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Endpoint estimates for homogeneous Littlewood-Paley g-functions with non-doubling measures

Dachun Yang and Dongyong Yang

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Abstract. Let μ be a nonnegative Radon measure on $\mathbb{R}^d$ which satisfies the growth condition that there exist constants $C_0 > 0$ and $n \in (0, d]$ such that for all $x \in \mathbb{R}^d$ and $r > 0$, $\mu(B(x, r)) \le C_0 r^n$, where $B(x, r)$ is the open ball centered at x and having radius r. In this paper, when $\mathbb{R}^d$ is not an initial cube which implies $\mu(\mathbb{R}^d) = \infty$, the authors prove that the homogeneous Littlewood-Paley g-function of Tolsa is bounded from the Hardy space $H^1(\mu)$ to $L^1(\mu)$, and furthermore, that if $f \in \mathrm{RBMO}\,(\mu)$, then $[\dot{g}(f)]^2$ is either infinite everywhere or finite almost everywhere, and in the latter case, $[\dot{g}(f)]^2$ belongs to $\mathrm{RBLO}\,(\mu)$ with norm no more than $C\|f\|^2_{\mathrm{RBMO}\,(\mu)}$, where $C > 0$ is independent of f.

1. Introduction

Recall that a *non-doubling measure* μ on $\mathbb{R}^d$ means that μ is a nonnegative Radon measure which only satisfies the following growth condition, namely, there exist constants $C_0 > 0$ and $n \in (0, d]$ such that for all $x \in \mathbb{R}^d$ and $r > 0$,

$$\mu\Big(B(x,r)\Big) \le C_0 r^n, \tag{1.1}$$

where $B(x,r)$ is the open ball centered at x and having radius r. Such a measure μ is not necessary to be doubling, which is a key assumption in the classical theory of harmonic analysis. In recent years, it was shown that many results on the Calderón-Zygmund theory remain valid for non-doubling measures; see, for example, [6, 7, 8, 9, 10, 11, 12, 5, 3]. One of the main motivations for extending the classical theory to the non-doubling context was the solution of several questions related to analytic capacity, like Vitushkin's conjecture or Painlevé's problem; see [13, 14, 16] or survey papers [15, 17, 18] for more details.

In particular, Tolsa [11] developed a Littlewood-Paley theory with non-doubling measures for functions in $L^p(\mu)$ when $p \in (1, \infty)$ and used this Littlewood-Paley decomposition to establish some $T(1)$ theorems. The main purpose of this paper is to investigate the behaviors of the homogeneous Littlewood-Paley g-functions of Tolsa in [11] at the extremal cases, namely, in the cases when $p = 1$ or $p = \infty$. To be precise, in this paper, *when* $\mathbb{R}^d$ *is not an initial cube which implies* $\mu(\mathbb{R}^d) = \infty$ (see [11]), we prove that the homogeneous Littlewood-Paley g-function $\dot{g}(f)$ of Tolsa is bounded from the Hardy space $H^1(\mu)$ to $L^1(\mu)$, and furthermore, we prove that if $f \in \mathrm{RBMO}\,(\mu)$, then $[\dot{g}(f)]^2$ is either infinite everywhere or finite almost everywhere, and in the latter case, $[\dot{g}(f)]^2$ is bounded from $\mathrm{RBMO}\,(\mu)$ to $\mathrm{RBLO}\,(\mu)$, where $\mathrm{RBMO}\,(\mu)$ was introduced by Tolsa in [10] and $\mathrm{RBLO}\,(\mu)$ was introduced by Jiang in [3]. Notice that $L^\infty(\mu) \subset \mathrm{RBMO}\,(\mu)$. The last above-mentioned result generalizes the corresponding result of Leckband [4] in replacing $L^\infty(\mathbb{R}^d)$ by $\mathrm{BMO}\,(\mathbb{R}^d)$, even when μ is the d-dimensional Lebesgue measure and $\dot{g}(f)$ is the classical homogeneous Littlewood-Paley g-function. *When* $\mu(\mathbb{R}^d) < \infty$, then $\mathbb{R}^d$ is an initial cube (see [11]) and the homogeneous Littlewood-Paley g-function degenerates into the inhomogeneous Littlewood-Paley g-function $g(f)$. We also obtain similar results for this inhomogeneous Littlewood-Paley g-function, by first establishing a new theory of local atomic Hardy space $h^{1,\infty}_{\mathrm{atb}}(\mu)$, $\mathrm{rbmo}\,(\mu)$ and $\mathrm{rblo}\,(\mu)$ in the sense of Goldberg [1]. To limit the length of this paper, we will present these results in [2]. An interesting open problem is if $\dot{g}(f)$ and $g(f)$ can characterize the Hardy space $H^1(\mu)$ and $h^{1,\infty}_{\mathrm{atb}}(\mu)$, respectively.

The organization of this paper is as follows. In Section 2, we recall some necessary definitions and notation, including the definitions of atomic Hardy spaces, $\mathrm{RBMO}\,(\mu)$, $\mathrm{RBLO}\,(\mu)$, approximations to the identity and the homogeneous Littlewood-Paley g-function $\dot{g}(f)$. In Section 3, we establish the boundedness of the homogeneous Littlewood-Paley g-function $\dot{g}(f)$ from $H^1(\mu)$ to $L^1(\mu)$, and prove that if f belongs to $\mathrm{RBMO}\,(\mu)$, then $[\dot{g}(f)]^2$ is either infinite everywhere or finite almost everywhere, and in the latter case, $[\dot{g}(f)]^2$ belongs to $\mathrm{RBLO}\,(\mu)$ with norm no more than $C\|f\|^2_{\mathrm{RBMO}\,(\mu)}$, where $C>0$ is independent of f. As a corollary, we also obtain the boundedness of the homogeneous Littlewood-Paley g-function $\dot{g}(f)$ from $\mathrm{RBMO}\,(\mu)$ to $\mathrm{RBLO}\,(\mu)$.

Throughout the paper, we always denote by C a *positive constant* which is independent of the main parameters, but it may vary from line to line. *Constant with subscript* such as C_1, does not change in different occurrences. The *symbol* $Y \lesssim Z$ means that there exists a constant $C>0$ such that $Y \le CZ$. *The symbol* $A \sim B$ means that $A \lesssim B \lesssim A$. Moreover, for any $D \subset \mathbb{R}^d$, we denote by $\chi_{_D}$ the *characteristic function of* D.

2. Preliminaries

In this section, we recall some necessary notions and notation. By a cube $Q \subset \mathbb{R}^d$, we mean a closed cube whose sides are parallel to the axes and centered at some point of $\mathrm{supp}\,(\mu)$, and we denote its side length by $l(Q)$ and its center by x_Q. If $\mu(\mathbb{R}^d)<\infty$, we also regard $\mathbb{R}^d$ as a cube. Let α, β be two positive constants, $\alpha \in (1,\infty)$ and $\beta \in (\alpha^n,\infty)$. A cube Q is said to be an (α,β)-*doubling cube* if it satisfies $\mu(\alpha Q) \le \beta\mu(Q)$, where and in what follows, given $\lambda>0$ and any cube Q, λQ denotes the *cube concentric with* Q *and having side length* $\lambda l(Q)$. It was pointed out by Tolsa (see [10, pp. 95-96] or [11, Remark 3.1]) that if $\beta>\alpha^n$, then for any $x \in \mathrm{supp}\,(\mu)$ and any $R>0$, there exists some (α,β)-doubling cube Q centered at x with $l(Q)\ge R$, and that if $\beta>\alpha^d$, then for μ-almost everywhere $x\in\mathbb{R}^d$, there exists a sequence of (α,β)-doubling cubes $\{Q_k\}_{k\in\mathbb{N}}$ centered at x with $l(Q_k)\to 0$ as $k\to\infty$. In what follows, by a *doubling cube*, we always mean a $(2,2^{d+1})$-doubling cube, and for any cube Q, we denote by $\widetilde{Q}$ the *smallest doubling cube* which has the form 2^kQ with $k\in\mathbb{N}\cup\{0\}$.

