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Two-weight inequalities for singular integral operators satisfying a variant of Hörmander's condition

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Abstract. In this paper, we present some sufficient conditions for the boundedness of convolution operators that their kernel satisfies a certain version of Hörmander's condition, in the weighted Lebesgue spaces $L_{p,\omega}(\mathbb{R}^n)$.

1. Introduction

Let $\mathbb{R}^n$ be n–dimensional Euclidean space, $x = (x_1, ..., x_n)$, $\xi = (\xi_1, ..., \xi_n)$ are vectors in $\mathbb{R}^n$, $x \cdot \xi = x_1\xi_1 + \ldots + x_n\xi_n$, $|x| = (x \cdot x)^{1/2}$, $\mathbb{R}^n_0 = \mathbb{R}^n \setminus \{0\}$.

Suppose that ω be a positive, measurable, and real function defined in $\mathbb{R}^n$, i.e., is a weight function. By $L_{p,\omega}(\mathbb{R}^n)$ we denote the space of measurable functions $f(x)$ on $\mathbb{R}^n$ with finite norm

$$\|f\|_{L_{p,\omega}(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} |f(x)|^p \, \omega(x) dx \right)^{1/p}, \ 1 \le p < \infty.$$

For $\omega = 1$, we obtain the nonweighted space L_p, i.e., $L_{p,1}(\mathbb{R}^n) = L_p(\mathbb{R}^n)$.

We write $f \in L_p^{loc}(\mathbb{R}^n)$, $1 \le p < \infty$, if f belongs to $L_p(F)$ on any closed bounded set $F \subset \mathbb{R}^n$.

Let $K : \mathbb{R}_0^n \to \mathbb{R}$, $K \in L_1^{loc}(\mathbb{R}_0^n)$, $\mathbb{R}_0^n = \mathbb{R}^n \setminus \{0\}$, be a function satisfying the following conditions:

1) $K(t\,x) \equiv K\,(t\,x_1, \dots, t\,x_n) = t^{-n}\,K(x)$ for any $t > 0$, $x \in R_0^n$;
2) $\displaystyle\int_{|x|=1} K(x)\,d\sigma(x) = 0$;
3) $\displaystyle\int_0^1 \frac{w(t)}{t}\,dt < \infty$, where $w(t) = \sup\limits_{|\xi-\eta|\le t} |K(\xi) - K(\eta)|$ for $|\xi| = |\eta| = 1$.

Let $f \in L_p(\mathbb{R}^n)$, $1 < p < \infty$, and consider the following singular integral (1)

$$Tf(x) = p.v.\int_{\mathbb{R}^n} K(x-y)f(y)dy = \lim_{\varepsilon\to 0}\int_{\{y\in\mathbb{R}^n : |x-y|>\varepsilon\}} K(x-y)f(y)dy.$$

In the following theorem Calderon and Zygmund [5] proved the boundedness of the operator T.

Theorem 1. *Suppose that the kernel K of the singular integral* (1) *satisfies conditions* $1) - 3)$ *and $f \in L_p(\mathbb{R}^n)$, $1 \le p < \infty$. Then the singular integral exists for $x \in \mathbb{R}^n$ almost everywhere and the following inequalities holds*

$$\begin{aligned} \|Tf\|_{L_p(\mathbb{R}^n)} &\le C_1\,\|f\|_{L_p(\mathbb{R}^n)},\ 1 < p < \infty, \\ \int_{\{x\in\mathbb{R}^n\,:\,|Tf(x)|>\lambda\}} dx &\le \frac{C_2}{\lambda}\int_{\mathbb{R}^n} |f(x)|dx, \end{aligned}$$

where C_1, $C_2 > 0$ is independent of f.

Hörmander [13] imposed a weaker constraint on the kernel of the singular integral (1), namely,

$$\int_{\{x\in\mathbb{R}^n\,:\,|x|>2|y|\}} |K(x-y) - K(x)|\,dx \le C, \tag{2}$$

where $K \in L_1^{loc}(\mathbb{R}_0^n)$ and $C > 0$ is a constant independent of y. By replacing condition 3) with condition (2), under conditions 1), 2) he proved Theorem 1 for singular integrals with kernels satisfying condition (2). This condition is related to condition 3), and under this condition, inequality (2) holds (see [19]).

On the other hand, singular integrals whose kernels do not satisfy Hörmander's condition (2) are widely considered, for example oscillatory and some other singular integrals (see [20]).

Suppose that $K \in L_2(\mathbb{R}^n)$ is a function, satisfying the following conditions:

(K1) $\|\widehat{K}\|_\infty \le C$;

(K2) $|K(x)| \le \dfrac{C}{|x|^n}$;

(K3) There exist functions $A_1, \dots, A_m \in L_1^{loc}(\mathbb{R}_0^n)$, and the finite family $\Phi = \{\phi_1, \dots, \phi_m\}$ of essentially bounded functions in $\mathbb{R}^n$ such that $|det\,[\phi_j(y_i))]|^2 \in RH_\infty(R^{nm}),\ y_i \in \mathbb{R}^n,\ i,j = 1, \dots, m$;

(K4) For a fixed $\gamma > 0$ and for any $|x| > 2|y| > 0$,

$$\left| K(x-y) - \sum_{i=1}^{m} A_i(x)\,\phi_i(y) \right| \le C\,\frac{|y|^\gamma}{|x-y|^{n+\gamma}}, \tag{3}$$

where $C > 0$ is a constant and $\widehat{K}(\xi) = \int_{\mathbb{R}^n} e^{-i\,(x,\xi)} K(x)dx$ is the Fourier transform of the function K. In general, the functions A_i, $\phi_i,\ i = 1, \dots, m$ defined in $\mathbb{R}_0^n$ are complex-valued.

Remark 1. Any kernel satisfying condition (3) also satisfies the condition

$$\int_{|x|>2|y|} \left| K(x-y) - \sum_{i=1}^{m} A_i(x)\,\phi_i(y) \right| dx \le C,\ |x| > 2|y|. \tag{4}$$

Note that conditions $(K1)-(K4)$ were imposed in [20] and condition (4) was studied in [10]. For example, for $m = 1,\ A_1(x) = K(x),\ \phi_1(y) \equiv 1$ condition (4) yields Hörmander's condition (2). Note that, in this sense, condition (4) is a generalization of Hörmander's condition (2).

