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Volterra composition operators from generalized weighted weighted Bergman spaces to μ-Bloch spaces

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Abstract. Let φ be a holomorphic self-map and g be a fixed holomorphic function on the unit ball B. The boundedness and compactness of the operator

$$T_{g,\varphi}f(z) = \int_0^1 f(\varphi(tz))\Re g(tz)\frac{dt}{t}$$

from the generalized weighted Bergman space into the μ-Bloch space are studied in this paper.

1. Introduction

Let B be the unit ball of $\mathbb{C}^n$. Let $z = (z_1, \dots, z_n)$ and $w = (w_1, \dots, w_n)$ be points in $\mathbb{C}^n$, we write

$$\langle z, w\rangle = z_1\bar{w}_1 + \cdots + z_n\bar{w}_n, \ |z| = \sqrt{|z_1|^2 + \cdots + |z_n|^2}.$$

Thus $B = \{z \in \mathbb{C}^n : |z| < 1\}$. Let dv be the normalized Lebesgue measure of B, i.e. $v(B) = 1$. Let $H(B)$ be the space of all holomorphic functions on B. For $f \in H(B)$, let $\Re f(z) = \sum_{j=1}^{n} z_j \frac{\partial f}{\partial z_j}(z)$ represent the radial derivative of $f \in H(B)$. We write $\Re^m f = \Re(\Re^{m-1} f)$.

A positive continuous function μ on $[0,1)$ is called normal, if there exist positive numbers s and t, $0 < s < t$, and $\delta \in [0,1)$ such that

$$\frac{\mu(r)}{(1-r)^s} \text{ is decreasing on } [\delta,1) \text{ and } \lim_{r\to 1} \frac{\mu(r)}{(1-r)^s} = 0;$$

$$\frac{\mu(r)}{(1-r)^t} \text{ is increasing on } [\delta,1) \text{ and } \lim_{r\to 1} \frac{\mu(r)}{(1-r)^t} = \infty$$

(see, e.g. [4]).

Let μ be a normal function on $[0,1)$. The μ-Bloch space, denoted by $\mathcal{B}_\mu = \mathcal{B}_\mu(B)$, is the set of all $f \in H(B)$ such that

$$b_\mu(f) = \sup_{z\in B} \mu(|z|)\, |\Re f(z)| < \infty.$$

$\mathcal{B}_\mu$ is a Banach space with the norm $\|f\|_{\mathcal{B}_\mu} = |f(0)| + b_\mu(f)$. Let $\mathcal{B}_{\mu,0}$ denote the subspace of $\mathcal{B}_\mu$ consisting of those $f \in \mathcal{B}_\mu$ for which

$$\lim_{|z|\to 1} \mu(|z|)|\Re f(z)| = 0.$$

We call $\mathcal{B}_{\mu,0}$ the little μ-Bloch space. When $\mu(r) = 1 - r^2$ and $\mu(r) = (1-r^2)^{1-\beta}(0 < \beta < 1)$, the induced spaces $\mathcal{B}_\mu$ are the Bloch spaces and the Lipschitz type spaces, respectively.

For any $p > 0$ and $\alpha \in \mathbb{R}$, let N be the smallest nonnegative integer such that $pN + \alpha > -1$. We say that an $f \in H(B)$ belongs to the generalized weighted Bergman space A^p_α, if

$$\|f\|_{A^p_\alpha} = |f(0)| + \Big[\int_B |\Re^N f(z)|^p (1-|z|^2)^{pN+\alpha} dv(z)\Big]^{1/p} < \infty.$$

The generalized weighted Bergman space A^p_α is introduced by Zhao and Zhu (see, e.g., [15]). This space covers the traditional weighted Bergman space($a > -1$), the Besov space, the Hardy space H^2 and the so-called Arveson space. For example, the space A^p_0 is the classical Bergman space; the space A^2_{-n} is the so-called Arveson space; the space $A^p_{-(n+1)}$ is the Besov space. See [15, 16] for some basic facts on the weighted Bergman space.

Let φ be a holomorphic self-map of B. The composition operator C_φ is defined by

$$(C_\varphi f)(z) = (f \circ \varphi)(z), \ f \in H(B).$$

The book [2] contains much information on this topic.

Suppose that $g : B \to \mathbb{C}^1$ is a holomorphic map, the extended Cesàro operator, which was introduced in [4], is defined as following

$$T_g f(z) = \int_0^1 f(tz) \frac{dg(tz)}{dt} = \int_0^1 f(tz) \Re g(tz) \frac{dt}{t}, \qquad f \in H(B),\ z \in B.$$

This operator is also called the Riemann-Stieltjes operator(see, e.g. [14]). See [1, 4, 5, 6, 8, 9, 10, 13, 14] for more information of the operator T_g on various spaces in the unit ball.

Motivated by the definition of operators C_φ and T_g, we define a more general operator

$$(1) \qquad T_{g,\varphi} f(z) = \int_0^1 f(\varphi(tz)) \Re g(tz) \frac{dt}{t}, \qquad f \in H(B),\ z \in B.$$

The operator $T_{g,\varphi}$ will be called the Volterra composition operator. In the setting of the unit disk D, this operator has the following form

$$T_{g,\varphi} f(z) = \int_0^z (f \circ \varphi)(\xi) g'(\xi) d\xi, \qquad f \in H(D),\ z \in D,$$

which was first studied in [7]. To the best of our knowledge, the operator $T_{g,\varphi}$ in the unit ball is studied in the present paper for the first time.

In this paper we study the boundedness and compactness of Volterra composition operators $T_{g,\varphi}$ from the generalized weighted Bergman space into $\mathcal{B}_\mu$ and $\mathcal{B}_{\mu,0}$. As some corollaries, we obtain characterizations of the extended Cesàro operator T_g from the generalized weighted Bergman space into $\mathcal{B}_\mu$ and $\mathcal{B}_{\mu,0}$.