Given two cubes Q, $R\subset\mathbb{R}^d$, let x_Q be the center of Q, and Q_R be the smallest cube concentric with Q containing Q and R. The following coefficients were first introduced by Tolsa in [10]; see also [11, 12].

Definition 2.1. Given two cubes Q, $R\subset\mathbb{R}^d$, we define

$$\delta(Q,R)=\max\left\{\int_{Q_R\setminus Q}\frac{1}{|x-x_Q|^n}\,d\mu(x),\ \int_{R_Q\setminus R}\frac{1}{|x-x_R|^n}\,d\mu(x)\right\}.$$

We may treat points $x \in \mathbb{R}^d$ as if they were cubes (with side length $l(x) = 0$). So, for x, $y \in \mathbb{R}^d$ and some cube Q, the notations $\delta(x, Q)$ and $\delta(x, y)$ make sense; see [11, 12] for some useful properties of $\delta(\cdot, \cdot)$. We now recall the notion of cubes of generations in [11, 12]; see [11, 12] for more details.

Definition 2.2. We say that $x \in \mathbb{R}^d$ is a *stopping point* (or *stopping cube*) if $\delta(x, Q) < \infty$ for some cube $Q \ni x$ with $0 < l(Q) < \infty$. We say that $\mathbb{R}^d$ is an *initial cube* if $\delta(Q, \mathbb{R}^d) < \infty$ for some cube Q with $0 < l(Q) < \infty$. The cubes Q such that $0 < l(Q) < \infty$ are called *transit cubes*.

Remark 2.1. In [11, p. 67], it was pointed out that if $\delta(x, Q) < \infty$ for some transit cube Q containing x, then $\delta(x, Q') < \infty$ for any other transit cube Q' containing x. Also, if $\delta(Q, \mathbb{R}^d) < \infty$ for some transit cube Q, then $\delta(Q', \mathbb{R}^d) < \infty$ for any transit cube Q'.

Throughout this paper, we *always assume that* $\mathbb{R}^d$ *is not an initial cube.*

Let A be some big positive constant. In particular, we assume that A is much bigger than the constants ϵ_0, ϵ_1 and γ_0, which appear, respectively, in Lemma 3.1, Lemma 3.2 and Lemma 3.3 of [11]. Moreover, the constants A, ϵ_0, ϵ_1 and γ_0 depend only on C_0, n and d. In what follows, for $\epsilon > 0$ and $a, b \in \mathbb{R}$, the *notation* $a = b \pm \epsilon$ does not mean any precise equality but the estimate $|a - b| \leq \epsilon$.

Definition 2.3. Assume that $\mathbb{R}^d$ is not an initial cube. We fix some doubling cube $R_0 \subset \mathbb{R}^d$. This will be our *'reference' cube.* For each $j \in N$, let R_{-j} be some doubling cube concentric with R_0, containing R_0, and such that $\delta(R_0, R_{-j}) = jA \pm \epsilon_1$ (which exists because of Lemma 3.3 of [11]). If Q is a transit cube, we say that Q is a *cube of generation* $k \in \mathbb{Z}$ if it is a doubling cube, and for some cube R_{-j} containing Q we have $\delta(Q, R_{-j}) = (j + k)A \pm \epsilon_1$. If $Q \equiv \{x\}$ is a stopping cube, we say that Q is a *cube of generation* $k \in \mathbb{Z}$ if for some cube R_{-j} containing x we have $\delta(Q, R_{-j}) \leq (j + k)A + \epsilon_1$.

Using Lemma 3.2 in [11], it is easy to verify that for any $x \in \operatorname{supp}(\mu)$ and $k \in \mathbb{Z}$, there exists a doubling cube of generation k; see [11, p. 68]. Moreover, the definition of cubes of generations is proved in [11, p. 68] to be independent of the chosen reference R_{-j} in the sense modulo some small errors. Throughout this paper, for any $x \in \operatorname{supp}(\mu)$ and $k \in \mathbb{Z}$, we denote by $Q_{x,k}$ a *fixed doubling cube centered at* x *of generation* k. On cubes of generations $\{Q_{x,k}\}_{k \in \mathbb{Z}}$, we have the following simple observation.

Proposition 2.1. *Suppose that* $\mathbb{R}^d$ *is not an initial cube. Then for any* $x \in \operatorname{supp}(\mu)$, $l(Q_{x,k}) \to \infty$ *as* $k \to -\infty$.

Proof. For any given $x \in \operatorname{supp}(\mu)$, we first assume that $\{x\}$ is not a stopping cube. Then for any $N \in \mathbb{N}$, $Q_{x,0}$ and $Q_{x,-N}$ are transit cubes (see [11, p. 68]) satisfying that $Q_{x,0} \subset Q_{x,-N}$ and $\delta(Q_{x,0}, Q_{x,-N}) = NA \pm 6\epsilon_1$. In fact, by Definition 2.3, there exist $j_1, j_2 \in \mathbb{N}$ such that $Q_{x,0} \subset R_{-j_1}$ with $\delta(Q_{x,0}, R_{-j_1}) = j_1 A \pm \epsilon_1$ and $Q_{x,-N} \subset R_{-j_2}$ with $\delta(Q_{x,-N}, R_{-j_2}) = (j_2 - N)A \pm \epsilon_1$. Choosing $j \geq \max(j_1, j_2)$ and using Lemma 3.1 (d) in [11] imply that $\delta(Q_{x,0}, R_{-j}) = jA \pm 3\epsilon_1$ and $\delta(Q_{x,-N}, R_{-j}) = (j-N)A \pm 3\epsilon_1$. By the fact that $Q_{x,0} \subset Q_{x,-N} \subset R_{-j}$, it follows from Lemma 3.1 (d) in [11] again that

$$\delta(Q_{x,0}, Q_{x,-N}) = \delta(Q_{x,0}, R_{-j}) - \delta(Q_{x,-N}, R_{-j}) = NA \pm 6\epsilon_1.$$

Since $\{l(Q_{x,k})\}_{k\in\mathbb{Z}}$ is decreasing, if the conclusion of Proposition 2.1 is not true, then there exists $M > 0$ such that for any $N \in \mathbb{N}$, $l(Q_{x,-N}) \leq Ml(Q_{x,0})$. Lemma 3.1 (c) in [11] shows that there exists a constant C_d depending only on d such that

$$\delta(Q_{x,0}, Q_{x,-N}) \leq C_d\Big(1 + \log \frac{l(Q_{x,-N})}{l(Q_{x,0})}\Big) \leq C_d(1 + \log M).$$

On the other hand, since $\epsilon_1 \ll A$, then $NA \pm 6\epsilon_1 > NA/2$. Therefore, if we take $N > 2C_d(1 + \log M)/A$, we then have a contradiction that

$$C_d(1 + \log M) < \frac{1}{2}NA < NA \pm 6\epsilon_1 = \delta(Q_{x,0}, Q_{x,-N}) \leq C_d(1 + \log M),$$

which implies that the conclusion of Proposition 2.1 is true in the case that $\{x\}$ is not a stopping cube.

