widehat{m}_k^{i-1}, t_k^{i-1})$, $j, l \in I(1,3)$. Hence, we can write $u_j(\widehat{m}_k^{i-1}, t_k^{i-1})$ as

$$\begin{cases} u_j\left(\widehat{m}_k^{i-1}, t_k^{i-1}\right) \equiv u_j\left(\widehat{m}_k^{i-1}, y_j\left(t_{k-\widehat{m}_k^{i-1}+1}^{i-1}\right), y_j\left(t_{k-\widehat{m}_k^{i-1}+2}^{i-1}\right), \cdots, y_j\left(t_{k-1}^{i-1}\right), y_j^f\left(\widehat{m}_k^{i-1}, t_k^{i-1}\right)\right), \\ \kappa_{j,l}\left(\widehat{m}_k^{i-1}, t_k^{i-1}\right) \equiv \kappa_{j,l}\left(\widehat{m}_k^{i-1}, y_j\left(t_{k-\widehat{m}_k^{i-1}+1}^{i-1}\right), y_j\left(t_{k-\widehat{m}_k^{i-1}+2}^{i-1}\right), \cdots, y_j\left(t_{k-1}^{i-1}\right), y_j^f\left(\widehat{m}_k^{i-1}, t_k^{i-1}\right)\right), \\ \delta_{j,l}\left(\widehat{m}_k^{i-1}, t_k^{i-1}\right) \equiv \delta_{j,l}\left(\widehat{m}_k^{i-1}, y_j\left(t_{k-\widehat{m}_k^{i-1}+1}^{i-1}\right), y_j\left(t_{k-\widehat{m}_k^{i-1}+2}^{i-1}\right), \cdots, y_j\left(t_{k-1}^{i-1}\right), y_j^f\left(\widehat{m}_k^{i-1}, t_k^{i-1}\right)\right), \\ \beta_{j,l}\left(\widehat{m}_k^{i-1}, t_k^{i-1}\right) \equiv \beta_{j,l}\left(\widehat{m}_k^{i-1}, p_j\left(t_{k-\widehat{m}_k^{i-1}+1}^{i-1}\right), p_j\left(t_{k-\widehat{m}_k^{i-1}+2}^{i-1}\right), \cdots, p_j\left(t_{k-1}^{i-1}\right), p_j^f\left(\widehat{m}_k^{i-1}, t_k^{i-1}\right)\right), \\ \gamma_{j,l}\left(\widehat{m}_k^{i-1}, t_k^{i-1}\right) \equiv \gamma_{j,l}\left(\widehat{m}_k^{i-1}, p_j\left(t_{k-\widehat{m}_k^{i-1}+1}^{i-1}\right), p_j\left(t_{k-\widehat{m}_k^{i-1}+2}^{i-1}\right), \cdots, p_j\left(t_{k-1}^{i-1}\right), p_j^f\left(\widehat{m}_k^{i-1}, t_k^{i-1}\right)\right), \\ \sigma_{j,l}\left(\widehat{m}_k^{i-1}, t_k^{i-1}\right) \equiv \sigma_{j,l}\left(\widehat{m}_k^{i-1}, p_j\left(t_{k-\widehat{m}_k^{i-1}+1}^{i-1}\right), p_j\left(t_{k-\widehat{m}_k^{i-1}+2}^{i-1}\right), \cdots, p_j\left(t_{,k-1}^{i-1}\right), p_j^f\left(\widehat{m}_k^{i-1}, t_k^{i-1}\right)\right), j, l \in I(1,n). \end{cases} \tag{91}$$

For $j, l \in I(1,n)$, to find $(y_j^f(\widehat{m}_{k+2}^{i-1}, t_{k+2}^{i-1}),\ p_j^f(\widehat{m}_{k+2}^{i-1}, t_{k+2}^{i-1}))$, we use the estimates