Throughout the paper, constants are denoted by C, they are positive and may differ from one occurrence to the other.

2. Main results and proofs

In this section we give our main results and proofs. We will consider three cases: $n+1+\alpha > 0$, $n+1+\alpha = 0$ and $n+1+\alpha < 0$. Before we formulate our main results, we state several auxiliary results which will be used in the proofs. They are incorporated in the lemmas which follows.

Lemma 1. [15] (i) *Suppose $p > 0$ and $\alpha + n + 1 > 0$. Then there exists a constant $C > 0$ such that*

$$|f(z)| \le \frac{C \|f\|_{A^p_\alpha}}{(1-|z|^2)^{\frac{n+\alpha+1}{p}}}$$

for all $f \in A^p_\alpha$ *and* $z \in B$.

(ii) *Suppose* $p > 0$ *and* $\alpha + n + 1 < 0$ *or* $0 < p \le 1$ *and* $\alpha + n + 1 = 0$. *Then every function in* A^p_α *is continuous on the closed unit ball and so is bounded.*

(iii) *Suppose* $p > 1$, $1/p + 1/q = 1$ *and* $\alpha + n + 1 = 0$. *Then there exists a constant* $C > 0$ *such that*

$$|f(z)| \le C\Big[\ln \frac{2}{1-|z|^2}\Big]^{1/q}$$

for all $f \in A^p_\alpha$ *and* $z \in B$.

Lemma 2. *A closed set* K *in* $\mathcal{B}_{\mu,0}$ *is compact if and only if it is bounded and satisfies*

$$\lim_{|z|\to 1} \sup_{f\in K} \mu(|z|)|\Re f(z)| = 0.$$

Proof. The proof is similar to the proof of Lemma 1 in [11]. We omit the details. □

The following criterion for compactness follows from standard arguments similar to those outlined in Proposition 3.11 of [2]. We omit the details of the proof.

Lemma 3. *Assume that* $p > 0$, α *is a real number,* $g \in H(B)$, φ *is a holomorphic self-map of* B *and* μ *is a normal function on* $[0,1)$. *Then* $T_{g,\varphi} : A^p_\alpha \to \mathcal{B}_\mu$ *is compact if and only if* $T_{g,\varphi} : A^p_\alpha \to \mathcal{B}_\mu$ *is bounded and for any bounded sequence* $(f_k)_{k\in\mathbb{N}}$ *in* A^p_α *which converges to zero uniformly on compact subsets of* B *as* $k \to \infty$, *we have* $\|T_{g,\varphi} f_k\|_{\mathcal{B}_\mu} \to 0$ *as* $k \to \infty$.

Especially, when $p > 0$ and $\alpha + n + 1 < 0$, we need the following criterion for compactness follows from arguments similar to those in Lemma 3.7 of [12].

Lemma 4. *Let* $p > 0$ *and* $\alpha + n + 1 < 0$. *Let* T *be a bounded linear operator from* A^p_α *into a normed linear space* Y. *Then* T *is compact if and only if* $\|Tf_k\|_Y \to 0$ *whenever* (f_k) *is a norm-bounded sequence in* A^p_α *that converges to* 0 *uniformly on* $\overline{B}$.

Proof. The necessity is obvious. Now we prove the sufficiency part. Suppose that T is not compact. Then there is a bounded sequence (g_k) in A^p_α such that (Tg_k) has no convergent subsequence. Note that when $p > 0$ and $\alpha + n + 1 < 0$, A^p_α are indeed Lipschitz continuous (see Theorem 66 of [15]). Similarly to the proof of Lemma 3.6 of [12], we see that every bounded sequence in A^p_α has a subsequence that converges uniformly on $\overline{B}$ by Lemma 1 and Arzela-Ascoli Theorem. Hence (g_k) has a subsequence (f_k) such that $f_k \to f$ uniformly on $\overline{B}$. By Fatou's lemma we see that

$f \in A^p_\alpha$. The sequence $(f_k - f)$ is bounded in A^p_α and converges to 0 uniformly on $\overline{B}$. By assumption $\|Tf_k - Tf\|_Y \to 0$ as $k \to \infty$. This implies that the subsequence (Tf_k) of (Tg_k) converges in Y (to Tf), a contradiction. □

2.1. Case $n+1+\alpha > 0$.

Theorem 1. *Assume that $p > 0$, α is a real number such that $n+\alpha+1 > 0$, $g \in H(B)$, φ is a holomorphic self-map of B and μ is a normal function on $[0,1)$. Then $T_{g,\varphi} : A^p_\alpha \to \mathcal{B}_\mu$ is bounded if and only if*

$$M := \sup_{z\in B} \frac{\mu(|z|)|\Re g(z)|}{(1-|\varphi(z)|^2)^{\frac{n+1+\alpha}{p}}} < \infty. \tag{2}$$

Proof. Suppose that (2) holds. A calculation with (1) gives the following fundamental and useful formula(see, e.g. [4])

$$\Re[T_{g,\varphi}(f)](z) = f(\varphi(z))\Re g(z).$$

Then for arbitrary $z \in B$ and $f \in A^p_\alpha$, by Lemma 1 we have

$$\begin{aligned} \mu(|z|)|\Re(T_{g,\varphi}f)(z)| &= \mu(|z|)|f(\varphi(z))||\Re g(z)| \\ &\le C\|f\|_{A^p_\alpha} \frac{\mu(|z|)|\Re g(z)|}{(1-|\varphi(z)|^2)^{\frac{n+1+\alpha}{p}}}. \end{aligned} \tag{3}$$

Using the condition (2), the boundedness of the operator $T_{g,\varphi} : A^p_\alpha \to \mathcal{B}_\mu$ follows by taking the supremum in (3) over B.

