JOURNAL OF
FUNCTION SPACES AND APPLICATIONS
Volume 7, Number 3 (2009), 241–250

http://www.jfsa.net

The boundedness of commutator of Riesz transform associated with Schrödinger operators on a Hardy space

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(*Communicated by Vladimir Maz'ya*)

2000 Mathematics Subject Classification. Primary: 42B20, 42B30.

Keywords and phrases. Commutator, BMO, Riesz transform, Schrödinger operators, Hardy space.

Abstract. In this paper, we study the boundedness of commutator $[b,T]$ of Riesz transform associated with Schrödinger operator and b is BMO type function, note that the kernel of T has no smoothness, and the boundedness from $H_b^1(R^n) \to L^1(R^n)$ is obtained.

1. Introduction

It is well know that the Calderón-Zygmmund singular operator is an important operator in Harmonic Analysis. The properties of the C-Z singular operator and its commutator are studied by many scholars. Such as in [1] [2] [3]. Among this, C. Perez [3] states the $H_b^1(R^n) \to L^1(R^n)$ boundedness of the commutator $[b,T]$, where T is a C-Z singular operator and $b \in BMO(R^n)$.

Schrödinger differential operator is another interesting topic in Harmonic Analysis. Let $A = -\Delta + V(x)$ be the Schrödinger differential operator

[1]This work is partially supported by NNSF 107010178

on $R^n, n \geq 3$. Throughout the paper we will assume that $V(x)$ is a non-zero, nonnegative potential, and belongs to B_q for some $q > n/2$. Let $T = \nabla(-\Delta + V(x))$, $[b,T]f = bTf - Tbf$. The L^p boundedness of T and the commutator $[b,T]$ is widely studied in [4] [5] when $V(x)$ satisfies some conditions. The basic idea in [4] is to find a pointwise estimate of the kernel and the comparison to the kernel of classical Riesz transform. But in [6], Z. Guo, P. Li and L. Peng adopt a different idea to get the L^p boundedness of some commutators of Riesz transforms associated to Schrödinger operator since the kernel no longer satisfied the regular condition of Calderón-Zygmmund kernel. Note that the kernels have some other kind of smoothness $H(m)$. Inspired by their work, we will consider $H^1_b(R^n) \to L^1(R^n)$ boundedness of commutator $[b,T]$ in this case, where $b \in BMO$.

2. Some preliminaries and notations

In this section, we first recall some definitions and lemmas we need in this paper.

Q will always denote a cube with sides parallel to the axes. $\lambda Q(\lambda > 0)$ denotes the cube has the same center as Q and dilated by λ. Also $B = B(x_B,\ r)$ will denote a ball centered at x_B with radius r and corresponding notation applies for λB . We adopt the idea of Strömberg. Recall that the sharp function of Fefferman-Stein is defined by

$$M^\sharp f(x) = \sup_{x \in B} \frac{1}{|B|} \int_B |f(y) - f_B| dy,$$

simultaneity, recall that BMO is defined by

$$BMO(R^n) = \{f \in L^1_{loc}(R^n) : \|f\|_{BMO} = \|M^\sharp f\|_\infty < \infty\}.$$

where $f_B = \frac{1}{|B|}\int_B f(y)dy$, and the supremum is taken on all balls B with $x \in B$. Two basic facts about BMO will be used in this paper,

$$|f_{2^k B} - f_B| \leq C(k+1)\|f\|_{BMO},\ k > 0$$

and the one due to John-Nirenberg

$$\|f\|_{BMO} \sim \sup_B \left(\frac{1}{|B|} \int_B |f(y) - f_B|^p dy \right)^{1/p},\ p > 1.$$

In this paper we will assume that V belongs to B_q for some $q > \frac{n}{2}$, that is,

$$\left(\frac{1}{|B|}\int_B V^q(x)dx\right)^{1/q} \leq C\left(\frac{1}{|B|}\int_B V(x)dx\right),$$

for every ball $B \subset \mathbb{R}^n$. Define auxiliary function

$$\rho(x,V) = \rho(x) = \frac{1}{m(x,V)} = \sup\left\{r > 0 : \frac{1}{r^{n-2}}\int_{B(x,r)} V(y)dy \leq 1\right\}.$$

