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Remark on the boundedness of the Cauchy singular integral operator on variable Lebesgue spaces with radial oscillating weights

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To Professor Kokilashvili on his seventieth birthday

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Abstract. Recently V. Kokilashvili, N. Samko, and S. Samko have proved a sufficient condition for the boundedness of the Cauchy singular integral operator on variable Lebesgue spaces with radial oscillating weights over Carleson curves. This condition is formulated in terms of Matuszewska-Orlicz indices of weights. We prove a partial converse of their result.

1. Introduction and main result

Let Γ be a rectifiable curve in the complex plane. We equip Γ with Lebesgue length measure $|d\tau|$. We say that a curve Γ is simple if it does not have self-intersections. In other words, Γ is said to be simple if it is homeomorphic either to a line segment or to to a circle. In the latter

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situation we will say that Γ is a Jordan curve. The *Cauchy singular integral* of $f \in L^1(\Gamma)$ is defined by

$$(Sf)(t) := \frac{1}{\pi i} \int_\Gamma \frac{f(\tau)}{\tau - t} d\tau \quad (t \in \Gamma).$$

This integral is understood in the principal value sense, that is,

$$\int_\Gamma \frac{f(\tau)}{\tau - t} d\tau := \lim_{R \to 0} \int_{\Gamma \setminus \Gamma(t,R)} \frac{f(\tau)}{\tau - t} d\tau,$$

where $\Gamma(t,R) := \{\tau \in \Gamma : |\tau - t| < R\}$ for $R > 0$. David [4] (see also [3, Theorem 4.17]) proved that the Cauchy singular integral generates the bounded operator S on the Lebesgue space $L^p(\Gamma)$, $1 < p < \infty$, if and only if Γ is a *Carleson* (*Ahlfors-David regular*) *curve*, that is,

$$\sup_{t \in \Gamma} \sup_{R > 0} \frac{|\Gamma(t,R)|}{R} < \infty,$$

where for any measurable set $\Omega \subset \Gamma$ the symbol $|\Omega|$ denotes its measure. To have a better idea about Carleson curves, consider the following example. Let $\alpha > 0$ and

$$\Gamma := \{0\} \cup \big\{\tau \in \mathbb{C} : \ \tau = x + ix^\alpha \sin(1/x), \ 0 < x \le 1\big\}.$$

One can show (see [3, Example 1.3]) that Γ is not rectifiable for $0 < \alpha \le 1$, Γ is rectifiable but not Carleson for $1 < \alpha < 2$, and Γ is a Carleson curve for $\alpha \ge 2$.

A measurable function $w : \Gamma \to [0, \infty]$ is referred to as a *weight function* or simply a *weight* if $0 < w(\tau) < \infty$ for almost all $\tau \in \Gamma$. Suppose $p : \Gamma \to [1, \infty]$ is a measurable a.e. finite function. Denote by $L^{p(\cdot)}(\Gamma, w)$ the set of all measurable complex-valued functions f on Γ such that

$$\int_\Gamma |f(\tau)w(\tau)/\lambda|^{p(\tau)} |d\tau| < \infty$$

for some $\lambda = \lambda(f) > 0$. This set becomes a Banach space when equipped with the Luxemburg-Nakano norm

$$\|f\|_{p(\cdot),w} := \inf \left\{ \lambda > 0 : \int_\Gamma |f(\tau)w(\tau)/\lambda|^{p(\tau)} |d\tau| \le 1 \right\}.$$

If p is constant, then $L^{p(\cdot)}(\Gamma, w)$ is nothing else but the weighted Lebesgue space. Therefore, it is natural to refer to $L^{p(\cdot)}(\Gamma, w)$ as a *weighted generalized Lebesgue space with variable exponent* or simply as a *weighted* *variable Lebesgue space*. This is a special case of Musielak-Orlicz spaces [19] (see also [13]). Nakano [20] considered these spaces (without weights) as examples of so-called modular spaces, and sometimes the spaces $L^{p(\cdot)}(\Gamma, w)$ are referred to as weighted Nakano spaces.