If $\{x\}$ is a stopping cube, recalling that there exists some $k_x \in \mathbb{Z}$ such that all the cubes of generation $k < k_x$ are transit cubes (see [11, p. 68]), we obtain that for $N \in \mathbb{N}$ large enough, $Q_{x,k_x-1} \subset Q_{x,-N}$ and $\delta(Q_{x,k_x-1}, Q_{x,-N}) = (N + k_x - 1)A \pm 6\epsilon_1$ via an argument as above. Furthermore, if there exists $M > 0$ such that for any $N \in \mathbb{N}$, $l(Q_{x,-N}) \leq Ml(Q_{x,k_x-1})$, then by taking $N > 2\max(k_x - 1, C_d(1 + \log M)/A)$ together with an argument as above, we also have a contradiction, which implies that $l(Q_{x,k}) \to \infty$ as $k \to -\infty$. This finishes the proof of Proposition 2.1. $\square$

In [11], Tolsa constructed a class of *approximations to the identity* $\{S_k\}_{k=-\infty}^{\infty}$ related to $\{Q_{x,k}\}_{x\in\mathbb{R}^d, k\in\mathbb{Z}}$, which are integral operators given by kernels $S_k(x, y)$ on $\mathbb{R}^d \times \mathbb{R}^d$ satisfying the following properties:

(A-1) $S_k(x, y) = S_k(y, x)$ for all $x, y \in \mathbb{R}^d$;

(A-2) For any $k \in \mathbb{Z}$ and any $x \in \operatorname{supp}(\mu)$, if $Q_{x,k}$ is a transit cube, then

$$\int_{\mathbb{R}^d} S_k(x,y)\,d\mu(y) = 1;$$

(A-3) If $Q_{x,k}$ is a transit cube, then $\operatorname{supp}(S_k(x,\cdot)) \subset Q_{x,k-1}$;

(A-4) If $Q_{x,k}$ and $Q_{y,k}$ are transit cubes, then there exists a constant $C > 0$ such that

$$0 \le S_k(x,y) \le \frac{C}{[l(Q_{x,k}) + l(Q_{y,k}) + |x-y|]^n}; \tag{2.1}$$

(A-5) If $Q_{x,k}$, $Q_{x',k}$ and $Q_{y,k}$ are transit cubes, and $x,\, x' \in Q_{x_0,k}$ for some $x_0 \in \operatorname{supp}(\mu)$, then there exists a constant $C > 0$ such that

$$|S_k(x,y) - S_k(x',y)| \le C \frac{|x-x'|}{l(Q_{x_0,k})} \frac{1}{[l(Q_{x,k}) + l(Q_{y,k}) + |x-y|]^n}. \tag{2.2}$$

Moreover, Tolsa [11] pointed out that Properties (A-1) through (A-5) also hold if any of $Q_{x,k}$, $Q_{x',k}$ and $Q_{y,k}$ is a stopping cube. In what follows, without loss of generality, for any $x \in \operatorname{supp}(\mu)$, we *always assume that* $Q_{x,k}$ *is not a stopping cube*, since the proofs for stopping cubes are similar.

For any $k \in \mathbb{Z}$, $f \in L^1_{\text{loc}}(\mu)$ and $x \in \operatorname{supp}(\mu)$, define

$$S_k f(x) = \int_{\mathbb{R}^d} S_k(x,y) f(y)\,d\mu(y).$$

Let $D_k = S_k - S_{k-1}$ for $k \in \mathbb{Z}$, and we also use D_k to denote the corresponding *integral operator with kernel* D_k. The *homogeneous Littlewood-Paley* g*-function* $\dot{g}(f)$ is then defined by

$$\dot{g}(f)(x) = \left[\sum_{k=-\infty}^{\infty} |D_k f(x)|^2\right]^{1/2}.$$

We next recall the notions of the atomic Hardy space $H^{1,p}_{\text{atb}}(\mu)$ for $p \in (1,\infty]$ and the BMO-type space $\operatorname{RBMO}(\mu)$ in [10] and $\operatorname{RBLO}(\mu)$ in [3].

Definition 2.4. Let $\eta > 1$ and $1 < p \le \infty$. A function $b \in L^1_{\text{loc}}(\mu)$ is called a p*-atomic block* if

(1) there exists some cube R such that $\operatorname{supp}(b) \subset R$,

(2) $\int_{\mathbb{R}^d} b(x)\,d\mu(x) = 0$,

(3) for $j = 1, 2$, there exist functions a_j supported on cubes $Q_j \subset R$ and numbers $\lambda_j \in \mathbb{R}$ such that $b = \lambda_1 a_1 + \lambda_2 a_2$, and

$$\|a_j\|_{L^p(\mu)} \le [\mu(\eta Q_j)]^{1/p-1}[1+\delta(Q_j,\, R)]^{-1}. \tag{2.3}$$

Then we define $|b|_{H^{1,\,p}_{\mathrm{atb}}(\mu)} = |\lambda_1| + |\lambda_2|$. We say that $f \in H^{1,\,p}_{\mathrm{atb}}(\mu)$ if there exist p-atomic blocks $\{b_i\}_{i\in\mathbb{N}}$ such that $f = \sum_{i=1}^\infty b_i$ with $\sum_{i=1}^\infty |b_i|_{H^{1,\,p}_{\mathrm{atb}}(\mu)} < \infty$. The $H^{1,\,p}_{\mathrm{atb}}(\mu)$ *norm of* f is defined by $\|f\|_{H^{1,\,p}_{\mathrm{atb}}(\mu)} = \inf\{\sum_{i=1}^\infty |b_i|_{H^{1,\,p}_{\mathrm{atb}}(\mu)}\}$, where the infimum is taken over all the possible decompositions of f in p-atomic blocks as above.

Remark 2.2. It was proved by Tolsa [10] that the definition of $H^{1,\,p}_{\mathrm{atb}}(\mu)$ is independent of the chosen constant $\eta > 1$, and for any $1 < p \le \infty$, all the atomic Hardy spaces $H^{1,\,p}_{\mathrm{atb}}(\mu)$ coincide with equivalent norms. Moreover, a maximal function characterization of $H^{1,\,p}_{\mathrm{atb}}(\mu)$ was also established in [12]. Thus, in the rest of this paper, we denote *the atomic Hardy space* $H^{1,\,p}_{\mathrm{atb}}(\mu)$ simply by $H^1(\mu)$, and when we use the atomic characterization of $H^1(\mu)$, we always assume $\eta = 2$ and $p = \infty$ in Definition 2.4.

Definition 2.5. Let $\eta \in (1, \infty)$. A function $f \in L^1_{\mathrm{loc}}(\mu)$ is said to be in the *space* $\mathrm{RBMO}\,(\mu)$ if there exists some constant $C_1 \ge 0$ such that for any cube Q centered at some point of $\mathrm{supp}\,(\mu)$,

$$\frac{1}{\mu(\eta Q)} \int_Q \left| f(y) - m_{\widetilde{Q}}(f) \right| \, d\mu(y) \le C_1,$$

and for any two doubling cubes $Q \subset R$,

$$|m_Q(f) - m_R(f)| \le C_1[1 + \delta(Q, R)],$$

where $m_Q(f)$ denotes the *mean of* f *over cube* Q, namely, $m_Q(f) = \frac{1}{\mu(Q)} \int_Q f(y)\, d\mu(y)$. Moreover, we define the $\mathrm{RBMO}\,(\mu)$ *norm of* f by the minimal constant C_1 as above and denote it by $\|f\|_{\mathrm{RBMO}\,(\mu)}$.

Remark 2.3. It was proved by Tolsa [10] that the definition of $\mathrm{RBMO}\,(\mu)$ is independent of the choices of η. As a result, throughout this paper, we always assume $\eta = 2$ in Definition 2.5.

The following space $\mathrm{RBLO}\,(\mu)$ was introduced in [3]. It is obvious that

$$L^\infty(\mu) \subset \mathrm{RBLO}\,(\mu) \subset \mathrm{RBMO}\,(\mu).$$

Definition 2.6. We say $f \in L^1_{\rm loc}(\mu)$ belongs to the *space* RBLO(μ) if there exists some constant $C_2 \ge 0$ such that for any doubling cube Q,

$$m_Q(f) - \operatorname*{essinf}_{x\in Q} f(x) \le C_2,$$

and for any two doubling cubes $Q \subset R$,

$$m_Q(f) - m_R(f) \le C_2[1 + \delta(Q, R)].$$

The minimal constant C_2 as above is defined to be the *norm of* f *in the space* RBLO(μ) and denoted by $\|f\|_{{\rm RBLO}\,(\mu)}$.

3. Main results and their proofs

We begin with the boundedness of the homogeneous Littlewood-Paley g-function $\dot g(f)$ from $H^1(\mu)$ to $L^1(\mu)$. Recall that $\mathbb{R}^d$ is assumed not to be an initial cube.