Definition 2.1. ([3]) A function a is a b - atom if there is a cube Q for which

$$(1)\,\mathrm{supp}\, a \subset Q,\ (2)\|a\|_{L^\infty} \leq \frac{1}{|\, Q\,|},\ (3)\int_Q a(y)dy = 0,\ (4)\int_Q a(y)b(y)dy = 0.$$

The space $H_b^1(\mathbb{R}^n)$ consists of the subspace of $L^1(\mathbb{R}^n)$ functions f which can be written as $f = \sum_j \lambda_j a_j$ where a_j are b - atom and λ_j are complex numbers with $\sum_j \mid \lambda_j \mid < \infty$ and define its space norm as $\|f\|_{H_b^1} = inf(\sum_j \mid \lambda_j \mid)$. Like the definition in [6], in our problem, we need the following smoothness of kernel.

Definition 2.2. $K(x,y)$ is said to satisfy $H(m)$ for some $m > 1$, if there exist a constant C such that, for any $l > 0,\ y,\ x_B \in R^n$ with $|y - x_B| \leq l$, then

$$\sum_{k=5}^{\infty} k(2^k l)^{n/m'}\left(\int_{2^k l \leq |x-x_B| < 2^{k+1} l} |K(x,y) - K(x,x_B)|^m dx\right)^{1/m} \leq C,$$

where $1/m' = 1 - 1/m$.

It is easy to prove that if $K(x,y)$ is the usual Calderón-Zygmund kernel, it satisfies $H(m)$ for any $m \geq 1$.

Lemma A. ([4]) *Suppose* $V \in B_{q_0}, q_0 > 1$. *Assume that* $-\Delta u + (V(x) + i\tau)u = 0$ *in* $B(x_0, 2R)$ *for some* $x_0 \in R^n,\ R > 0$. *Then*

(a) *for* $x \in B(x_0, R)$,

$$|\nabla u(x)| \leq C \sup_{B(x_0,2R)} |u| \cdot \int_{B(x_0,2R)} \frac{V(y)}{|x-y|^{n-1}}dy + \frac{C}{R^{n+1}}\int_{B(x_0,2R)} |u(y)|dy,$$

(b) *if* $(n/2) < q_0 < n,\ (1/t) = (1/q_0) - (1/n),\ k_0 > \log_2 C_0 + 1,$ then

$$\left(\int_{B(x_0,R)} |\nabla u|^t dx\right)^{1/t} \leq CR^{(n/q_0)-2}\{1+Rm(x_0,V)\}^{k_0} \sup_{B(x_0,2R)} |u|.$$

Lemma B. ([6]) *Suppose* $V \in B_q$ *for some* $q > n/2$. *Let* $N > \log_2 C_0 + 1$, *where* C_0 *is the constant in doubling measure inequality* $\int_{B(x,2r)} V(y)dy \leq C_0 \int_{B(x,r)} V(y)dy$. *Then for any* $x_0 \in R^n, R > 0$,

$$\frac{1}{\{1+m(x_0,V)R\}^N} \int_{B(x_0,R)} V(\xi)d\xi \leq CR^{n-2}.$$

Let $\Gamma(x,y,\tau)$ denote the fundamental solution for the Schrödinger operator $-\Delta + (V(x)+i\tau)$. A pointwise estimate of $\Gamma(x,y,\tau)$ given in [4] is a key result to our calculus.

Theorem A. ([4]) *Suppose* $V \in B_{n/2}$. *Then, for any* $x,\ y \in R^n,\ \tau \in R$, *and integer* $k > 0$,

$$\Gamma(x,y,\tau) \leq \frac{C_k}{\{1+|\tau|^{1/2}|x-y|\}^k\{1+m(x,V)|x-y|\}^k} \cdot \frac{1}{|x-y|^{n-2}}.$$

3. Main results and proofs

Let $T = \nabla(-\Delta+V)^{-1/2}$, we'll study the boundedness of commutator of T in Hardy space.