$$\begin{cases} u_j\left(\widehat{m}_{k+1}^{i-1}, t_{k+1}^{i-1}\right) \equiv u_j\left(\widehat{m}_{k+1}^{i-1}, y_j\left(t_{k-\widehat{m}_k^{i-1}+2}^{i-1}\right), y_j\left(t_{k-\widehat{m}_k^{i-1}+3}^{i-1}\right), \cdots, y_j\left(t_{k-1}^{i-1}\right), y_j^f\left(\widehat{m}_k^{i-1}, t_k^{i-1}\right), y_j^f\left(\widehat{m}_{k+1}^{i-1}, t_{k+1}^{i-1}\right)\right), \\ \kappa_{j,l}\left(\widehat{m}_{k+1}^{i-1}, t_{k+1}^{i-1}\right) \equiv \kappa_{j,l}\left(\widehat{m}_{k+1}^{i-1}, y_j\left(t_{k-\widehat{m}_k^{i-1}+2}^{i-1}\right), y_j\left(t_{k-\widehat{m}_k^{i-1}+3}^{i-1}\right), \cdots, y_j\left(t_{k-1}^{i-1}\right), y_j^f\left(\widehat{m}_k^{i-1}, t_k^{i-1}\right), y_j^f\left(\widehat{m}_{k+1}^{i-1}, t_{k+1}^{i-1}\right)\right), \\ \delta_{j,l}\left(\widehat{m}_{k+1}^{i-1}, t_{k+1}^{i-1}\right) \equiv \delta_{j,l}\left(\widehat{m}_{k+1}^{i-1}, y_j\left(t_{k-\widehat{m}_k^{i-1}+2}^{i-1}\right), y_j\left(t_{k-\widehat{m}_k^{i-1}+3}^{i-1}\right), \cdots, y_j\left(t_{k-1}^{i-1}\right), y_j^f\left(\widehat{m}_k^{i-1}, t_k^{i-1}\right), y_j^f\left(\widehat{m}_{k+1}^{i-1}, t_{k+1}^{i-1}\right)\right), \\ \beta_{j,l}\left(\widehat{m}_{k+1}^{i-1}, t_{k+1}^{i-1}\right) \equiv \beta_{j,l}\left(\widehat{m}_{k+1}^{i-1}, p_j\left(t_{k-\widehat{m}_k^{i-1}+2}^{i-1}\right), p_j\left(t_{k-\widehat{m}_k^{i-1}+3}^{i-1}\right), \cdots, p_j\left(t_{k-1}^{i-1}\right), p_j^f\left(\widehat{m}_k^{i-1}, t_k^{i-1}\right), p_j^f\left(\widehat{m}_{k+1}^{i-1}, t_{k+1}^{i-1}\right)\right), \\ \gamma_{j,l}\left(\widehat{m}_{k+1}^{i-1}, t_{k+1}^{i-1}\right) \equiv \gamma_{j,l}\left(\widehat{m}_{k+1}^{i-1}, p_j\left(t_{k-\widehat{m}_k^{i-1}+2}^{i-1}\right), p_j\left(t_{k-\widehat{m}_k^{i-1}+3}^{i-1}\right), \cdots, p_j\left(t_{k-1}^{i-1}\right), p_j^f\left(\widehat{m}_k^{i-1}, t_k^{i-1}\right), p_j^f\left(\widehat{m}_{k+1}^{i-1}, t_{k+1}^{i-1}\right)\right), \\ \sigma_{j,l}\left(\widehat{m}_{k+1}^{i-1}, t_{k+1}^{i-1}\right) \equiv \sigma_{j,l}\left(\widehat{m}_{k+1}^{i-1}, p_j\left(t_{k-\widehat{m}_k^{i-1}+2}^{i-1}\right), p_j\left(t_{k-\widehat{m}_k^{i-1}+3}^{i-1}\right), \cdots, p_j\left(t_{k-1}^{i-1}\right), p_j^f\left(\widehat{m}_k^{i-1}, t_k^{i-1}\right), p_j^f\left(\widehat{m}_{k+1}^{i-1}, t_{k+1}^{i-1}\right)\right). \end{cases} \tag{92}$$

Continuing this process in this manner, to find $(y_j^f(\widehat{m}_{k+l}^{i-1}, t_{k+l}^{i-1}), p_j^f(\widehat{m}_{k+l}^{i-1}, t_{k+l}^{i-1}))$, we use the estimates

