Reiterated homogenization of nonlinear monotone operators in a general deterministic setting

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Abstract. We study reiterated homogenization of a nonlinear non-periodic elliptic differential operator in a general deterministic setting as opposed to the usual stochastic setting. Our approach proceeds from an appropriate notion of convergence termed reiterated Σ-convergence. A general deterministic homogenization theorem is proved and several concrete examples are studied under various structure hypotheses ranging from the classical periodicity hypothesis to more complicated, but realistic, structure hypotheses.

1. Introduction

We study the homogenization (as $0 < \varepsilon \to 0$) of the boundary value problem

$$(1.1) \qquad -\mathrm{div}\, a\left(\frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}, Du_\varepsilon\right) = f \text{ in } \Omega,\ u_\varepsilon \in W_0^{1,p}(\Omega; \mathbb{R})$$

where Ω is a bounded open set in $\mathbb{R}_x^N$ (the N-dimensional numerical space of variables $x = (x_1, \cdots, x_N)$), f is given in $W^{-1,p'}(\Omega; \mathbb{R})$ with $p' = \frac{p}{p-1}$, $1 < p < \infty$, D and div denote the usual gradient and divergence operators, respectively, in Ω, and finally, a is a given function $(y, z, \xi) \to a(y, z, \xi)$ from $\mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N$ to $\mathbb{R}^N$ $(N \geq 1)$ with the following properties :

(1.2)

For any arbitrary $\xi \in \mathbb{R}^N$, the function $(y, z) \to a(y, z, \xi)$ possesses the Caratheodory property, i.e.,

(i) for each $z \in \mathbb{R}^N$, the function $y \to a(y, z, \xi)$ is measurable from $\mathbb{R}^N$ (with Lebesgue measure) into $\mathbb{R}^N$,

(ii) for almost every $y \in \mathbb{R}^N$, the function $z \to a(y, z, \xi)$ maps $\mathbb{R}^N$ continuously into $\mathbb{R}^N$.

(1.3)

$a(y, z, \omega) = \omega$ for almost all $y \in \mathbb{R}^N$ and for all $z \in \mathbb{R}^N$, where ω denotes the origin in $\mathbb{R}^N$.

(1.4)

There are four constants $c_1, c_2 > 0$, $0 < \alpha_1 \leq \min(1, p-1)$, and $\alpha_2 \geq \max(p, 2)$ such that for almost every $y \in \mathbb{R}^N$ and for every $z \in \mathbb{R}^N$ we have

(iii) $|a(y, z, \xi_1) - a(y, z, \xi_2)| \leq$
$$c_1(1 + |\xi_1| + |\xi_2|)^{p-1-\alpha_1} |\xi_1 - \xi_2|^{\alpha_1}$$

(iv) $(a(y, z, \xi_1) - a(y, z, \xi_2)) \cdot (\xi_1 - \xi_2) \geq$
$$c_2(1 + |\xi_1| + |\xi_2|)^{p-\alpha_2} |\xi_1 - \xi_2|^{\alpha_2}$$

for $\xi_1, \xi_2 \in \mathbb{R}^N$, where the dot denotes the usual Euclidean inner product in $\mathbb{R}^N$, and $|\cdot|$ the associated norm.

For the sake of clearness it is well to note that the equation in (1.1) actually writes as

$$-\text{div } a^\varepsilon (\cdot, \cdot, Du_\varepsilon) = f \text{ in } \Omega,$$

where $a^\varepsilon (\cdot, \cdot, Du_\varepsilon)$ stands for the function $x \to a(\frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}, Du_\varepsilon(x))$ from Ω to $\mathbb{R}^N$. However, since the set $Q_\varepsilon = \{(x, \frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}) : x \in \Omega\}$ is negligible in $\mathbb{R}_x^N \times \mathbb{R}_y^N \times \mathbb{R}_z^N$ (for Lebesgue measure $dxdydz$), it is not clear that this function is well defined. Indeed, unless the function $(y, z) \to a(y, z, \xi)$ (for fixed ξ) is a continuous mapping of $\mathbb{R}^N \times \mathbb{R}^N$ into $\mathbb{R}^N$ and $\mathbf{v}$ is taken in $\mathcal{C}(\Omega; \mathbb{R})^N = \mathcal{C}(\Omega; \mathbb{R}) \times \cdots \times \mathcal{C}(\Omega; \mathbb{R})$ (N times), it would be naive to state that $a(\frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}, \mathbf{v}(x))$ is the value taken by $a(y, z, \mathbf{v}(x))$ at point $(y = \frac{x}{\varepsilon}, z = \frac{x}{\varepsilon^2})$. All that will be clarified in Section 2.

Provided the differential operator $u \to \operatorname{div} a^\varepsilon (\cdot, \cdot, Du)$, $u \in W^{1,p}(\Omega; \mathbb{R})$, is well defined and has suitable properties (see Corollary 2.1), it is a classical matter to prove an existence and uniqueness result for (1.1) (see, e.g., [16]). Thus, we have a generalized sequence $(u_\varepsilon)_{\varepsilon>0}$ at our disposal, and the problem is to study, under a suitable condition on $a(y, z, \xi)$ (for fixed ξ) - called a *structure hypothesis*, the limiting behaviour of u_ε as $\varepsilon \to 0$. This lies within the class of so-called *reiterated homogenization problems*. Reiterated homogenization was introduced in [4] for linear operators. Multiscale convergence was first applied to reiterated homogenization in [8]. The reiterated homogenization of nonlinear elliptic operators was first studied in [14, 15], and latter in [16], in the usual periodic setting.

In this study we investigate the homogenization of (1.1) not under the periodicity hypothesis as in the previous references, but in a general deterministic setting including the periodicity, almost periodicity, convergence at infinity hypotheses, and others. Our approach proceeds from an appropriate notion of convergence termed *reiterated Σ-convergence*. A general deterministic homogenization theorem for (1.1) is established, and several examples considered in various concrete settings are presented by way of illustration. Reiterated Σ-convergence is likely to carry over to other settings. In particular, by a suitable adaptation of the approach carried out in [2, 3], it is possible to frame, using reiterated Σ-convergence, a reiterated homogenization theory of integral functionals in a general deterministic setting similar to that which is introduced in the present study.

The study is organized as follows. Section 2 deals with preliminary notions and results about the traces $a(\frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}, \mathbf{v}(x))$ $(x \in \Omega)$ and reiterated Σ-convergence. In Section 3 we study the abstract deterministic homogenization problem for (1.1). The periodicity hypothesis stated in [14, 15, 16] is here replaced by an abstract assumption covering a great variety of concrete structure hypotheses. Finally, Section 4 is concerned with a few concrete examples of homogenization problems for (1.1). More precisely, we consider the problem of investigating the limiting behaviour (as $\varepsilon \to 0$) of u_ε (the solution of (1.1)) under various concrete structure hypotheses ranging from the classical periodicity condition to more complicated (but realistic) structure hypotheses, and we show how each of them can reduce to the abstract hypothesis in Section 3.

Except where otherwise stated, vector spaces are considered over the complex field, $\mathbb{C}$, and scalar functions assume complex values. If X and F denote a locally compact space and a Banach space, respectively, then $\mathcal{C}(X; F)$ stands for the space of continuous functions of X into F, and $\mathcal{B}(X; F)$ stands for those functions in $\mathcal{C}(X; F)$ that are bounded. We equip $\mathcal{B}(X; F)$ with the supremum norm $\|u\|_\infty = \sup_{x \in X} \|u(x)\|$, where $\|\cdot\|$ denotes the norm in F. For shortness we write $\mathcal{C}(X) = \mathcal{C}(X; \mathbb{C})$

and $\mathcal{B}(X) = \mathcal{B}(X; \mathbb{C})$. Likewise the spaces $L^p(X; F)$ and $L^p_{loc}(X; F)$ (X provided with a positive Radon measure) are denoted by $L^p(X)$ and $L^p_{loc}(X)$, respectively, when $F = \mathbb{C}$ (we refer to [6, 7, 9] for integration theory). Finally, it is always assumed that the numerical space $\mathbb{R}^d$ (d a positive integer) is equipped with Lebesgue measure $dx = dx_1 \cdots dx_d$.

2.   Preliminaries

2.1.   Traces. Let Ω be a bounded open set in $\mathbb{R}^N_x$. Let $\varepsilon > 0$. For $u \in L^1_{loc}(\Omega \times \mathbb{R}^N_y \times \mathbb{R}^N_z) = L^1_{loc}(\Omega; L^1_{loc}(\mathbb{R}^N_y \times \mathbb{R}^N_z))$, we set

$$(2.1) \qquad u^\varepsilon(x) = u\left(x, \frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}\right) \quad (x \in \Omega)$$

whenever the right-hand side has meaning. This is obviously the case if u is continuous on $\Omega \times \mathbb{R}^N_y \times \mathbb{R}^N_z$, since the right of (2.1) is then none other than the value of $u(x, y, z)$ at $(y = \frac{x}{\varepsilon}, z = \frac{x}{\varepsilon^2})$, x being given in Ω. If u lies in $L^p(\Omega; B)$ ($1 \leq p \leq \infty$), where B is a closed vector subspace of $\mathcal{B}(\mathbb{R}^N_y \times \mathbb{R}^N_z)$, there is no serious difficulty in verifying that the right-hand side of (2.1) still has meaning (though in a generalized sense), which determines a function $u^\varepsilon \in L^p(\Omega)$ with $\|u^\varepsilon\|_{L^p(\Omega)} \leq \|u\|_{L^p(\Omega;B)}$. Our next purpose is to define u^ε for $u \in \mathcal{C}(\overline{\Omega}; L^\infty(\mathbb{R}^N_y; \mathcal{B}(\mathbb{R}^N_z)))$. First of all, for $\psi \in L^\infty(\mathbb{R}^N_y; \mathcal{B}(\mathbb{R}^N_z))$, put $^\varepsilon\psi(y) = \psi\left(y, \frac{y}{\varepsilon}\right)$ $(y \in \mathbb{R}^N)$, which gives a function $^\varepsilon\psi \in L^\infty(\mathbb{R}^N_y)$. Next, define $\psi^\varepsilon(x) = {}^\varepsilon\psi(\frac{x}{\varepsilon})$ for $x \in \mathbb{R}^N$, or more explicitly

$$(2.2) \qquad \psi^\varepsilon(x) = \psi\left(\frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}\right) \quad (x \in \mathbb{R}^N).$$

Clearly $\psi^\varepsilon \in L^\infty(\mathbb{R}^N)$ with $\|\psi^\varepsilon\|_{L^\infty(\mathbb{R}^N)} \leq \|\psi\|_{L^\infty(\mathbb{R}^N_y; \mathcal{B}(\mathbb{R}^N_z))}$. This being so, let $u \in \mathcal{C}(\overline{\Omega}) \otimes L^\infty(\mathbb{R}^N_y; \mathcal{B}(\mathbb{R}^N_z))$, i.e.,

$$(2.3) \qquad u = \sum_{\text{finite}} \varphi_i \otimes \psi_i, \ \varphi_i \in \mathcal{C}(\overline{\Omega}), \ \psi_i \in L^\infty(\mathbb{R}^N_y; \mathcal{B}(\mathbb{R}^N_z)).$$

Put

$$(2.4) \qquad u^\varepsilon(x) = \sum_{\text{finite}} \varphi_i(x)\psi_i^\varepsilon(x) \ (x \in \Omega)$$

where $\psi_i^\varepsilon(x)$ is defined as in (2.2). We have $u^\varepsilon \in L^\infty(\Omega)$ with $\|u^\varepsilon\|_{L^\infty(\Omega)} \leq \sup_{x \in \overline{\Omega}} \|u(x)\|_{L^\infty(\mathbb{R}^N_y; \mathcal{B}(\mathbb{R}^N_z))}$ (proceed as in [17, Proposition 1.5]). Combining this with the fact that $\mathcal{C}(\overline{\Omega}) \otimes L^\infty(\mathbb{R}^N_y; \mathcal{B}(\mathbb{R}^N_z))$ is dense in $\mathcal{C}(\overline{\Omega}; L^\infty(\mathbb{R}^N_y; \mathcal{B}(\mathbb{R}^N_z)))$, we get immediately the following

Proposition 2.1. *The operator $u \to u^\varepsilon$ (u^ε given by (2.3)-(2.4)) of $\mathcal{C}(\overline{\Omega}) \otimes L^\infty(\mathbb{R}_y^N; \mathcal{B}(\mathbb{R}_z^N))$ into $L^\infty(\Omega)$ extends by continuity to a continuous linear mapping, still denoted by $u \to u^\varepsilon$, of $\mathcal{C}(\overline{\Omega}; L^\infty(\mathbb{R}_y^N; \mathcal{B}(\mathbb{R}_z^N)))$ into $L^\infty(\Omega)$ such that $\|u^\varepsilon\|_{L^\infty(\Omega)} \leq \sup_{x \in \overline{\Omega}} \|u(x)\|_{L^\infty(\mathbb{R}_y^N; \mathcal{B}(\mathbb{R}_z^N))}$ for all $u \in \mathcal{C}(\overline{\Omega}; L^\infty(\mathbb{R}_y^N; \mathcal{B}(\mathbb{R}_z^N)))$. Furthermore, if u verifies $u(x,y,z) \geq 0$ for all $(x,z) \in \overline{\Omega} \times \mathbb{R}^N$ and for almost all $y \in \mathbb{R}^N$, then $u^\varepsilon(x) \geq 0$ for almost all $x \in \Omega$.*

Thus, for $u \in \mathcal{C}(\overline{\Omega}; L^\infty(\mathbb{R}_y^N; \mathcal{B}(\mathbb{R}_z^N)))$, the function u^ε is defined in the sense of Proposition 2.1 and hence we are justified in still making use of the notation in (2.1).

Let us now try to give a meaning to the notation $a(\frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}, Du(x))$ ($x \in \Omega$) with $u \in W^{1,p}(\Omega, \mathbb{R})$. We will need the following lemma, which is of interest in itself.

Lemma 2.1. *Let A_z be a closed subalgebra of $\mathcal{B}(\mathbb{R}_z^N)$ and f be a function of $\mathbb{R}_y^N \times \mathbb{R}_z^N$ into $\mathbb{R}$ with the following properties :*

(i) *Provided with the supremum norm, A_z is separable*

(ii) *A_z contains the constants*

(iii) *A_z is stable under complex conjugation*

(iv) *For each fixed $z \in \mathbb{R}^N$, the function $y \to f(y,z)$ is measurable from $\mathbb{R}^N$ to $\mathbb{R}$*

(v) *For almost every $y \in \mathbb{R}^N$, the function $z \to f(y,z)$ of $\mathbb{R}^N$ into $\mathbb{R}$, denoted below by $f(y, \cdot)$, lies in A_z.*

Then, the function $y \to f(y, \cdot)$ is measurable from $\mathbb{R}^N$ into $\mathcal{B}(\mathbb{R}_z^N)$.