There exist other conditions stronger than condition (2) (see [9, 21]). The function $K(x) = (sin\,x)/x$ satisfies conditions $(K1) - (K4)$ and does not satisfy conditions 1), 2), and Hörmander's condition (2) (see [3]).

Definition 1. [17] It is said that a locally integrable weight function ω *belongs to* $A_p(\mathbb{R}^n)$, where $1 < p < \infty$, if

$$\sup_B \left(|B|^{-1} \int_B \omega(x)dx \right) \left(|B|^{-1} \int_B \omega(x)^{1-p'} dx \right)^{p-1} < \infty,$$

where the supremum is taken over all balls $B \subset \mathbb{R}^n$ and $p' = \frac{p}{p-1}$.

For $p=1$, we say $\omega \in A_1(\mathbb{R}^n)$, if

$$\sup_B \left(|B|^{-1} \int_B \omega(x)dx \right) \operatorname*{ess\,sup}_B \frac{1}{\omega(x)} < \infty,$$

or

$$|B|^{-1} \int_B \omega(x)\, dx \le C\omega(x) \text{ a.e. } x \in B$$

for any balls $B \subset \mathbb{R}^n$.

Suppose that the function K satisfies conditions $(K1)-(K4)$. For $f \in L_p(\mathbb{R}^n)$, $1 \le p < \infty$ define the following convolution operator generated by the kernel K as

$$Af(x) = \int_{\mathbb{R}^n} K(x-y)\, f(y)\, dy. \tag{5}$$

For the convolution operator (5), the following theorem holds.

Theorem 2. [20] *Suppose that* $w \in A_p(\mathbb{R}^n)$, $1 \le p < \infty$, *and the kernel of the convolution operator* (5) *satisfies conditions* $(K1)-(K4)$. *Then the following inequalities holds:*

$$\|Af\|_{L_{p,w}(\mathbb{R}^n)} \le C_3 \, \|f\|_{L_{p,w}(\mathbb{R}^n)}, \; 1 < p < \infty,$$

$$\int_{\{x\in\mathbb{R}^n \,:\, |Af(x)|>\lambda\}} \omega(x)dx \le \frac{C_4}{\lambda} \int_{\mathbb{R}^n} |f(x)|\omega(x)dx,$$

where C_3, $C_4 > 0$ *is independent of* f.

Note that in the "nonweighted" case, when condition $(K2)$ is not imposed and condition (3) is replaced by condition (4), Theorem 2 was proved in [10].

Lemma 1. *Suppose that* $1 \le p \le q \le \infty$ *and* $u(t)$ *and* $v(t)$ *are positive functions defined on* $(0,\infty)$.

(i) *For the validity of the inequality*

$$\left(\int_0^\infty u(t) \left| \int_0^t \varphi(\tau)d\tau \right|^q dt \right)^{1/q} \le K_1 \left(\int_0^\infty |\varphi(t)|^p v(t)dt \right)^{1/p}$$

with a constant K_1, *not depending on* φ, *it is necessary and sufficient that*

$$\sup_{t>0} \left(\int_t^\infty u(\tau)d\tau \right)^{p/q} \left(\int_0^t v(\tau)^{1-p'} d\tau \right)^{p-1} < \infty.$$

(ii) *For the validity of the inequality*

$$\left(\int_0^\infty u(t)\left|\int_t^\infty \varphi(\tau)d\tau\right|^q dt\right)^{1/q} \le K_2\left(\int_0^\infty |\varphi(t)|^p\, v(t)dt\right)^{1/p}$$

with a constant K_2, *not depending on* φ, *it is necessary and sufficient that*

$$\sup_{t>0}\left(\int_0^t u(\tau)d\tau\right)^{p/q}\left(\int_t^\infty v(\tau)^{1-p'}d\tau\right)^{p-1} < \infty.$$

Lemma 1 was established by Muckenhoupt [18] for $1 \le p = q \le \infty$ and J.S. Bradley [4], V.M. Kokilashvili [14], V.G. Maz'ya [16] for $p < q$.

Lemma 2. [15] *Let* $u(t)$ *and* $v(t)$ *be positive functions on* $(0,\infty)$.

(i) *If the following condition is satisfied*

$$\sup_{t>0}\left(\int_t^\infty v(\tau)d\tau\right)\operatorname*{ess\,sup}_{\tau\in(0,2t)}\frac{1}{u(\tau)} < \infty,$$

then the inequality

$$\int_0^\infty v(t)\left|\int_0^t F(\tau)d\tau\right| dt \le c\int_0^\infty u(t)|F(t)|dt$$

holds, where the constant $c > 0$ *does not depend on* F.

(ii) *If the following condition is satisfied*

$$\sup_{t>0}\left(\int_0^t v(\tau)d\tau\right)\operatorname*{ess\,sup}_{\tau\in\left(\frac{t}{2},\infty\right)}\frac{1}{u(\tau)} < \infty,$$

then the inequality

$$\int_0^\infty v(t)\left|\int_t^\infty F(\tau)d\tau\right| dt \le c\int_0^\infty u(t)|F(t)|dt$$

holds, where the constant $c > 0$ *does not depend on* F.

Lemma 3. [1, 6] *Suppose that* $1 \le p \le q \le \infty$ *and* $u(x)$ *and* $v(x)$ *are positive functions defined on* $\mathbb{R}^n$.