$$\begin{cases} u_j(\widehat{m}_{k+l-1}^{i-1}, t_{k+l-1}^{i-1}) \equiv u_j\left(\widehat{m}_{k+l-1}^{i-1}, y_j\left(t_{k-\widehat{m}_k^{i-1}+l}^{i-1}\right), y_j\left(t_{k-\widehat{m}_k^{i-1}+l+1}^{i-1}\right), \cdots, y_j(t_{k-1}^{i-1}), y_j^f(\widehat{m}_k^{i-1}, t_k^{i-1}), \cdots, y_j^f(\widehat{m}_{k+l-1}^{i-1}, t_{k+l-1}^{i-1})\right), \\ \kappa_{j,l}(\widehat{m}_{k+l-1}^{i-1}, t_{k+l-1}^{i-1}) \equiv \kappa_{j,l}\left(\widehat{m}_{k+l-1}^{i-1}, y_j\left(t_{k-\widehat{m}_k^{i-1}+l}^{i-1}\right), y_j\left(t_{k-\widehat{m}_k^{i-1}+l+1}^{i-1}\right), \cdots, y_j(t_{k-1}^{i-1}), y_j^f(\widehat{m}_k^{i-1}, t_k^{i-1}), \cdots, y_j^f(\widehat{m}_{k+l-1}^{i-1}, t_{k+l-1}^{i-1})\right), \\ \delta_{j,l}(\widehat{m}_{k+l-1}^{i-1}, t_{k+l-1}^{i-1}) \equiv \delta_{j,l}\left(\widehat{m}_{k+l-1}^{i-1}, y_j\left(t_{k-\widehat{m}_k^{i-1}+l}^{i-1}\right), y_j\left(t_{k-\widehat{m}_k^{i-1}+l+1}^{i-1}\right), \cdots, y_j(t_{k-1}^{i-1}), y_j^f(\widehat{m}_k^{i-1}, t_k^{i-1}), \cdots, y_j^f(\widehat{m}_{k+l-1}^{i-1}, t_{k+l-1}^{i-1})\right), \\ \beta_{j,l}(\widehat{m}_{k+l-1}^{i-1}, t_{k+l-1}^{i-1}) \equiv \beta_{j,l}\left(\widehat{m}_{k+l-1}^{i-1}, p_j\left(t_{k-\widehat{m}_k^{i-1}+l}^{i-1}\right), p_j\left(t_{k-\widehat{m}_k^{i-1}+l+1}^{i-1}\right), \cdots, p_j(t_{k-1}^{i-1}), p_j^f(\widehat{m}_k^{i-1}, t_k^{i-1}), \cdots, p_j^f(\widehat{m}_{k+l-1}^{i-1}, t_{k+l-1}^{i-1})\right), \\ \gamma_{j,l}(\widehat{m}_{k+l-1}^{i-1}, t_{k+l-1}^{i-1}) \equiv \gamma_{j,l}\left(\widehat{m}_{k+l-1}^{i-1}, p_j\left(t_{k-\widehat{m}_k^{i-1}+l}^{i-1}\right), p_j\left(t_{k-\widehat{m}_k^{i-1}+l+1}^{i-1}\right), \cdots, p_j(t_{k-1}^{i-1}), p_j^f(\widehat{m}_k^{i-1}, t_k^{i-1}), \cdots, p_j^f(\widehat{m}_{k+l-1}^{i-1}, t_{k+l-1}^{i-1})\right), \\ \sigma_{j,l}(\widehat{m}_{k+l-1}^{i-1}, t_{k+l-1}^{i-1}) \equiv \sigma_{j,l}\left(\widehat{m}_{k+l-1}^{i-1}, p_j\left(t_{k-\widehat{m}_k^{i-1}+i}^{i-1}\right), p_j\left(t_{k-\widehat{m}_k^{i-1}+l+1}^{i-1}\right), \cdots, p_j(t_{k-1}^{i-1}), p_j^f(\widehat{m}_k^{i-1}, t_k^{i-1}), \cdots, p_j^f(\widehat{m}_{k+l-1}^{i-1}, t_{k+l-1}^{i-1})\right). \end{cases} \tag{93}$$

8.1.1. Prediction/Confidence Interval for Energy Commodities. In order to be able to assess the future certainty, we also discuss about the prediction/confidence interval. We define the $100(1-\alpha)\%$ confidence interval for the forecast of the state $(\mathbf{y}^f_{\widehat{m}_l^{i-1}, t_l^{i-1}}, \mathbf{p}^f_{\widehat{m}_l^{i-1}, t_l^{i-1}})$ at time t_l^{i-1}, $l \geq k$, $j \in I(1,n)$ as

$$\begin{cases} y_j^f(\widehat{m}_l^{i-1}, t_l^{i-1}) \pm z_{1-\alpha/2}\left(s^{j,j}_{m\wedge_{l-1}^{i-1}, t_{l-1}^{i-1}}\left(y_j^f\right)\right)^{1/2}, \\ p_j^f(\widehat{m}_l^{i-1}, t_l^{i-1}) \pm z_{1-\alpha/2}\left(s^{j,j}_{m\wedge_{l-1}^{i-1}, t_{l-1}^{i-1}}\left(p_j^f\right)\right)^{1/2}, \end{cases} \tag{94}$$