Theorem 3.1. *Let* $V \in B_q$ *and* $n/2 < q < n,\ b \in BMO$. *Then* $[b,\ T]$ *is a bounded operator from* $H_b^1(R^n)$ *to* $L^1(R^n)$.

To prove Theorem 3.1, we need the following lemmas.

Lemma 3.1. *Let* $T = \nabla(-\Delta+V)^{-1/2},\ K(x,y)$ *be kernel of* T. *Suppose* $V \in B_q$ *for some* $n/2 < q < n$. *Then there exist* $\delta > 0$ *and for any integer* $k > 0,\ 0 < h < |x-y|/16$,

$$(1)\qquad |K(x,y)| \leq \frac{C_k}{\{1+m(y,V)|x-y|\}^k} \cdot \frac{1}{|x-y|^{n-1}} \times \left(\int_{B(y,|x-y|)} \frac{V(\xi)}{|x-\xi|^{n-1}} d\xi + \frac{1}{|x-y|} \right)$$

$$(2)\qquad |K(x,y+h) - K(x,y)| \leq \frac{C_k}{\{1+m(y,V)|x-y|\}^k} \cdot \frac{|h|^\delta}{|x-y|^{n-1+\delta}} \times \left(\int_{B(y,|x-y|)} \frac{V(\xi)}{|x-\xi|^{n-1}} d\xi + \frac{1}{|x-y|} \right).$$

Proof. By partial integral, we know that

$$K(x,y) = -\frac{1}{2\pi} \int_R (-i\tau)^{-1/2} \nabla_x \Gamma(x,y,\tau) d\tau.$$

Fix $x,\ y \in R^n$, $R = \dfrac{|x-y|}{8}$, $1/t = 1/q - 1/n$, $\delta = 2 - n/q > 0$, $0 < h < R/2$, we have

$$|K(x,y+h) - K(x,y)| \leq \frac{1}{2\pi} \int_R |\tau|^{-1/2} |\nabla_x \Gamma(x,y+h,\tau) - \nabla_x \Gamma(x,y,\tau)| d\tau.$$

It follows from the imbedding theorem of Morrey and Lemma A(b) that

$$\begin{aligned} &|\nabla_x \Gamma(x,y+h,\tau) - \nabla_x \Gamma(x,y,\tau)| \\ &\quad \leq C|h|^{1-n/t} \left(\int_{B(x,R)} |\nabla_y \nabla_x \Gamma(x,z,\tau)|^t dz \right)^{1/t} \\ &\quad \leq C|h|^{1-n/t} R^{n/q-2} \{1 + Rm(y,V)\}^{R_0} \sup_{z \in B(y,2R)} |\nabla_x \Gamma(x,z,\tau)|. \end{aligned}$$

Since $\Gamma(x,z,\tau) = \Gamma(z,x,-\tau)$, we have $\nabla_x \Gamma(x,z,\tau) = \nabla_y \Gamma(z,x-\tau)$. It follows from Lemma A(a) that,

$$\begin{aligned} &\sup_{z \in B(y,2R)} |\nabla_x \Gamma(x,z,\tau)| \leq \sup_{z \in B(y,2R)} |\nabla_y \Gamma(z,x,-\tau)| \\ &\quad \leq \sup_{z \in B(y,2R)} \Bigg\{ \sup_{\eta \in B(z,|z-x|/4)} |\Gamma(\eta,y,-\tau)| \int_{B(z,|z-x|/2)} \frac{V(\xi)}{|z-\xi|^{n-1}} d\xi \\ &\qquad + \frac{C}{|z-x|^{n+1}} \int_{B(z,|z-x|/2)} \Gamma(\xi,x,-\tau) d\xi \Bigg\}. \end{aligned}$$