Conversely, suppose that $T_{g,\varphi} : A^p_\alpha \to \mathcal{B}_\mu$ is bounded. Assume that

$$t > n\max(1, \frac{1}{p}) + \frac{\alpha+1}{p}. \tag{4}$$

For $a \in B$, set

$$f_a(z) = \frac{(1-|a|^2)^{t-\frac{n+1+\alpha}{p}}}{(1-\langle z,a\rangle)^t}.$$

Then from Theorem 32 of [15] we see that $f_a \in A^p_\alpha$ and $\sup_{a\in B}\|f_a\|_{A^p_\alpha} < \infty$. Therefore

$$\begin{aligned} C\|T_{g,\varphi}\|_{A^p_\alpha\to\mathcal{B}_\mu} &\ge \|T_{g,\varphi}f_{\varphi(b)}\|_{\mathcal{B}_\mu} \ge \sup_{z\in B}\mu(|z|)|\Re(T_{g,\varphi}f_{\varphi(b)})(z)| \\ &\ge \frac{\mu(|b|)|\Re g(b)|}{(1-|\varphi(b)|^2)^{\frac{n+1+\alpha}{p}}}, \end{aligned} \tag{5}$$

from which we get (2). This completes the proof of Theorem 1. □

Theorem 2. *Assume that* $p>0$, α *is a real number such that* $n+\alpha+1>0$, $g\in H(B)$, φ *is a holomorphic self-map of* B *and* μ *is a normal function on* $[0,1)$. *Then* $T_{g,\varphi}: A^p_\alpha \to \mathcal{B}_\mu$ *is compact if and only if* $g\in\mathcal{B}_\mu$ *and*

$$\lim_{|\varphi(z)|\to 1} \frac{\mu(|z|)|\Re g(z)|}{(1-|\varphi(z)|^2)^{\frac{n+1+\alpha}{p}}}=0. \tag{6}$$

Proof. Suppose that $g\in\mathcal{B}_\mu$ and (6) holds. From $g\in\mathcal{B}_\mu$ and (6), it is easy to see that (2) holds. Hence $T_{g,\varphi}: A^p_\alpha\to\mathcal{B}_\mu$ is bounded by Theorem 1. From (6), for given $\varepsilon>0$, there is a constant $\delta\in(0,1)$, such that

$$\sup_{\{z\in B\,:\,\delta<|\varphi(z)|<1\}} \frac{\mu(|z|)|\Re g(z)|}{(1-|\varphi(z)|^2)^{\frac{n+1+\alpha}{p}}}<\varepsilon. \tag{7}$$

Let $(f_k)_{k\in\mathbb{N}}$ be a bounded sequence in A^p_α such that $f_k\to 0$ uniformly on compact subsets of B as $k\to\infty$. Let $G=\{w\in B: |w|\le\delta\}$. From the fact that $g\in\mathcal{B}_\mu$ and (7), we have

$$\begin{aligned}
&\|T_{g,\varphi}f_k\|_{\mathcal{B}_\mu} = \sup_{z\in B}\mu(|z|)|f_k(\varphi(z))\Re g(z)|\\
=\;& \Big(\sup_{\{z\in B:\,|\varphi(z)|\le\delta\}} + \sup_{\{z\in B\,:\,\delta<|\varphi(z)|<1\}}\Big)\mu(|z|)|\Re g(z)||f_k(\varphi(z))|\\
=\;& \|g\|_{\mathcal{B}_\mu}\sup_{w\in G}|f_k(w)| + C\|f_k\|_{A^p_\alpha}\sup_{\{z\in B\,:\,\delta<|\varphi(z)|<1\}}\frac{\mu(|z|)|\Re g(z)|}{(1-|\varphi(z)|^2)^{\frac{n+1+\alpha}{p}}}\\
\le\;& \|g\|_{\mathcal{B}_\mu}\sup_{w\in G}|f_k(w)| + C\varepsilon. \qquad (8)
\end{aligned}$$

Observe that G is a compact subset of B, then it gives $\lim_{k\to\infty}\sup_{w\in G}|f_k(w)|=0$. Using this fact and letting $k\to\infty$ in (8), we obtain $\limsup_{k\to\infty}\|T_{g,\varphi}f_k\|_{\mathcal{B}_\mu}\le C\varepsilon$. Since ε is an arbitrary positive number, we obtain $\limsup_{k\to\infty}\|T_{g,\varphi}f_k\|_{\mathcal{B}_\mu}=0$. Employing Lemma 3, we get that $T_{g,\varphi}: A^p_\alpha\to\mathcal{B}_\mu$ is compact.

Conversely, suppose that $T_{g,\varphi}: A^p_\alpha\to\mathcal{B}_\mu$ is compact, then $T_{g,\varphi}: A^p_\alpha\to\mathcal{B}_\mu$ is bounded. It follows from the proof of Theorem 1 that $g\in\mathcal{B}_\mu$. Let $(z_k)_{k\in\mathbb{N}}$ be a sequence in B such that $|\varphi(z_k)|\to 1$ as $k\to\infty$. Set

$$f_k(z)=\frac{(1-|\varphi(z_k)|^2)^{t-\frac{n+\alpha+1}{p}}}{(1-\langle z,\varphi(z_k)\rangle)^t},\ k\in\mathbb{N},$$

where t satisfies (4). From Theorem 32 of [15] we see that $(f_k)_{k\in\mathbb{N}}$ is a bounded sequence in A^p_α. Moreover, it is easy to see that f_k converges to zero uniformly on compact subsects of B. In view of Lemma 3 it follows that

$$\limsup_{k\to\infty} \|T_{g,\varphi} f_k\|_{\mathcal{B}_\mu} = 0. \tag{9}$$

In addition, we have

$$\|T_{g,\varphi} f_k\|_{\mathcal{B}_\mu} = \sup_{z\in B} \mu(|z|)|\Re(T_{g,\varphi} f_k)(z)| \geq \frac{\mu(|z_k|)|\Re g(z_k)|}{(1-|\varphi(z_k)|^2)^{\frac{n+1+\alpha}{p}}}. \tag{10}$$