where $(s^{j,j}(m\wedge_{l-1}^{i-1}, t_{l-1}^{i-1}))^{1/2}$ is the estimate for the sample standard deviation for the forecasted state y_j, and $(s^{j,j}(m\wedge_{l-1}^{i-1}, t_{l-1}^{i-1}))^{1/2}$ is the estimate for the sample standard deviation for the forecasted state p_j derived from the following iterative process

$$\begin{cases} y_j^f(\widehat{m}_l^{i-1}, t_l^{i-1}) = y_j^f(\widehat{m}_{l-1}^{i-1}, t_{l-1}^{i-1}) + \left(u_j(\widehat{m}_{l-1}^{i-1}, t_{l-1}^{i-1}) - y_j^f(\widehat{m}_{l-1}^{i-1}, t_{l-1}^{i-1})\right)\left[\kappa_{j,j}(\widehat{m}_{l-1}^{i-1}, t_{l-1}^{i-1}) y_j^f(\widehat{m}_{l-1}^{i-1}, t_{l-1}^{i-1}) + \sum_{l \neq j}^{n} \kappa_{j,l} y_l^f(\widehat{m}_{l-1}^{i-1}, t_{l-1}^{i-1})\right]\Delta t \\ \quad + \delta_{j,j}(\widehat{m}_{l-1}^{i-1}, t_{l-1}^{i-1})\left(u_j(\widehat{m}_{l-1}^{i-1}, t_{l-1}^{i-1}) - y_j^f(\widehat{m}_{l-1}^{i-1}, t_{l-1}^{i-1})\right) W_{j,j}(l) + \left(u_j(\widehat{m}_{l-1}^{i-1}, t_{l-1}^{i-1}) - y_j^f(\widehat{m}_{l-1}^{i-1}, t_{l-1}^{i-1})\right) \sum_{l \neq j}^{n} \delta_{j,l} y_l^f(\widehat{m}_{l-1}^{i-1}, t_{l-1}^{i-1}) W_{j,l}(l), \\ p_j^f(\widehat{m}_l^{i-1}, t_l^{i-1}) = p_j^f(\widehat{m}_{l-1}^{i-1}, t_{l-1}^{i-1}) + p_j^f(\widehat{m}_{l-1}^{i-1}, t_{l-1}^{i-1})\left[\gamma_{j,j}(\widehat{m}_{l-1}^{i-1}, t_{l-1}^{i-1})\left(y_j^f(\widehat{m}_{l-1}^{i-1}, t_{l-1}^{i-1}) - p_j^f(\widehat{m}_{l-1}^{i-1}, t_{l-1}^{i-1})\right) + \beta_j(\widehat{m}_{l-1}^{i-1}, t_{l-1}^{i-1}) + \sum_{l \neq j}^{n} \gamma_{j,l}(\widehat{m}_{l-1}^{i-1}, t_{l-1}^{i-1}) p_l^f(\widehat{m}_{l-1}^{i-1}, t_{l-1}^{i-1})\right]\Delta t \\ \quad + \sigma_{j,j}(\widehat{m}_{l-1}^{i-1}, t_{l-1}^{i-1}) p_j^f(\widehat{m}_{l-1}^{i-1}, t_{l-1}^{i-1}) Z_{j,j}(l) + p_j^f(\widehat{m}_{l-1}^{i-1}, t_{l-1}^{i-1}) \sum_{l \neq j}^{n} \sigma_{j,l}(\widehat{m}_{l-1}^{i-1}, t_{l-1}^{i-1}) p_l^f(\widehat{m}_{l-1}^{i-1}, t_{l-1}^{i-1}) Z_{j,l}(l). \end{cases} \tag{95}$$

It is clear that the 95% confidence interval for the forecast at time t_k is

$$\begin{cases} \left(y_j^f(\widehat{m}_k^{i-1}, t_k^{i-1}) - 1.96\left(s^{j,j}_{m\wedge_{k-1}^{i-1}, k-1}\left(y_j^f\right)\right)^{1/2}, y_j^f(\widehat{m}_k^{i-1}, t_k^{i-1}) + 1.96\left(s^{j,j}_{m\wedge_{k-1}^{i-1}, k-1}\left(y_j^f\right)\right)^{1/2}\right), \\ \left(p_j^f(\widehat{m}_k^{i-1}, t_k^{i-1}) - 1.96\left(s^{j,j}_{m\wedge_{k-1}^{i-1}, k-1}\left(p_j^f\right)\right)^{1/2}, p_j^f(\widehat{m}_k^{i-1}, t_k^{i-1}) + 1.96\left(s^{j,j}_{m\wedge_{k-1}^{i-1}, k-1}\left(p_j^f\right)\right)^{1/2}\right), \end{cases} \tag{96}$$

where the lower ends denote the lower bounds of the state estimate and the upper ends denote the upper bounds of the state estimate.