(i) *For the* n*-dimensional Hardy inequality*

$$\left(\int_{\mathbb{R}^n}\left(\int_{|y|<|x|/2}|f(y)|\,dy\right)^q \omega(x)\,dx\right)^{1/q} \le C_5\left(\int_{\mathbb{R}^n}|f(x)|^p v(x)\,dx\right)^{1/p}$$

with a constant C_5, *independent on* f, *to hold, it is necessary and sufficient that the following condition be satisfied:*

$$\sup_{R>0}\left(\int_{|x|>2R}\omega(x)\ dx\right)^{1/q}\left(\int_{|x|<R}v^{1-p'}(x)\ dx\right)^{1/p'}<\infty.$$

(ii) *For the n-dimensional (dual) Hardy inequality*

$$\left(\int_{\mathbb{R}^n}\left(\int_{|y|>2|x|}|f(y)|\ dy\right)^q u(x)\ dx\right)^{1/q}\leq C_6\left(\int_{\mathbb{R}^n}|f(x)|^p\omega(x)\ dx\right)^{1/p}$$

with a constant C_6, *independent on* f, *to hold, it is necessary and sufficient that the following condition be satisfied:*

$$\sup_{R>0}\left(\int_{|x|<R}u(x)\ dx\right)^{1/q}\left(\int_{|x|>2R}\omega^{1-p'}(x)\ dx\right)^{1/p'}<\infty.$$

Lemma 4. [8, 15] *Suppose that* $1\leq p<\infty$, $\beta>1$, $\varphi\in A_p(\mathbb{R}^n)$, *and suppose that* u, u_1 *are positive increasing (decreasing) functions defined on* $(0,\infty)$. *Suppose that* $\omega(x)=u(|x|)\varphi(x)$, $\omega_1(x)=u_1(|x|)\varphi(x)$ *and the weighted pair* $(\omega(x),\omega_1(x))$ *satisfies the following condition:*

(i) *For* $1<p<\infty$, $\mathcal{A}_p(\omega,\omega_1)<\infty$, *where*

$$\mathcal{A}_p(\omega,\omega_1):=\sup_{r>0}\Big(\int\limits_{|x|>2r}\omega_1(x)|x|^{-np}dx\Big)\Big(\int\limits_{|x|<r}\omega^{1-p'}(x)dx\Big)^{p-1}$$

(ii) *For* $p=1$, $\mathcal{A}_1(\omega,\omega_1)<\infty$, *where*

$$\mathcal{A}_1(\omega,\omega_1):=\sup_{r>0}\Big(\int_{|x|>2r}\omega_1(x)|x|^{-n}dx\Big)\operatorname*{ess\,sup}_{|x|<r}\frac{1}{\omega(x)}$$

(iii) *For* $1<p<\infty$, $\mathcal{B}_p(\omega,\omega_1)<\infty$, *where*

$$\mathcal{B}_p(\omega,\omega_1):=\sup_{r>0}\Big(\int\limits_{|x|<r}\omega_1(x)dx\Big)\Big(\int\limits_{|x|>2r}\omega^{1-p'}(x)|x|^{-np'}dx\Big)^{p-1}$$

(iv) *For* $p=1$, $\mathcal{B}_1(\omega,\omega_1)<\infty$, *where*

$$\mathcal{B}_1(\omega,\omega_1):=\sup_{r>0}\Big(\int_{|x|<r}\omega_1(x)dx\Big)\operatorname*{ess\,sup}_{|x|>2r}\frac{1}{\omega(x)|x|^n}$$

Then there exists a positive constant C depending only on p, n such that, for any $t > 0$, the following inequality holds:

$$u_1(2t) \le C\mathcal{A}_p(\omega,\omega_1)\, u(t) \ \ (u_1\,(t/2) \le C\mathcal{B}_p(\omega,\omega_1)\, u(t))\,.$$

In the case $\varphi = 1$ Lemma 4 was proved also in [11].

2. Main results

Theorem 3. *Suppose that the kernel K of the convolution operator* (5) *satisfies the conditions $(K1)-(K4)$ and $\phi \in A_p(\mathbb{R}^n)$, $1 \le p < \infty$. If $\omega(x) = u(x)\phi(x)$ and $\omega_1(x) = u_1(x)\phi(x)$ are weight functions on $\mathbb{R}^n$, satisfies the conditions*

$$\mathcal{A}_p(\omega,\omega_1) < \infty,\ \mathcal{B}_p(\omega,\omega_1) < \infty,$$

and there exist $b > 0$ such that

$$\sup_{|x|/4<|y|\le 4|x|} u_1(y) \le b\, u(x) \quad \textit{for a.e. } x \in \mathbb{R}^n. \tag{6}$$

Then there exists a $C_7 > 0$ such that, for any $f \in L_{p,\omega}(\mathbb{R}^n)$, $1 < p < \infty$ the following inequality holds

$$\int_{\mathbb{R}^n} |Af(x)|^p\, \omega_1(x)\, dx \le C_7 \int_{\mathbb{R}^n} |f(x)|^p\, \omega(x)\, dx. \tag{7}$$

Moreover, the condition (6) *can be replaced by the condition : there exist $b > 0$ such that*

$$u_1(x) \left(\sup_{|x|/4\le|y|\le|x|} \frac{1}{u(y)} \right) \le b \quad \textit{for a.e. } x \in \mathbb{R}^n.$$

Proof. For $k \in Z$ we define $E_k = \{x \in \mathbb{R}^n : 2^k < |x| \le 2^{k+1}\}$, $E_{k,1} = \{x \in \mathbb{R}^n : |x| \le 2^{k-1}\}$, $E_{k,2} = \{x \in \mathbb{R}^n : 2^{k-1} < |x| \le 2^{k+2}\}$, $E_{k,3} = \{x \in \mathbb{R}^n : |x| > 2^{k+2}\}$. Then $E_{k,2} = E_{k-1} \cup E_k \cup E_{k+1}$ and the multiplicity of the covering $\{E_{k,2}\}_{k\in Z}$ is equal to 3.

Let $1<p<\infty$. Given $f\in L_{p,\omega}(\mathbb{R}^n)$, we write

$$\begin{aligned}|Af(x)| &= \sum_{k\in Z}|Af(x)|\chi_{E_k}(x)\\ &\le \sum_{k\in Z}|Af_{k,1}(x)|\,\chi_{E_k}(x)+\sum_{k\in Z}|Af_{k,2}(x)|\,\chi_{E_k}(x)\\ &\quad+\sum_{k\in Z}|Af_{k,3}(x)|\,\chi_{E_k}(x)\\ &\equiv A_1f(x)+A_2f(x)+A_3f(x),\end{aligned}\tag{8}$$

where χ_{E_k} is the characteristic function of the set E_k, $f_{k,i}=f\chi_{E_{k,i}}$, $i=1,2,3$.