Combining (9) with (10) we get the desired result. The proof is completed. □

Theorem 3. *Assume that $p > 0$, α is a real number such that $n+\alpha+1 > 0$, $g \in H(B)$, φ is a holomorphic self-map of B and μ is a normal function on $[0,1)$. Then $T_{g,\varphi} : A^p_\alpha \to \mathcal{B}_{\mu,0}$ is bounded if and only if $T_{g,\varphi} : A^p_\alpha \to \mathcal{B}_\mu$ is bounded and $g \in \mathcal{B}_{\mu,0}$.*

Proof. Suppose that $T_{g,\varphi} : A^p_\alpha \to \mathcal{B}_{\mu,0}$ is bounded. Then it is clear that $T_{g,\varphi} : A^p_\alpha \to \mathcal{B}_\mu$ is bounded. Taking $f(z) = 1$ and employing the boundedness of $T_{g,\varphi} : A^p_\alpha \to \mathcal{B}_{\mu,0}$, we see that $g \in \mathcal{B}_{\mu,0}$.

Conversely, suppose that $T_{g,\varphi} : A^p_\alpha \to \mathcal{B}_\mu$ is bounded and $g \in \mathcal{B}_{\mu,0}$. Suppose that $f \in A^p_\alpha$ with $\|f\|_{A^p_\alpha} \leq L$, using polynomial approximations we obtain (see, e.g., [15])

$$\lim_{|z|\to 1} (1-|z|^2)^{\frac{n+1+\alpha}{p}} |f(z)| = 0.$$

From the above equality and $g \in \mathcal{B}_{\mu,0}$, for every $\varepsilon > 0$, there exists a $\delta \in (0,1)$ such that when $\delta < |z| < 1$,

$$(1-|z|^2)^{\frac{n+1+\alpha}{p}} |f(z)| < \varepsilon/M \tag{11}$$

and

$$\mu(|z|)|\Re g(z)| < \frac{\varepsilon(1-\delta^2)^{\frac{n+1+\alpha}{p}}}{L}, \tag{12}$$

where M is defined in (2). Therefore if $\delta < |z| < 1$ and $\delta < |\varphi(z)| < 1$, from (2) and (11) we have

$$\begin{aligned} \mu(|z|)|\Re(T_{g,\varphi} f)(z)| &= \frac{\mu(|z|)|\Re g(z)|}{(1-|\varphi(z)|^2)^{\frac{n+1+\alpha}{p}}} (1-|\varphi(z)|^2)^{\frac{n+1+\alpha}{p}} |f(\varphi(z))| \\ &\leq M(1-|\varphi(z)|^2)^{\frac{n+1+\alpha}{p}} |f(\varphi(z))| < \varepsilon. \qquad (13) \end{aligned}$$

If $\delta < |z| < 1$ and $|\varphi(z)| \le \delta$, using Lemma 1 and (12) we have

$$\begin{aligned} \mu(|z|)|\Re(T_{g,\varphi}f)(z)| &= \frac{\mu(|z|)|\Re g(z)|}{(1-|\varphi(z)|^2)^{\frac{n+1+\alpha}{p}}}(1-|\varphi(z)|^2)^{\frac{n+1+\alpha}{p}}|f(\varphi(z))| \\ &\le C\|f\|_{A^p_\alpha}\frac{\mu(|z|)|\Re g(z)|}{(1-|\varphi(z)|^2)^{\frac{n+1+\alpha}{p}}} \\ &\le \frac{C\|f\|_{A^p_\alpha}}{(1-\delta^2)^{\frac{n+1+\alpha}{p}}}\mu(|z|)|\Re g(z)| < C\varepsilon. \qquad (14) \end{aligned}$$

Combining (13) with (14) we get that $T_{g,\varphi}f \in \mathcal{B}_{\mu,0}$. Since f is arbitrary we see that $T_{g,\varphi}(A^p_\alpha) \subset \mathcal{B}_{\mu,0}$, which together with the boundedness of $T_{g,\varphi}: A^p_\alpha \to \mathcal{B}_\mu$, we get the desired result. This completes the proof of the theorem. □

Theorem 4. *Assume that $p > 0$, α is a real number such that $n+\alpha+1 > 0$, $g \in H(B)$, φ is a holomorphic self-map of B and μ is a normal function on $[0,1)$. Then $T_{g,\varphi}: A^p_\alpha \to \mathcal{B}_{\mu,0}$ is compact if and only if*

$$\lim_{|z|\to 1}\frac{\mu(|z|)|\Re g(z)|}{(1-|\varphi(z)|^2)^{\frac{n+1+\alpha}{p}}} = 0. \qquad (15)$$

Proof. Suppose that $T_{g,\varphi}: A^p_\alpha \to \mathcal{B}_{\mu,0}$ is compact. Then $T_{g,\varphi}: A^p_\alpha \to \mathcal{B}_{\mu,0}$ is bounded and $T_{g,\varphi}: A^p_\alpha \to \mathcal{B}_\mu$ is compact. By Theorems 2 and 3 we obtain

$$\lim_{|\varphi(z)|\to 1}\frac{\mu(|z|)|\Re g(z)|}{(1-|\varphi(z)|^2)^{\frac{n+1+\alpha}{p}}} = 0. \qquad (16)$$

and

$$\lim_{|z|\to 1}\mu(|z|)|\Re g(z)| = 0, \qquad (17)$$

By (16), for every $\varepsilon > 0$, there exists a $\delta \in (0,1)$,

$$\frac{\mu(|z|)|\Re g(z)|}{(1-|\varphi(z)|^2)^{\frac{n+1+\alpha}{p}}} < \varepsilon$$

when $\delta < |\varphi(z)| < 1$. By (17), for the above ε, there exists $r \in (0,1)$,

$$\mu(|z|)|\Re g(z)| \le \varepsilon(1-|\delta|^2)^{\frac{n+1+\alpha}{p}}$$

when $r < |z| < 1$.