Proof. It is clear that A_z is a commutative $\mathcal{C}^*$-algebra with identity. Its spectrum, $\Delta(A_z)$, is a metrizable compact space admitting $\{\delta_z\}_{z \in \mathbb{R}^N}$ (δ_z the Dirac measure on $\mathbb{R}^N$ at z) as a dense subset. Furthermore, the Gelfand transformation on A_z, denoted below by $\mathcal{G}$, is an isometric isomorphism of the $\mathcal{C}^*$-algebra A_z onto the $\mathcal{C}^*$-algebra $\mathcal{C}(\Delta(A_z))$ (see, e.g., [13] for further details). Having made these preliminaries, let us fix an arbitrary point $s \in \Delta(A_z)$. Let $(z_n)_{n \in \mathbb{N}}$ ($\mathbb{N}$ denotes the nonnegative integers) be a sequence in $\mathbb{R}^N$ such that $\delta_{z_n} \to s$ in $\Delta(A_z)$ as $n \to \infty$. Then, as $n \to \infty$, we have $\langle \delta_{z_n}, f(y, \cdot) \rangle \to \langle s, f(y, \cdot) \rangle = \mathcal{G}(f(y, \cdot))(s)$ for almost all $y \in \mathbb{R}^N$, where the brackets denote the duality pairing between A_z' (topological dual of A_z) and A_z. It follows that the function $y \to \mathcal{G}(f(y, \cdot))(s)$ is measurable from $\mathbb{R}^N$ to $\mathbb{R}$, since the same is true of each of the functions $y \to \langle \delta_{z_n}, f(y, \cdot) \rangle = f(y, z_n)$ (n ranging over $\mathbb{N}$), according to property (iv). In other words, if δ_s denotes the Dirac measure on $\Delta(A_z)$ at s, and g denotes the function $y \to \mathcal{G}(f(y, \cdot))$ from $\mathbb{R}^N$ to $\mathcal{C}(\Delta(A_z))$, then the

function $y \to \langle \delta_s, g(y) \rangle$ is measurable from $\mathbb{R}^N$ into $\mathbb{R}$ and that for any arbitrary $s \in \Delta(A_z)$. Therefore, given any arbitrary Radon measure ρ on $\Delta(A_z)$ with finite support, i.e., ρ of the form

$$\rho = \sum_{\text{finite}} c_i \delta_{s_i} \ (c_i \in \mathbb{C} \text{ and } s_i \in \Delta(A_z)),$$

the function $y \to \langle \rho, g(y) \rangle$ is measurable from $\mathbb{R}^N$ into $\mathbb{C}$, the brackets denoting this time the duality pairing between $\mathcal{M}(\Delta(A_z)) = \mathcal{C}'(\Delta(A_z))$ (space of complex Radon measures on $\Delta(A_z)$) and $\mathcal{C}(\Delta(A_z))$. With this in mind, fix freely a Radon measure η on $\Delta(A_z)$. Note that η is bounded, since $\Delta(A_z)$ is compact. Thus, we may assume without loss of generality that η lies in the closed unit ball $B' \subset \mathcal{M}(\Delta(A_z))$. But B' with the relative weak $*$ topology on $\mathcal{M}(\Delta(A_z))$ is a metrizabble compact space (the compacity is classical, the metrizability follows by property (i)). Hence, recalling a classical result (see, e.g., [6, Chap.III, p.71, Corol.1]), we may consider a sequence

$$(\rho_n)_{n \in \mathbb{N}}, \ \rho_n \in B', \ \rho_n \text{ with finite support}$$

such that $\rho_n \to \eta$ in $\mathcal{M}(\Delta(A_z))$ (with the weak $*$ topology) as $n \to \infty$. Consequently, a.e. in $y \in \mathbb{R}^N$, $\langle \rho_n, g(y) \rangle \to \langle \eta, g(y) \rangle$ as $n \to \infty$. We deduce by a classical argument that the function $y \to \langle \eta, g(y) \rangle$ is measurable from $\mathbb{R}^N$ into $\mathbb{C}$. Therefore, in view of the arbitrariness of η, the lemma follows by [6, Chap.IV; p.182, Corol.2, and p.174, Thm 1]. $\qquad\square$

At the present time, we assume that
(2.5)
$$a_i(y, \cdot, \xi) \in A_z \ (1 \leq i \leq N) \text{ for all } \xi \in \mathbb{R}^N \text{ and for almost all } y \in \mathbb{R}^N$$

where $a_i(y, \cdot, \xi)$ stands for the function $z \to a_i(y, z, \xi)$ of $\mathbb{R}^N$ into $\mathbb{R}$, and A_z is a closed subalgebra of $\mathcal{B}(\mathbb{R}_z^N)$ with the properties (i)-(iii) of Lemma 2.1. We will see that condition (2.5) is fulfilled in practice (see the forthcoming remark).

Remark 2.1. By combining [part (ii) of] (1.2) with (1.3) and [part (iii) of] (1.4), we have immediately $a_i(y, \cdot, \xi) \in \mathcal{B}(\mathbb{R}_z^N)$ $(1 \leq i \leq N)$ for all $\xi \in \mathbb{R}^N$ and for almost all $y \in \mathbb{R}^N$. Unfortunately Lemma 2.1 does not apply because the space $\mathcal{B}(\mathbb{R}_z^N)$ is not separable. We will see that a condition such as (2.5) is nevertheless fulfilled in practice as a consequence of the concrete structure hypothesis on $a_i(y, z, \xi)$ (for fixed ξ); see Section 4.

Now, recalling (1.2)-(1.4), we see that Lemma 2.1 applies with $f(y, z) = a_i(y, z, \xi)$, where i and ξ are freely fixed. Hence

$$a_i(\cdot, \cdot, \xi) \in L^\infty(\mathbb{R}_y^N; \mathcal{B}(\mathbb{R}_z^N)) \ (1 \leq i \leq N)$$

for any fixed $\xi \in \mathbb{R}^N$, where $a_i(\cdot, \cdot, \xi)$ denotes the function $(y, z) \to a_i(y, z, \xi)$ of $\mathbb{R}^N \times \mathbb{R}^N$ into $\mathbb{R}$. With this in mind, let $\mathbf{v} \in \mathcal{C}_\mathbb{R}(\overline{\Omega})^N = \mathcal{C}_\mathbb{R}(\overline{\Omega}) \times \cdots \times \mathcal{C}_\mathbb{R}(\overline{\Omega})$ (N times), where $\mathcal{C}_\mathbb{R}(\overline{\Omega}) = \mathcal{C}(\overline{\Omega}; \mathbb{R})$. The function $(x, y, z) \to a_i(y, z, \mathbf{v}(x))$ of $\overline{\Omega} \times \mathbb{R}_y^N \times \mathbb{R}_z^N$ into $\mathbb{R}$ lies in $\mathcal{C}(\overline{\Omega}; L^\infty(\mathbb{R}_y^N; \mathcal{B}(\mathbb{R}_z^N)))$ (the verification is an easy matter). Hence, according to Proposition 2.1, we can define the function $x \to a_i(\frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}, \mathbf{v}(x))$ from Ω into $\mathbb{R}$, which belongs to $L^\infty_\mathbb{R}(\Omega) = L^\infty(\Omega; \mathbb{R})$ and is denoted below by $a_i^\varepsilon(\cdot, \cdot, \mathbf{v})$.

Proposition 2.2. *Let $1 < p < \infty$. Suppose (2.5) holds. For each fixed integer $1 \leq i \leq N$, the transformation $\mathbf{v} \to a_i^\varepsilon(\cdot, \cdot, \mathbf{v})$ of $\mathcal{C}_\mathbb{R}(\overline{\Omega})^N$ into $L^\infty_\mathbb{R}(\Omega)$ extends by continuity to a continuous mapping, still denoted by $\mathbf{v} \to a_i^\varepsilon(\cdot, \cdot, \mathbf{v})$, of $L^p_\mathbb{R}(\Omega)^N$ into $L^{p'}_\mathbb{R}(\Omega)$ ($p' = \frac{p}{p-1}$). Furthermore, on letting*

$$a^\varepsilon(\cdot, \cdot, \mathbf{v}) = (a_i^\varepsilon(\cdot, \cdot, \mathbf{v}))_{1 \leq i \leq N} \text{ for } \mathbf{v} \in L^p_\mathbb{R}(\Omega)^N,$$

we have

$$\|a^\varepsilon(\cdot, \cdot, \mathbf{v}) - a^\varepsilon(\cdot, \cdot, \mathbf{w})\|_{L^{p'}(\Omega)^N} \leq$$
$$c_1 \|1 + |\mathbf{v}| + |\mathbf{w}|\|_{L^p(\Omega)}^{p-1-\alpha_1} \|\mathbf{v} - \mathbf{w}\|_{L^p(\Omega)^N}^{\alpha_1}$$

and

$$(a^\varepsilon(\cdot, \cdot, \mathbf{v}) - a^\varepsilon(\cdot, \cdot, \mathbf{w})) \cdot (\mathbf{v} - \mathbf{w}) \geq$$
$$c_2(1 + |\mathbf{v}| + |\mathbf{w}|)^{p-\alpha_2} |\mathbf{v} - \mathbf{w}|^{\alpha_2} \ a.e. \ in \ \Omega$$

for all $\mathbf{v}, \mathbf{w} \in L^p_\mathbb{R}(\Omega)^N$.

Proof. This follows in view of the proof of [19, Proposition 2.1]. □

As a direct consequence of this, we have the following

Corollary 2.1. *Let the hypotheses of Proposition 2.2 be satisfied. For $u \in W^{1,p}(\Omega; \mathbb{R})$, let $a^\varepsilon(\cdot, \cdot, Du) = (a_i^\varepsilon(\cdot, \cdot, Du))_{1 \leq i \leq N}$ be defined as in Proposition 2.2, which gives a mapping $u \to a^\varepsilon(\cdot, \cdot, Du)$ of $W^{1,p}(\Omega; \mathbb{R})$ into $L^{p'}(\Omega)^N$. We have*

$$\|a^\varepsilon(\cdot, \cdot, Du) - a^\varepsilon(\cdot, \cdot, Dv)\|_{L^{p'}(\Omega)^N} \leq$$
$$c_1 \|1 + |Du| + |Dv|\|_{L^p(\Omega)}^{p-1-\alpha_1} \|Du - Dv\|_{L^p(\Omega)^N}^{\alpha_1}$$

and

$$(a^\varepsilon(\cdot, \cdot, Du) - a^\varepsilon(\cdot, \cdot, Dv)) \cdot (Du - Dv) \geq$$
$$c_2(1 + |Du| + |Dv|)^{p-\alpha_2} |Du - Dv|^{\alpha_2} \ a.e. \ in \ \Omega$$

for all $u, v \in W^{1,p}(\Omega; \mathbb{R})$.

Remark 2.2. It is sometimes convenient to denote the function $a_i^\varepsilon(\cdot, \cdot, Du)$ (with $u \in W^{1,p}(\Omega; \mathbb{R})$) by $x \to a_i^\varepsilon(\frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}\cdot, Du(x))$. However, though entirely justified above, this is merely a formal notation.

2.2. Reiterated Σ-convergence. Let us first state some fundamentals of homogenization structures beyond the classical two-scale setting.

Let d be a positive integer. Let $\mathcal{H} = (H_\varepsilon)_{\varepsilon>0}$ be an action of $\mathbb{R}_+^*$ (the multiplicative group of positive real numbers) on the numerical space $\mathbb{R}^d$, i.e., $\mathcal{H}$ is a family, indexed by $\mathbb{R}_+^*$, of permutations H_ε of $\mathbb{R}^d$ such that

(i) $H_\varepsilon \circ H_{\varepsilon'} = H_{\varepsilon\varepsilon'}$ for all $\varepsilon, \varepsilon' > 0$

(ii) $H_{\varepsilon=1} = Id_{\mathbb{R}^d}$,

where $\circ$ and $Id_{\mathbb{R}^d}$ denote usual composition and the identical mapping, respectively, of $\mathbb{R}^d$. We assume further that :

$(H)_1$ Each H_ε maps continuously $\mathbb{R}^d$ into itself.

$(H)_2$ $\lim_{\varepsilon\to 0} |H_\varepsilon(x)| = +\infty$ for any $x \in \mathbb{R}^d$ with $x \neq \omega$, where $|\cdot|$ and ω denote the Euclidean norm and the origin in $\mathbb{R}^d$, respectively.

$(H)_3$ The Lebesgue measure λ on $\mathbb{R}^d$ is quasi-invariant under $\mathcal{H}$, i.e., to each $\varepsilon > 0$ there is attached some $\gamma(\varepsilon) > 0$ such that $H_\varepsilon(\lambda) = \gamma(\varepsilon)\lambda$.

Remark 2.3. In view of $(H)_1$, the mapping H_ε is a homeomorphism of $\mathbb{R}^d$ onto itself and therefore the image measure $H_\varepsilon(\lambda)$ is well defined (see, e.g., [7]). We recall that $H_\varepsilon(\lambda)$ is the Radon measure on $\mathbb{R}^d$ given by $\langle H_\varepsilon(\lambda), \varphi \rangle = \int \varphi(H_\varepsilon(x))dx$ for $\varphi \in \mathcal{K}(\mathbb{R}^d)$ (space of compactly supported continuous complex functions on $\mathbb{R}^d$), or equivalently by $H_\varepsilon(\lambda)(B) = \lambda(H_\varepsilon^{-1}(B))$ for any bounded Borel set $B \subset \mathbb{R}^d$. In view of $(H)_3$, the transformation $u \to u \circ H_\varepsilon$ (usual composition) maps $L_{loc}^p(\mathbb{R}^d)$ (resp. $L^p(\mathbb{R}^d)$) into itself, $1 \leq p \leq \infty$.

We denote by $\Pi^\infty(\mathbb{R}^d, \mathcal{H})$, or simply Π^∞ when there is no danger of confusion, the space of those $u \in \mathcal{B}(\mathbb{R}^d)$ for which a complex number $M(u)$ exists such that $u \circ H_\varepsilon \to M(u)$ in $L^\infty(\mathbb{R}^d)$-weak $*$ as $\varepsilon \to 0$. Π^∞ is a closed vector subspace of $\mathcal{B}(\mathbb{R}^d)$, Π^∞ contains the constants, Π^∞ is stable under complex conjugation. Furthermore, the mapping $u \to M(u)$ of Π^∞ into $\mathbb{C}$ is a positive linear form on Π^∞ attaining the value 1 on the constant function 1. Such a linear form is necessarily continuous and of norm exactly one. We call M the *mean value* on $\mathbb{R}^d$ for $\mathcal{H}$.

We are now in a position to introduce the notion of a homogenization structure in the present general setting. We begin by setting an underlying

notion. By a *structural representation* on $\mathbb{R}^d$ for the action $\mathcal{H}$ is meant any set $\Gamma \subset \mathcal{B}(\mathbb{R}^d)$ with the properties :

(HS1) Γ is a group under multiplication

(HS2) Γ is countable

(HS3) Γ is stable under complex conjugation

(HS4) $\Gamma \subset \Pi^\infty$.

Next, in the collection of all structural representations on $\mathbb{R}^d$ for $\mathcal{H}$, we consider the equivalence relation $\sim$ defined as : $\Gamma \sim \Gamma'$ if and only if $CLS(\Gamma) = CLS(\Gamma')$, where $CLS(\Gamma)$ denotes the closed vector subspace of $\mathcal{B}(\mathbb{R}^d)$ spanned by Γ. By an *H-structure* on $\mathbb{R}^d$ for $\mathcal{H}$ (H stands for *homogenization*) we shall understand any equivalence class modulo $\sim$.

The notion of an H-structure is intimately connected with that of an H-algebra. Specifically, let Σ be an H-structure on $\mathbb{R}^d$ for $\mathcal{H}$. Let $A = CLS(\Gamma)$, where Γ is any equivalence class representative of Σ (such a Γ is termed a *representation* of Σ). The space A is a so-called H-algebra on $\mathbb{R}^d$ for $\mathcal{H}$, i.e., a closed subalgebra of $\mathcal{B}(\mathbb{R}^d)$ with the features :

(HA1) A with the supremum norm is separable

(HA2) A contains the constants

(HA3) A is stable under complex conjugation

(HA4) $A \subset \Pi^\infty$.