First we shall estimate $\|A_1f\|_{L_{p,\omega_1}}$. Note that for $x\in E_k$, $y\in E_{k,1}$ we have $|y|\le 2^{k-1}\le|x|/2$. Moreover, $E_k\cap supp f_{k,1}=\emptyset$ and $|x-y|\ge|x|/2$. Hence by condition $(K2)$

$$\begin{aligned}A_1f(x) &\le C\sum_{k\in Z}\left(\int_{\mathbb{R}^n}\frac{|f_{k,1}(y)|}{|x-y|^n}dy\right)\chi_{E_k}\\ &\le C\int_{|y|\le|x|/2}|x-y|^{-n}|f(y)|dy\le 2^nC|x|^{-n}\int_{|y|\le|x|/2}|f(y)|dy\end{aligned}$$

for any $x\in E_k$. Hence we have

$$\int_{\mathbb{R}^n}|A_1f(x)|^p\omega_1(x)dx\le(2^nC)^p\int_{\mathbb{R}^n}\left(\int_{|y|<|x|/2}|f(y)|dy\right)^p|x|^{-np}\omega_1(x)dx.$$

Since $\mathcal{A}_p(\omega,\omega_1)<\infty$, the Hardy inequality

$$\int_{\mathbb{R}^n}\omega_1(x)|x|^{-np}\left(\int_{|y|<|x|/2}|f(y)|dy\right)^p dx\le C_8\int_{\mathbb{R}^n}|f(x)|^p\omega(x)dx$$

holds and $C_9\le c'\mathcal{A}_p(\omega,\omega_1)$, where c' depends only on n and p. In fact the condition $\mathcal{A}_p(\omega,\omega_1)<\infty$ is necessary and sufficient for the validity of this inequality (see [1], [6]). Hence, we obtain

$$\int_{\mathbb{R}^n}|A_1f(x)|^p\omega_1(x)dx\le C_8\int_{\mathbb{R}^n}|f(x)|^p\omega(x)dx.\tag{9}$$

where C_9 is independent of f.

Next we estimate $\|A_3f\|_{L_{p,\omega_1}}$. It is easy to verify, for $x\in E_k$, $y\in E_{k,3}$ we have $|y|>2|x|$ and $|x-y|\ge|y|/2$. Since $E_k\cap supp f_{k,3}=\emptyset$, for $x\in E_k$

by condition $(K2)$ we obtain

$$A_3 f(x) \le C \int_{|y|>2|x|} \frac{|f(y)|}{|x-y|^n} dy \le 2^n C \int_{|y|>2|x|} |f(y)|\, |y|^{-n} dy.$$

Hence we have

$$\int_{\mathbb{R}^n} |A_3 f(x)|^p \omega_1(x) dx \le (2^n C)^p \int_{\mathbb{R}^n} \left(\int_{|y|>2|x|} |f(y)|\, |y|^{-n} dy \right)^p \omega_1(x) dx.$$

Since $\mathcal{B}_p(\omega, \omega_1) < \infty$, the Hardy inequality

$$\int_{\mathbb{R}^n} \left(\int_{|y|>2|x|} |f(y)|\, |y|^{-n} dy \right)^p \omega_1(x) dx \le C_6 \int_{\mathbb{R}^n} |f(x)|^p \omega(x) dx$$

holds and $C_6 \le c'' \mathcal{B}_p(\omega, \omega_1)$, where c'' depends only on n and p. In fact the condition $\mathcal{B}_p(\omega, \omega_1) < \infty$ is necessary and sufficient for the validity of this inequality (see [1], [6]). Hence, we obtain

$$\int_{\mathbb{R}^n} |A_3 f(x)|^p \omega_1(x) dx \le C_9 \int_{\mathbb{R}^n} |f(x)|^p \omega(x) dx, \tag{10}$$

where C_9 is independent of f.

Finally, we estimate $\|A_2 f\|_{L_{p,\omega_1}}$. From the $L_{p,\phi}(\mathbb{R}^n)$ boundedness of T and condition (6) we have

$$\begin{aligned}
\int_{\mathbb{R}^n} |A_2 f(x)|^p \omega_1(x) dx &= \int_{\mathbb{R}^n} \left(\sum_{k \in Z} |A f_{k,2}(x)|\, \chi_{E_k}(x) \right)^p \omega_1(x) dx \\
&= \int_{\mathbb{R}^n} \left(\sum_{k \in Z} |A f_{k,2}(x)|^p\, \chi_{E_k}(x) \right) \omega_1(x) dx \\
&= \sum_{k \in Z} \int_{E_k} |A f_{k,2}(x)|^p\, u_1(x) \phi(x) dx \\
&\le \sum_{k \in Z} \sup_{x \in E_k} u_1(x) \int_{\mathbb{R}^n} |A f_{k,2}(x)|^p\, \phi(x) dx \\
&\le \|A\|_\phi^p \sum_{k \in Z} \sup_{x \in E_k} u_1(x) \int_{\mathbb{R}^n} |f_{k,2}(x)|^p\, \phi(x) dx \\
&= \|A\|_\phi^p \sum_{k \in Z} \sup_{y \in E_k} u_1(y) \int_{E_{k,2}} |f(x)|^p \phi(x) dx,
\end{aligned}$$

where $\|A\|_\phi \equiv \|A\|_{L_{p,\phi}\to L_{p,\phi}}$. Since $2^{k-1} < |x| \le 2^{k+2}$, $x \in E_{k,2}$, we have by condition (a)

$$\sup_{y\in E_k} u_1(y) = \sup_{2^{k-1}<|y|\le 2^{k+2}} u_1(y) \le \sup_{|x|/4<|y|\le 4|x|} u_1(y) \le b\, u(x)$$

for almost all $x \in E_{k,2}$. Therefore we get

$$\begin{aligned}\int_{\mathbb{R}^n} |A_2 f(x)|^p \omega_1(x)dx &\le \|A\|_\phi^p b \sum_{k\in Z} \int_{E_{k,2}} |f(x)|^p u(x)\phi(x)dx \\ &\le C_{10} \int_{\mathbb{R}^n} |f(x)|^p \omega(x)dx \qquad (11)\end{aligned}$$

since the multiplicity of covering $\{E_{k,2}\}_{k\in Z}$ is equal to 3, where $C_{10} = 3\|A\|_\phi^p b$.