Therefore, when $r < |z| < 1$ and $\delta < |\varphi(z)| < 1$, we have that

$$\frac{\mu(|z|)|\Re g(z)|}{(1-|\varphi(z)|^2)^{\frac{n+1+\alpha}{p}}} < \varepsilon. \tag{18}$$

If $|\varphi(z)| \le \delta$ and $r < |z| < 1$, we obtain

$$\frac{\mu(|z|)|\Re g(z)|}{(1-|\varphi(z)|^2)^{\frac{n+1+\alpha}{p}}} \le \frac{1}{(1-|\delta|^2)^{\frac{n+1+\alpha}{p}}}\mu(|z|)|\Re g(z)| < \varepsilon. \tag{19}$$

Combing (18) with (19) we get (15) as desired.

Conversely, suppose that (15) holds. It follows from Lemma 2 that $T_{g,\varphi} : A^p_\alpha \to \mathcal{B}_{\mu,0}$ is compact if and only if

$$\lim_{|z|\to 1} \sup_{\|f\|_{A^p_\alpha}\le 1} \mu(|z|)|\Re(T_{g,\varphi}f)(z)| = 0. \tag{20}$$

For any $f \in A^p_\alpha$ with $\|f\|_{A^p_\alpha} \le 1$, by (3) we have

$$\mu(|z|)|\Re(T_{g,\varphi}f)(z)| \le C\|f\|_{A^p_\alpha}\frac{\mu(|z|)|\Re g(z)|}{(1-|\varphi(z)|^2)^{\frac{n+1+\alpha}{p}}}.$$

Using (15) we get

$$\lim_{|z|\to 1} \sup_{\|f\|_{A^p_\alpha}\le 1} \mu(|z|)|\Re(T_{g,\varphi}f)(z)| \le C \lim_{|z|\to 1}\frac{\mu(|z|)|\Re g(z)|}{(1-|\varphi(z)|^2)^{\frac{n+1+\alpha}{p}}} = 0,$$

as desired. This completes the proof of the theorem. □

Let $\varphi(z) = z$, $\mu(r) = (1-r^2)^\beta$. From Theorems 1-4 we have the following result (see [8, 9] for the case of $\alpha > -1$).

Corollary 1. *Assume that $p > 0$, α is a real number such that $n+\alpha+1 > 0$, $\frac{n+1+\alpha}{p} \le \beta < \infty$ and $g \in H(B)$. Then the following statements hold.*

(i) *$T_g : A^p_\alpha \to \mathcal{B}^\beta$ is bounded if and only if $T_g : A^p_\alpha \to \mathcal{B}^\beta_0$ is bounded if and only if*

$$\sup_{z\in B}(1-|z|^2)^{\beta-\frac{n+1+\alpha}{p}}|\Re g(z)| < \infty;$$

(ii) *$T_g : A^p_\alpha \to \mathcal{B}^\beta$ is compact if and only if $T_g : A^p_\alpha \to \mathcal{B}^\beta_0$ is compact if and only if*

$$\lim_{|z|\to 1}(1-|z|^2)^{\beta-\frac{n+1+\alpha}{p}}|\Re g(z)| = 0.$$

2.2 Case $n+1+\alpha=0$**.** First, we consider the case $p>1$.

Theorem 5. *Assume that* $p>1$, $1/p+1/q=1$ *and* $n+1+\alpha=0$, $g\in H(B)$, φ *is a holomorphic self-map of* B *and* μ *is a normal function on* $[0,1)$. *Then* $T_{g,\varphi}: A^p_\alpha\to\mathcal{B}_\mu$ *is bounded if and only if*

$$M_3 := \sup_{z\in B}\mu(|z|)|\Re g(z)|\Big(\ln\frac{2}{1-|\varphi(z)|^2}\Big)^{1/q}<\infty. \tag{21}$$

Proof. Suppose that (21) holds. Then for arbitrary $z\in B$ and $f\in A^p_\alpha$, by Lemma 1 we have

$$\begin{aligned}\mu(|z|)|\Re(T_{g,\varphi}f)(z)| &= \mu(|z|)|f(\varphi(z))||\Re g(z)|\\ &\le C\|f\|_{A^p_\alpha}\mu(|z|)|\Re g(z)|\Big(\ln\frac{2}{1-|\varphi(z)|^2}\Big)^{1/q},\end{aligned} \tag{22}$$

from which we see that $T_{g,\varphi}: A^p_\alpha\to\mathcal{B}_\mu$ is bounded.