Also from Theorem A, [4, Lemma 1.4(b)], using the fact that $|\eta - x| \sim |z-x|, |\xi - x| \sim |z-x|,\ |z-x| \sim |x-y|, |z-\xi| \sim |x-\xi|$ and choosing k_1 sufficiently large, we obtain

$$\begin{aligned}
&\sup_{z\in B(y,2R)} |\nabla_x \Gamma(x,z,\tau)| \\
\le &\sup_{z\in B(y,2R)} \frac{C_{k_1}}{\{1+|\tau|^{1/2}|\eta-x|\}^{k_1}\{1+m(\eta,V)|\eta-x|\}^{k_1}} \\
&\times \frac{1}{|\eta-x|^{n-2}} \int_{B(z,|z-x|/2)} \frac{V(\xi)}{|z-\xi|^{n-1}} d\xi \\
&+ \frac{C_{k_1}}{|z-x|^{n+1}} \int_{B(z,|z-x|/2)} \frac{C_{k_1}}{\{1+|\tau|^{1/2}|\xi-x|\}^{k_1}\{1+m(\xi,V)|\xi-x|\}^{k_1}} \\
&\times \frac{1}{|\xi-x|^{n-2}} \\
\le &\sup_{z\in B(y,2R)} \frac{C_{k_1}}{\{1+|\tau|^{1/2}|x-y|\}^{k_1}\{1+m(y,V)|x-y|\}^{k_1}} \\
&\times \frac{1}{|x-y|^{n-2}} \int_{B(y,|x-y|)} \frac{V(\xi)}{|x-\xi|^{n-1}} d\xi \\
&+ \frac{C_{k_1}}{|x-y|^{n-1}} \cdot \frac{C_{k_1}}{\{1+|\tau|^{1/2}|x-y|\}^{k_1}\{1+m(y,V)|x-y|\}^{k_1}}.
\end{aligned}$$

Computing as in the proof of Lemma 4 in [6], the assertion is proved. □

Lemma 3.2. Let $T = \nabla(-\Delta+V)^{-1/2},\ V \in B_q$ for some $n/2 < q < n$. $K(x,y)$ be a kernel of T. Then $K(x,y)$ satisfies $H(m)$, where $1/m = 1/q - 1/n$.

Proof. For any $l > 0, y, x_B \in R^n$ with $|y - x_B| \le l$, choosing N sufficiently large, by Lemma B, (3.1) and $V \in B_q$, we have

$$\begin{aligned}
&\left(\int_{2^k l \le |x-x_B| < 2^{k+1} l} |K(x,y) - K(x,x_B)|^m dx\right)^{1/m} \\
\le &\left(\int_{2^k l \le |x-x_B| < 2^{k+1} l} \Big| \frac{C_N}{\{1+m(x_B,V)|x-x_B|\}^N} \right. \\
&\left. \times \frac{|y-x_B|^\delta}{|x-x_B|^{n-1+\delta}} \Big(\int_{B(x,|x-y|)} \frac{V(\xi)}{|x-\xi|^{n-1}} d\xi + \frac{1}{|x-x_B|} \Big) \Big|^m dx \right)^{1/m}
\end{aligned}$$

$$
\begin{aligned}
\leq\quad & C_N \frac{l^\delta}{(2^k l)^{n-1+\delta}} \cdot \frac{1}{\{1+m(x_B,V)2^k l\}^N} \left\| \int \frac{V(\xi)\chi_{B(x_B,2^{k+3}l)}}{|x-\xi|^{n-1}} d\xi \right\|_{L^m_x} \\
& + \frac{l^\delta}{(2^k l)^{n/m'+\delta}} \\
\leq\quad & C_N \frac{l^\delta}{(2^k l)^{n-1+\delta}} \cdot \frac{1}{\{1+m(x_B,V)2^k l\}^N} \left(\int_{B(x_B,2^{k+3}l)} V(\xi)^q d\xi \right)^{1/q} \\
& + \frac{l^\delta}{(2^k l)^{n/m'+\delta}} \\
\leq\quad & C_N \frac{l^\delta}{(2^k l)^{n-1+\delta}} \cdot \frac{1}{\{1+m(x_B,V)2^k l\}^N} \left(\int_{B(x_B,2^{k+3}l)} V(\xi) d\xi \right) \cdot (2^k l)^{-n/q'} \\
& + \frac{l^\delta}{(2^k l)^{n/m'+\delta}} \\
\leq\quad & C_N \frac{l^\delta}{(2^k l)^{n-1+\delta}} (2^k l)^{n/q-2} + \frac{l^\delta}{(2^k l)^{n/m'+\delta}} \\
\leq\quad & C \frac{l^\delta}{(2^k l)^{n/m'+\delta}}.
\end{aligned}
$$