Following [12, Section 2.3], denote by W the class of all continuous functions $\varrho : [0, |\Gamma|] \to [0, \infty)$ such that $\varrho(0) = 0$, $\varrho(x) > 0$ if $0 < x \le |\Gamma|$, and ϱ is almost increasing, that is, there is a universal constant $C > 0$ such that $\varrho(x) \le C\varrho(y)$ whenever $x \le y$. Further, let $\mathbb{W}$ be the set of all functions $\varrho : [0, |\Gamma|] \to [0, \infty]$ such that $x^\alpha \varrho(x) \in W$ and $x^\beta/\varrho(x) \in W$ for some $\alpha, \beta \in \mathbb{R}$. Clearly, the functions $\varrho(x) = x^\gamma$ belong to $\mathbb{W}$ for all $\gamma \in \mathbb{R}$. For $\varrho \in \mathbb{W}$, put

$$\Phi_\varrho^0(x) := \limsup_{y \to 0} \frac{\varrho(xy)}{\varrho(y)}, \quad x \in (0, \infty).$$

Since $\varrho \in \mathbb{W}$, one can show that the limits

$$m(\varrho) := \lim_{x \to 0} \frac{\log \Phi_\varrho^0(x)}{\log x}, \quad M(\varrho) := \lim_{x \to \infty} \frac{\log \Phi_\varrho^0(x)}{\log x}$$

exist and $-\infty < m(\varrho) \le M(\varrho) < +\infty$. These numbers were defined by Matuszewska and Orlicz [17, 18] (see also [15] and [16, Chapter 11]). We refer to $m(\varrho)$ (resp. $M(\varrho)$) as the *lower* (resp. *upper*) *Matuszewska-Orlicz index* of ϱ. For $\varrho(x) = x^\gamma$ one has $m(\varrho) = M(\varrho) = \gamma$. Examples of functions $\varrho \in \mathbb{W}$ with $m(\varrho) < M(\varrho)$ can be found, for instance, in [1], [16, p. 93], [21, Section 2].

Fix pairwise distinct points $t_1, \dots, t_n \in \Gamma$ and functions $w_1, \dots, w_n \in \mathbb{W}$. Consider the following weight

$$w(t) := \prod_{k=1}^{n} w_k(|t - t_k|), \quad t \in \Gamma. \tag{1.1}$$

Each function $w_k(|t - t_k|)$ is a radial oscillating weight. The weight (1.1) is a continuous function on $\Gamma \setminus \{t_1, \dots, t_n\}$. This is a natural generalization of so-called Khvedelidze weights $w(t) = \prod_{k=1}^{n} |t - t_k|^{\lambda_k}$, where $\lambda_k \in \mathbb{R}$ (see, e.g., [3, Section 2.2], [9], [10]). Recently V. Kokilashvili, N. Samko, and S. Samko have proved the following (see [12, Theorem 4.3] and also [11] for similar results for maximal functions).

Theorem 1.1 ([12, Theorem 4.3]). *Suppose* Γ *is a simple Carleson curve and* $p : \Gamma \to (1, \infty)$ *is a continuous function satisfying*

$$|p(\tau) - p(t)| \le -A_\Gamma / \log|\tau - t| \quad \textit{whenever} \quad |\tau - t| \le 1/2, \tag{1.2}$$

where A_Γ *is a positive constant depending only on* Γ. *Let* $w_1, \dots, w_n \in \mathbb{W}$ *and the weight* w *be given by* (1.1). *If*

(1.3)

$$0 < 1/p(t_k) + m(w_k), \quad 1/p(t_k) + M(w_k) < 1 \quad \textit{for all} \quad k \in \{1, \dots, n\},$$

then the Cauchy singular integral operator S *is bounded on* $L^{p(\cdot)}(\Gamma, w)$.