Furthermore, A depends only on Σ and not on the chosen representation Γ of Σ; so that we may set $A = \mathcal{J}(\Sigma)$ (the *image* of Σ), which yields a mapping $\Sigma \to \mathcal{J}(\Sigma)$ that carries the collection of all H-structures (for $\mathcal{H}$) bijectively over the collection of all H-algebras (for $\mathcal{H}$) (see [18, Theorem 3.1]).

It is an easy matter to see that the theory of H-structures developed earlier in the particular setting of [18] carries over to the present general setting. Thus, basic notions such as the partial derivatives on $\Delta(A)$ (A a given H-algebra on $\mathbb{R}^d$ for $\mathcal{H}$), the Sobolev spaces $W^{1,p}(\Delta(A))$, the Σ-convergence, etc., remain valid and hence are not worth repeating here. We refer the reader to [18, 19] for further details.

In the present work we are concerned with three specific actions of $\mathbb{R}_+^*$: the action $\mathcal{H} = (H_\varepsilon)_{\varepsilon>0}$ on $\mathbb{R}^N$ (integer $N \geq 1$) given by $H_\varepsilon(x) = \frac{x}{\varepsilon}$ ($x \in \mathbb{R}^N$), the action $\mathcal{H}' = (H_\varepsilon')_{\varepsilon>0}$ on $\mathbb{R}^N$ given by $H_\varepsilon'(x) = \frac{x}{\varepsilon^2}$ ($x \in \mathbb{R}^N$), and their product $\mathcal{H}^* = \mathcal{H} \times \mathcal{H}'$, which is precisely the action of $\mathbb{R}_+^*$ on $\mathbb{R}^{2N} = \mathbb{R}^N \times \mathbb{R}^N$ defined by $\mathcal{H}^* = (H_\varepsilon^*)_{\varepsilon>0}$, $H_\varepsilon^* = H_\varepsilon \times H_\varepsilon'$ (direct product), i.e.,

$$H_\varepsilon^*(x, x') = \left(\frac{x}{\varepsilon}, \frac{x'}{\varepsilon^2} \right) \text{ for } (x, x') \in \mathbb{R}^N \times \mathbb{R}^N.$$

Each of these three actions satisfies properties $(H)_1$-$(H)_3$, mutatis mutandis.

Now, let Σ_y be an H-structure of class $\mathcal{C}^\infty$ on $\mathbb{R}_y^N$ for $\mathcal{H}$, and Σ_z be an H-structure of class $\mathcal{C}^\infty$ on $\mathbb{R}_z^N$ for $\mathcal{H}'$. Their product $\Sigma = \Sigma_y \times \Sigma_z$ is defined exactly as in [18, Definition 3.4], and is an H-structure of class $\mathcal{C}^\infty$ on $\mathbb{R}^N \times \mathbb{R}^N$ for the product action $\mathcal{H}^*$. It is an elementary exercise to verify that Proposition 3.2, Theorem 3.2 and Corollaries 3.1-3.2 of [18] carry over mutatis mutandis to the present context. We will put $A_y = \mathcal{J}(\Sigma_y)$ (image of Σ_y), $A_z = \mathcal{J}(\Sigma_z)$ and $A = \mathcal{J}(\Sigma)$, and use the same letter, $\mathcal{G}$, to denote the Gelfand transformation on A_y, A_z and A, as well. Points in $\Delta(A_y)$ (resp. $\Delta(A_z)$) are denoted by s (resp. r). The compact space $\Delta(A_y)$ (resp. $\Delta(A_z)$) is equipped with the so-called M-measure, β_y (resp. β_z) for A_y (resp. A_z). It is fundamental to recall that $\Delta(A) = \Delta(A_y) \times \Delta(A_z)$ (cartesian product) and further the M-measure for A, with which $\Delta(A)$ is equipped, is precisely the product measure $\beta = \beta_y \otimes \beta_z$ (see [18]).

Before we can introduce the concept of reiterated Σ-convergence, we require one further notion.

Definition 2.1. The H-structure $\Sigma = \Sigma_y \times \Sigma_z$ is called a *reiteration* H-structure if for each $\psi \in A = \mathcal{J}(\Sigma)$ (image of Σ), we have $\psi^\varepsilon \to M(\psi)$ in $L^\infty(\mathbb{R}_x^N)$-weak $*$ as $\varepsilon \to 0$, where ψ^ε is defined (in an obvious manner) in (2.2), and M is the mean value on $\mathbb{R}^{2N} = \mathbb{R}^N \times \mathbb{R}^N$ for the product action $\mathcal{H}^*$.

Remark 2.4. According to (HA4), for each $\psi \in A = \mathcal{J}(\Sigma)$ we have $\psi \circ H_\varepsilon^* \to M(\psi)$ in $L^\infty(\mathbb{R}_x^N \times \mathbb{R}_{x'}^N)$-weak $*$, which is very different from the convergence property in the above definition.

We give below a few examples of reiteration H-structures.

Example 2.1. Let $\Gamma = \{\gamma_k : k \in \mathbb{Z}^N\}$ ($\mathbb{Z}$ denotes the integers), where for each $k \in \mathbb{R}^N$, we write γ_k for the usual exponential function on $\mathbb{R}^N$, i.e., $\gamma_k(y) = \exp(2i\pi k \cdot y)$ ($y \in \mathbb{R}^N$). The set Γ is a structural representation on $\mathbb{R}^N$ for $\mathcal{H}$ and $\mathcal{H}'$, as well. We define $\Sigma_{\mathbb{Z}^N}$ (resp. $\Sigma'_{\mathbb{Z}^N}$) to be the unique H-structure on $\mathbb{R}^N$ for $\mathcal{H}$ (resp. $\mathcal{H}'$) of which Γ is one representation. $\Sigma_{\mathbb{Z}^N}$ is referred to as the periodic H-structure on $\mathbb{R}^N$ represented by $\mathbb{R}^N$ (see [18, Example 3.2]). We have $\mathcal{J}(\Sigma_{\mathbb{Z}^N}) = \mathcal{J}(\Sigma'_{\mathbb{Z}^N}) = \mathcal{P}(\mathbb{R}^N)$, where $\mathcal{P}(\mathbb{R}^N)$ is the space of functions $u \in \mathcal{C}(\mathbb{R}^N)$ such that $u(y + k) = u(y)$ for all $y \in \mathbb{R}^N$ and all $k \in \mathbb{Z}^N$ (such functions are said to be $\mathbb{Z}^N$-periodic). Hence $\Sigma'_{\mathbb{Z}^N} = \Sigma_{\mathbb{Z}^N}$. Finally, we consider the product H-structure $\Sigma = \Sigma_{\mathbb{Z}^N} \times \Sigma_{\mathbb{Z}^N}$ on $\mathbb{R}_y^N \times \mathbb{R}_z^N$ for $\mathcal{H}^*$. It can be proved that Σ is a reiteration H-structure (the verification is left to the reader).

Example 2.2. Let $\Gamma_y = \{\gamma_k : k \in \mathcal{R}_y\}$ and $\Gamma_z = \{\gamma_k : k \in \mathcal{R}_z\}$, where $\mathcal{R}_y$ and $\mathcal{R}_z$ are countable subgroups of $\mathbb{R}^N$. The set Γ_y (resp. Γ_z) is a structural representation on $\mathbb{R}^N$ for $\mathcal{H}$ (resp. $\mathcal{H}'$). We define $\Sigma_{\mathcal{R}_y}$ (resp. $\Sigma_{\mathcal{R}_z}$) to be the unique H-structure on $\mathbb{R}^N$ for $\mathcal{H}$ (resp. $\mathcal{H}'$) of which Γ_y (resp. Γ_z) is one representation. $\Sigma_{\mathcal{R}_y}$ (resp. $\Sigma_{\mathcal{R}_z}$) is referred to as the almost periodic H-structure on $\mathbb{R}^N$ represented by $\mathcal{R}_y$ (resp. $\mathcal{R}_z$), see [18, Example 3.3]. According to [18, Example 3.6], the product $\Sigma = \Sigma_{\mathcal{R}_y} \times \Sigma_{\mathcal{R}_z}$, which is an H-structure on $\mathbb{R}^N \times \mathbb{R}^N$ for $\mathcal{H}^*$, is precisely the almost periodic H-structure on $\mathbb{R}^{2N} = \mathbb{R}^N \times \mathbb{R}^N$ represented by the countable subgroup $\mathcal{R} = \mathcal{R}_y \times \mathcal{R}_z$ of $\mathbb{R}^{2N}$. Thus, $\Sigma = \Sigma_{\mathcal{R}}$, and further there is no serious difficulty in verifying that Σ is a reiteration H-stucture.

Example 2.3. Let Σ_∞ be the so-called H-structure of the convergence at infinity on $\mathbb{R}^N$ [18, Example 3.4]. This is an H-structure on $\mathbb{R}^N$ for $\mathcal{H}$ and $\mathcal{H}'$, as well. The product $\Sigma = \Sigma_\infty \times \Sigma_\infty$ is an H-structure on $\mathbb{R}^N \times \mathbb{R}^N$ for the product action $\mathcal{H}^*$. In view of [18, Proposition 3.2], we have $\mathcal{J}(\Sigma) = \mathcal{B}_\infty(\mathbb{R}_y^N; \mathcal{B}_\infty(\mathbb{R}_z^N))$, from which one can easily check that Σ is a reiteration H-structure.

Example 2.4. Let $\Sigma = \Sigma_y \times \Sigma_\infty$, where $\Sigma_z = \Sigma_\infty$ is as in Example 2.3 and Σ_y is any H-structure of class $\mathcal{C}^\infty$ on $\mathbb{R}^N$ for $\mathcal{H}$. Σ is an H-structure on $\mathbb{R}^N \times \mathbb{R}^N$ for $\mathcal{H}^*$. Furthermore, $\mathcal{J}(\Sigma) = \mathcal{B}_\infty(\mathbb{R}_z^N; A_y)$ with $A_y = \mathcal{J}(\Sigma_y)$ (proceed as in [18, Example 3.7]), from which one can easily deduce that Σ is a reiteration H-structure.

Example 2.5. Let $\Sigma = \Sigma_{\mathcal{R}_y} \times \Sigma_{\infty, \mathcal{R}_z}$ where $\mathcal{R}_y$, $\mathcal{R}_z$ and $\Sigma_{\mathcal{R}_y}$ are as in Example 2.2, and $\Sigma_{\infty, \mathcal{R}_z}$ is the H-structure on $\mathbb{R}^N$ for $\mathcal{H}'$ defined in [18, Example 3.5]. This is an H-structure on $\mathbb{R}^N \times \mathbb{R}^N$ for $\mathcal{H}^*$. It can be shown that Σ is a reiteration H-structure.

Example 2.6. Let $\Sigma = \Sigma_{\infty, \mathcal{R}_y} \times \Sigma_{\infty, \mathcal{R}_z}$ where $\mathcal{R}_y$ and $\mathcal{R}_z$ are as above. This is clearly an H-structure on $\mathbb{R}^N \times \mathbb{R}^N$ for $\mathcal{H}^*$, and there is no serious difficulty in verifying that Σ is a reiteration H-stucture.

Returning now to the preceding general framework, we assume from now that the H-structure $\Sigma = \Sigma_y \times \Sigma_z$ is a reiteration H-structure. The letter Ω throughout will denote a bounded open set in $\mathbb{R}_x^N$. Here is a fundamental result.

Proposition 2.3. *As $\varepsilon \to 0$, we have $u^\varepsilon \to \widetilde{u}$ in $L^\infty(\Omega)$-weak $*$ for $u \in \mathcal{C}(\overline{\Omega}; A)$ and $u^\varepsilon \to \widetilde{u}$ in $L^p(\Omega)$-weak for $u \in L^p(\Omega; A)$ $(1 \leq p < \infty)$, where u^ε is defined in (2.1) and $\widetilde{u}$ denotes the complex function on Ω given by $\widetilde{u}(x) = M(u(x))$ $(x \in \Omega)$, M as in Definition 2.1.*

Proof. Starting from the convergence property in Definition 2.1, we see immediately that the proposition follows by the density of $\mathcal{C}(\overline{\Omega}) \otimes A$ in $\mathcal{C}(\overline{\Omega}; A)$ and by that of $\mathcal{C}(\overline{\Omega}; A)$ in $L^p(\Omega; A)$. □

We are now ready to introduce the concepts of reiterated weak and strong Σ-convergence. The letter E throughout will denote a family of positive real numbers admitting 0 as an accumulation point. For example $E = \mathbb{R}_+^*$. Attention is drawn to the especial case where $E = (\varepsilon_n)$ (integers $n \geq 0$) with $\varepsilon_n > 0$ and $\varepsilon_n \to 0$ as $n \to \infty$; E is then referred to as a *fundamental sequence*.

Definition 2.2. A sequence $(u_\varepsilon)_{\varepsilon \in E} \subset L^p(\Omega)$ $(1 \leq p < \infty)$ is said to :
(i) *weakly Σ-converge reiteratively in* $L^p(\Omega)$ to some $u_0 \in L^p(\Omega \times \Delta(A))$ if as $E \ni \varepsilon \to 0$, we have

$$(2.6) \qquad \int_\Omega u_\varepsilon v^\varepsilon dx \to \int\int_{\Omega \times \Delta(A)} u_0 \widehat{v} dx d\beta$$

for every $v \in L^{p'}(\Omega; A)$ $(\frac{1}{p'} = 1 - \frac{1}{p})$, where v^ε is defined as in (2.1), and $\widehat{v} = \mathcal{G} \circ v$ (i.e., $\widehat{v}$ denotes the function in $L^{p'}(\Omega; \mathcal{C}(\Delta(A)))$ given by $\widehat{v}(x) = \mathcal{G}(v(x))$, $x \in \Omega$);
(ii) *strongly Σ-converge reiteratively in* $L^p(\Omega)$ to some $u_0 \in L^p(\Omega \times \Delta(A))$ if the following condition is fulfilled :

(SC) Given $\eta > 0$ and $v \in L^p(\Omega; A)$ with $\|u_0 - \widehat{v}\|_{L^p(\Omega \times \Delta(A))} \leq \frac{\eta}{2}$, there is some $\alpha > 0$ such that $\|u_\varepsilon - v^\varepsilon\|_{L^p(\Omega)} \leq \eta$ provided $E \ni \varepsilon \leq \alpha$.

We express this by writing $u_\varepsilon \to u_0$ *reiteratively in* $L^p(\Omega)$*-weak* Σ in case (i), and $u_\varepsilon \to u_0$ *reiteratively in* $L^p(\Omega)$*-strong* Σ in case (ii).
There is no difficulty in verifying the following results.

(1) Suppose $u_0 = \widehat{v}_0$ with $v_0 \in L^p(\Omega; A)$. Then $u_\varepsilon \to u_0$ reiteratively in $L^p(\Omega)$-strong Σ if and only if $\|u_\varepsilon - v_0^\varepsilon\|_{L^p(\Omega)} \to 0$ as $E \ni \varepsilon \to 0$.

(2) For $u \in L^p(\Omega; A)$ we have $u^\varepsilon \to \widehat{u}$ reiteratively in $L^p(\Omega)$-strong Σ.

(3) If $u_\varepsilon \to u$ in $L^p(\Omega)$ (strong) as $E \ni \varepsilon \to 0$, then $u_\varepsilon \to u$ reiteratively in $L^p(\Omega)$-strong Σ.

Also, the proof of the next proposition is a simple exercise left to the reader.