Inequalities (8), (9), (10), (11) imply (7) which completes the proof. □

Analogously proved the following theorem.

Theorem 4. *Suppose that the kernel K of the convolution operator* (5) *satisfies the conditions $(K1)-(K4)$, and $\omega(x) = u(x)\phi(x)$, $\omega_1(x) = u_1(x)\phi(x)$ are weight functions on $\mathbb{R}^n$, $\phi \in A_1(\mathbb{R}^n)$. If the weighted pair $(\omega(x), \omega_1(x))$ satisfies condition* (6) *and*

$$\mathcal{A}_1(\omega,\omega_1) \equiv \sup_{r>0} \left(\int_{|x|>2r} \omega_1(x)|x|^{-n} dx \right) \operatorname*{ess\,sup}_{|x|<r} \frac{1}{\omega(x)} < \infty,$$

$$\mathcal{B}_1(\omega,\omega_1) \equiv \sup_{r>0} \left(\int_{|x|<r} \omega_1(x) dx \right) \operatorname*{ess\,sup}_{|x|>2r} \frac{1}{\omega(x)|x|^n} < \infty.$$

Then there exists a $C_{11} > 0$ such that, for any $f \in L_{1,\omega}(\mathbb{R}^n)$, the following inequality holds

$$\int_{\{x\in\mathbb{R}^n:\,|Af(x)|>\lambda\}} \omega_1(x)\,dx \le \frac{C_{11}}{\lambda} \int_{\mathbb{R}^n} |f(x)|\,\omega(x)\,dx. \qquad (12)$$

Theorem 5. *Suppose that the kernel K of the convolution operator* (5) *satisfies the conditions $(K1)-(K4)$, and $\varphi \in A_1(\mathbb{R}^n)$. Let u and u_1 are positive increasing functions on $(0,\infty)$, such that the weights functions $\omega(x) = u(|x|)\varphi(x)$ and $\omega_1(x) = u_1(|x|)\varphi(x)$ satisfy the condition*

$$\mathcal{A}_1(\omega,\omega_1) < \infty$$

Then inequality (12) *is valid.*

Proof. Suppose that $f \in L_{1,\omega}(\mathbb{R}^n)$. Let u_1 are positive increasing functions on $(0,\infty)$ and $\mathcal{A}_1(\omega,\omega_1) < \infty$.

Without loss of generality we can suppose that u_1 may be represented by

$$u_1(t) = u_1(0+) + \int_0^t \psi(\tau) d\tau,$$

where $u_1(0+) = \lim_{t\to 0} u_1(t)$ and $u_1(t) \geq 0$ on $(0,\infty)$. In fact there exists a sequence of increasing absolutely continuous functions ϖ_n such that $\varpi_n(t) \leq \omega_1(t)$ and $\lim\limits_{n\to\infty} \varpi_n(t) = \omega_1(t)$ for any $t \in (0,\infty)$ (see [2, 11, 7, 8, 12] for details).

We have

$$\begin{aligned}
\int_{\{x\in\mathbb{R}^n : |Af(x)|>\lambda\}} \omega_1(x)dx &= u_1(0+) \int_{\{x\in\mathbb{R}^n : |Af(x)|>\lambda\}} \phi(x)dx \\
&\quad + \int_{\{x\in\mathbb{R}^n : |Af(x)|>\lambda\}} \left(\int_0^{|x|} \psi(\tau)d\tau \right) \phi(x)dx \\
&= J_1 + J_2.
\end{aligned}$$

If $u_1(0+) = 0$, then $J_1 = 0$. If $u_1(0+) \neq 0$ by the weak L_1 boundedness of A, $\phi \in A_1(\mathbb{R}^n)$ thanks to Lemma 4

$$\begin{aligned}
J_1 &\leq \frac{1}{\lambda} \|A\|_\phi u_1(0+) \int_{\mathbb{R}^n} |f(x)|\phi(x)dx \\
&\leq \frac{1}{\lambda} \|A\|_\phi \int_{\mathbb{R}^n} |f(x)| u_1(|x|)\phi(x)dx \\
&\leq \frac{b}{\lambda} \|A\|_\phi \int_{\mathbb{R}^n} |f(x)|\omega(x)dx.
\end{aligned}$$

After changing the order of integration in J_2 we have

$$\begin{aligned}
J_2 &= \int_0^\infty \psi(t) \left(\int_{|x|>t} \chi\{x : |Af(x)| > \lambda\} \phi(x)dx \right) dt \\
&\leq \int_0^\infty \psi(t) \left(\int_{|x|>t} \chi\left\{x : |A(f\chi_{\{|y|>t/2\}})(x)| > \lambda\right\} \phi(x)dx \right. \\
&\quad \left. + \int_{|x|>t} \chi\left\{x : |A(f\chi_{\{|y|\leq t/2\}})(x)| > \lambda\right\} \phi(x)dx \right) dt \\
&= J_{21} + J_{22}.
\end{aligned}$$

Using the weak L_1 boundeedness of A and Lemma 4 we have

$$\begin{aligned} J_{21} &\le \frac{\|A\|}{\lambda}\int_0^\infty \psi(t)\left(\int_{|y|>t/2}|f(y)|\phi(y)dy\right)dt \\ &= \frac{\|A\|}{\lambda}\int_{\mathbb{R}^n}|f(y)|\left(\int_0^{2|y|}\psi(t)dt\right)\phi(y)dy \\ &\le \frac{\|A\|}{\lambda}\int_{\mathbb{R}^n}|f(y)|u_1(2|y|)\phi(y)dy \\ &\le b\,\frac{\|A\|}{\lambda}\int_{\mathbb{R}^n}|f(y)|\omega(y)dy. \end{aligned}$$