Conversely, suppose that $T_{g,\varphi}: A^p_\alpha\to\mathcal{B}_\mu$ is bounded. For $a\in B$, set

$$f_a(z)=\Big(\ln\frac{2}{1-|a|^2}\Big)^{-1/p}\Big(\ln\frac{2}{1-\langle z,a\rangle}\Big). \tag{23}$$

Using Theorem 1.12 of [16], it is easy to check that $f_a\in A^p_{-(n+1)}$. Therefore

$$\begin{aligned}C\|T_{g,\varphi}\|_{A^p_\alpha\to\mathcal{B}_\mu} &\ge \|T_{g,\varphi}f_{\varphi(b)}\|_{\mathcal{B}_\mu}\ge\sup_{z\in B}\mu(|z|)|\Re(T_{g,\varphi}f_{\varphi(b)})(z)|\\ &\ge \mu(|b|)|\Re g(b)|\Big(\ln\frac{2}{1-|\varphi(b)|^2}\Big)^{1/q}.\end{aligned} \tag{24}$$

From the last inequality we get the desired result. □

Theorem 6. *Assume that* $p>1$, $1/p+1/q=1$ *and* $n+1+\alpha=0$, $g\in H(B)$, φ *is a holomorphic self-map of* B *and* μ *is a normal function on* $[0,1)$. *Then* $T_{g,\varphi}: A^p_\alpha\to\mathcal{B}_\mu$ *is compact if and only if* $g\in\mathcal{B}_\mu$ *and*

$$\lim_{|\varphi(z)|\to 1}\mu(|z|)|\Re g(z)|\Big(\ln\frac{2}{1-|\varphi(z)|^2}\Big)^{1/q}=0. \tag{25}$$

Proof. Suppose that (25) holds. In this case, the proof of Theorem 2 still works with minor changes and therefore the details are omitted.

Conversely, suppose that $T_{g,\varphi}: A^p_\alpha\to\mathcal{B}_\mu$ is compact, then it is clear that $T_{g,\varphi}: A^p_\alpha\to\mathcal{B}_\mu$ is bounded. It follows from the proof of Theorem 5 that $g\in\mathcal{B}_\mu$. Let $(z_k)_{k\in\mathbb{N}}$ be a sequence in B such that $|\varphi(z_k)|\to 1$ as $k\to\infty$.

Set

$$(26)\quad f_k(z) = \Big(\ln \frac{2}{1-|\varphi(z_k)|^2}\Big)^{-1/p}\Big(\ln \frac{2}{1-\langle z, \varphi(z_k)\rangle}\Big),\ k\in\mathbb{N}.$$

Using Theorem 1.12 of [16], we see that $(f_k)_{k\in\mathbb{N}}$ is a bounded sequence in A^p_α. Moreover, $f_k \to 0$ uniformly on compact subsects of B. It follows from Lemma 3 that $\|T_{g,\varphi}f_k\|_{\mathcal{B}_\mu} \to 0$ as $k\to\infty$. Since

$$\begin{aligned}\|T_{g,\varphi}f_k\|_{\mathcal{B}_\mu} &= \sup_{z\in B}\mu(|z|)|\Re(T_{g,\varphi}f_k)(z)|\\ &\geq \mu(|z_k|)|\Re g(z_k)|\Big(\ln\frac{2}{1-|\varphi(z_k)|^2}\Big)^{1/q},\end{aligned} \tag{27}$$

we obtain

$$\lim_{k\to\infty}\mu(|z_k|)|\Re g(z_k)|\Big(\ln\frac{2}{1-|\varphi(z_k)|^2}\Big)^{1/q} = 0,$$

from which we get the desired result. □

Theorem 7. *Assume that* $p>1$, $1/p+1/q=1$ *and* $n+1+\alpha=0$, $g\in H(B)$, φ *is a holomorphic self-map of* B *and* μ *is a normal function on* $[0,1)$. *Then* $T_{g,\varphi}: A^p_\alpha \to \mathcal{B}_{\mu,0}$ *is bounded if and only if* $T_{g,\varphi}: A^p_\alpha\to\mathcal{B}_\mu$ *is bounded and* $g\in\mathcal{B}_{\mu,0}$.

Proof. Suppose that $T_{g,\varphi}: A^p_\alpha\to\mathcal{B}_{\mu,0}$ is bounded, then $T_{g,\varphi}: A^p_\alpha\to\mathcal{B}_\mu$ is bounded. Taking $f(z)=1$, then employing the boundedness of $T_{g,\varphi}: A^p_\alpha\to\mathcal{B}_{\mu,0}$, we get $g\in\mathcal{B}_{\mu,0}$, as desired.

Conversely, suppose that $T_{g,\varphi}: A^p_\alpha\to\mathcal{B}_\mu$ is bounded and $g\in\mathcal{B}_{\mu,0}$. For each polynomial $p(z)$,

$$(28)\quad \mu(|z|)|\Re(T_{g,\varphi}p)(z)| = \mu(|z|)|p(\varphi(z))||\Re g(z)| \leq \|p\|_\infty \mu(|z|)|\Re g(z)|.$$

From the above inequality, it follows that for each polynomial p, $T_{g,\varphi}(p)\in\mathcal{B}_{\mu,0}$. Since the set of all polynomials is dense in A^p_α, for every $f\in A^p_\alpha$ there is a sequence of polynomials $(p_k)_{k\in\mathbb{N}}$ such that $\|p_k - f\|_{A^p_\alpha}\to 0$ as $k\to\infty$. From the boundedness of $T_{g,\varphi}: A^p_\alpha\to\mathcal{B}_\mu$, we have that

$$(29)\quad \|T_{g,\varphi}p_k - T_{g,\varphi}f\|_{\mathcal{B}_\mu} \leq \|T_{g,\varphi}\|\,\|p_k - f\|_{A^p_\alpha}\to 0,\quad \text{as } k\to\infty.$$

From this and since $\mathcal{B}_{\mu,0}$ is a closed subset of $\mathcal{B}_\mu$, we obtain

$$T_{g,\varphi}f = \lim_{k\to\infty} T_{g,\varphi}p_k \in \mathcal{B}_{\mu,0}. \tag{30}$$

Therefore $T_{g,\varphi}: A^p_\alpha\to\mathcal{B}_{\mu,0}$ is bounded. The proof is completed. □

Using Theorems 6 and 7, similarly to the proof of Theorem 4, we obtain the following result. We omit the proof.