Theorem 3.1. *There exists a constant* $C > 0$ *such that for all* $f \in H^1(\mu)$,

$$\|\dot g(f)\|_{L^1(\mu)} \le C\|f\|_{H^1(\mu)}.$$

Proof. Let b be any ∞-atomic block as in Definition 2.4. To be precise, assume that $b = \lambda_1 a_1 + \lambda_2 a_2$. By the Fatou lemma, to prove Theorem 3.1, it is enough to show that $\dot g(b)$ is in $L^1(\mu)$ and

$$\|\dot g(b)\|_{L^1(\mu)} \lesssim |\lambda_1| + |\lambda_2|.$$

Assume that ${\rm supp}\,(b) \subset R$ and ${\rm supp}\,(a_j) \subset Q_j$ for $j = 1, 2$ as in Definition 2.4. Since $\dot g$ is sublinear, we write

$$\begin{aligned}
&\int_{\mathbb{R}^d} \dot g(b)(x)\,d\mu(x)\\
&\quad= \int_{4R} \dot g(b)(x)\,d\mu(x) + \int_{\mathbb{R}^d\setminus 4R} \dot g(b)(x)\,d\mu(x)\\
&\quad\le \sum_{j=1}^2 |\lambda_j| \int_{2Q_j} \dot g(a_j)(x)\,d\mu(x) + \sum_{j=1}^2 |\lambda_j| \int_{4R\setminus 2Q_j} \dot g(a_j)(x)\,d\mu(x)\\
&\qquad+ \int_{\mathbb{R}^d\setminus 4R} \dot g(b)(x)\,d\mu(x) \equiv {\rm I}_1 + {\rm I}_2 + {\rm I}_3.
\end{aligned}$$

Recalling that $\dot{g}$ is bounded on $L^2(\mu)$ (see Theorem 6.1 in [11]), by the Hölder inequality and (2.3), we then see that

$$\begin{aligned} \mathrm{I}_1 &\le \sum_{j=1}^{2}|\lambda_j|\left\{\int_{2Q_j}[\dot{g}(a_j)(x)]^2\,d\mu(x)\right\}^{\frac12}[\mu(2Q_j)]^{\frac12}\\ &\lesssim \sum_{j=1}^{2}|\lambda_j|\left\{\int_{Q_j}[a_j(x)]^2\,d\mu(x)\right\}^{\frac12}[\mu(2Q_j)]^{\frac12}\\ &\lesssim \sum_{j=1}^{2}|\lambda_j|\|a_j\|_{L^\infty(\mu)}\mu(2Q_j)\le\sum_{j=1}^{2}|\lambda_j|, \end{aligned}$$

which is a desired estimate.

For $j=1,2$, let x_j be the center of Q_j. Notice that for $x\notin 2Q_j$ and $y\in Q_j$, $|x-y|\sim|x-x_j|$. From this fact, the Hölder inequality, the fact that for any $x\neq y$,

$$\left[\sum_{k=-\infty}^{\infty}|D_k(x,y)|^2\right]^{1/2}\lesssim\frac{1}{|x-y|^n} \tag{3.1}$$

(see [11, p. 82]) and (2.3), it follows that

$$\begin{aligned} \dot{g}(a_j)(x) &\le \left[\int_{Q_j}\sum_{k=-\infty}^{\infty}|D_k(x,y)|^2|a_j(y)|^2\,d\mu(y)\right]^{\frac12}[\mu(Q_j)]^{\frac12}\\ &\lesssim \left[\int_{Q_j}\frac{|a_j(y)|^2}{|x-y|^{2n}}\,d\mu(y)\right]^{\frac12}[\mu(Q_j)]^{\frac12}.\\ &\lesssim \frac{\|a_j\|_{L^\infty(\mu)}}{|x-x_j|^n}\mu(Q_j)\lesssim\frac{1}{|x-x_j|^n}\frac{1}{1+\delta(Q_j,\,R)}. \end{aligned}$$

By Lemma 3.1 (d) in [11], $\delta(2Q_j,4R)\lesssim 1+\delta(Q_j,R)$, which in turn implies that

$$\mathrm{I}_2\lesssim\sum_{j=1}^{2}\frac{|\lambda_j|}{1+\delta(Q_j,\,R)}\delta(2Q_j,4R)\lesssim\sum_{j=1}^{2}|\lambda_j|.$$

We now estimate I_3. Let $x_0\in\operatorname{supp}(\mu)\cap R$. By the vanishing moment of b, the Minkowski inequality and the Hölder inequality, for $x\notin 4R$,

$$\begin{aligned}
&\left\{\sum_{k=-\infty}^{\infty}|D_kb(x)|^2\right\}^{1/2}\\
&=\left\{\sum_{k=-\infty}^{\infty}\left|\int_R[D_k(x,\,y)-D_k(x,\,x_0)]b(y)\,d\mu(y)\right|^2\right\}^{1/2}\\
&\le\left\{\sum_{k=-\infty}^{\infty}\left[\sum_{j=1}^{2}|\lambda_j|\int_{Q_j}|D_k(x,\,y)-D_k(x,\,x_0)||a_j(y)|\,d\mu(y)\right]^2\right\}^{1/2}\\
&\lesssim\sum_{j=1}^{2}|\lambda_j|\,[\mu(Q_j)]^{1/2}\left\{\int_{Q_j}\left(\sum_{k=-\infty}^{\infty}|D_k(x,\,y)-D_k(x,\,x_0)|^2\right)|a_j(y)|^2\,d\mu(y)\right\}^{1/2}.
\end{aligned}$$

Therefore, Theorem 3.1 is reduced to showing that

$$\int_{\mathbb{R}^d\setminus 4R}\left[\int_{Q_j}\left(\sum_{k=-\infty}^{\infty}|D_k(x,\,y)-D_k(x,\,x_0)|^2\right)|a_j(y)|^2\,d\mu(y)\right]^{\frac{1}{2}}\,d\mu(x)\lesssim[\mu(2Q_j)]^{-1/2}.$$

For any transit cube R and any $x\in R\cap\operatorname{supp}(\mu)$, let H_R^x be the *largest integer* k such that $R\subset Q_{x,\,k}$. By Proposition 2.1, we know that H_Q^x exists and is unique. We now claim that for any $y\in Q_j$, any integer $i\ge 3$ and $k\ge H_R^{x_0}-i+4$,

$$\operatorname{supp}(D_k(\cdot,y)-D_k(\cdot,x_0))\subset Q_{x_0,\,H_R^{x_0}-i+1}. \tag{3.2}$$

In fact, by (A-3) and the fact that $\{Q_{x,\,k}\}_k$ is decreasing, $\operatorname{supp}(D_k(\cdot,y)-D_k(\cdot,x_0))\subset Q_{y,\,k-2}\cup Q_{x_0,\,k-2}\subset Q_{y,\,H_R^{x_0}-i+2}\cup Q_{x_0,\,H_R^{x_0}-i+2}$. Since $i\ge 3$, then $y\in Q_j$ and the decreasing property of $\{Q_{x_0,\,k}\}_k$ imply that $y\in Q_{x_0,\,H_R^{x_0}-i+2}$, which together with Lemma 4.2 (c) in [11] implies that $Q_{y,\,H_R^{x_0}-i+2}\subset Q_{x_0,\,H_R^{x_0}-i+1}$. Thus, (3.2) holds.