Let us estimate J_{22}. For $|x|>t$ and $|y|\le t/2$ we have $|x|/2\le |x-y|\le 3|x|/2$, and so

$$\begin{aligned} J_{22} &\le c_4\int_0^\infty \psi(t)\left(\int_{|x|>t}\chi\left\{y:\int_{|y|\le t/2}|f(y)|\,|x-y|^{-n}dy>\lambda\right\}\phi(x)dx\right)dt \\ &\le c_5\int_0^\infty \psi(t)\,\chi\left\{y:\int_{|y|\le t/2}|f(y)|\,|y|^{-n}dy>\lambda\right\}\left(\int_{|x|>t}\phi(x)|x|^{-n}dx\right)dt \\ &= \frac{c_6}{\lambda}\int_0^\infty \psi(t)\left(\int_{|x|>t}\phi(x)|x|^{-n}dx\right)\left(\int_{|y|\le t/2}|f(y)|dy\right)dt. \end{aligned}$$

The Hardy inequality

$$\int_0^\infty \psi(t)\left(\int_{|y|\le t/2}|f(y)|dy\right)dt\le C\int_{\mathbb{R}^n}|f(y)|\omega(|y|)dy$$

for $p=1$ is characterized by the condition $C\le c'\mathcal{A}'_1$ (see [4], [14]), where

$$\begin{aligned} \mathcal{A}'_1 &\equiv \sup_{r>0}\left(\int_{2r}^\infty\left(\int_{|x|>t}\phi(x)|x|^{-n}dx\right)\psi(t)dt\right)\operatorname*{ess\,sup}_{|x|<r}\frac{1}{\omega(x)} \\ &= \sup_{r>0}\left(\int_{|x|>2r}\phi(x)|x|^{-n}\left(\int_{2r}^{|x|}\psi(t)dt\right)dx\right)\operatorname*{ess\,sup}_{|x|<r}\frac{1}{\omega(x)} \\ &\le \sup_{r>0}\left(\int_{|x|>2r}\phi(x)|x|^{-n}u_1(|x|)dx\right)\operatorname*{ess\,sup}_{|x|<r}\frac{1}{\omega(x)} \\ &= \sup_{r>0}\left(\int_{|x|>2r}\omega_1(|x|)|x|^{-n}dx\right)\operatorname*{ess\,sup}_{|x|<r}\frac{1}{\omega(x)}=\mathcal{A}_1(\omega,\omega_1)<\infty. \end{aligned}$$

Hence, applying the Hardy inequality, we obtain

$$J_{22} \leq \frac{C_{12}}{\lambda} \int_{\mathbb{R}^n} |f(x)|\omega(|x|)dx.$$

Combining the estimates of J_1 and J_2, we get (12) for $\omega_1(t) = \omega_1(0+) + \int_0^t \psi(\tau)d\tau$. By Fatou's theorem on passing to the limit under the Lebesgue integral sign, this implies (12). The theorem is proved. □

Analogously proved the following theorem.

Theorem 6. *Suppose that* $1 < p < \infty$*, the kernel* K *of the convolution operator* (5) *satisfies the conditions* $(K1) - (K4)$ *and* $\varphi \in A_p(\mathbb{R}^n)$*. Let* u*,* u_1 *are positive increasing functions on* $(0, \infty)$*,* $\omega(x) = u(|x|)\varphi(x)$*,* $\omega_1(x) = u_1(|x|)\varphi(x)$ *and* $\mathcal{A}_p(\omega, \omega_1) < \infty$*. Then inequality* (7) *is valid.*

Theorem 7. *Suppose that the kernel* K *of the convolution operator* (5) *satisfies the conditions* $(K1) - (K4)$ *and* $\varphi \in A_1(\mathbb{R}^n)$*. Let* u *and* u_1 *are positive decreasing functions on* $(0, \infty)$*, such that the weights functions* $\omega(x) = u(|x|)\varphi(x)$ *and* $\omega_1(x) = u_1(|x|)\varphi(x)$ *satisfy the condition*

$$\mathcal{B}_1(\omega, \omega_1) < \infty$$

Then inequality (12) *is valid.*

Proof. Without loss of generality we can suppose that ω_1 may be represented by

$$\omega_1(t) = \omega_1(+\infty) + \int_t^\infty \psi(\tau)d\tau,$$

where $\omega_1(+\infty) = \lim_{t\to\infty} \omega_1(t)$ and $\omega_1(t) \geq 0$ on $(0, \infty)$. In fact there exists a sequence of decreasing absolutely continuous fuctions ϖ_n such that $\varpi_n(t) \leq \omega_1(t)$ and $\lim_{n\to\infty} \varpi_n(t) = \omega_1(t)$ for any $t \in (0, \infty)$(see [2, 11, 7, 8, 12] for details).

We have

$$\begin{aligned}
\int_{\{x\in\mathbb{R}^n : |Af(x)|>\lambda\}} \omega_1(x)dx &= u_1(+\infty) \int_{\{x\in\mathbb{R}^n : |Af(x)|>\lambda\}} \phi(x)dx \\
&\quad + \int_{\{x\in\mathbb{R}^n : |Af(x)|>\lambda\}} \left(\int_{|x|}^\infty \psi(\tau)d\tau \right) \phi(x)dx \\
&= I_1 + I_2.
\end{aligned}$$

If $u_1(+\infty) = 0$, then $I_1 = 0$. If $u_1(+\infty) \neq 0$, by the weak L_1 boundedness of A, $\phi \in A_1(\mathbb{R}^n)$ thanks to Lemma 4

$$\begin{aligned} J_1 &\leq \frac{1}{\lambda}\|A\|_\phi u_1(+\infty) \int_{\mathbb{R}^n} |f(x)|\phi(x)dx \\ &\leq \frac{1}{\lambda}\|A\|_\phi \int_{\mathbb{R}^n} |f(x)|u_1(|x|)\phi(x)dx \\ &\leq \frac{b}{\lambda}\|A\|_\phi \int_{\mathbb{R}^n} |f(x)|\omega(|x|)dx. \end{aligned}$$