Theorem 8. *Assume that* $p > 1$, $1/p + 1/q = 1$ *and* $n + 1 + \alpha = 0$, $g \in H(B)$, φ *is a holomorphic self-map of* B *and* μ *is a normal function on* $[0,1)$. *Then* $T_{g,\varphi} : A^p_\alpha \to \mathcal{B}_{\mu,0}$ *is compact if and only if*

$$\lim_{|z|\to 1} \mu(|z|)|\Re g(z)|\Big(\ln \frac{2}{1-|\varphi(z)|^2}\Big)^{1/q} = 0. \tag{31}$$

From Theorems 5-8, we have the following corollary.

Corollary 2. *Suppose* $p > 1$, $1/p + 1/q = 1$ *and* $n + 1 + \alpha = 0$. *Let* $g \in H(B)$ *and* $0 < \beta < \infty$. *Then the following statements hold.*

(i) $T_g : A^p_\alpha \to \mathcal{B}^\beta$ *is bounded if and only if* $T_g : A^p_\alpha \to \mathcal{B}^\beta_0$ *is bounded if and only if*

$$\sup_{z\in B}(1-|z|^2)^\beta|\Re g(z)|\Big(\ln \frac{2}{1-|z|^2}\Big)^{1/q} < \infty;$$

(ii) $T_g : A^p_\alpha \to \mathcal{B}^\beta$ *is compact if and only if* $T_g : A^p_\alpha \to \mathcal{B}^\beta_0$ *is compact if and only if*

$$\lim_{|z|\to 1}(1-|z|^2)^\beta|\Re g(z)|\Big(\ln \frac{2}{1-|z|^2}\Big)^{1/q} = 0.$$

Next we consider the case of $0 < p \le 1$.

Theorem 9. *Assume that* $n + 1 + \alpha = 0$ *and* $0 < p \le 1$, $g \in H(B)$, φ *is a holomorphic self-map of* B *and* μ *is a normal function on* $[0,1)$. *Then* $T_{g,\varphi} : A^p_\alpha \to \mathcal{B}_\mu$ *is bounded if and only if* $g \in \mathcal{B}_\mu$.

Proof. Suppose that $g \in \mathcal{B}_\mu$. For an $f \in A^p_\alpha$, by Lemma 1 we have

$$\begin{aligned}\sup_{z\in B}\mu(|z|)|\Re(T_{g,\varphi}f)(z)| &= \sup_{z\in B}\mu(|z|)|f(\varphi(z))||\Re g(z)|\\ &\le C\|f\|_{A^p_\alpha}\sup_{z\in B}\mu(|z|)|\Re g(z)|.\end{aligned}$$

From the above inequality we see that $T_{g,\varphi} : A^p_\alpha \to \mathcal{B}_\mu$ is bounded.

Conversely, suppose that $T_{g,\varphi} : A^p_\alpha \to \mathcal{B}_\mu$ is bounded. Taking $f(z) = 1$, then using the boundedness of $T_{g,\varphi} : A^p_\alpha \to \mathcal{B}_\mu$, we get the desired result. □

Theorem 10. *Assume that* $n + 1 + \alpha = 0$ *and* $0 < p \le 1$, $g \in H(B)$, φ *is a holomorphic self-map of* B *and* μ *is a normal function on* $[0,1)$. *Then*

$T_{g,\varphi} : A^p_\alpha \to \mathcal{B}_\mu$ is compact if and only if $g \in \mathcal{B}_\mu$ and

$$\lim_{|\varphi(z)|\to 1} \mu(|z|)|\Re g(z)| = 0. \tag{32}$$

Proof. Suppose that $g \in \mathcal{B}_\mu$ and that (32) holds. In this case, the proof is similar to the proof of Theorem 2 and hence we omit it.

Conversely, suppose that $T_{g,\varphi} : A^p_\alpha \to \mathcal{B}_\mu$ is compact. Then it is clear that $T_{g,\varphi} : A^p_\alpha \to \mathcal{B}_\mu$ is bounded. It follows from Theorem 9 that $g \in \mathcal{B}_\mu$. Let $(z_k)_{k\in\mathbb{N}}$ be a sequence in B such that $|\varphi(z_k)| \to 1$ as $k \to \infty$. Set

$$f_k(z) = \frac{1 - |\varphi(z_k)|^2}{1 - \langle z, \varphi(z_k)\rangle},\ k \in \mathbb{N}.$$

From Theorem 6.6 of [16] we see that $(f_k)_{k\in\mathbb{N}}$ is a bounded sequence in A^p_α. Moreover, f_k converges to zero uniformly on compact subsects of B. In view of Lemma 3 it follows that

$$\limsup_{k\to\infty} \|T_{g,\varphi} f_k\|_{\mathcal{B}_\mu} = 0. \tag{33}$$

On the other hand, we have

$$\|T_{g,\varphi} f_k\|_{\mathcal{B}_\mu} = \sup_{z\in B} \mu(|z|)|\Re(T_{g,\varphi} f_k)(z)| \ge \mu(|z_k|)|\Re g(z_k)|. \tag{34}$$

Combining (33) with (34) we see that (32) holds. The proof is completed. □

Theorem 11. *Assume that $n + 1 + \alpha = 0$ and $0 < p \le 1$, $g \in H(B)$, φ is a holomorphic self-map of B and μ is a normal function on $[0, 1)$. Then the following statements are equivalent.*

(i) *$T_{g,\varphi} : A^p_\alpha \to \mathcal{B}_{\mu,0}$ is bounded;*

(ii) *$T_{g,\varphi} : A^p_\alpha \to \mathcal{B}_{\mu,0}$ is compact;*

(iii) *$g \in \mathcal{B}_{\mu,0}$.*

Proof. $(ii) \Rightarrow (i)$. This implication is obvious.