Proposition 2.4. *Suppose a sequence* $(u_\varepsilon)_{\varepsilon \in E} \subset L^p(\Omega)$ $(1 \leq p < \infty)$ *weakly Σ-converges reiteratively in* $L^p(\Omega)$ *to some* $u_0 \in L^p(\Omega \times \Delta(A))$. *Define* $u_0^* \in L^p(\Omega \times \Delta(A_y))$ *as* $u_0^*(x, s) = \int_{\Delta(A_z)} u_0(x, s, r) d\beta_z(r)$ $(x \in \Omega,$

$s \in \Delta(A_y)$), and $\widetilde{u} \in L^p(\Omega)$ as $\widetilde{u}(x) = \int_{\Delta(A_z)} \int_{\Delta(A_y)} u_0(x,s,r) d\beta_y(s) d\beta_z(r)$ ($x \in \Omega$). Then, as $E \ni \varepsilon \to 0$,

    (i) $u_\varepsilon \to u_0^*$ in $L^p(\Omega)$-weak Σ_y [18, Definition 4.1]

    (ii) $u_\varepsilon \to \widetilde{u}_0$ in $L^p(\Omega)$-weak.

The results of the Σ-convergence setting [18] carry over mutatis mutandis, together with their proofs, to the present setting. Let us state the most important of such results.

Proposition 2.5. *Assume that* $1 < p < \infty$. *Given a fundamental sequence* E *and a sequence* $(u_\varepsilon)_{\varepsilon \in E}$ *which is bounded in* $L^p(\Omega)$, *a subsequence* E' *can be extracted from* E *such that the sequence* $(u_\varepsilon)_{\varepsilon \in E'}$ *weakly* Σ-*converges reiteratively in* $L^p(\Omega)$.

Proposition 2.6. *Suppose a sequence* $(u_\varepsilon)_{\varepsilon \in E}$ *strongly* Σ-*converges reiteratively in* $L^p(\Omega)$ *to some* $u_0 \in L^p(\Omega \times \Delta(A))$. *Then, as* $E \ni \varepsilon \to 0$,

    (i) $u_\varepsilon \to u_0$ *reiteratively in* $L^p(\Omega)$-*weak* Σ

    (ii) $\|u_\varepsilon\|_{L^p(\Omega)} \to \|u_0\|_{L^p(\Omega \times \Delta(A))}$.

Reciprocally, if $p = 2$ *and if assertions (i)-(ii) hold, then* $u_\varepsilon \to u_0$ *reiteratively in* $L^p(\Omega)$-*strong* Σ.

Proposition 2.7. *Suppose the two real numbers* $p, q \geq 1$ *are such that* $\frac{1}{\sigma} = \frac{1}{p} + \frac{1}{q} \leq 1$. *Let* $u_0 \in L^p(\Omega \times \Delta(A))$ *and* $v_0 \in L^q(\Omega \times \Delta(A))$, *and let* $u_\varepsilon \in L^p(\Omega)$ *and* $v_\varepsilon \in L^q(\Omega)$ *for* $\varepsilon \in E$. *Finally, assume that* $u_\varepsilon \to u_0$ *reiteratively in* $L^p(\Omega)$-*strong* Σ *and* $v_\varepsilon \to v_0$ *reiteratively in* $L^q(\Omega)$-*weak* Σ. *Then* $u_\varepsilon v_\varepsilon \to u_0 v_0$ *reiteratively in* $L^\sigma(\Omega)$-*weak* Σ.

The notion of a $W^{1,p}(\Omega)$-proper reiteration H-structure will play a fundamental role in this study. We refer to, e.g., [1] for the classical Sobolev space $W^{1,p}(\Omega)$, to [19] (see also [18]) for special Sobolev spaces such as $W^{1,p}(\Delta(A_y))$ and $W_{\#}^{1,p}(\Delta(A_y))$ together with the various associated derivative operators.

Definition 2.3. The reiteration H-structure $\Sigma = \Sigma_y \times \Sigma_z$ is termed $W^{1,p}(\Omega)$-*proper* (p a given real number with $p \geq 1$) if the following three conditions are satisfied.

(PR)$_1$ Σ_y is total for p, i.e., $\mathcal{D}(\Delta(A_y))$ is dense in $W^{1,p}(\Delta(A_y))$

(PR)$_2$ Σ_z is total for p

(PR)$_3$ Given a fundamental sequence E and a sequence $(u_\varepsilon)_{\varepsilon \in E}$ which is bounded in $W^{1,p}(\Omega)$, there are a subsequence E' extracted from

E and three functions $u_0 \in W^{1,p}(\Omega)$, $u_1 \in L^p(\Omega; W_{\#}^{1,p}\Delta(A_y))$ and $u_2 \in L^p(\Omega; L^p(\Delta(A_y); W_{\#}^{1,p}\Delta(A_z)))$, such that, as $E' \ni \varepsilon \to 0$,

$$u_\varepsilon \to u_0 \text{ in } W^{1,p}(\Omega)-\text{weak}$$

$$\frac{\partial u_\varepsilon}{\partial x_j} \to \frac{\partial u_0}{\partial x_j} + \partial_j u_1 + \partial_j u_2 \text{ reit.in } L^p(\Omega)-\text{weak } \Sigma (1 \le j \le N),$$

where reit. stands for reiteratively.

Our next purpose is to present a few examples of $W^{1,p}(\Omega)$-proper reiteration H-structures.

Example 2.7. Our goal here is to show that the reiteration H-structure $\Sigma = \Sigma_{\mathbb{Z}^N} \times \Sigma_{\mathbb{Z}^N}$ of Example 2.1 is $W^{1,p}(\Omega)$-proper for any arbitrary real $p > 1$. Let us first recall that when dealing with periodic H-structures, it is possible to do without the Gelfand representation theory (see, e.g., [19, Example 3.1]). So, let $Y = (0,1)^N$ (the open unit cube in $\mathbb{R}_y^N$) and $Z = (0,1)^N$ (the open unit cube in $\mathbb{R}_z^N$). We denote by $\mathcal{C}_{per}(Y \times Z)$ the space of functions $\psi \in \mathcal{C}(\mathbb{R}_y^N \times \mathbb{R}_z^N)$ that are $Y \times Z$-periodic, i.e., that verify $\psi(y+k, z+\ell) = \psi(y,z)$ for $(y,z) \in \mathbb{R}^N \times \mathbb{R}^N$ and $(k,\ell) \in \mathbb{Z}^N \times \mathbb{Z}^N$. Note that $\mathcal{C}_{per}(Y \times Z) = \mathcal{J}(\Sigma)$ (the image of $\Sigma = \Sigma_{\mathbb{Z}^N} \times \Sigma_{\mathbb{Z}^N}$). On the other hand, we need the space $L_{per}^p(Y \times Z)$ of $Y \times Z$-periodic functions in $L_{loc}^p(\mathbb{R}_y^N \times \mathbb{R}_z^N)$, and the space $\mathcal{V}_{\text{div}}(Y \times Z)$ of those $\mathbf{u} \in \mathcal{C}_{per}^\infty(Y \times Z)^N$ such that $\text{div}_y \mathbf{u} = 0$ and $\text{div}_z \mathbf{u} = 0$, where $\mathcal{C}_{per}^\infty(Y \times Z) = \mathcal{C}_{per}(Y \times Z) \cap \mathcal{C}^\infty(\mathbb{R}_y^N \times \mathbb{R}_z^N)$. As a preliminary step, we have the following

Lemma 2.2. *Assume that $1 < p < \infty$. Let $\mathbf{f} = (f_j) \in L_{per}^p(Y \times Z)^N$. Suppose we have*

$$\sum_{j=1}^N \int \int_{Y \times Z} f_j v_j dy dz = 0$$

for all $\mathbf{v} = (v_j) \in \mathcal{V}_{\text{div}}(Y \times Z)$. Then, there exists a unique couple of functions $u_1 \in W_{\#}^{1,p}(Y) = \{v \in W_{loc}^{1,p}(\mathbb{R}_y^N) : v \ Y\text{-periodic}, \int_Y v(y) dy = 0\}$, $u_2 \in L^p(Y; W_{\#}^{1,p}(Z))$, such that $\mathbf{f} = D_y u_1 + D_z u_2$.

Proof. This is a simple adaptation of the proof of [19, Lemma 3.4].   □

We are now able to state the desired result.

Proposition 2.8. *The reiteration H-structure $\Sigma = \Sigma_{\mathbb{Z}^N} \times \Sigma_{\mathbb{Z}^N}$ is $W^{1,p}(\Omega)$-proper for each real $p > 1$.*

Proof. Properties $(PR)_1$-$(PR)_2$ of Definition 2.3 follow by [19, Proposition 3.2], and $(PR)_3$ follows by a classical approach using Lemma 2.2 (see, e.g., [16, Theorem 13]). $\qquad\square$

Example 2.8. The aim here is to verify that the reiteration H-structure $\Sigma = \Sigma_{\mathcal{R}_y} \times \Sigma_{\mathcal{R}_z}$ of Example 2.2 is $W^{1,2}(\Omega)$-proper. The first step will be to frame a preliminary lemma analogous to Lemma 2.2. For $\mathbf{u} = (u_i) \in \mathcal{D}(\Delta(A))^N = \mathcal{G}(A^\infty)^N$, where $\mathcal{G}$ is the Gelfand transformation on A (we recall that $\mathcal{G}$ maps A^∞ isomorphically over $\mathcal{D}(\Delta(A))$), let $\widehat{\mathrm{div}}_y \mathbf{u} = \mathcal{G}(\mathrm{div}_y \mathcal{G}^{-1}(\mathbf{u}))$, and $\widehat{\mathrm{div}}_z \mathbf{u} = \mathcal{G}(\mathrm{div}_z \mathcal{G}^{-1}(\mathbf{u}))$, where $\mathcal{G}^{-1}(\mathbf{u}) = (\mathcal{G}^{-1} u_i)_{1 \le i \le N}$.

Lemma 2.3. *Let* $\mathbf{f} = (f_j) \in L^2(\Delta(A))^N$. *Suppose*

$$\sum_{j=1}^N \int_{\Delta(A)} f_j v_j d\beta = 0$$

for all $\mathbf{v} = (v_j) \in \mathcal{D}(\Delta(A))^N$ *with* $\widehat{\mathrm{div}}_y \mathbf{v} = 0$ *and* $\widehat{\mathrm{div}}_z \mathbf{v} = 0$. *There exists a couple of functions* $u_1 \in W^{1,2}_\#(\Delta(A_y))$, $u_2 \in L^2(\Delta(A_y); W^{1,2}_\#(\Delta(A_z)))$ *such that* $\mathbf{f} = \partial_s u_1 + \partial_r u_2$, *where* $\partial_s u_1 = (\partial_i u_1)_{1 \le i \le N}$ *and* $\partial_r u_2 = (\partial_i u_2)_{1 \le i \le N}$.

Proof. This follows in view of the proof of [17, Lemma 4.2]. $\qquad\square$

This leads to the next proposition.

Proposition 2.9. *The reiteration* H-*structure* $\Sigma = \Sigma_{\mathcal{R}_y} \times \Sigma_{\mathcal{R}_z}$ *is* $W^{1,2}(\Omega)$-*proper.*

Proof. Properties $(PR)_1$-$(PR)_2$ of Definition 2.3 follow by [17, Proposition 4.3] whereas $(PR)_3$ is deduced from Lemma 2.3 in a classical way (see, e.g., the proof of [17, Theorem 4.1]). $\qquad\square$

Further useful examples of $W^{1,p}(\Omega)$-proper reiteration H-structures can be deduced from Examples 2.7 and 2.8 by means of the following lemma.

Lemma 2.4. *Consider a further reiteration* H-*structure* $\Sigma_2 = \Sigma_{2y} \times \Sigma_{2z}$ *on* $\mathbb{R}^N_y \times \mathbb{R}^N_z$ *for* $\mathcal{H}^*$, *where* Σ_{2y} (*resp.* Σ_{2z}) *is a further* H-*structure of class* $\mathcal{C}^\infty$ *on* $\mathbb{R}^N_y$ (*resp.* $\mathbb{R}^N_z$) *for* $\mathcal{H}$ (*resp.* $\mathcal{H}'$). *Suppose each of the couples* (Σ_y, Σ_{2y}) *and* (Σ_z, Σ_{2z}) *satisfies a hypothesis similar to hypothesis* (H) *of* [19, Section 3]. *If the reiteration* H-*structure* Σ_2 *is* $W^{1,p}(\Omega)$-*proper for some real* $p \ge 1$, *then so also is* Σ.

Proof. Assuming that Σ_2 is $W^{1,p}(\Omega)$-proper for some given real $p \geq 1$, we see immediately by [19, Lemma 3.1] that conditions $(PR)_1$-$(PR)_2$ of Definition 2.3 are fulfilled. Thus, it remains to verify $(PR)_3$. But this follows by an adaptation of a well-known line of argument (see, e.g., [19, Theorem 3.1]). $\qquad\square$

We are now able to provide further examples of $W^{1,p}(\Omega)$-proper reiteration H-structures.

Example 2.9. The H-structure of Example 2.5 is $W^{1,p}(\Omega)$-proper for $p = 2$ in general, for any real $p > 1$ in the particular case where $\mathcal{R}_y = \mathbb{Z}^N$ and $\mathcal{R}_z = \mathbb{Z}^N$. Indeed, this follows by Lemma 2.4 with $\Sigma = \Sigma_{\mathcal{R}_y} \times \Sigma_{\infty,\mathcal{R}_z}$, $\Sigma_2 = \Sigma_{\mathcal{R}_y} \times \Sigma_{\mathcal{R}_z}$, and use of Propositions 2.8-2.9 and the fact that the couple $\{\Sigma_{\mathcal{R}_z}, \Sigma_{\infty,\mathcal{R}_z}\}$ satisfies hypothesis (H) (see [18, 19]).

Example 2.10. Let $\mathcal{R}_y$ be a countable subgroup of $\mathbb{R}_y^N$. The reiteration H-structure $\Sigma = \Sigma_{\mathcal{R}_y} \times \Sigma_\infty$ (Example 2.4) is $W^{1,2}(\Omega)$-proper. This follows by the same argument as used to prove [18, Corollary 4.4 and Example 4.4]. The details are left to the reader.

Example 2.11. The reiteration H-structure of Example 2.6 is $W^{1,p}(\Omega)$-proper for $p = 2$ in general, and for any $p > 1$ in the particular case when $\mathcal{R}_y = \mathbb{Z}^N$, $\mathcal{R}_z = \mathbb{Z}^N$. Indeed, this follows by applying Lemma 2.4 with $\Sigma = \Sigma_{\infty,\mathcal{R}_y} \times \Sigma_{\infty,\mathcal{R}_z}$, $\Sigma_2 = \Sigma_{\mathcal{R}_y} \times \Sigma_{\mathcal{R}_z}$, and proceeding as in Example 2.9.

Example 2.12. The H-structure of Example 2.3 is $W^{1,2}(\Omega)$-proper. Indeed, since $\Sigma_\infty = \Sigma_{\infty,\{\omega\}}$, this is nothing else but a particular case of Example 2.11.