Observe that for any $y\in Q_j$, we have $y\in Q_{x_0,\,k}$ for $k\le H_R^{x_0}-i+3$. Then the symmetry of S_k and (2.2) imply that

$$|D_k(x,y)-D_k(x,x_0)|\lesssim\frac{|x_0-y|}{l(Q_{x_0,\,k})}\frac{1}{[l(Q_{x_0,\,k})+|x-x_0|]^n}. \tag{3.3}$$

On the other hand, since $l(Q_{x_0,\,H_R^{x_0}}) \le \frac{1}{10} l(Q_{x_0,\,H_R^{x_0}-1})$ (see [11, p. 69]), we then have $4R \subset Q_{x_0,\,H_R^{x_0}-1}$ and

$$\mathbb{R}^d \setminus (4R) = \Big(Q_{x_0,\,H_R^{x_0}-2} \setminus (4R)\Big) \bigcup \bigcup_{i=3}^{\infty} \Big(Q_{x_0,\,H_R^{x_0}-i} \setminus Q_{x_0,\,H_R^{x_0}-i+1}\Big).$$

Suppose that $x \in Q_{x_0,\,H_R^{x_0}-i} \setminus Q_{x_0,\,H_R^{x_0}-i+1}$ for $i \ge 3$, then (3.2) and (3.3) along with Lemma 3.4 in [11] yield that for any $y \in Q_j$,

$$\begin{aligned}
\sum_{k=-\infty}^{\infty} |D_k(x,\,x_0) - D_k(x,\,y)|^2 &= \sum_{k=-\infty}^{H_R^{x_0}-i+3} |D_k(x,\,x_0) - D_k(x,\,y)|^2 \\
&\lesssim \sum_{k=-\infty}^{H_R^{x_0}-i+3} \frac{|x_0 - y|^2}{[l(Q_{x_0,\,k})]^2} \frac{1}{[l(Q_{x_0,\,k}) + |x - x_0|]^{2n}} \\
&\lesssim \sum_{k=-\infty}^{H_R^{x_0}-i+3} \frac{[l(R)]^2}{[l(Q_{x_0,\,k})]^2} \frac{1}{|x - x_0|^{2n}} \\
&\lesssim \frac{[l(R)]^2}{|x - x_0|^{2n}} \frac{1}{[l(Q_{x_0,\,H_R^{x_0}-i+3})]^2}.
\end{aligned}$$

Notice that for any $k \in \mathbb{Z}$ and $x \in \operatorname{supp}(\mu)$,

$$\delta(Q_{x,\,k}, Q_{x,\,k-1}) \lesssim 1. \tag{3.4}$$

As a consequence, another application of (2.3) together with $R \subset Q_{x_0,\,H_R^{x_0}}$ shows that

$$\begin{aligned}
&\sum_{i=-\infty}^{H_R^{x_0}-3} \int_{Q_{x_0,\,i} \setminus Q_{x_0,\,i+1}} \left[\int_{Q_j} \left(\sum_{k=-\infty}^{\infty} |D_k(x,y) - D_k(x,x_0)|^2\right) |a_j(y)|^2 \, d\mu(y)\right]^{\frac{1}{2}} d\mu(x) \\
&\quad \lesssim \sum_{i=-\infty}^{H_R^{x_0}-3} \int_{Q_{x_0,\,i} \setminus Q_{x_0,\,i+1}} \left[\int_{Q_j} \frac{[l(R)]^2}{|x - x_0|^{2n}} \frac{|a_j(y)|^2}{[l(Q_{x_0,\,i+3})]^2} \, d\mu(y)\right]^{\frac{1}{2}} d\mu(x) \\
&\quad \lesssim \sum_{i=-\infty}^{H_R^{x_0}-3} \frac{l(R)}{l(Q_{x_0,\,i+3})} \|a_j\|_{L^\infty(\mu)} [\mu(Q_j)]^{1/2} \delta(Q_{x_0,\,i+1},\, Q_{x_0,\,i}) \lesssim [\mu(2Q_j)]^{-1/2}.
\end{aligned}$$

On the other hand, since $Q_{x_0,\,H_R^{x_0}+2} \subset 4R$ (see [11, p. 69]), it follows, from (2.3), (3.1) and the fact that for any $x \notin 4R$ and $y \in R$, $|x - x_0| \sim |x - y|$, that

$$\int_{Q_{x_0,\,H_R^{x_0}-2}\setminus 4R}\left[\int_{Q_j}\left(\sum_{k=-\infty}^{\infty}|D_k(x,y)-D_k(x,x_0)|^2\right)|a_j(y)|^2\,d\mu(y)\right]^{\frac{1}{2}}\,d\mu(x)$$
$$\lesssim \int_{Q_{x_0,\,H_R^{x_0}-2}\setminus 4R}\left[\int_{Q_j}\frac{|a_j(y)|^2}{|x-x_0|^{2n}}\,d\mu(y)\right]^{\frac{1}{2}}\,d\mu(x)$$
$$\lesssim \sum_{i=H_R^{x_0}-2}^{H_R^{x_0}+1}\|a_j\|_{L^\infty(\mu)}[\mu(Q_j)]^{1/2}\delta(Q_{x_0,\,i+1},\,Q_{x_0,\,i})\lesssim[\mu(2Q_j)]^{-1/2}.$$

Therefore, $\mathrm{I}_3\lesssim\sum_{j=1}^2|\lambda_j|$, which completes the proof of Theorem 3.1. $\square$

To establish the boundedness of the homogeneous Littlewood-Paley g-function $\dot{g}(f)$ from $\mathrm{RBMO}\,(\mu)$ to $\mathrm{RBLO}\,(\mu)$, we need the following estimate.

Lemma 3.1. *There exists a constant $C>0$ such that for any two cubes $Q\subset R$ and $f\in\mathrm{RBMO}\,(\mu)$,*

$$\int_R\frac{|f(y)-m_{\widetilde{Q}}(f)|}{[|y-x_Q|+l(Q)]^n}\,d\mu(y)\le C[1+\delta(Q,R)]^2\|f\|_{\mathrm{RBMO}\,(\mu)}.$$

Proof. Without loss of generality, we may assume that $\|f\|_{\mathrm{RBMO}\,(\mu)}=1$. For any $Q\subset R$, set

$$K_{Q,\,R}\equiv 1+\sum_{k=1}^{N_{Q,\,R}}\frac{\mu\left(2^kQ\right)}{[l\left(2^kQ\right)]^n},$$

where $N_{Q,\,R}$ is the smallest integer k such that $l\left(2^kQ\right)\ge l(R)$ (see [10]). It is trivial to check that

$$K_{Q,\,R}\sim 1+\delta(Q,R). \tag{3.5}$$

Notice that from (1.1) and Definition 2.5, it follows that

$$\int_Q\frac{|f(y)-m_{\widetilde{Q}}(f)|}{[|y-x_Q|+l(Q)]^n}\,d\mu(y)\le\frac{1}{[l(Q)]^n}\int_Q\left|f(y)-m_{\widetilde{Q}}(f)\right|\,d\mu(y)\lesssim 1.$$

Therefore, to show Lemma 3.1, it suffices to verify that

$$\int_{R\setminus Q}\frac{|f(y)-m_{\widetilde{Q}}(f)|}{|y-x_Q|^n}\,d\mu(y)\lesssim[1+\delta(Q,R)]^2. \tag{3.6}$$

By (1.1) and Lemma 2.1 in [10] together with Definition 2.5,

$$\begin{aligned}
&\int_{R\setminus Q} \frac{|f(y)-m_{\widetilde{Q}}(f)|}{|y-x_Q|^n}\,d\mu(y)\\
&\lesssim \sum_{k=0}^{N_{Q,\,R}} \frac{1}{[l(2^{k+1}Q)]^n}\int_{2^{k+1}Q\setminus 2^kQ}\left|f(y)-m_{\widetilde{Q}}(f)\right|\,d\mu(y)\\
&\le \sum_{k=0}^{N_{Q,\,R}} \frac{1}{[l(2^{k+1}Q)]^n}\int_{2^{k+1}Q\setminus 2^kQ}\left|f(y)-m_{\widetilde{2^{k+1}Q}}(f)\right|\,d\mu(y)\\
&\quad+\sum_{k=0}^{N_{Q,\,R}} \frac{\mu\left(2^{k+1}Q\right)}{[l(2^{k+1}Q)]^n}\left|m_{\widetilde{2^{k+1}Q}}(f)-m_{\widetilde{Q}}(f)\right|\\
&\lesssim \sum_{k=1}^{N_{Q,\,R}+1} \frac{\mu(2^kQ)}{\left[l\left(2^kQ\right)\right]^n}+\sum_{k=1}^{N_{Q,\,R}+1} \frac{\mu(2^kQ)}{\left[l\left(2^kQ\right)\right]^n}\left[1+\delta\left(Q,2^kQ\right)\right]\\
&\lesssim K_{Q,\,R}+K_{Q,\,R}[1+\delta(Q,\,R)]\lesssim[1+\delta(Q,\,R)]^2,
\end{aligned}$$

which completes the proof of Lemma 3.1. □

The following conclusion is a slight variant of Lemma 9.3 in [10], which can be proved by a slight modification of the proof of Lemma 9.3 in [10]. We omit the details.