For the weight $w(t) = \prod_{k=1}^{n} |t - t_k|^{\lambda_k}$, (1.3) reads as $0 < 1/p(t_k) + \lambda_k < 1$ for all $k \in \{1, \dots, n\}$. This condition is also necessary for the boundedness of S on the variable Lebesgue space $L^{p(\cdot)}(\Gamma, w)$ with the Khvedelidze weight w (see [10]).

The author have proved in [8] that for Jordan curves condition (1.3) is necessary for the boundedness of the operator S.

Theorem 1.2 ([8, Corollary 4.3])**.** *Suppose* Γ *is a rectifiable Jordan curve and* $p : \Gamma \to (1, \infty)$ *is a continuous function satisfying* (1.2). *Let* $w_1, \dots, w_n \in \mathbb{W}$ *and the weight* w *be given by* (1.1). *If the Cauchy singular integral operator* S *is bounded on* $L^{p(\cdot)}(\Gamma, w)$, *then* Γ *is a Carleson curve and* (1.3) *is fulfilled.*

The proof of this result given in [8] essentially uses that Γ is closed. In this paper we embark on the situation of non-closed curves. Our main result is a partial converse of Theorem 1.1. It follows from our results [6, 8] based on further development of ideas from [3, Chap. 1–3].

Theorem 1.3 (Main result)**.** *Let* Γ *be a rectifiable curve homeomorphic to a line segment and* $p : \Gamma \to (1, \infty)$ *be a continuous function satisfying* (1.2). *Suppose* $w_1, \dots, w_n \in \mathbb{W}$ *and the weight* w *is given by* (1.1). *If the Cauchy singular integral operator* S *is bounded on* $L^{p(\cdot)}(\Gamma, w)$, *then* Γ *is a Carleson curve and*

$$0 \le 1/p(t_k) + m(w_k), \quad 1/p(t_k) + M(w_k) \le 1 \quad \textit{for all} \quad k \in \{1, \dots, n\}.$$

Moreover, if there exists an $\varepsilon_0 > 0$ *such that the Cauchy singular integral operator* S *is bounded on* $L^{p(\cdot)}(\Gamma, w^{1+\varepsilon})$ *for all* $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$, *then*

$$0 < 1/p(t_k) + m(w_k), \quad 1/p(t_k) + M(w_k) < 1 \quad \textit{for all} \quad k \in \{1, \dots, n\}.$$

For standard Lebesgue spaces, the boundedness of the operator S on $L^p(\Gamma, w)$, $1 < p < \infty$, implies that S is also bounded on $L^p(\Gamma, w^{1+\varepsilon})$ for all ε in a sufficiently small neighborhood of zero (see [3, Theorems 2.31 and 4.15]). Hence if $1 < p < \infty$, Γ is a simple Carleson curve, $w_1, \dots, w_n \in \mathbb{W}$, and the weight w is given by (1.1), then S is bounded on the standard Lebesgue space $L^p(\Gamma, w)$, $1 < p < \infty$, if and only if

$$0 < 1/p + m(w_k), \quad 1/p + M(w_k) < 1 \quad \text{for all} \quad k \in \{1, \dots, n\}.$$

We believe that all weighted variable Lebesgue spaces have this stability property.

Conjecture 1.4. *Let* Γ *be a simple rectifiable curve,* $p:\Gamma\to[1,\infty]$ *be a measurable a.e. finite function, and* $w:\Gamma\to[0,\infty]$ *be a weight such that the Cauchy singular integral operator* S *is bounded on* $L^{p(\cdot)}(\Gamma,w)$. *Then there is a number* $\varepsilon_0>0$ *such that* S *is bounded on* $L^{p(\cdot)}(\Gamma,w^{1+\varepsilon})$ *for all* $\varepsilon\in(-\varepsilon_0,\varepsilon_0)$.