3.   The abstract homogenization problem

3.1. Setting of the abstract problem and preliminaries. Before we can state the abstract homogenization problem for (1.1) we need further details. The basic notation is as in Section 2. Let $1 \leq p < \infty$. We denote by $\Xi^p(\mathbb{R}_y^N; \mathcal{B}(\mathbb{R}_z^N))$, or simply Ξ^p when there is no danger of confusion, the space of those functions $u \in L_{loc}^p(\mathbb{R}_y^N; \mathcal{B}(\mathbb{R}_z^N))$ for which

$$\sup_{0 < \varepsilon \leq 1} \int_U \left\| u\left(\frac{x}{\varepsilon}, \cdot\right) \right\|_\infty^p dx < \infty$$

for any bounded open set U in $\mathbb{R}_x^N$. This is a Banach space with norm

$$\|u\|_{\Xi^p} = \sup_{0 < \varepsilon \leq 1} \left(\int_{B_N} \left\| u\left(\frac{x}{\varepsilon}, \cdot\right) \right\|_\infty^p dx \right)^{\frac{1}{p}} \quad (u \in \Xi^p)$$

where B_N is the open unit ball in $\mathbb{R}^N_x$.

Remark 3.1. It should be noted that the way we defined ψ^ε (in (2.2)) for $\psi \in L^\infty(\mathbb{R}^N_y; \mathcal{B}(\mathbb{R}^N_z))$ remains rigorously valid when $L^\infty(\mathbb{R}^N_y; \mathcal{B}(\mathbb{R}^N_z))$ is replaced by $L^p_{loc}(\mathbb{R}^N_y; \mathcal{B}(\mathbb{R}^N_z))$, so that (2.2) extends to all $\psi \in L^p_{loc}(\mathbb{R}^N_y; \mathcal{B}(\mathbb{R}^N_z))$. Furthermore, for each bounded open set $U \subset \mathbb{R}^N_x$, we have

$$\sup_{0 < \varepsilon \leq 1} \left(\int_U \left| u\left(\frac{x}{\varepsilon}, \frac{x}{\varepsilon^2} \right) \right|^p dx \right)^{\frac{1}{p}} \leq n^{\frac{N}{p}} \|u\|_{\Xi^p} \quad (u \in \Xi^p),$$

where n is some positive integer such that $U \subset nB_N$. Thus, as ε ranges over $(0, 1)$, the mappings $u \to u^\varepsilon$ are uniformly equicontinuous from Ξ^p into $L^p_{loc}(\mathbb{R}^N_x)$.

Now, in the notation of Subsection 2.2, let $\Sigma = \Sigma_y \times \Sigma_z$ be a reiteration H-structure on $\mathbb{R}^N_y \times \mathbb{R}^N_z$ for $\mathcal{H}^*$. We define $\mathfrak{X}^p_\Sigma(\mathbb{R}^N_y; \mathcal{B}(\mathbb{R}^N_z))$ (or $\mathfrak{X}^p_\Sigma$, or simply $\mathfrak{X}^p$ when there is no danger of confusion) to be the closure of $A = \mathcal{J}(\Sigma)$ in Ξ^p. We provide $\mathfrak{X}^p_\Sigma$ with the Ξ^p-norm, which makes it a Banach space. Propositions 2.3-2.4 and Corollaries 2.1-2.2 of [18] carry over mutatis mutandis, together with their proofs, to the present context. Let us especially attract attention to the following two fundamental propositions.

Proposition 3.1. *The mean value M on $\mathbb{R}^N \times \mathbb{R}^N$ for $\mathcal{H}^*$, viewed as defined on $A = \mathcal{J}(\Sigma)$, extends by continuity to a positive continuous linear form (still denoted by M) on $\mathfrak{X}^p_\Sigma$. Furthermore, for each $u \in \mathfrak{X}^p_\Sigma$, we have $u^\varepsilon \to M(u)$ in $L^p(\Omega)$-weak as $\varepsilon \to 0$, where u^ε (defined as in (2.2)) is considered as a function in $L^p(\Omega)$, and Ω is any fixed bounded open set in $\mathbb{R}^N_x$.*

Proposition 3.2. *The Gelfand transformation $\mathcal{G} : A \to \mathcal{C}(\Delta(A))$ extends by continuity to a (unique) continuous linear mapping, still denoted by $\mathcal{G}$, of $\mathfrak{X}^p_\Sigma$ into $L^p(\Delta(A))$.*

Remark 3.2. As a direct consequence of the preceding propositions, we have $M(u) = \int_{\Delta(A)} \mathcal{G}(u) d\beta$ for all $u \in \mathfrak{X}^p_\Sigma$.

At the present time, let $\mathfrak{X}^{p,\infty}_\Sigma = \mathfrak{X}^p_\Sigma \cap L^\infty(\mathbb{R}^N_y; \mathcal{B}(\mathbb{R}^N_z))$ be provided with the $L^\infty(\mathbb{R}^N_y; \mathcal{B}(\mathbb{R}^N_z))$-norm. It can be shown (as in part (v) of [18, Corollary 2.2]) that for $u \in \mathfrak{X}^{p,\infty}_\Sigma$, we have $\mathcal{G}(u) \in L^\infty(\Delta(A))$ and $\|\mathcal{G}(u)\|_{L^\infty(\Delta(A))} \leq \|u\|_{L^\infty(\mathbb{R}^N_y; \mathcal{B}(\mathbb{R}^N_z))}$. Another result worth mentioning is the following, where Ω is as in Proposition 3.1.

Proposition 3.3. *Let $1 < p < \infty$. Let E be a fundamental sequence. Suppose a sequence $(u_\varepsilon)_{\varepsilon \in E}$ weakly Σ-converges reiteratively in $L^p(\Omega)$ to some $u_0 \in L^p(\Omega \times \Delta(A))$ (Definition 2.2). Then, as $E \ni \varepsilon \to 0$, (2.6) holds for $v \in \mathcal{C}(\overline{\Omega}; \mathfrak{X}_\Sigma^{p', \infty})$ $(p' = \frac{p}{p-1})$.*

Proof. There is no difficulty in showing that (2.6) holds for $v \in \mathcal{C}(\overline{\Omega}) \otimes \mathfrak{X}_\Sigma^{p', \infty}$. Hence the desired result follows by mere routine and use of the density of $\mathcal{C}(\overline{\Omega}) \otimes \mathfrak{X}_\Sigma^{p', \infty}$ in $\mathcal{C}(\overline{\Omega}; \mathfrak{X}_\Sigma^{p', \infty})$. $\qquad\square$

The next result is a direct consequence of Proposition 3.3.

Corollary 3.1. *Let the hypotheses be as in Proposition 3.3. If $v \in \mathcal{C}(\overline{\Omega}; \mathfrak{X}_\Sigma^{p', \infty})$, then $v^\varepsilon \to \widehat{v}$ reiteratively in $L^p(\Omega)$-weak Σ as $E \ni \varepsilon \to 0$.*

We turn now to the statement of the abstract homogenization problem for (1.1).

Throughout the remainder of the present section it is assumed that $1 < p < \infty$. Our main goal is to investigate the limiting behaviour, as $\varepsilon \to 0$, of u_ε (the solution of (1.1) for fixed $\varepsilon > 0$) under the abstract hypothesis

$$(3.1) \qquad a_i(\cdot, \cdot, \psi) \in \mathfrak{X}_\Sigma^{p'}(\mathbb{R}_y^N; \mathcal{B}(\mathbb{R}_z^N)) \text{ for all } \psi \in (A_\mathbb{R})^N \quad (1 \le i \le N)$$

where $p' = \frac{p}{p-1}$ and

$$A_\mathbb{R} \quad = \quad A \cap \mathcal{C}(\mathbb{R}_y^N \times \mathbb{R}_z^N; \mathbb{R}) \text{ with } A = \mathcal{J}(\Sigma),$$

where $a_i(\cdot, \cdot, \psi)$ stands for the function $(y, z) \to a_i(y, z, \psi(y, z))$ from $\mathbb{R}^N \times \mathbb{R}^N$ into $\mathbb{R}$.

This is referred to as the *homogenization problem for* (1.1) *in the abstract deterministic setting associated with the reiteration H-structure* $\Sigma = \Sigma_y \times \Sigma_z$.

The resolution of this problem requires a few preliminaries. We start with one fundamental lemma.

Lemma 3.1. *Let (3.1) hold. Then, $a_i(\cdot, \cdot, \psi) \in L^\infty(\mathbb{R}_y^N; A_z)$ $(1 \le i \le N)$ for any $\psi \in (A_\mathbb{R})^N$.*

Proof. Thanks to part (iii) of (1.4), the lemma is proved if we can check that for $\psi \in (A_\mathbb{R})^N$ and $1 \le i \le N$, the function $a_i(\cdot, \cdot, \psi)$ is measurable from $\mathbb{R}_y^N$ into A_z. Thus, in view of (3.1), it is enough to establish that

$$(3.2) \qquad \mathfrak{X}_\Sigma^q(\mathbb{R}_y^N; \mathcal{B}(\mathbb{R}_z^N)) \subset L_{loc}^q(\mathbb{R}_y^N; A_z) \ (1 \le q < \infty).$$

But for any fixed real $q \geq 1$, we clearly have $A_y \otimes A_z \subset L_{loc}^q(\mathbb{R}_y^N; A_z)$, where $A_y = \mathcal{J}(\Sigma_y)$ and $A_z = \mathcal{J}(\Sigma_z)$, of course. Hence (3.2) follows by three facts : 1) $A_y \otimes A_z$ is dense in $\mathfrak{X}_\Sigma^q(\mathbb{R}_y^N; \mathcal{B}(\mathbb{R}_z^N))$, as a consequence of the fact that $A_y \otimes A_z$ is dense in A [18, Proposition 3.2]; 2) $\Xi^q(\mathbb{R}_y^N; \mathcal{B}(\mathbb{R}_z^N))$ is continuously embedded in $L_{loc}^q(\mathbb{R}_y^N; \mathcal{B}(\mathbb{R}_z^N))$; 3) $L_{loc}^q(\mathbb{R}_y^N; A_z)$ is closed in $L_{loc}^q(\mathbb{R}_y^N; \mathcal{B}(\mathbb{R}_z^N))$. $\hfill\square$

Remark 3.3. Lemma 3.1 shows immediately that (3.1) implies (2.5).

At the present time, let the index $1 \leq i \leq N$ be fixed. For $\varphi = (\varphi_j)_{1 \leq j \leq N}$ in $\mathcal{C}_\mathbb{R}(\Delta(A))^N$, where $\mathcal{C}_\mathbb{R}(\Delta(A)) = \mathcal{C}(\Delta(A); \mathbb{R})$, let

$$(3.3) \qquad b_i(\varphi) = \mathcal{G}(a_i(\cdot, \cdot, \mathcal{G}^{-1}\varphi))$$

where $\mathcal{G}^{-1}\varphi = (\mathcal{G}^{-1}\varphi_j)_{1 \leq j \leq N}$. Assuming (3.1), we see by Lemma 3.1 that $a_i(\cdot, \cdot, \mathcal{G}^{-1}\varphi)$ lies in $\mathfrak{X}_\Sigma^{p', \infty}(\mathbb{R}_y^N; \mathcal{B}(\mathbb{R}_z^N))$; hence (3.3) defines a mapping b_i of $\mathcal{C}_\mathbb{R}(\Delta(A))^N$ into $L^\infty(\Delta(A))$. The next proposition and corollary can be achieved using Proposition 2.2 and Corollary 3.1 and reasoning as in the proof of [19, Proposition 4.1 and Corollary 4.1].

Proposition 3.4. *Let* $1 < p < \infty$*. Let* Ω *be a bounded open set in* $\mathbb{R}_x^N$*. Suppose (3.1) holds. Then the following assertions are true :*

(i) *Let the index* $1 \leq i \leq N$ *be fixed. For* $\psi = (\psi_j)_{1 \leq j \leq N}$ *in* $\mathcal{C}(\overline{\Omega}; (A_\mathbb{R})^N)$*, the function* $b_i \circ \widehat{\psi}$ *(usual composition) of* $\overline{\Omega}$ *into* $L^\infty(\Delta(A))$ *lies in* $\mathcal{C}(\overline{\Omega}; L^\infty(\Delta(A)))$ *and further* $a_i^\varepsilon(\cdot, \cdot, \psi^\varepsilon) \to b_i \circ \widehat{\psi}$ *reit. in* $L^{p'}(\Omega)$*-weak* Σ *as* $E \ni \varepsilon \to 0$*, where* $\psi^\varepsilon = (\psi_j^\varepsilon)_{1 \leq j \leq N}$ (ψ_j^ε *defined as in (2.1)).*

(ii) *The mapping* $\phi \to b(\phi) = (b_i \circ \phi)_{1 \leq i \leq N}$ *of* $\mathcal{C}(\overline{\Omega}; \mathcal{C}_\mathbb{R}(\Delta(A))^N)$ *into* $L^{p'}(\Omega \times \Delta(A))^N$ *extends by continuity to a mapping, still denoted by* b*, of* $L^p(\Omega; L_\mathbb{R}^p(\Delta(A))^N)$ *into* $L^{p'}(\Omega \times \Delta(A))^N$ *such that*

$$\|b(\mathbf{u}) - b(\mathbf{v})\|_{L^{p'}(\Omega \times \Delta(A))^N}$$
$$\leq \quad c_1 \|1 + |\mathbf{u}| + |\mathbf{v}|\|_{L^p(\Omega \times \Delta(A))}^{p-1-\alpha_1} \|\mathbf{u} - \mathbf{v}\|_{L^p(\Omega; L^p(\Delta(A))^N)}^{\alpha_1}$$

and

$$(b(\mathbf{u}) - b(\mathbf{v})) \cdot (\mathbf{u} - \mathbf{v})$$
$$(3.4) \qquad \geq \quad c_2(1 + |\mathbf{u}| + |\mathbf{v}|)^{p-\alpha_2} |\mathbf{u} - \mathbf{v}|^{\alpha_2} \text{ a.e. in } \Omega \times \Delta(A)$$

for all $\mathbf{u}, \mathbf{v} \in L^p(\Omega; L_\mathbb{R}^p(\Delta(A))^N)$*.*

Corollary 3.2. *Let the hypotheses be those of Proposition* 3.4*. For each real* $\varepsilon > 0$*, Let* $\phi_\varepsilon \in \mathcal{D}_\mathbb{R}(\Omega) = \mathcal{C}_0^\infty(\Omega; \mathbb{R})$ *be given by*

$$(3.5) \qquad \phi_\varepsilon(x) = \psi_0(x) + \varepsilon\psi_1\left(x, \frac{x}{\varepsilon}\right) + \varepsilon^2\psi_2\left(x, \frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}\right) \quad (x \in \Omega)$$

with $\psi_0 \in \mathcal{D}_\mathbb{R}(\Omega)$, $\psi_1 \in \mathcal{D}_\mathbb{R}(\Omega) \otimes {}_\mathbb{R}A_y^\infty$, $\psi_2 \in \mathcal{D}_\mathbb{R}(\Omega) \otimes {}_\mathbb{R}A_y^\infty \otimes {}_\mathbb{R}A_z^\infty$, where ${}_\mathbb{R}A_y^\infty = A_y^\infty \cap \mathcal{C}(\mathbb{R}^N; \mathbb{R})$ and a similar definition for ${}_\mathbb{R}A_z^\infty$. Let the index $1 \le i \le N$ be fixed. Then, as $\varepsilon \to 0$,

$$a_i^\varepsilon(\cdot, \cdot, D\phi_\varepsilon) \to b_i(D\psi_0 + \partial_s\widehat{\psi}_1 + \partial_r\widehat{\psi}_2) \text{ reit. in } L^{p'}(\Omega)\text{-weak } \Sigma,$$

where : $\partial_s\widehat{\psi}_1 = (\partial_j\widehat{\psi}_1)_{1 \le j \le N}$ with $\partial_j\widehat{\psi}_1 = \partial_j \circ \widehat{\psi}_1$, the partial derivative ∂_j being here taken on $\Delta(A) = \Delta(A_y) \times \Delta(A_z)$ with respect to $\Delta(A_y)$; $\partial_r\widehat{\psi}_2 = (\partial_j\widehat{\psi}_2)_{1 \le j \le N}$ with $\partial_j\widehat{\psi}_2 = \partial_j \circ \widehat{\psi}_2$, ∂_j being this time taken on $\Delta(A)$ with respect to $\Delta(A_z)$; the functions $\widehat{\psi}_1$ and $\widehat{\psi}_2$ are viewed as defined on $\overline{\Omega}$ with values in $\mathcal{D}(\Delta(A))$.