Lemma 3.2. *There exists some constant P_0 (big enough) depending on C_0 and n such that if $x \in \mathbb{R}^d$ is some fixed point and $\{f_Q\}_{Q\ni x}$ is a collection of numbers such that $f_Q - f_R \le [1+\delta(Q,R)]C_x$ for all doubling cubes $Q \subset R$ with $x \in Q$ such that $1+\delta(Q,R)\le P_0$, then*

$$f_Q - f_R \le C[1+\delta(Q,R)]C_x \quad \textit{for all doubling cubes } Q\subset R \textit{ with } x\in Q,$$

where C depends on C_0, n and P_0.

Theorem 3.2. *For any $f \in \mathrm{RBMO}\,(\mu)$, $\dot{g}(f)$ is either infinite everywhere or finite almost everywhere, and in the latter case,*

$$\|[\dot{g}(f)]^2\|_{\mathrm{RBLO}\,(\mu)} \le C\|f\|^2_{\mathrm{RBMO}\,(\mu)}, \tag{3.7}$$

where $C>0$ is independent of f.

Proof. We first claim that for any $f \in \mathrm{RBMO}\,(\mu)$, if there exists a point $x_0 \in \mathbb{R}^d$ such that $\dot{g}(f)(x_0) < \infty$, then for any doubling cube $Q \ni x_0$,

$$\frac{1}{\mu(Q)}\int_Q\left\{[\dot{g}(f)(x)]^2-\inf_{y\in Q}[\dot{g}(f)(y)]^2\right\}d\mu(x)\lesssim\|f\|^2_{\mathrm{RBMO}\,(\mu)}. \tag{3.8}$$

Without loss of generality, we may assume that $\|f\|_{\mathrm{RBMO}\,(\mu)} = 1$. For any $x \in \operatorname{supp}(\mu) \cap Q$, set

$$\left[\dot{g}^{H_Q^x}(f)(x)\right]^2 \equiv \sum_{k=H_Q^x+4}^{\infty} |D_k f(x)|^2 \text{ and } \left[\dot{g}_{H_Q^x}(f)(x)\right]^2 \equiv \sum_{k=-\infty}^{H_Q^x+3} |D_k f(x)|^2.$$

Notice that $Q_{x,\,j} \subset \frac{4}{3}Q$ when $j \geq H_Q^x + 2$ (see [11, p. 69]). This fact together with $\operatorname{supp}(D_k(x,\cdot)) \subset Q_{x,\,k-2}$ and $\int_{\mathbb{R}^d} D_k(x,y)\,d\mu(y) = 0$ implies that when $k \geq H_Q^x + 4$,

$$D_k f(x) = D_k \left[\left(f - m_{\widetilde{\frac{4}{3}Q}}(f)\right)\chi_{\frac{4}{3}Q}\right](x).$$

It follows from the doubling property of Q along with Remark 2.3, the $L^2(\mu)$-boundedness of $\dot{g}(f)$ (see [11, Theorem 6.1]) and Corollary 3.5 in [10] that

$$\begin{aligned}(3.9)\qquad \frac{1}{\mu(Q)}\int_Q \left[\dot{g}^{H_Q^x}(f)(x)\right]^2 d\mu(x) &\leq \frac{1}{\mu(Q)}\int_Q \left\{\dot{g}\left[\left(f - m_{\widetilde{\frac{4}{3}Q}}(f)\right)\chi_{\frac{4}{3}Q}\right](x)\right\}^2 d\mu(x)\\ &\lesssim \frac{1}{\mu(2Q)}\int_{\frac{4}{3}Q}\left|f(x) - m_{\widetilde{\frac{4}{3}Q}}(f)\right|^2 d\mu(x) \lesssim 1.\end{aligned}$$

Now observe that for any $x, y \in Q$,

$$\left[\dot{g}_{H_Q^x}(f)(x)\right]^2 - [\dot{g}(f)(y)]^2 \leq \left[\dot{g}_{H_Q^x}(f)(x)\right]^2 - \left[\dot{g}_{H_Q^x}(f)(y)\right]^2.$$

Thus taking (3.9) into account, to show (3.8), we only need to verify that for μ-a.e. $y \in Q$,

$$\left[\dot{g}_{H_Q^x}(f)(x)\right]^2 - \left[\dot{g}_{H_Q^x}(f)(y)\right]^2 \lesssim 1. \tag{3.10}$$

We assert that for each $k \in \mathbb{Z}$ and $z \in \mathbb{R}^d$,

$$|D_k f(z)| \lesssim 1. \tag{3.11}$$

Indeed, (2.1) implies that

$$|D_k(z,y)| \lesssim \frac{1}{[l(Q_{z,\,k}) + l(Q_{y,\,k}) + |z-y|]^n}. \tag{3.12}$$

Then since $\operatorname{supp}(D_k(z,\cdot)) \subset Q_{z,k-2}$, by the vanishing moment of D_k, Lemma 3.1 and (3.12), we have

$$\begin{aligned} |D_k f(z)| &\le \int_{Q_{z,\,k-2}} |D_k(z,y)||f(y) - m_{Q_{z,\,k}}(f)|\,d\mu(y) \\ &\lesssim \int_{Q_{z,\,k-2}} \frac{|f(y) - m_{Q_{z,\,k}}(f)|}{[|z-y| + l(Q_{z,\,k})]^n}\,d\mu(z) \lesssim 1. \end{aligned}$$

Thus, (3.11) holds. From this assertion we see that for $x,\ y \in Q$,

$$\begin{aligned} &\left[\dot{g}_{H_Q^x}(f)(x)\right]^2 - \left[\dot{g}_{H_Q^x}(f)(y)\right]^2 \\ &\le \sum_{k=-\infty}^{H_Q^x-3} |D_k f(x) - D_k f(y)||D_k f(x) + D_k f(y)| + \sum_{k=H_Q^x-2}^{H_Q^x+3} |D_k f(x)|^2 \\ &\lesssim \sum_{k=-\infty}^{H_Q^x-3} |D_k f(x) - D_k f(y)| + 1. \end{aligned}$$

By the symmetry of D_k and (3.2), we see that for any fixed integer $i \ge 3$ and $k \ge H_Q^x - i + 4$, and all $z \in Q_{x,\,H_Q^x-i} \setminus Q_{x,\,H_Q^x-i+1}$,

$$D_k(x,z) - D_k(y,z) = 0.$$

Therefore, from the vanishing moment of D_k, we see that

$$\begin{aligned} &\sum_{k=-\infty}^{H_Q^x-3} |D_k f(x) - D_k f(y)| \\ &\quad\le \int_{\mathbb{R}^d} \left(\sum_{k=-\infty}^{H_Q^x-3} |D_k(x,z) - D_k(y,z)|\right) \left|f(z) - m_{Q_{x,\,H_Q^x}}(f)\right|\,d\mu(z) \\ &\quad\le \sum_{i=3}^{\infty} \int_{Q_{x,\,H_Q^x-i}\setminus Q_{x,\,H_Q^x-i+1}} \left(\sum_{k=-\infty}^{H_Q^x-i+3} |D_k(x,z) - D_k(y,z)|\right) \\ &\qquad\times \left|f(z) - m_{Q_{x,\,H_Q^x}}(f)\right|\,d\mu(z) \\ &\qquad+ \int_{Q_{x,\,H_Q^x-2}} \left(\sum_{k=-\infty}^{H_Q^x-3} |D_k(x,z) - D_k(y,z)|\right) \left|f(z) - m_{Q_{x,\,H_Q^x}}(f)\right|\,d\mu(z) \\ &\quad\equiv \mathrm{J}_1 + \mathrm{J}_2. \end{aligned}$$