8.1.2. Illustration: Prediction/Confidence Interval for Energy Commodities with No Jump. For the case of no jump, the following graphs show the simulated, forecasted, and 95 percent confidence limit for the daily Henry Hub natural gas data set [36], daily crude oil data set [37], and daily coal data set [35], respectively.

Figures 15(a)–15(c) show the graph of the forecast and 95 percent confidence limit for the case where there is no jump for the daily Henry Hub natural gas data set [36], daily crude oil data set [37], and daily coal data set [35], respectively. The figure shows two regions: the simulation region S and the forecast region F. For the simulation region S, we plot the real data set together with the simulated data set as described in Figure 14. For forecast region F, we plot the estimate of the forecast as explained in Section 8.1.1.

8.1.3. Illustration: Prediction/Confidence Interval for Energy Commodities with Jump. For the jump process, the following graphs show the forecast and 95 percent confidence limit for the daily Henry Hub natural gas data set [36], daily crude oil data set [37], and daily coal data set [35], respectively.

Figures 16(a)–16(c) show the graphs of the forecast and 95 percent confidence limit for the case where there is jump for the daily Henry Hub Natural gas data set [36], daily Crude Oil data set [37], and daily Coal data set [35], respectively. Figures 16(a)–16(c) show two regions: the simulation region S and the forecast region F. For the simulation region S, we plot the real data set together with the simulated data set as described in Figure 14. For the forecast region F, we plot the estimate of the forecast as explained in Section 8.1.1.

9. Conclusion and Future Work

We examine the relationship between the prices of the energy commodities: natural gas, crude oil, and coal. To do this, multivariate stochastic models with and without external random interventions are developed to describe the relationship between the price of energy commodities. The time-varying parameters in the stochastic model are estimated using the LLGMM method. Figures 7 and 14 suggest that the model with jump fits better than the model without jump. For $j, l \in I(1,3)$, the estimates for the drift interaction parameters $\gamma_{j,l}(\widehat{m}_k, t_k^{i-1})$ for the case where jump is incorporated suggest that there are definitely interactions between the spot price of these three commodities. As discussed in (57) and (58), the sign of these parameters suggest that there is competition or cooperation between commodities l and j. The estimate of the parameter $\gamma_{j,l}(\widehat{m}_k, t_k^{i-1})$ in Tables 3 and 10 and Figures 4 and 11 suggests that these commodities either compete or cooperate with each other depending on the time period. We can also describe the relationship between any two commodity j and l, $j \neq l \in I(1,3)$, based on the overall average $\widehat{\gamma}_{j,l} = 1/N\sum_{k=1}^{N}\gamma_{j,l}(\widehat{m}_k, t_k^{i-1})$. For example, for the case where jump is incorporated, $\widehat{\gamma}_{1,3} = -0.0017$ and $\widehat{\gamma}_{3,1} = -0.0095$. This suggests that on the average, there is competition between these two commodities. Also, $\widehat{\gamma}_{1,2} = 0.0018$ and $\bar{\gamma}_{2,1} = 0.0083$. This indicates that on the average, there is cooperation between natural gas and crude oil. Finally, $\widehat{\gamma}_{2,3} = -0.0146$ and $\widehat{\gamma}_{3,2} = -0.0013$. Therefore, on the average, there is competition between crude oil and coal. In the future, we plan to apply the local lagged adapted generalized method of moments to interconnected nonlinear stochastic dynamic model for log-spot price, expected log-spot price, and volatility process. Also, we plan to incorporate delay in the multivariate interconnected nonlinear stochastic model. We plan to be able to apply the extended local lagged adapted generalized method of moments to other multivariate interconnected nonlinear dynamic model different from energy commodity model.

Data Availability

We will provide the data upon request.

Disclosure

We would also like to mention that this research paper was presented in the Ph.D. Dissertation [15] and at the American Mathematical Society (AMS) conference at San Antonio, Texas.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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