Furthermore, if $(v_\varepsilon)_{\varepsilon \in E}$ is a sequence in $L^p(\Omega)$ such that $v_\varepsilon \to v_0$ reit. in $L^p(\Omega)$-weak Σ as $E \ni \varepsilon \to 0$, then, as $E \ni \varepsilon \to 0$,

$$\int_\Omega a_i^\varepsilon(\cdot, \cdot, D\phi_\varepsilon)v_\varepsilon dx \to \int\int_{\Omega \times \Delta(A)} b_i(D\psi_0 + \partial_s\widehat{\psi}_1 + \partial_r\widehat{\psi}_2)v_0 dxd\beta.$$

Remark 3.4. Let $\varphi \in \mathcal{C}^1(\Delta(A))$. By $\partial_j\varphi$ to be the *partial derivative* *(of index j) on* $\Delta(A) = \Delta(A_y) \times \Delta(A_z)$ *with respect to* $\Delta(A_y)$ we mean the function $(s, r) \to [\partial_j\varphi(\cdot, r)](s)$ of $\Delta(A)$ into $\mathbb{C}$, where $\partial_j\varphi(\cdot, r)$ (for fixed $r \in \Delta(A_z)$) denotes the partial derivative (of index j) of $\varphi(\cdot, r)$ on $\Delta(A_y)$.

3.2. The abstract homogenization result. Let the basic notation be as above. Let $1 < p < \infty$. We assume that the reiteration H-structure $\Sigma = \Sigma_y \times \Sigma_z$ is $W^{1,p}(\Omega)$-proper (Definiton 2.3). Let

$$\mathbb{F}_0^{1,p} = W_0^{1,p}(\Omega; \mathbb{R}) \times L^p(\Omega; W_\#^{1,p}(\Delta(A_y); \mathbb{R}))$$
$$\times L^p(\Omega; L^p(\Delta(A_y); W_\#^{1,p}(\Delta(A_z); \mathbb{R})))$$

where

$$W_\#^{1,p}(\Delta(A_y); \mathbb{R}) = \{v \in W_\#^{1,p}(\Delta(A_y)) : \partial_j v \in L_\mathbb{R}^p(\Delta(A_y)) \, (1 \le j \le N)\}$$

and an analogous definition for $W_\#^{1,p}(\Delta(A_z); \mathbb{R})$. We provide $\mathbb{F}_0^{1,p}$ with the norm

$$\|\mathbf{u}\|_{\mathbb{F}_0^{1,p}} = \sum_{i=1}^N \left[\|D_{x_i}u_0\|_{L^p(\Omega)} + \|\partial_i u_1\|_{L^p(\Omega \times \Delta(A_y))} + \|\partial_i u_2\|_{L^p(\Omega \times \Delta(A))}\right],$$
$$\mathbf{u} = (u_0, u_1, u_2) \in \mathbb{F}_0^{1,p} \quad (\text{with } D_{x_i} = \tfrac{\partial}{\partial x_i})$$

which makes it a Banach space. We will need the space

$$F_0^\infty = \quad \mathcal{D}_\mathbb{R}(\Omega) \times [\mathcal{D}_\mathbb{R}(\Omega) \otimes J_y(\mathcal{D}_\mathbb{R}(\Delta(A_y))/\mathbb{R})] \times$$
$$[\mathcal{D}_\mathbb{R}(\Omega) \otimes \mathcal{D}_\mathbb{R}(\Delta(A_y)) \otimes J_z(\mathcal{D}_\mathbb{R}(\Delta(A_z))/\mathbb{R})],$$

where

$$\mathcal{D}_\mathbb{R}(\Delta(A_y)) = \mathcal{D}(\Delta(A_y)) \cap \mathcal{C}(\Delta(A_y); \mathbb{R}),$$

$$\mathcal{D}_\mathbb{R}(\Delta(A_y))/\mathbb{C} = \left\{ \varphi \in \mathcal{D}_\mathbb{R}(\Delta(A_y)) : \int_{\Delta(A_y)} \varphi d\beta_y = 0 \right\},$$

and J_y denotes the canonical mapping of $W^{1,p}(\Delta(A_y))/\mathbb{C}$ into its separated completion $W_\#^{1,p}(\Delta(A_y))$ (of course, all that remaining valid with z in place of index y).

In view of $(\text{PR})_1$-$(\text{PR})_2$ in Definition 2.3 (use also [19, Remark 3.5]), F_0^∞ is dense in $\mathbb{F}_0^{1,p}$.

Remark 3.5. It amounts to the same thing to define F_0^∞ as the space of all $\phi = (\psi_0, J_y \circ \widehat{\psi}_1, J_z \circ \widehat{\psi}_2)$ with $\psi_0 \in \mathcal{D}_\mathbb{R}(\Omega)$, $\psi_1 \in \mathcal{D}_\mathbb{R}(\Omega) \otimes ({}_\mathbb{R}A_y^\infty/\mathbb{C})$ and $\psi_2 \in \mathcal{D}_\mathbb{R}(\Omega) \otimes {}_\mathbb{R}A_y^\infty \otimes ({}_\mathbb{R}A_z^\infty/\mathbb{C})$, where : ${}_\mathbb{R}A_y^\infty/\mathbb{C} = \{\psi \in {}_\mathbb{R}A_y^\infty : M(\psi) = 0\}$ (M the mean value on $\mathbb{R}_y^N$ for $\mathcal{H}$), ${}_\mathbb{R}A_z^\infty/\mathbb{C} = \{\psi \in {}_\mathbb{R}A_z^\infty : M(\psi) = 0\}$ (M the mean value on $\mathbb{R}_z^N$ for $\mathcal{H}'$), $\widehat{\psi}_1 = \mathcal{G} \circ \psi_1$ is viewed as a mapping of Ω into $\mathcal{D}(\Delta(A_y))$ and $\widehat{\psi}_2 = \mathcal{G} \circ \psi_2$ as a mapping of $\Omega \times \Delta(A_y)$ into $\mathcal{D}(\Delta(A_z))$.

We will also need the variational problem

(3.6)
$$\mathbf{u} = (u_0, u_1, u_2) \in \mathbb{F}_0^{1,p} :$$
$$\int \int_{\Omega \times \Delta(A)} b(\mathbb{D}\mathbf{u}) \cdot \mathbb{D}\mathbf{v} dx d\beta = \langle f, v_0 \rangle$$
$$\text{for all } \mathbf{v} = (v_0, v_1, v_2) \in \mathbb{F}_0^{1,p}$$

where $f \in W^{-1,p'}(\Omega; \mathbb{R})$, $\langle , \rangle$ stands for the duality pairing between $W^{-1,p'}(\Omega; \mathbb{R})$ and $W_0^{1,p}(\Omega; \mathbb{R})$, and where for each $\mathbf{v} = (v_0, v_1, v_2)$ in $\mathbb{F}_0^{1,p}$, we denote $\mathbb{D}\mathbf{v} = Dv_0 + \partial_s v_1 + \partial_r v_2$ with $\partial_s v_1 = (\partial_j v_1)_{1 \le j \le N}$ and $\partial_r v_2 = (\partial_j v_2)_{1 \le j \le N}$. According to (3.4), there is at most one $\mathbf{u} = (u_0, u_1, u_2)$ verifying (3.6).

We are now in a position to prove the basic deterministic homogenization theorem for (1.1).

Theorem 3.1. Let $1 < p < \infty$. Suppose (3.1) holds and further $\Sigma = \Sigma_y \times \Sigma_z$ is $W^{1,p}(\Omega)$-proper. For each fixed real number $\varepsilon > 0$, let u_ε be the solution of (1.1). Then, as $\varepsilon \to 0$,

(3.7)
$$u_\varepsilon \to u_0 \text{ in } W_0^{1,p}(\Omega)\text{-weak,}$$

(3.8) $\dfrac{\partial u_\varepsilon}{\partial x_i} \to \dfrac{\partial u_0}{\partial x_i} + \partial_i u_1 + \partial_i u_2$ *reit. in $L^p(\Omega)$-weak Σ ($1 \le i \le N$),*

where $\mathbf{u} = (u_0, u_1, u_2)$ is uniquely defined by (3.6).

Proof. It is an easy matter to check using Corollary 2.1 that the (generalized) sequence $(u_\varepsilon)_{\varepsilon>0}$ is bounded in $W_0^{1,p}(\Omega)$. Thus, given an arbitrary fundamental sequence E, appeal to the $W^{1,p}(\Omega)$-properness of Σ yields a subsequence E' from E and some triple $\mathbf{u} = (u_0, u_1, u_2) \in \mathbb{F}_0^{1,p}$ such that (3.7) and (3.8) hold as $E' \ni \varepsilon \to 0$. Therefore, recalling that the variational problem (3.6) admits at most one solution, we see that the theorem is proved if we can show that $\mathbf{u} = (u_0, u_1, u_2)$ verifies the variational equation in (3.6). For this purpose, let us fix freely $\phi = (\psi_0, J_y \circ \widehat{\psi}_1, J_z \circ \widehat{\psi}_2) \in F_0^\infty$ as in Remark 3.5, and let us attach to ϕ the sequence $(\phi_\varepsilon)_{\varepsilon>0}$, ϕ_ε given by (3.5). It is a simple exercise to verify that

(3.9) $\langle f, u_\varepsilon - \phi_\varepsilon \rangle - \displaystyle\int_\Omega a^\varepsilon(\cdot, \cdot, D\phi_\varepsilon) \cdot (Du_\varepsilon - D\phi_\varepsilon)dx \ge 0.$

The next step is to pass to the limit in (3.9) when $E' \ni \varepsilon \to 0$. Before we can do this, however, we need to know that as $\varepsilon \to 0$,

(3.10) $D_{x_i}\phi_\varepsilon \to D_{x_i}\psi_0 + \partial_i\widehat{\psi}_1 + \partial_i\widehat{\psi}_2$ reit. in $L^p(\Omega)$-weak Σ ($1 \le i \le N$)

and

(3.11) $\phi_\varepsilon \to \psi_0$ in $W_0^{1,p}(\Omega)$-weak,

where $D_{x_i} = \frac{\partial}{\partial x_i}$. Let us verify this. Considering the obvious equality

$D_{x_i}\phi_\varepsilon = D_{x_i}\psi_0 + (D_{y_i}\psi_1)^\varepsilon + (D_{z_i}\psi_2)^\varepsilon + \varepsilon[(D_{x_i}\psi_1)^\varepsilon + (D_{y_i}\psi_2)^\varepsilon] + \varepsilon^2(D_{x_i}\psi_2)^\varepsilon$

where the index i is freely fixed and the notation u^ε is as in (2.1), one is immediately led to

$$\|D_{x_i}\phi_\varepsilon - D_{x_i}\psi_0 - (D_{y_i}\psi_1)^\varepsilon - (D_{z_i}\psi_2)^\varepsilon\|_{L^p(\Omega)} \to 0$$

as $\varepsilon \to 0$. Since $(D_{y_i}\psi_1)^\varepsilon \to \partial_i\widehat{\psi}_1$ reit. in $L^p(\Omega)$-strong Σ as $\varepsilon \to 0$ and $(D_{z_i}\psi_2)^\varepsilon \to \partial_i\widehat{\psi}_2$ reit. in $L^p(\Omega)$-strong Σ as $\varepsilon \to 0$, we deduce that

$$D_{x_i}\phi_\varepsilon \to D_{x_i}\psi_0 + \partial_i\widehat{\psi}_1 + \partial_i\widehat{\psi}_2 \text{ reit. in } L^p(\Omega)\text{-strong }\Sigma.$$

Hence (3.10) follows by Proposition 2.6, whereas (3.11) follows from (3.10) by [part (ii) of] Proposition 2.4 and use of the fact that $\int_{\Delta(A)}(\partial_i\widehat{\psi}_1 +$

$\partial_i \widehat{\psi_2}) d\beta = 0$. Having made this point and recalling on the other hand (3.7)-(3.8) as $E' \ni \varepsilon \to 0$, we can now pass to the limit in (3.9) using Corollary 3.2 and we get

$$\langle f, u_0 - \psi_0 \rangle - \int\int_{\Omega \times \Delta(A)} b(\mathbb{D}\phi) \cdot (\mathbb{D}\mathbf{u} - \mathbb{D}\phi) dx d\beta \geq 0,$$

where ϕ ranges over F_0^∞ and hence over $\mathbb{F}_0^{1,p}$ too, since the former space is dense in the latter. Therefore, the theorem follows by mere routine (see the proof of [19, Theorem 4.1]). $\square$

4.   Concrete homogenization problems for (1.1)

In this section, we consider a few examples of homogenization problems for (1.1) in a concrete setting (as opposed to the abstract framework of Section 3) and we show how their study leads naturally to the abstract setting of Section 3 and so we may conclude by merely applying Theorem 3.1.

4.1. Problem I (Periodic setting). We assume here that for each fixed $\xi \in \mathbb{R}^N$, the function $(y, z) \to a(y, z, \xi)$ satisfies the following condition, commonly known as the *periodicity hypothesis* :

(4.1)   For each $k \in \mathbb{Z}^N$ and each $\ell \in \mathbb{Z}^N$, we have
$a(y + k, z + \ell, \xi) = a(y, z, \xi)$ a.e. in $(y, z) \in \mathbb{R}^N \times \mathbb{R}^N$.

One also expresses (4.1) by saying that $a(y, z, \xi)$ (for fixed $\xi \in \mathbb{R}^N$) is Y-periodic in y and Z-periodic in z, or simply that $a(y, z, \xi)$ is $Y \times Z$-periodic in (y, z), where $Y = (0, 1)^N$ and $Z = (0, 1)^N$ (see Example 2.7).

Our purpose is to study the homogenization of (1.1) under the periodicity hypothesis. This problem was studied in [16] from a different point of view. Here we present an approach which is in keeping with the general pattern of deterministic homogenization theory.