Here we have used the fact that $B(x,|x-y|) \subset B(x_B, 2^{k+3}l)$. In fact, for all $\xi \in B(x,|x-y|)$, $|\xi - x_B| \leq |\xi - x| + |x - x_B| \leq |x-y| + |x-x_B| \leq |x-x_B| + |y-x_B| + |x-x_B| \leq 2^{k+3}$. Therefore,

$$
\sum_{k=5}^{\infty} k(2^k l)^{n/m'} \left(\int_{2^k l \leq |x-x_B| < 2^{k+1} l} |K(x,y) - K(x,x_B)|^m dx \right)^{1/m} \leq \sum_{k=5}^{\infty} \frac{Ck}{(2^k)^\delta} \leq C
$$

and we are done. □

Proof of Theorem 3.1. Let $b \in BMO$. By the atomic decomposition of Hardy space, we only need to prove that there exists a constant C such that for each $b-atom\ a$

$$
\int_{R^n} |[b,\ T]a(y)| dy \leq C \|b\|_{BMO} \|a\|_{H^1_b(R^n)}.
$$

Suppose $\operatorname{supp} a \subset B(x_B, l)$ for some ball B. Then

$$
\int_{R^n} |\,[b,\ T]a(y)\,|\, dy = \int_{2B} |\,[b,\ T]a(y)\,|\, dy + \int_{R^n \setminus 2B} |\,[b,\ T]a(y)\,|\, dy = \mathrm{I} + \mathrm{II}.
$$

The estimate of I follows by the boundedness of $[b,\ T]$ on $L^2(R^n)$ (see [5]) and the size condition of atom a, i.e.,

$$\begin{aligned}
\mathrm{I} &\le C|B|\Big(\frac{1}{|2B|}\int_{2B}|[b,\ T]a(y)|^2dy\Big)^{1/2} \le C\|b\|_{BMO}|B|\cdot\Big(\frac{1}{|B|}\int_B|a(y)|^2dy\Big)^{1/2}\\
&\le C\|b\|_{BMO}|B|\cdot\|a\|_\infty \le C\|b\|_{BMO}.
\end{aligned}$$

To estimate II, we split $[b,\ T]$ as $[b,\ T]a = (b-b_B)Ta - T((b-b_B)a)$. Then

$$\mathrm{II} \le \int_{R^n\backslash 2B}|(b(x)-b_B)Ta(x)|dx + \int_{R^n\backslash 2B}|T((b-b_B)a)(x)|dx = \mathrm{III}+\mathrm{IV}.$$