If this conjecture would be true, we were able to prove the complete converse of Theorem 1.1 for non-closed curves, too.

2. Proof

In this section we formulate several results from [3, 6, 8] and show that Theorem 1.3 easily follows from them.

2.1 Muckenhoupt type condition. Suppose Γ is a simple rectifiable curve and $p:\Gamma\to(1,\infty)$ is a continuous function. Since Γ is compact, one has

$$1<\min_{\tau\in\Gamma}p(\tau),\quad \max_{\tau\in\Gamma}p(\tau)<\infty$$

and the conjugate exponent

$$q(\tau):=p(\tau)/(p(\tau)-1)\quad(\tau\in\Gamma)$$

is well defined and also bounded and bounded away from zero. We say that a weight $w:\Gamma\to[0,\infty]$ belongs to $A_{p(\cdot)}(\Gamma)$ if

$$\sup_{t\in\Gamma}\sup_{R>0}\frac{1}{R}\|w\chi_{\Gamma(t,R)}\|_{p(\cdot)}\|w^{-1}\chi_{\Gamma(t,R)}\|_{q(\cdot)}<\infty.$$

If $p=const\in(1,\infty)$, then this class coincides with the well known Muckenhoupt class. From the Hölder inequality for $L^{p(\cdot)}(\Gamma)$ (see e.g. [19, Theorems 13.12 and 13.13] for Muslielak-Orlicz spaces over arbitrary measure spaces and also [13, Theorem 2.1] for variable Lebesgue spaces over domains in $\mathbb{R}^n$) it follows that if $w\in A_{p(\cdot)}(\Gamma)$, then Γ is a Carleson curve.

Since $L^{p(\cdot)}(\Gamma,w)$ is a Banach function space in the sense of [2, Definition 1.1], the next result follows from [6, Theorem 6.1] (stated in [6] for Jordan curves, however its proof remains the same for curves homeomorphic to line segments, see also [7, Theorem 3.2]).

Theorem 2.1. *Let* Γ *be a simple rectifiable curve and let* $p:\Gamma\to(1,\infty)$ *be a continuous function. If* $w:\Gamma\to[0,\infty]$ *is an arbitrary weight such that the operator* S *is bounded on* $L^{p(\cdot)}(\Gamma,w)$, *then* $w\in A_{p(\cdot)}(\Gamma)$.

If $p = const \in (1, \infty)$, then $w \in A_p(\Gamma)$ is also sufficient for the boundedness of S on the weighted Lebesgue space $L^p(\Gamma, w)$ (see e.g. [3, Theorem 4.15]).

2.2 Submultiplicative functions. Following [3, Section 1.4], we say a function $\Phi : (0, \infty) \to (0, \infty]$ is *regular* if it is bounded in an open neighborhood of 1. A function $\Phi : (0, \infty) \to (0, \infty]$ is said to be *submultiplicative* if

$$\Phi(xy) \le \Phi(x)\Phi(y) \quad \text{for all} \quad x, y \in (0, \infty).$$

It is easy to show that if Φ is regular and submultiplicative, then Φ is bounded away from zero in some open neighborhood of 1. Moreover, in this case $\Phi(x)$ is finite for all $x \in (0, \infty)$. Given a regular and submultiplicative function $\Phi : (0, \infty) \to (0, \infty)$, one defines

$$\alpha(\Phi) := \sup_{x \in (0,1)} \frac{\log \Phi(x)}{\log x}, \quad \beta(\Phi) := \inf_{x \in (1,\infty)} \frac{\log \Phi(x)}{\log x}.$$

Clearly, $-\infty < \alpha(\Phi)$ and $\beta(\Phi) < \infty$.