It is clear that the right reiteration H-structure for this problem is the periodic H-structure $\Sigma = \Sigma_{\mathbb{Z}^N} \times \Sigma_{\mathbb{Z}^N}$ of Example 2.1. So we have here $\Sigma_y = \Sigma_{\mathbb{Z}^N}$, $\Sigma_z = \Sigma_{\mathbb{Z}^N}$; thus, $A_y = \mathcal{C}_{per}(Y)$, $A_z = \mathcal{C}_{per}(Z)$ and $A = \mathcal{C}_{per}(Y \times Z)$ (see Example 2.7). In this setting, (3.6) takes a rather simple form :

(4.2) $\mathbf{u} = (u_0, u_1, u_2) \in \mathbb{F}_0^{1,p}$:
$\int_\Omega \int_Y \int_Z a(y, z, \mathbb{D}\mathbf{u}(x, y, z)) \cdot \mathbb{D}\mathbf{v}(x, y, z) dx dy dz = \langle f, v_0 \rangle$
for all $\mathbf{v} = (v_0, v_1, v_2) \in \mathbb{F}_0^{1,p}$,

where :

$$\mathbb{F}_0^{1,p} = W_0^{1,p}(\Omega;\mathbb{R}) \times L^p(\Omega;W_\#^{1,p}(Y;\mathbb{R})) \times L^p(\Omega;L^p(Y;W_\#^{1,p}(Z;\mathbb{R}))),$$
$$W_\#^{1,p}(Y;\mathbb{R}) = \{v \in W_{loc}^{1,p}(\mathbb{R}_y^N;\mathbb{R}) : v \; Y\text{-periodic}, \; \int_Y v dy = 0\},$$
$$\mathbb{D}\mathbf{v} = D_x v_0 + D_y v_1 + D_z v_2 \text{ for } \mathbf{v} = (v_0, v_1, v_2) \in \mathbb{F}_0^{1,p}.$$

Given an arbitrary real $p > 1$, the aim is to show that as $\varepsilon \to 0$, we have $u_\varepsilon \to u_0$ in $W_0^{1,p}(\Omega)$-weak and $D_{x_i} u_\varepsilon \to D_{x_i} u_0 + D_{y_i} u_1 + D_{z_i} u_2$ reiteratively in $L^p(\Omega)$-weak Σ ($1 \le i \le N$), where u_ε is the solution of (1.1) (for fixed $\varepsilon > 0$) and $\mathbf{u} = (u_0, u_1, u_2)$ is (uniquely) defined by (4.2). But since Σ is $W^{1,p}(\Omega)$-proper (Proposition 2.8), we see by Theorem 3.1 that the whole problem reduces to verifying that (3.1) holds.

Let $1 \le i \le N$ and $\psi \in (A_\mathbb{R})^N$ be freely fixed. Let $z \in \mathbb{R}^N$ be arbitrarily fixed. In view of (1.2) and part (iii) of (1.4), the function $h : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ given by $h(y, \xi) = a_i(y, z, \xi)$ ($y, \xi \in \mathbb{R}^N$) has the Caratheodory property. Hence, a classical result (see, e.g., [11, p.75]) reveals that if u is any measurable function from $\mathbb{R}^N$ into $\mathbb{R}^N$, then the function $y \to h(y, u(y))$ is measurable from $\mathbb{R}^N$ into $\mathbb{R}$. Choosing in particular $u(y) = \psi(y, z)$ ($y \in \mathbb{R}^N$), we see that the function $y \to a_i(y, z, \psi(y, z))$ is measurable from $\mathbb{R}^N$ into $\mathbb{R}$, and that for any arbitrary z in $\mathbb{R}^N$. On the other hand, by a routine calculation using [part (ii) of] (1.2) and [part (iii) of] (1.4) one can easily show that the function $z \to a_i(y, z, \psi(y, z))$ (for fixed y) is continuous on $\mathbb{R}^N$. Taking account of (4.1), we deduce that, a.e. in $y \in \mathbb{R}^N$, the function $z \to a_i(y, z, \psi(y, z))$ of $\mathbb{R}^N$ into $\mathbb{R}$, denoted below by $a_i(y, \cdot, \psi(y, \cdot))$, lies in $\mathcal{C}_{per}(Z)$. Hence it follows by Lemma 2.1 that the function $y \to a_i(y, \cdot, \psi(y, \cdot))$, denoted by $a_i(\cdot, \cdot, \psi)$, is measurable from $\mathbb{R}^N$ into $\mathcal{C}_{per}(Z)$. From this we deduce using (1.3), part (iii) of (1.4), and (4.1), that $a_i(\cdot, \cdot, \psi)$ lies in $L_{per}^\infty(Y;\mathcal{C}_{per}(Z))$. But $L_{per}^\infty(Y;\mathcal{C}_{per}(Z)) \subset L_{per}^{p'}(Y;\mathcal{C}_{per}(Z))$. Hence (3.1) follows by the facts that $A = \mathcal{C}_{per}(Y \times Z)$ is dense in $L_{per}^{p'}(Y;\mathcal{C}_{per}(Z))$ and the latter space is continuously embedded in $\Xi^{p'}(\mathbb{R}_y^N;\mathcal{B}(\mathbb{R}_z^N))$.

4.2. Problem II. We study here the homogenization of (1.1) under the structure hypothesis

$$(4.3) \qquad a_i(\cdot, \cdot, \xi) \in \mathcal{B}_\infty(\mathbb{R}_z^N;\mathcal{C}_{per}(Y)) \text{ for any } \xi \in \mathbb{R}^N \; (1 \le i \le N),$$

where $\mathcal{B}_\infty(\mathbb{R}_z^N;\mathcal{C}_{per}(Y))$ denotes the space of continuous functions $\psi : \mathbb{R}_z^N \to \mathcal{C}_{per}(Y)$ such that $\psi(z) = \psi(\cdot, z)$ has a limit in $\mathcal{B}(\mathbb{R}_y^N)$ when $|z| \to \infty$.

Proposition 4.1. *Let* (4.3) *be satisfied. Then* (3.1) *holds with* $\Sigma = \Sigma_{\mathbb{Z}^N} \times \Sigma_\infty$ *(i.e., with* $\Sigma_y = \Sigma_{\mathbb{Z}^N}$ *and* $\Sigma_z = \Sigma_\infty$ *) as in Example 2.10, so that the conclusion of Theorem 3.1 holds for* $p = 2$ *.*

Proof. First of all, we note that $\mathcal{B}_\infty(\mathbb{R}_z^N; \mathcal{C}_{per}(Y))$ is exactly the image of the reiteration H-structure $\Sigma = \Sigma_{\mathbb{Z}^N} \times \Sigma_\infty$ (Example 2.4) and further the latter is $W^{1,2}(\Omega)$-proper. Therefore, the proof is complete once we have established that

$$(4.4) \quad a_i(\cdot, \cdot, \psi) \in A = \mathcal{B}_\infty(\mathbb{R}_z^N; \mathcal{C}_{per}(Y)) \text{ for all } \psi \in (A_\mathbb{R})^N \ (1 \le i \le N).$$

To do this, let $1 \le i \le N$ and $\psi \in (A_\mathbb{R})^N$ be freely fixed. Let K be a compact set in $\mathbb{R}^N$ such that $\psi(y, z) \in K$ for all $(y, z) \in \mathbb{R}^N \times \mathbb{R}^N$. According to (4.3), we may view a_i as a function $\xi \to a_i(\cdot, \cdot, \xi)$ of $\mathbb{R}^N$ into $A_\mathbb{R}$, which function lies in $\mathcal{C}(\mathbb{R}^N; A_\mathbb{R})$ (use part (iii) of (1.4)). Still calling a_i the restriction of the latter function to K, we have therefore $a_i \in \mathcal{C}(K; A_\mathbb{R})$. Recalling that $\mathcal{C}_\mathbb{R}(K) \otimes A_\mathbb{R}$ is dense in $\mathcal{C}(K; A_\mathbb{R})$ (see, e.g., [6, p.46]), we see that we may consider a sequence $(q_n)_{n \ge 1}$ in $\mathcal{C}_\mathbb{R}(K) \otimes A_\mathbb{R}$ such that

$$\sup_{(y,z,\xi) \in \mathbb{R}^N \times \mathbb{R}^N \times K} |q_n(y, z, \xi) - a_i(y, z, \xi)| \to 0 \text{ as } n \to \infty.$$

Hence $q_n(\cdot, \cdot, \psi) \to a_i(\cdot, \cdot, \psi)$ in $\mathcal{B}(\mathbb{R}_y^N \times \mathbb{R}_z^N)$ as $n \to \infty$. Thus, (4.4) is proved if we can check that each $q_n(\cdot, \cdot, \psi)$ lies in A. However, it is enough to verify that we have $q(\cdot, \cdot, \psi) \in A$ for any function $q : \mathbb{R}_y^N \times \mathbb{R}_z^N \times \mathbb{R}_\xi^N \to \mathbb{R}$ of the form

$$q(y, z, \xi) = \chi(\xi)\phi(y, z) \ (y, z, \xi \in \mathbb{R}^N) \text{ with } \chi \in \mathcal{C}(K; \mathbb{R}) \text{ and } \phi \in A_\mathbb{R}.$$

Given one such q, by the Stone-Weierstrass theorem we may consider a sequence (f_n) of polynomials in $\xi = (\xi_1, \cdots, \xi_N) \in K$ such that $f_n \to \chi$ in $\mathcal{C}(K)$ as $n \to \infty$, hence $f_n(\psi) \to \chi(\psi)$ in $\mathcal{B}(\mathbb{R}_y^N \times \mathbb{R}_z^N)$ as $n \to \infty$, where $f_n(\psi)$ stands for $f_n \circ \psi$ (usual composition) and $\chi(\psi)$ stands for $\chi \circ \psi$. We deduce that $\chi(\psi)$ lies in $A_\mathbb{R}$, since the same is true of each $f_n(\psi)$ ($A_\mathbb{R}$ being an algebra). The proposition follows thereby. $\qquad\square$

4.3. Problem III. We investigate here the limiting behaviour, as $\varepsilon \to 0$, of u_ε (defined for each $\varepsilon > 0$ by (1.1)) under the structure hypothesis

$$(4.5) \qquad a_i(\cdot, \cdot, \xi) \in \mathcal{B}_\infty(\mathbb{R}_z^N; AP(\mathbb{R}_y^N)) \text{ for any } \xi \in \mathbb{R}^N \ (1 \le i \le N)$$

where $AP(\mathbb{R}_y^N)$ is the space of almost periodic continuous complex functions on $\mathbb{R}_y^N$ [5, 12, 13].

Proposition 4.2. *Let* (4.5) *be satisfied. There is a countable subgroup* $\mathcal{R}_y$ *of* $\mathbb{R}_y^N$ *such that* (3.1) *holds with* $\Sigma = \Sigma_{\mathcal{R}_y} \times \Sigma_\infty$ *as in Example 2.10, so that the conclusion of Theorem 3.1 holds for* $p = 2$.

Proof. As shown in [20, Corollary 4.1], there exists a countable subgroup, $\mathcal{R}_y$, of $\mathbb{R}_y^N$ such that $a_i(\cdot, \cdot, \xi) \in \mathcal{B}_\infty(\mathbb{R}_z^N; AP_{\mathcal{R}_y}(\mathbb{R}_y^N))$ for all $\xi \in \mathbb{R}^N$ and all $1 \leq i \leq N$, where $AP_{\mathcal{R}_y}(\mathbb{R}_y^N)$ denotes the space of those $u \in AP(\mathbb{R}_y^N)$ such that $M(\overline{\gamma}_k u) = 0$ for all $k \in \mathbb{R}^N \backslash \mathcal{R}_y$ (γ_k is as in Example 2.1 and M is the mean value on $\mathbb{R}^N$ for the action $\mathcal{H}$). But then the space $A = \mathcal{B}_\infty(\mathbb{R}_z^N; AP_{\mathcal{R}_y}(\mathbb{R}_y^N))$ is nothing else but the image of the $W^{1,2}(\Omega)$-proper H-structure $\Sigma = \Sigma_{\mathcal{R}_y} \times \Sigma_\infty$. Hence, we are through with the proof if we can show that $a_i(\cdot, \cdot, \psi) \in A$ for all $\psi \in (A_\mathbb{R})^N$ and all $1 \leq i \leq N$. But this follows by repeating word for word the proof of (4.4). $\square$

4.4. Problem IV.

The case to be examined in the present subsection states as above except that in place of (4.5) we have here the almost periodicity hypothesis

$$(4.6) \qquad a_i(\cdot, \cdot, \xi) \in AP(\mathbb{R}_y^N \times \mathbb{R}_z^N) \text{ for all } \xi \in \mathbb{R}^N \ (1 \leq i \leq N).$$

Proposition 4.3. *Under assumption* (4.6), *condition* (3.1) *is fulfilled with* $\Sigma = \Sigma_{\mathcal{R}_y} \times \Sigma_{\mathcal{R}_z}$ *(Example 2.2), where* $\mathcal{R}_y$ *(resp.* $\mathcal{R}_z$) *is a suitable countable subgroup of* $\mathbb{R}_y^N$ *(resp.* $\mathbb{R}_z^N$). *Since* Σ *is* $W^{1,2}(\Omega)$-*proper (Proposition* 2.9), *the conclusion of Theorem 3.1 holds for* $p = 2$.

Proof. Appeal to [20, Corollary 4.1], once again, yields some countable subgroup R of $\mathbb{R}^{2N} = \mathbb{R}^N \times \mathbb{R}^N$ such that $a_i(\cdot, \cdot, \xi) \in AP_R(\mathbb{R}^{2N})$ for all $\xi \in \mathbb{R}^N$ and all $1 \leq i \leq N$. Letting $\mathcal{R}_y = pr_y(R)$ and $\mathcal{R}_z = pr_z(R)$, where pr_y (resp. pr_z) denotes the natural projection of $\mathbb{R}^{2N} = \mathbb{R}_y^N \times \mathbb{R}_z^N$ onto $\mathbb{R}_y^N$ (resp. $\mathbb{R}_z^N$), we deduce

$$a_i(\cdot, \cdot, \xi) \in A = AP_\mathcal{R}(\mathbb{R}_y^N \times \mathbb{R}_z^N) \text{ for all } \xi \in \mathbb{R}^N \ (1 \leq i \leq N)$$

where $\mathcal{R} = \mathcal{R}_y \times \mathcal{R}_z$ (a countable subgroup of $\mathbb{R}^N \times \mathbb{R}^N$). But then $AP_\mathcal{R}(\mathbb{R}_y^N \times \mathbb{R}_z^N)$ is precisely the image of $\Sigma = \Sigma_{\mathcal{R}_y} \times \Sigma_{\mathcal{R}_z}$ (see [18, Examples 3.3 and 3.6]), and the latter is $W^{1,2}(\Omega)$-proper. Therefore the proposition is proved if we can show that $a_i(\cdot, \cdot, \psi) \in A = AP_\mathcal{R}(\mathbb{R}_y^N \times \mathbb{R}_z^N)$ for all $\psi \in (A_\mathbb{R})^N$ and all $1 \leq i \leq N$. But the proof of this is a *verbatim* copy of that of (4.4) and therefore is not worth repeating. $\square$

4.5. Problem V.

Let $\mathcal{B}_{\infty, \mathbb{Z}^N}(\mathbb{R}_y^N)$ denote the closure in $\mathcal{B}(\mathbb{R}_y^N)$ of the space of all finite sums $\sum_{\text{finite}} \varphi_i u_i$ with $\varphi_i \in \mathcal{B}_\infty(\mathbb{R}_y^N)$, $u_i \in \mathcal{C}_{per}(Y)$, where

$\mathcal{B}_\infty(\mathbb{R}_y^N)$ is the space of those continuous complex functions on $\mathbb{R}_y^N$ that converge at infinity.

We intend to study the homogenization of (1.1) under the structure hypothesis

$$(4.7) \qquad a_i(\cdot, \cdot, \xi) \in \mathcal{B}_\infty(\mathbb{R}_z^N; \mathcal{B}_{\infty, \mathbb{Z}^N}(\mathbb{R}_y^N)) \text{ for all } \xi \in \mathbb{R}^N \ (1 \le i \le N).$$

Proposition 4.4. *Under hypothesis* (4.7), *we have* (3.1) *with* $\Sigma = \Sigma_{\infty, \mathbb{Z}^N} \times \Sigma_\infty$. *Furtheremore, the H-structure Σ is $W^{1,2}(\Omega)$-proper, so that the conclusion of Theorem* 3.1 *holds for* $p = 2$.