After changing the order of integration in J_2 we have

$$\begin{aligned} J_2 &= \int_0^\infty \psi(t) \left(\int_{|x|<t} \chi\{x : |Af(x)| > \lambda\}\phi(x)dx \right) dt \\ &\leq \int_0^\infty \psi(t) \left(\int_{|x|<2t} \chi\left\{x : |A(f\chi_{\{|y|>t/2\}})(x)| > \lambda\right\} \phi(x)dx \right. \\ &\quad \left. + \int_{|x|<t} \chi\left\{x : |A(f\chi_{\{|y|\leq 2t\}})(x)| > \lambda\right\} \phi(x)dx \right) dt \\ &= I_{21} + I_{22}. \end{aligned}$$

Using the weak L_1 boundedness of A and Lemma 4 we obtain

$$\begin{aligned} I_{21} &\leq \|A\| \int_0^\infty \psi(t) \left(\int_{|x|<2t} |f(x)|\phi(x)dx \right) dt \\ &= \|A\| \int_{\mathbb{R}^n} |f(x)|\phi(x) \left(\int_{|x|/2}^\infty \psi(t)dt \right) dx \\ &\leq \|A\| \int_{\mathbb{R}^n} |f(x)|\, u_1(|x|/2)\, \phi(x)\, dx \\ &\leq b\,\|A\| \int_{\mathbb{R}^n} |f(x)|\, u(|x|)\, \phi(x)\, dx \\ &= b\,\|A\| \int_{\mathbb{R}^n} |f(x)|\, \omega(x)\, dx. \end{aligned}$$

Let us estimate J_{22}. For $|x|<t$ and $|y|\geq 2t$ we have $|y|/2\leq |x-y|\leq 3|y|/2$, and so

$$\begin{aligned} I_{22} &\leq c_8\int_0^\infty \psi(t)\left(\int_{|x|<t}\chi\left\{y:\int_{|y|\geq 2t}|f(y)|\,|x-y|^{-n}dy>\lambda\right\}\phi(x)dx\right)dt \\ &\leq c_9\int_0^\infty \psi(t)\;\chi\left\{y:\int_{|y|\geq 2t}|f(y)|\,|y|^{-n}dy>\lambda\right\}\left(\int_{|x|<t}\phi(x)dx\right)dt \\ &= \frac{c_9}{\lambda}\int_0^\infty \psi(t)\left(\int_{|x|<t}\phi(x)dx\right)\left(\int_{|y|\geq 2t}|f(y)|\,|y|^{-n}dy\right)dt. \end{aligned}$$

The Hardy inequality

$$\int_0^\infty \psi(t)\left(\int_{|y|\geq 2t}|y|^{-n}|f(y)|dy\right)dt\leq C\int_{\mathbb{R}^n}|f(y)|\omega(|y|)dy$$

for $p=1$ is characterized by the condition $C\leq c'\mathcal{B}'_1$ (see [4], [14]), where

$$\begin{aligned} \mathcal{B}'_1 &\equiv \sup_{r>0}\left(\int_0^r\left(\int_{|x|<t}\phi(x)dx\right)\psi(t)dt\right)\operatorname*{ess\,sup}_{|x|>2r}\frac{1}{\omega(x)} \\ &= \sup_{r>0}\left(\int_{|x|<r}\phi(x)\left(\int_{|x|}^r\psi(t)dt\right)dx\right)\operatorname*{ess\,sup}_{|x|>2r}\frac{1}{\omega(x)} \\ &\leq \sup_{r>0}\left(\int_{|x|<r}\phi(x)u_1(|x|)dx\right)\operatorname*{ess\,sup}_{|x|>2r}\frac{1}{\omega(x)} \\ &= \sup_{r>0}\left(\int_{|x|<r}\omega_1(|x|)dx\right)\operatorname*{ess\,sup}_{|x|>2r}\frac{1}{\omega(x)}<\infty. \end{aligned}$$

Condition (c') of the theorem guarantees that $\mathcal{B}'\leq\mathcal{B}<\infty$. Hence, applying the Hardy inequality, we obtain

$$I_{22}\leq\frac{C_{13}}{\lambda}\int_{\mathbb{R}^n}|f(x)|\omega(|x|)dx.$$

Combining the estimates of I_1 and I_2, we get (12) for $\omega_1(t)=\omega_1(+\infty)+\int_t^\infty\psi(t)dt$. By Fatou's theorem on passing to the limit under the Lebesgue integral sign, this implies (12). The theorem is proved. □

Analogously proved the following theorem.

Theorem 8. *Suppose that* $1<p<\infty$, *the kernel* K *of the convolution operator* (5) *satisfies the conditions* $(K1)-(K4)$ *and* $\varphi\in A_p(\mathbb{R}^n)$. *Suppose* *that u, u_1 are positive decreasing functions on $(0,\infty)$, $\omega(x) = u(|x|)\varphi(x)$, $\omega_1(x) = u_1(|x|)\varphi(x)$ and $\mathcal{B}_p(\omega,\omega_1) < \infty$. Then inequality (7) is valid.*

Remark 2. Note that for the case in which $u = u_1 = 1$, Theorem 3 was proved in [20] by using different methods. Further, in the case $1 < p < \infty$ Theorems 6 and 8 was proved in [3].

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References

[1] E. Adams, *On weighted norm inequalities for the Riesz transforms of functions with vanishing moments,* Studia Math., 78 (1984), 107–153.

[2] I.A. Aliev and A.D. Gadjiev, *Weighted estimates of multidimensional singilars generated by the generalized shift operator,* English, translated into Russian *Acad. Sci. Sb. Math.,* 77 (1994), 37–55.

[3] R.A. Bandaliyev, *Two-weight inequalities for convolution operators in Lebesgue space.* Mathematical Notes, 80 (2006), 3–10.

[4] J.S. Bradley, *The Hardy's inequalities with mixed norms,* Canad. Math. Bull., 21 (1978), 405–408.

[5] A.P. Calderon, A. Zygmund, *On the existence of certain singular integrals,* Acta. Math., 88 (1952), 85–139.

[6] D. Edmunds, P. Gurka and L. Pick, *Compactness of Hardy-type integral operators in weighted Banach function spaces,* Studia Math., 109 (1994), 73–90.

[7] D. Edmunds and V. Kokilashvili, *Two-weight inequalities for singular integrals*, Canad. Math. Bull., 38 (1995), 119–125.