Theorem 2.2 (see [3, Theorem 1.13] or [14, Chap. 2, Theorem 1.3])**.** *If a function* $\Phi : (0, \infty) \to (0, \infty)$ *is regular and submultiplicative, then*

$$\alpha(\Phi) = \lim_{x \to 0} \frac{\log \Phi(x)}{\log x}, \quad \beta(\Phi) = \lim_{x \to \infty} \frac{\log \Phi(x)}{\log x}$$

and $-\infty < \alpha(\Phi) \le \beta(\Phi) < +\infty$.

The quantities $\alpha(\Phi)$ and $\beta(\Phi)$ are called the *lower* and *upper indices of the regular and submultiplicative function* Φ, respectively.

2.3 Indices of powerlikeness. Fix $t \in \Gamma$ and put $d_t := \max_{\tau \in \Gamma} |\tau - t|$. Suppose $w : \Gamma \to [0, \infty]$ is a weight such that $\log w \in L^1(\Gamma(t, R))$ for every $R \in (0, d_t]$. Put

$$H_{w,t}(R_1, R_2) := \frac{\exp\left(\dfrac{1}{|\Gamma(t, R_1)|} \displaystyle\int_{\Gamma(t,R_1)} \log w(\tau)|d\tau|\right)}{\exp\left(\dfrac{1}{|\Gamma(t, R_2)|} \displaystyle\int_{\Gamma(t,R_2)} \log w(\tau)|d\tau|\right)}, \quad R_1, R_2 \in (0, d_t].$$

Consider the function

$$(V_t^0 w)(x) := \limsup_{R \to 0} H_{w,t}(xR, R), \quad x \in (0, \infty).$$

Combining Lemmas 4.8–4.9 and Theorem 5.9 of [6] with Theorem 3.4, Lemma 3.5 of [3], we arrive at the following.

Theorem 2.3. *Let* Γ *be a simple rectifiable curve,* $p:\Gamma\to(1,\infty)$ *be a continuous function satisfying* (1.2)*, and* $w:\Gamma\to[0,\infty]$ *be a weight such that* $w\in A_{p(\cdot)}(\Gamma)$*. Then, for every* $t\in\Gamma$*, the function* $V_t^0 w$ *is regular and submultiplicative and*

$$0\le 1/p(t)+\alpha(V_t^0 w),\quad 1/p(t)+\beta(V_t^0 w)\le 1.$$

The numbers $\alpha(V_t^0 w)$ and $\beta(V_t^0 w)$ are called the *lower* and *upper indices of powerlikeness* of w at $t\in\Gamma$, respectively (see [3, Chap. 3]). This terminology can be explained by the simple fact that for the power weight $w(\tau):=|\tau-t|^\lambda$ its indices of powerlikeness coincide and are equal to λ.

2.4 Matuszewska-Orlicz indices as indices of powerlikeness. If $\varrho\in\mathbb{W}$, then Φ_ϱ^0 is a regular and submultiplicative function and its indices are nothing else but the Matuszewska-Orlicz indices $m(\varrho)$ and $M(\varrho)$. The next result shows that for radial oscillating weights indices of powerlikeness and Matuszewska-Orlicz indices coincide.

Theorem 2.4 (see [8, Theorem 2.8])**.** *Suppose* Γ *is a simple Carleson curve. If* $w_1,\dots,w_n\in\mathbb{W}$ *and* $w(\tau)=\prod_{k=1}^n w_k(|\tau-t_k|)$*, then for every* $t\in\Gamma$ *the function* $V_t^0 w$ *is regular and submultiplicative and*

$$\begin{aligned}&\alpha(V_{t_k}^0 w)=m(w_k), &&\beta(V_{t_k}^0 w)=M(w_k) &&\textit{for}\quad k\in\{1,\dots,n\},\\ &\alpha(V_t^0 w)=0, &&\beta(V_t^0 w)=0 &&\textit{for}\quad t\in\Gamma\setminus\{t_1,\dots,t_n\}.\end{aligned}$$

Note that in [8], Theorem 2.4 is proved for Jordan curves. But the proof does not use the assumption that Γ is closed. It works also for non-closed curves considered in this paper.