Since $x,\, y \in Q$ implies that $x,\, y \in Q_{x,\,k}$ for $k \le H_Q^x$, by (2.2) and Lemma 3.4 in [11], we further obtain

$$\sum_{k=-\infty}^{H_Q^x-i+3} |D_k(x,z) - D_k(y,z)| \lesssim \sum_{k=-\infty}^{H_Q^x-i+3} \frac{|x-y|}{l(Q_{x,\,k})[l(Q_{x,\,k}) + |x-z|]^n} \lesssim \frac{l(Q)}{l(Q_{x,\,H_Q^x-i+3})}\frac{1}{|x-z|^n}.$$

Moreover, by (3.6), we have

$$\int_{Q_{x,\,H_Q^x-i}\setminus Q_{x,\,H_Q^x-i+1}} \frac{|f(z) - m_{Q_{x,\,H_Q^x-i+1}}(f)|}{|x-z|^n}\, d\mu(z) \lesssim \left[1+\delta\Big(Q_{x,\,H_Q^x-i+1}, Q_{x,\,H_Q^x-i}\Big)\right]^2 \lesssim 1.$$

Therefore, these facts, together with Definition 2.5, (3.4) and [11, Lemma 3.4] imply that

$$\begin{aligned}
&\mathrm{J}_1 \lesssim \\
&\lesssim \sum_{i=3}^{\infty} \frac{l(Q)}{l(Q_{x,H_Q^x-i+3})} \int_{Q_{x,\,H_Q^x-i}\setminus Q_{x,\,H_Q^x-i+1}} \frac{|f(z) - m_{Q_{x,\,H_Q^x}}(f)|}{|x-z|^n}\, d\mu(z) \\
&\le \sum_{i=3}^{\infty} \frac{l(Q)}{l(Q_{x,H_Q^x-i+3})} \int_{Q_{x,\,H_Q^x-i}\setminus Q_{x,\,H_Q^x-i+1}} \frac{|f(z) - m_{Q_{x,\,H_Q^x-i+1}}(f)|}{|x-z|^n}\, d\mu(z) \\
&\quad + \sum_{i=3}^{\infty} \frac{l(Q)}{l(Q_{x,H_Q^x-i+3})} \int_{Q_{x,\,H_Q^x-i}\setminus Q_{x,\,H_Q^x-i+1}} \frac{|m_{Q_{x,\,H_Q^x-i+1}}(f) - m_{Q_{x,\,H_Q^x}}(f)|}{|x-z|^n}\, d\mu(z) \\
&\lesssim \sum_{i=3}^{\infty} \frac{l(Q)}{l(Q_{x,H_Q^x-i+3})} + \sum_{i=3}^{\infty} \frac{l(Q)}{l(Q_{x,H_Q^x-i+3})} [1+\delta(Q_{x,H_Q^x}, Q_{x,H_Q^x-i+1})]^2 \\
&\lesssim \sum_{i=3}^{\infty} \frac{l(Q)}{l(Q_{x,H_Q^x-i+3})} (1+i)^2 \lesssim 1.
\end{aligned}$$

Now we turn our attention to J_2. The estimate (3.12), Lemma 3.4 in [11], (1.1), Definition 2.5 and (3.4) yield

$$\int_{Q_{x,\,H_Q^x-2}} \sum_{k=-\infty}^{H_Q^x-3} |D_k(x,z)| \left|f(z) - m_{Q_{x,\,H_Q^x}}(f)\right| d\mu(z)$$

$$
\begin{aligned}
&\lesssim \int_{Q_{x,\,H_Q^x-2}} \sum_{k=-\infty}^{H_Q^x-3} \frac{|f(z)-m_{Q_{x,\,H_Q^x}}(f)|}{[|x-z|+l(Q_{x,\,k})]^n}\,d\mu(z)\\
&\lesssim \int_{Q_{x,\,H_Q^x-2}} \frac{|f(z)-m_{Q_{x,\,H_Q^x}}(f)|}{[l(Q_{x,\,H_Q^x-2})]^n}\,d\mu(z)\\
&\lesssim \int_{Q_{x,\,H_Q^x-2}} \frac{|f(z)-m_{Q_{x,\,H_Q^x-2}}(f)|}{[l(Q_{x,\,H_Q^x-2})]^n}\,d\mu(z)+\left|m_{Q_{x,\,H_Q^x-2}}(f)-m_{Q_{x,\,H_Q^x}}(f)\right|\\
&\lesssim 1.
\end{aligned}
$$

On the other hand, notice that by Lemma 3.4 in [11], for $z\in Q_{y,\,H_Q^x-3}$,

$$
\sum_{k=-\infty}^{H_Q^x-3} \frac{1}{[|y-z|+l(Q_{y,\,k})]^n} \lesssim \frac{1}{[l(Q_{y,\,H_Q^x-3})]^n}.
$$

Since $y\in Q\subset Q_{x,H_Q^x}$, we have that $Q_{x,H_Q^x-2}\subset Q_{y,H_Q^x-3}$ as a result of Lemma 4.2 (c) in [11]. Then it follows from these observations and (3.12) together with Definition 2.5 that

$$
\begin{aligned}
&\int_{Q_{x,\,H_Q^x-2}} \sum_{k=-\infty}^{H_Q^x-3} |D_k(y,z)|\left|f(z)-m_{Q_{x,\,H_Q^x}}(f)\right|\,d\mu(z)\\
&\quad\lesssim \int_{Q_{x,\,H_Q^x-2}} \sum_{k=-\infty}^{H_Q^x-3} \frac{|f(z)-m_{Q_{x,\,H_Q^x}}(f)|}{[|y-z|+l(Q_{y,\,k})]^n}\,d\mu(z)\\
&\quad\lesssim \int_{Q_{x,\,H_Q^x-2}} \frac{|f(z)-m_{Q_{x,\,H_Q^x}}(f)|}{[l(Q_{y,\,H_Q^x-3})]^n}\,d\mu(z)\\
&\quad\lesssim \int_{Q_{x,\,H_Q^x-2}} \frac{|f(z)-m_{Q_{x,\,H_Q^x}}(f)|}{[l(Q_{x,\,H_Q^x-2})]^n}\,d\mu(z)\lesssim 1.
\end{aligned}
$$

Combining these estimates above implies

$$
\mathrm{J}_2 \le \int_{Q_{x,\,H_Q^x-2}} \left\{\sum_{k=-\infty}^{H_Q^x-3} [|D_k(x,z)|+|D_k(y,z)|]\right\}\left|f(z)-m_{Q_{x,\,H_Q^x}}(f)\right|\,d\mu(z)\lesssim 1.
$$

Thus (3.10) holds.