Proof. It is worth recalling that $\mathcal{B}_{\infty, \mathbb{Z}^N}(\mathbb{R}_y^N)$ is an H-algebra on $\mathbb{R}_y^N$ for $\mathcal{H}$, and $\Sigma_{\infty, \mathbb{Z}^N}$ denotes the H-structure of which $\mathcal{B}_{\infty, \mathbb{Z}^N}(\mathbb{R}_y^N)$ is the image (see [18, Example 3.5]). Now, there is no difficulty in showing that $\mathcal{B}_\infty(\mathbb{R}_z^N) \otimes \mathcal{B}_{\infty, \mathbb{Z}^N}(\mathbb{R}_y^N)$ is dense in $\mathcal{B}_\infty(\mathbb{R}_z^N; \mathcal{B}_{\infty, \mathbb{Z}^N}(\mathbb{R}_y^N))$. Hence, recalling that Σ_∞ is the H-structure on $\mathbb{R}_z^N$ (for $\mathcal{H}'$) of which $\mathcal{B}_\infty(\mathbb{R}_z^N)$ is the image (see [18, Example 3.4]), it follows by [18, Proposition 3.2] that the space $A = \mathcal{B}_\infty(\mathbb{R}_z^N; \mathcal{B}_{\infty, \mathbb{Z}^N}(\mathbb{R}_y^N))$ is exactly the image of the H-structure $\Sigma = \Sigma_{\infty, \mathbb{Z}^N} \times \Sigma_\infty$ on $\mathbb{R}_y^N \times \mathbb{R}_z^N$ (for $\mathcal{H}^*$). On the other hand, the latter is $W^{1,2}(\Omega)$-proper, as is immediate by Example 2.11 and use of the fact that $\Sigma_\infty = \Sigma_{\infty, \mathcal{R}_z = \{\omega\}}$ (ω the origin in $\mathbb{R}_z^N$). Therefore, the proof is complete once we have verified that $a_i(\cdot, \cdot, \psi) \in A$ for all $\psi \in (A_\mathbb{R})^N$ and all $1 \le i \le N$. But this follows by repeating word for word the proof of (4.4). $\qquad\square$

Corollary 4.1. *Suppose we have* $a_i(\cdot, \cdot, \xi) \in \mathcal{B}_\infty(\mathbb{R}_z^N; \mathcal{B}_\infty(\mathbb{R}_y^N))$ *for all* $\xi \in \mathbb{R}^N$ *and all* $1 \le i \le N$. *Then, the conclusion of Proposition* 4.4 *stands.*

Proof. Indeed, we have $\mathcal{B}_\infty(\mathbb{R}_z^N; \mathcal{B}_\infty(\mathbb{R}_y^N)) \subset \mathcal{B}_\infty(\mathbb{R}_z^N; \mathcal{B}_{\infty, \mathbb{Z}^N}(\mathbb{R}_y^N))$, hence the desired result follows by the preceding proposition. $\qquad\square$

4.6. Problem VI. The problem to be worked out here states as in Subsection 4.5 except that (4.7) is replaced by the structure hypothesis

$$(4.8) \qquad a_i(\cdot, \cdot, \xi) \in \mathcal{C}_{per}(Z; \mathcal{B}_{\infty, \mathbb{Z}^N}(\mathbb{R}_y^N)), \text{ for all } \xi \in \mathbb{R}^N \ (1 \le i \le N)$$

where $\mathcal{C}_{per}(Z; \mathcal{B}_{\infty, \mathbb{Z}^N}(\mathbb{R}_y^N))$ is the space of Z-periodic continuous functions of $\mathbb{R}_z^N$ into $\mathcal{B}_{\infty, \mathbb{Z}^N}(\mathbb{R}_y^N)$.

Proposition 4.5. *Under hypothesis* (4.8), *we have* (3.1) *with* $\Sigma = \Sigma_{\infty, \mathbb{Z}^N} \times \Sigma_{\mathbb{Z}^N}$. *Furtheremore, the H-structure Σ is $W^{1,p}(\Omega)$-proper, so that the conclusion of Theorem* 3.1 *holds.*

Proof. The space $A = \mathcal{C}_{per}(Z; \mathcal{B}_{\infty, \mathbb{Z}^N}(\mathbb{R}_y^N))$ coincides with the closure of $\mathcal{B}_{\infty, \mathbb{Z}^N}(\mathbb{R}_y^N) \otimes \mathcal{C}_{per}(Z)$ in $\mathcal{B}(\mathbb{R}_y^N \times \mathbb{R}_z^N)$. Hence it follows by [18, Proposition

3.2] that A is nothing but the image of the H-structure $\Sigma = \Sigma_{\infty,\mathbb{Z}^N} \times \Sigma_{\mathbb{Z}^N}$. On the other hand, by a procedure similar to that followed in Example 2.9 it can be shown that Σ is $W^{1,p}(\Omega)$-proper. Therefore, the proposition is proved if we can check that $a_i(\cdot,\cdot,\psi)$ lies in A for all $\psi \in (A_{\mathbb{R}})^N$ and all $1 \leq i \leq N$ which can be obtained by repeating the proof of (4.4). $\square$

4.7. Problem VII. We assume that the following condition is satisfied :

(4.9)   For each bounded set $\Lambda \subset \mathbb{R}^N$ and each real $\eta > 0$, there is some $\rho > 0$ such that $|a(y - t, z - \sigma, \xi) - a(y, z, \xi)| \leq \eta$ for all $(z, \xi) \in \mathbb{R}^N \times \Lambda$ and for almost all $y \in \mathbb{R}^N$ provided $|t| + |\sigma| \leq \rho$;

and we want to investigate the homogenization of (1.1) under the structure hypothesis

(4.10) $a_i(\cdot,\cdot,\xi) \in \mathcal{B}_\infty(\mathbb{R}^N_z; L^2_{per}(Y))$ for all $\xi \in \mathbb{R}^N$ $(1 \leq i \leq N)$

which generalizes (4.3).

For this purpose, let us introduce a suitable function space generalizing the usual amalgam of L^p and ℓ^∞ on $\mathbb{R}^N$ (see [10]). Let $1 \leq p < \infty$. We define $(L^p, \ell^\infty)(\mathbb{R}^N_y; \mathcal{B}(\mathbb{R}^N_z))$ to be the space of functions $u \in L^p_{loc}(\mathbb{R}^N_y; \mathcal{B}(\mathbb{R}^N_z))$ such that

$$\|u\|_{p,\infty} = \sup_{k \in \mathbb{Z}^N} \left(\int_{k+Y} \|u(y, \cdot)\|^p_\infty \, dy \right)^{\frac{1}{p}} < \infty.$$

This is a vector space over $\mathbb{C}$ and further $\|\cdot\|_{p,\infty}$ is a norm under which $(L^p, \ell^\infty)(\mathbb{R}^N_y; \mathcal{B}(\mathbb{R}^N_z))$ is a Banach space. One fundamental result is that $(L^p, \ell^\infty)(\mathbb{R}^N_y; \mathcal{B}(\mathbb{R}^N_z))$ is continuously embedded in $\Xi^p(\mathbb{R}^N_y; \mathcal{B}(\mathbb{R}^N_z))$ (this is immediate by [20, inequality (4.1)]). Let us turn now to the proof of

Proposition 4.6. *Let* (4.9)-(4.10) *hold. Then* (3.1) *is satisfied with* $p = 2$ *and* $\Sigma = \Sigma_{\mathbb{Z}^N} \times \Sigma_\infty$ $(W^{1,2}(\Omega)$-*proper reiteration* H-*stucture on* $\mathbb{R}^N \times \mathbb{R}^N$ *for* $\mathcal{H}^*$ *), so that the conclusion of Theorem* 3.1 *holds for* $p = 2$.

Proof. Let $(\theta_n)_{n \geq 1}$ be a mollifier on $\mathbb{R}^N_y$, i.e., $(\theta_n)_{n \geq 1} \subset C^\infty_0(\mathbb{R}^N_y)$ with $\theta_n \geq 0$, $\int \theta_n(y) dy = 1$, θ_n has support in $\varepsilon_n \overline{B}_N$, where $\overline{B}_N$ is the closed unit ball in $\mathbb{R}^N_y$ and $0 < \varepsilon_n \leq 1$ with $\varepsilon_n \to 0$ as $n \to \infty$. Let n be freely fixed. For any arbitrary $1 \leq i \leq N$, let

$$q^i_n(y, z, \xi) = \int \theta_n(t) a_i(y - t, z, \xi) dt \quad (y, z, \xi \in \mathbb{R}^N),$$

which gives a function $(y, z, \xi) \to q_n^i(y, z, \xi)$ of $\mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N$ into $\mathbb{R}$ with

$$q_n^i(\cdot, \cdot, \xi) \in \mathcal{C}(\mathbb{R}_z^N; \mathcal{C}_{per}(Y)) \text{ for any } \xi \in \mathbb{R}^N,$$

as it follows by (4.9). Furthermore, on letting $q_n = (q_n^i)_{1 \leq i \leq N}$ and recalling (1.3)-(1.4), we have for all $z \in \mathbb{R}^N$ and almost all $y \in \mathbb{R}^N$, $q_n(y, z, \omega) = 0$ and

$$|q_n(y, z, \xi_1) - q_n(y, z, \xi_2)| \leq c_1(1 + |\xi_1| + |\xi_2|)^{p-1-\alpha_1} |\xi_1 - \xi_2|^{\alpha_1},$$

where ξ_1 and ξ_2 are arbitrary.

The next point is to show that $q_n^i(\cdot, z, \xi)$ (where ξ is fixed) has a limit in $\mathcal{B}(\mathbb{R}_y^N)$ when $|z| \to \infty$. To this end, let us introduce the mapping $\xi \to a_i^*(\cdot, \xi)$ of $\mathbb{R}^N$ into $L^2_{per}(Y)$ such that $a_i(\cdot, z, \xi) \to a_i^*(\cdot, \xi)$ in $L^2(Y)$ as $|z| \to \infty$ (see (4.10)), where ξ is arbitrarily fixed. Put

$$p_n^i(y, \xi) = \int \theta_n(t) a_i^*(y - t, \xi) dt \quad (y, \xi \in \mathbb{R}^N).$$

This defines a function $(y, \xi) \to p_n^i(y, \xi)$ of $\mathbb{R}^N \times \mathbb{R}^N$ into $\mathbb{R}$ with $p_n^i(\cdot, \xi) \in \mathcal{C}_{per}(Y)$ for each fixed ξ. Let us show that $q_n^i(\cdot, z, \xi) \to p_n^i(\cdot, \xi)$ as $|z| \to \infty$.

To do this, we start from

$$q_n^i(y, z, \xi) - p_n^i(y, \xi) = \int \theta_n(t)[a_i(y - t, z, \xi) - a_i^*(y - t, \xi)] dt.$$

We apply Holder's inequality to get

$$\left| q_n^i(y, z, \xi) - p_n^i(y, \xi) \right| \leq c_n \left(\int_{B_N} |a_i(y - t, z, \xi) - a_i^*(y - t, \xi)|^2 dt \right)^{\frac{1}{2}},$$

where $c_n = \|\theta_n\|_\infty^{\frac{1}{2}}$. Now, note that as k ranges over $\mathbb{Z}^N$, the sets $k + \overline{Y}$ form a covering of $\mathbb{R}^N$. Thus, we may consider a finite set $S \subset \mathbb{Z}^N$ such that B_N is contained in the union $\bigcup_{k \in S}(k + \overline{Y})$. Then $y - B_N$ is contained in $\bigcup_{k \in S}(k + y + \overline{Y})$ (where y is freely fixed in $\mathbb{R}^N$). Hence, by the change of variable $\sigma = y - t$ in the preceding integral and use of the Y-periodicity we arrive at

$$\left| q_n^i(y, z, \xi) - p_n^i(y, \xi) \right| \leq c_n |S|^{\frac{1}{2}} \left(\int_Y |a_i(\sigma, z, \xi) - a_i^*(\sigma, \xi)|^2 d\sigma \right)^{\frac{1}{2}},$$

where $|S|$ denotes the cardinality of S. This being so, let $\eta > 0$. Let $\rho > 0$ be such that

$$\left(\int_Y |a_i(\sigma, z, \xi) - a_i^*(\sigma, \xi)|^2 \, d\sigma \right)^{\frac{1}{2}} \leq \frac{\eta}{c_n \, |S|^{\frac{1}{2}}}$$

for any $z \in \mathbb{R}^N$ verifying $|z| \geq \rho$. Then, clearly

$$\sup_{y \in \mathbb{R}^N} \left| q_n^i(y, z, \xi) - p_n^i(y, \xi) \right| \leq \eta \quad (z \in \mathbb{R}^N, \ |z| \geq \rho).$$

Therefore

$$q_n^i(\cdot, \cdot, \xi) \in \mathcal{B}_\infty(\mathbb{R}_z^N; \mathcal{C}_{per}(Y)), \quad \text{for all } \xi \in \mathbb{R}^N \ (1 \leq i \leq N)$$

where the integer $n \geq 1$ is arbitrary. So we are justified in carrying over to the present situation the line of argument leading to (4.4). This yields

(4.11) $q_n^i(\cdot, \cdot, \psi) \in A = \mathcal{B}_\infty(\mathbb{R}_z^N; \mathcal{C}_{per}(Y))$ for all $\psi \in (A_\mathbb{R})^N$ $(1 \leq i \leq N)$.

Let us next show the following assertion

(4.12)   Let $\psi \in (A_\mathbb{R})^N$ and $1 \leq i \leq N$ be fixed. To each $\eta > 0$ there is assigned some integer $\nu \geq 1$ such that $\left\| q_n^i(\cdot, \cdot, \psi) - a_i(\cdot, \cdot, \psi) \right\|_{2,\infty} \leq \eta$ for any integer $n \geq \nu$.

First, by Holder's inequality we have

$$\left| q_n^i(y, z, \psi(y, z)) - a_i(y, z, \psi(y, z)) \right|^2$$
$$\leq \int \theta_n(t) \left| a_i(y - t, z, \psi(y, z)) - a_i(y, z, \psi(y, z)) \right|^2 dt.$$

On the other hand, given $\eta > 0$, let $\rho > 0$ be such that $|a_i(y - t, z, \xi) - a_i(y, z, \xi)| \leq \eta$ for almost all $y \in \mathbb{R}^N$ and for all $(z, \xi) \in \mathbb{R}^N \times \Lambda$ provided $|t| \leq \rho$, where Λ is a compact set in $\mathbb{R}^N$ containing the range of ψ. Finally, let ν be a positive integer such that $\varepsilon_n \leq \rho$ for any $n \geq \nu$. Then, recalling that integration above is actually taken over $\varepsilon_n B_N$, one quickly arrives at

$$\left| q_n^i(y, z, \psi(y, z)) - a_i(y, z, \psi(y, z)) \right|^2 \leq \eta^2$$

for any integer $n \geq \nu$, for all $z \in \mathbb{R}^N$ and for almost all $y \in \mathbb{R}^N$. Hence (4.2) follows in an obvious manner. Combining this with (4.11) and recalling that $(L^2, \ell^\infty)(\mathbb{R}_y^N; \mathcal{B}(\mathbb{R}_z^N))$ is continuously embedded in $\Xi^2(\mathbb{R}_y^N; \mathcal{B}(\mathbb{R}_z^N))$, we obtain (3.1) with $p = 2$ and $\Sigma = \Sigma_{\mathbb{Z}^N} \times \Sigma_\infty$, which completes the proof. $\square$

Remark 4.1. It can be proved by a similar approach that Proposition 4.2 remains valid if in (4.5) $AP(\mathbb{R}_y^N)$ is substituted by $L_{AP}^2(\mathbb{R}_y^N)$ (the space of those functions in $L_{loc}^2(\mathbb{R}_y^N)$ that are almost periodic in the Stepanoff sense [10, 17]) provided (4.9) is satisfied.

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