By Lemma 3.2 and cancelation condition, $\int_B a(y)dy = 0$, so that

$$\begin{aligned}
\mathrm{III} &= \int_{R^n\backslash 2B}|(b(x)-b_B)Ta(x)|dx\\
&\le \int_{R^n\backslash 2B}\Big|(b(x)-b_B)\int_B a(y)(K(x,y)-K(x,x_B))dy\Big|dx\\
&\le \int_B\int_{R^n\backslash 2B}|(b(x)-b_B)a(y)(K(x,y)-K(x,x_B))|dxdy\\
&\le \int_B\sum_{k=1}^{\infty}\int_{2^kl\le|x-x_B|<2^{k+1}l}|(b(x)-b_B)a(y)(K(x,y)-K(x,x_B))|dxdy\\
&\le \int_B\sum_{k=1}^{\infty}(2^kl)^{n/p}k\left(\int_{2^kl\le|x-x_B|<2^{k+1}l}|K(x,y)-K(x,x_B)|^qdx\right)^{1/q}\\
&\quad\times\frac{1}{(2^kl)^{n/p}k}\left(\int_{2^kl\le|x-x_B|<2^{k+1}l}|b(x)-b_B|^pdx\right)^{1/p}|a(y)|dy\\
&\le C\sup_{k\ge1}\frac{1}{k}\int_B|a(y)|dy\cdot\frac{1}{(2^kl)^{n/p}}\left(\int_{2^kl\le|x-x_B|<2^{k+1}l}|b(x)-b_B|^pdx\right)^{1/p}\\
&\le C\sup_{k\ge1}\frac{1}{k}\int_B|a(y)|dy\\
&\quad\times\left(\frac{1}{(2^{k+1}l)^n}\int_{B(x_B,2^{k+1}l)}|b(x)-b_{2^{k+1}B}+b_{2^{k+1}B}-b_B|^pdx\right)^{1/p}
\end{aligned}$$

$$\begin{aligned}
&\le C\sup_{k\ge 1}\frac{1}{k}\int_B |a(y)|dy\\
&\quad\times\left\{\left(\frac{1}{(2^{k+1}l)^n}\int_{B(x_B,2^{k+1}l)}|b(x)-b_{2^{k+1}B}|^p dx\right)^{1/p}+|b_{2^{k+1}B}-b_B|\right\}\\
&\le C\sup_{k\ge 1}\frac{1}{k}\int_B |a(y)|dy\cdot(k+2)\|b\|_{BMO}\\
&\le C\|b\|_{BMO},
\end{aligned}$$

where $1/p+1/q=1$.

By the definition of a, we have

$$\int_B (b(y)-b_B)a(y)dy=\int_B a(y)b(y)dy-b_B\int_B a(y)=0,$$

For IV, by Lemma 3.2 and Hölder inequality, we have

$$\begin{aligned}
\mathrm{IV} &= \int_{R^n\setminus 2B}|T((b-b_B)a)(x)|dx\\
&= \int_{R^n\setminus 2B}\Big|\int_B K(x,y)(b(y)-b_B)a(y)dy\Big|dx\\
&\le \int_{R^n\setminus 2B}\Big|\int_B (K(x,y)-K(x,x_B))(b(y)-b_B)a(y)dy\Big|dx\\
&\le \int_B |b(y)-b_B)||a(y)|\int_{R^n\setminus 2B}|K(x,y)-K(x,x_B)|dxdy\\
&\le \int_B |b(y)-b_B||a(y)|\sum_{k=1}^{\infty}\int_{2^k l\le|x-x_B|<2^{k+1}l}|K(x,y)-K(x,x_B)|dxdy\\
&\le \int |b(y)-b_B||a(y)|\sum_{k=1}^{\infty}\left(\int_{2^k l\le|x-x_B|<2^{k+1}l}|K(x,y)-K(x,x_B)|^p dx\right)^{1/p}\\
&\quad\times\left(\int_{2^k l\le|x-x_B|<2^{k+1}l}dx\right)^{1/q}dy
\end{aligned}$$

$$\begin{aligned}
&\leq \int_B |b(y)-b_B||a(y)| \sum_{k=1}^{\infty} k(2^k l)^{n/q} \\
&\quad \times \left(\int_{2^k l \leq |x-x_B| < 2^{k+1} l} |K(x,y)-K(x,x_B)|^p dx \right)^{1/p} \cdot \frac{1}{k} dy \\
&\leq C \sup_{k\geq 1} \frac{1}{k} \cdot \frac{1}{|B|} \int_B |b(y)-b_B| dy \leq C\|b\|_{BMO}.
\end{aligned}$$

This is the proof of Theorem 3.3. □

Acknowledgment. The authors wish to express their deep thanks to the referee for many valuable comments.

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(*Received : April 2008*)


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