To finish the proof of Theorem 3.2, by Lemma 3.2, it suffices to show that for any doubling cubes $Q \subset R$,

$$\left[\dot{g}(f)^2\right]_Q - \left[\dot{g}(f)^2\right]_R \lesssim [1+\delta(Q,R)]^4. \tag{3.13}$$

For any $x \in \operatorname{supp}(\mu) \cap Q$, we first consider the case that $H_Q^x \geq H_R^x + 10$ by writing

$$\begin{aligned}\left[\dot{g}(f)^2\right]_Q - \left[\dot{g}(f)^2\right]_R &\leq \frac{1}{\mu(Q)} \int_Q \left[\dot{g}^{H_Q^x} f(x)\right]^2 d\mu(x) + \frac{1}{\mu(Q)} \int_Q \sum_{k=H_R^x+4}^{H_Q^x+3} |D_k f(x)|^2 \, d\mu(x) \\ &\quad + \frac{1}{\mu(Q)} \frac{1}{\mu(R)} \int_Q \int_R \left(\left[\dot{g}_{H_R^x} f(x)\right]^2 - \left[\dot{g}_{H_R^x} f(y)\right]^2 \right) d\mu(y)\, d\mu(x).\end{aligned}$$

By (3.10) with Q replaced by R, we see that

$$\frac{1}{\mu(Q)} \frac{1}{\mu(R)} \int_Q \int_R \left(\left[\dot{g}_{H_R^x} f(x)\right]^2 - \left[\dot{g}_{H_R^x} f(y)\right]^2 \right) d\mu(y)\, d\mu(x) \lesssim 1.$$

Therefore by (3.9) and (3.11), the estimate (3.13) is reduced to proving that

$$\frac{1}{\mu(Q)} \int_Q \sum_{k=H_R^x+4}^{H_Q^x-1} |D_k f(x)|^2 \, d\mu(x) \lesssim [1+\delta(Q,R)]^4. \tag{3.14}$$

By splitting

$$Q_{x,\,k-2} = (Q_{x,\,k-2} \setminus Q_{x,\,k-1}) \cup (Q_{x,\,k-1} \setminus Q_{x,\,k}) \cup Q_{x,\,k},$$

it follows from the vanishing moment of D_k, $\operatorname{supp}(D_k(x,\cdot)) \subset Q_{x,\,k-2}$ and (3.12) that

$$\begin{aligned}\sum_{k=H_R^x+4}^{H_Q^x-1} |D_k f(x)| &\leq \sum_{k=H_R^x+4}^{H_Q^x-1} \int_{Q_{x,\,k-2}} \frac{|f(z) - m_{Q_{x,\,H_Q^x-1}}(f)|}{[|x-z| + l(Q_{x,\,k})]^n} \, d\mu(z) \\ &\leq 2 \int_{Q_{x,\,H_R^x+2} \setminus Q_{x,\,H_Q^x-1}} \frac{|f(z) - m_{Q_{x,\,H_Q^x-1}}(f)|}{|x-z|^n} \, d\mu(z) \\ &\quad + \sum_{k=H_R^x+4}^{H_Q^x-1} \int_{Q_{x,\,k}} \frac{|f(z) - m_{Q_{x,\,H_Q^x-1}}(f)|}{[|x-z| + l(Q_{x,\,k})]^n} \, d\mu(z) \equiv \mathrm{L}_1 + \mathrm{L}_2.\end{aligned}$$

By $Q \subset Q_{x,\, H_Q^x - 1}$, $Q_{x,\, H_R^x + 2} \subset 2R$ and Lemma 3.1 in [11],

$$\delta\left(Q_{x,\, H_Q^x-1}, Q_{x,\, H_R^x+2}\right) \lesssim 1 + \delta(Q, R). \tag{3.15}$$

Thus, by (3.15), (3.6) and Definition 2.5, we have

$$\mathrm{L}_1 \lesssim \left[1 + \delta\left(Q_{x,\, H_Q^x-1}, Q_{x,\, H_R^x+2}\right)\right]^2 \lesssim [1 + \delta(Q, R)]^2.$$

To estimate L_2, by Lemma 3.4 in [11], we first see that for any integer

$$k \in \left[H_R^x + 4, H_Q^x - 1\right],$$

there exists a unique integer $j_k \in [0, N_{Q_{x,\, H_Q^x-1}, Q_{x,\, H_R^x+4}}]$ such that $2^{j_k} Q_{x,\, H_Q^x-1} \subset Q_{x,\, k} \subset 2^{j_k+1} Q_{x,\, H_Q^x-1}$, and for different k, j_k is different. It then follows from Definition 2.5, the decreasing property of $Q_{x,\, k}$, (3.15) and (3.5) that

$$\begin{aligned}
\mathrm{L}_2 &\le \sum_{k=H_R^x+4}^{H_Q^x-1} \int_{Q_{x,\, k}} \frac{|f(z) - m_{Q_{x,\, k}}(f)|}{[l(Q_{x,\, k})]^n}\, d\mu(z) \\
&\quad + \sum_{k=H_R^x+4}^{H_Q^x-1} \frac{\mu(Q_{x,\, k})}{[l(Q_{x,\, k})]^n} \left| m_{Q_{x,\, k}}(f) - m_{Q_{x,\, H_Q^x-1}}(f)\right| \\
&\lesssim \sum_{k=H_R^x+4}^{H_Q^x-1} \frac{\mu(Q_{x,\, k})}{[l(Q_{x,\, k})]^n} + \sum_{k=H_R^x+4}^{H_Q^x-1} \frac{\mu(Q_{x,\, k})}{[l(Q_{x,\, k})]^n} [1 + \delta(Q, R)] \\
&\lesssim \sum_{k=H_R^x+4}^{H_Q^x-1} \frac{\mu\left(2^{j_k+1} Q_{x,\, H_Q^x-1}\right)}{\left[l\left(2^{j_k} Q_{x,\, H_Q^x-1}\right)\right]^n} [1 + \delta(Q, R)] \\
&\lesssim K_{Q,\, 2R} [1 + \delta(Q, R)] \lesssim [1 + \delta(Q, R)]^2.
\end{aligned}$$

Consequently, (3.14) follows by combining the estimates for L_1 and L_2.

If $H_R^x \le H_Q^x \le H_R^x + 9$, then by the estimates (3.9) through (3.11), we also see that (3.13) holds, which completes the proof of Theorem 3.2. $\square$

From Theorem 3.2, we can easily deduce the following result.

Corollary 3.1. *For any* $f \in \mathrm{RBMO}(\mu)$, $\dot{g}(f)$ *is either infinite everywhere or finite almost everywhere, and in the latter case,*

$$\|\dot{g}(f)\|_{\mathrm{RBLO}\,(\mu)} \le C \|f\|_{\mathrm{RBMO}\,(\mu)},$$

where $C > 0$ *is independent of* f.

Proof. First, with the aid of (3.8) and the inequality that for any $a,\, b \geq 0$,

$$a - b \leq \left|a^2 - b^2\right|^{1/2}, \tag{3.16}$$

it is easy to see that if $\operatorname*{essinf}_{y\in Q} \dot{g}(f)(y) < \infty$,

$$\frac{1}{\mu(Q)} \int_Q \left[\dot{g}(f)(x) - \operatorname*{essinf}_{y\in Q} \dot{g}(f)(y)\right] d\mu(x) \lesssim \|f\|_{\mathrm{RBMO}\,(\mu)}.$$

Moreover, in the argument of (3.13), we see that for any doubling cubes $Q \subset R$, $x \in Q$, and $y \in R$,

$$\begin{aligned}\left|\left[\dot{g}(f)(x)\right]^2 - \left[\dot{g}(f)(y)\right]^2\right| \;&\leq \left[\dot{g}^{H_Q^x} f(x)\right]^2 + \left[\dot{g}^{H_R^x} f(y)\right]^2 + \sum_{k=H_R^x+4}^{H_Q^x+3} |D_k f(x)|^2 \\ &\quad + \left|\left[\dot{g}_{H_R^x} f(x)\right]^2 - \left[\dot{g}_{H_R^x} f(y)\right]^2\right|.\end{aligned}$$

From this fact with (3.9) through (3.11), (3.14) and (3.16), we obtain that for any doubling cubes $Q \subset R$,

$$m_Q[\dot{g}(f)] - m_R[\dot{g}(f)] \lesssim [1 + \delta(Q, R)]^2 \|f\|_{\mathrm{RBMO}\,(\mu)}.$$

An application of Lemma 3.2 leads to the conclusion of Corollary 3.1. □

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School of Mathematical Sciences, Beijing Normal University
Laboratory of Mathematics and Complex Systems
Ministry of Education, Beijing 100875
People's Republic of China
(*E-mail : dcyang@bnu.edu.cn*)
(*E-mail : dyyang623@126.com*)

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