$(i) \Rightarrow (iii)$. Taking $f(z) = 1$ and employing the boundedness of $T_{g,\varphi} : A^p_\alpha \to \mathcal{B}_{\mu,0}$ we get that $g \in \mathcal{B}_{\mu,0}$.

$(iii) \Rightarrow (ii)$. Suppose that $g \in \mathcal{B}_{\mu,0}$. For any $f \in A^p_\alpha$ with $\|f\|_{A^p_\alpha} \le 1$, we have

$$\mu(|z|)|\Re(T_{g,\varphi} f)(z)| \le C\|f\|_{A^p_\alpha} \mu(|z|)|\Re g(z)| \le C\mu(|z|)|\Re g(z)|,$$

from which we obtain

$$\lim_{|z|\to 1} \sup_{\|f\|_{A^p_\alpha}\le 1} \mu(|z|)|\Re(T_{g,\varphi}f)(z)| \le C \lim_{|z|\to 1} \mu(|z|)|\Re g(z)| = 0.$$

Using Lemma 2 we see that $T_{g,\varphi} : A^p_\alpha \to \mathcal{B}_{\mu,0}$ is compact and the assertion follows. □

From Theorems 9-11, we obtain the following corollary.

Corollary 3. *Suppose $0 < p \le 1$ and $n+1+\alpha = 0$. Let $g \in H(B)$ and $0 < \beta < \infty$. Then the following statements hold.*

(i) $T_g : A^p_\alpha \to \mathcal{B}^\beta$ *is bounded if and only if* $g \in \mathcal{B}^\beta$;

(ii) $T_g : A^p_\alpha \to \mathcal{B}^\beta$ *is compact if and only if* $T_g : A^p_\alpha \to \mathcal{B}^\beta_0$ *is bounded if and only if* $T_g : A^p_\alpha \to \mathcal{B}^\beta_0$ *is compact if and only if* $g \in \mathcal{B}^\beta_0$.

2.3. Case $n+1+\alpha < 0$.

Theorem 12. *Assume that $p > 0$ and $n+1+\alpha < 0$, $g \in H(B)$, φ is a holomorphic self-map of B and μ is a normal function on $[0,1)$. Then the following statements are equivalent.*

(i) $T_{g,\varphi} : A^p_\alpha \to \mathcal{B}_\mu$ *is bounded*;

(ii) $T_{g,\varphi} : A^p_\alpha \to \mathcal{B}_\mu$ *is compact*;

(iii) $g \in \mathcal{B}_\mu$.

Proof. $(ii) \Rightarrow (i)$. It is obvious.

$(i) \Rightarrow (iii)$. Taking $f(z) = 1$, then using the boundedness of $T_{g,\varphi} : A^p_\alpha \to \mathcal{B}_\mu$ we get the desired result.

$(iii) \Rightarrow (ii)$. Suppose that $g \in \mathcal{B}_\mu$. For an $f \in A^p_\alpha$, by Lemma 1 we see that f is continuous on the closed unit ball and so is bounded in B. Therefore

$$\mu(|z|)|\Re(T_{g,\varphi}f)(z)| = \mu(|z|)|f(\varphi(z))||\Re g(z)| \le C\|f\|_{A^p_\alpha}\mu(|z|)|\Re g(z)|. \tag{35}$$

From the above inequality we see that $T_{g,\varphi} : A^p_\alpha \to \mathcal{B}_\mu$ is bounded. Let $(f_k)_{k\in\mathbb{N}}$ be any bounded sequence in A^p_α and $f_k \to 0$ uniformly on $\overline{B}$ as $k \to \infty$. We have

$$\|T_{g,\varphi}f_k\|_{\mathcal{B}_\mu} = \sup_{z\in B} \mu(|z|)|f_k(\varphi(z))\Re g(z)| \le \|g\|_{\mathcal{B}_\mu} \sup_{z\in B} |f_k(\varphi(z))| \to 0,$$

as $k \to \infty$. Employing Lemma 4, the implication follows. □

Theorem 13. *Assume that $p > 0$ and $n+1+\alpha < 0$, $g \in H(B)$, φ is a holomorphic self-map of B and μ is a normal function on $[0,1)$. Then the following statements are equivalent.*

(i) $T_{g,\varphi} : A^p_\alpha \to \mathcal{B}_{\mu,0}$ *is bounded*;

(ii) $T_{g,\varphi} : A^p_\alpha \to \mathcal{B}_{\mu,0}$ *is compact*;

(iii) $g \in \mathcal{B}_{\mu,0}$.

Proof. The proof is similar to the proof of Theorem 11 and therefore we omit the details. □

From Theorems 12 and 13, we get the following corollary.

Corollary 4. *Suppose* $p > 0$ *and* $n + 1 + \alpha < 0$. *Let* $g \in H(B)$ *and* $0 < \beta < \infty$. *Then the following statements hold.*

(i) $T_g : A^p_\alpha \to \mathcal{B}^\beta$ *is bounded if and only if* $T_g : A^p_\alpha \to \mathcal{B}^\beta$ *is compact if and only if* $g \in \mathcal{B}^\beta$;

(ii) $T_g : A^p_\alpha \to \mathcal{B}^\beta_0$ *is bounded if and only if* $T_g : A^p_\alpha \to \mathcal{B}^\beta_0$ *is compact if and only if* $g \in \mathcal{B}^\beta_0$.

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