[8] D. Edmunds, V. Kokilashvili, and A. Meskhi, *Bounded and Compact Integral Operators*, Kluwer, Dordrecht, Boston, London, 2002.

[9] J. Garcia-Cuerva and J.L. Rubio de Francia, *Weighted Norm Inequalities and Related Topics,* North- Holland Math. Studies 116, Amsterdam, 1985.

[10] D.J. Grubb and C.N. Moore, *A variant of Hörmander condition for singular integrals,* Colloq. Math., 73 (1997), 165–172.

[11] V.S. Guliyev, *Two-weighted inequalities for integral operators in L_p-spaces, and applications,* Trudy Math. Inst. Steklov, 204 (1993), 113–136. English trans. in Proc. Steklov Inst. Math., 204 (1994), 97–116.

[12] E.G. Guseinov, *Singular integrals in spaces of functions that are summable with a monotone weight*,(Russian) Mat. Sb. (N.S.) 132(174) (1987), 28–44, 142; translation in Math. USSR-Sb. 60 (1988), 29–46

[13] L. Hörmander, *Estimates for translation invariant operators in L^p spaces,* Acta. Math., 104 (1960), 93–140.

[14] V.M. Kokilashvili, *On Hardy's inequalities in weighted spaces,* (Russian) Bull. Acad. Sci. Georgian SSR, 96 (1979), 37–40.

[15] V.M. Kokilashvili and A. Meskhi, *Two-weight inequalities for singular integrals defined on homogeneous groups,* Proc. A. Razmadze Math.Inst., 112 (1997), 57–90.

[16] V.G. Maz'ya, *Einbettingssatze fur Sobolewsche Raume* I, Leipzig, Teubner, 1979.

[17] B. Muckenhoupt, *Weighted norm inequalities for Hardy maximal functions,* Trans. Amer. Math. Soc., 165 (1972), 207–226.

[18] B. Muckenhoupt, *Hardy's inequality with weights,* Studia Math., 44 (1972), 31–38.

[19] E.M. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton Univ. Press, Princeton, NJ, 1970.

[20] R. Trujillo-Gonzalez, *Weighted norm inequalities for singular integrals operators satisfying a variant of Hörmander condition,* Comment. Math. Univ. Carolin., 44 (2003), 137–152.

[21] D.K. Watson, *Weighted estimates for singular integrals via Fourier transform estimates,* Duke Math. J., 60 (1990), 389–399.

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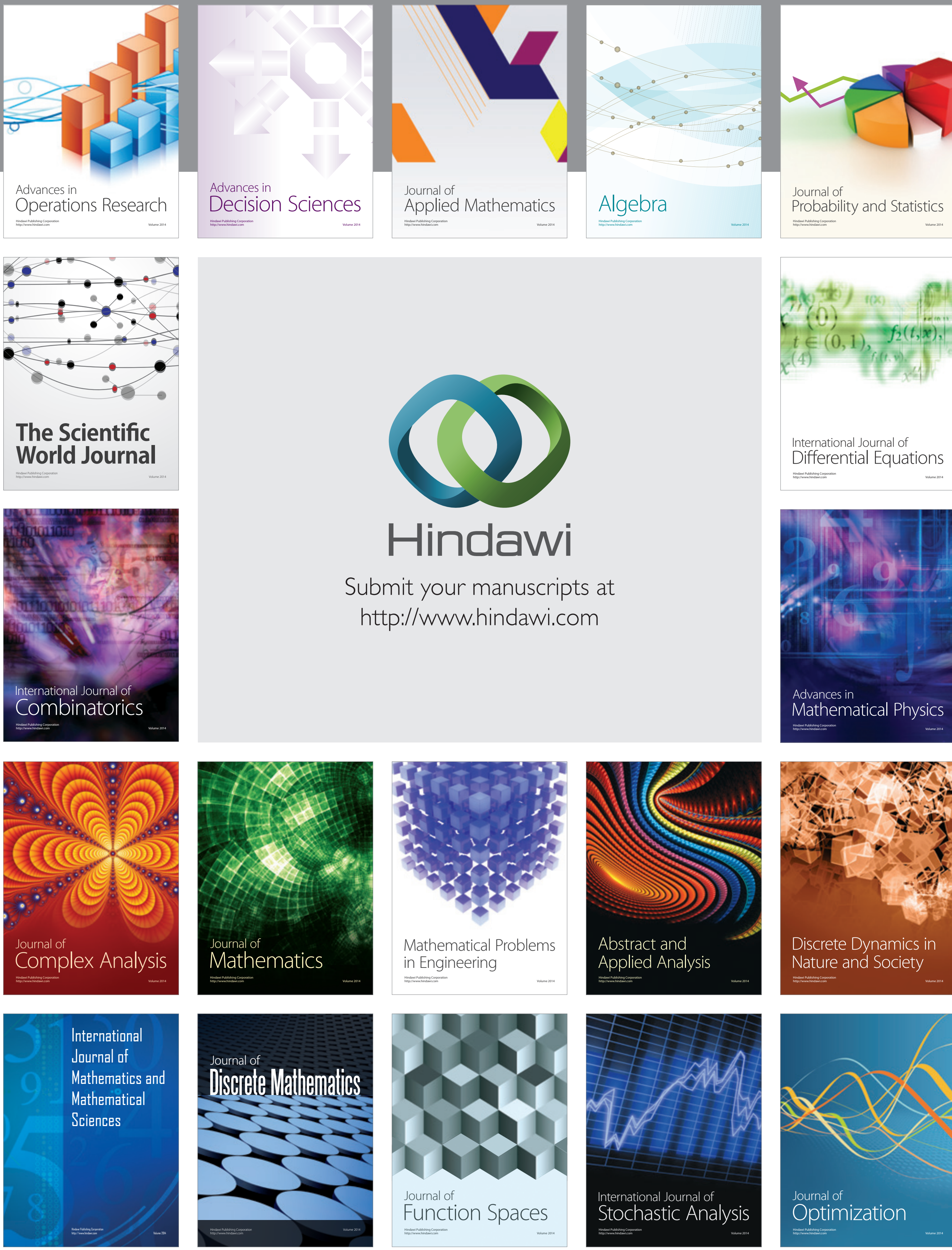
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