2.5 Proof of Theorem 1.3. Suppose S is bounded on $L^{p(\cdot)}(\Gamma,w)$. From Theorem 2.1 it follows that $w\in A_{p(\cdot)}(\Gamma)$. By Hölder's inequality this implies that Γ is a Carleson curve. Fix an arbitrary $t\in\Gamma$. Then, in view of Theorems 2.2 and 2.3 the function $V_t^0 w$ is regular and submultiplicative, so its indices are well defined and satisfy $0\le 1/p(t)+\alpha(V_t^0 w)$ and $1/p(t)+\beta(V_t w)\le 1$. From these inequalities and Theorem 2.4 it follows that

$$0\le 1/p(t_k)+m(w_k),\quad 1/p(t_k)+M(w_k)\le 1 \tag{2.1}$$

for all $k\in\{1,\dots,n\}$.

If S is bounded on all spaces $L^{p(\cdot)}(\Gamma, w^{1+\varepsilon})$ for all ε in a neighborhood of zero, then as before

$$0 \le 1/p(t_k) + m(w_k^{1+\varepsilon}), \quad 1/p(t_k) + M(w_k^{1+\varepsilon}) \le 1$$

for every $k \in \{1, \dots, n\}$. It is easy to see that $m(w_k^{1+\varepsilon}) = (1+\varepsilon)m(w_k)$ and $M(w_k^{1+\varepsilon}) = (1+\varepsilon)M(w_k)$. Therefore

$$0 \le 1/p(t_k) + (1+\varepsilon)m(w_k), \quad 1/p(t_k) + (1+\varepsilon)M(w_k) \le 1$$

for all ε in a neighborhood of zero and for all $k \in \{1, \dots, n\}$. These inequalities immediately imply that $0 < 1/p(t_k) + m(w_k)$ and $1/p(t_k) + M(w_k) < 1$ for all k. □

Remark 2.5. The presented proof involves the notion of indices of powerlikeness, which were invented to treat general Muckenhoupt weights (see [3]). Weights considered in the present paper are continuous except for a finite number of points. So, it would be rather interesting to find a direct proof of the fact that $w \in A_{p(\cdot)}(\Gamma)$ implies (2.1), which does not involve the indices of powerlikeness $\alpha(V_t^0 w)$ and $\beta(V_t^0 w)$.

2.6 Final remarks. In connection with Conjecture 1.4, we would like to note that for standard Lebesgue spaces $L^p(\Gamma, w)$ there are two different proofs of the stability of the boundedness of S on $L^p(\Gamma, w^{1+\varepsilon})$ for small ε. Simonenko's proof [22] is based on the stability of the Fredholm property of some singular integral operators related to the Riemann boundary value problem. Another proof is based on the self-improving property of Muckenhoupt weights (see e.g. [3, Theorem 2.31]). One may ask whether does $w \in A_{p(\cdot)}(\Gamma)$ imply $w^{1+\varepsilon} \in A_{p(\cdot)}(\Gamma)$ for all $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$ with some fixed $\varepsilon > 0$? The positive answer would give a proof of the complete converse of (1.3). The author does not know any stability result for the boundedness of S or a self-improving property for $w \in A_{p(\cdot)}(\Gamma)$.

After this paper had been submitted, P. Hästö and L. Diening [5] have found a necessary and sufficient condition for the boundedness of the classical Hardy-Littlewood maximal function on weighted variable Lebesgue spaces in the setting of $\mathbb{R}^n$. Note that they write a weight as a measure (outside of $|\cdot|^{p(\tau)}$). Their condition is another generalization of the classical Muckenhoupt condition. In the setting of Carleson curves (and the weight written inside of $|\cdot|^{p(\tau)}$), the Hästö-Diening condition takes the form

(2.2)
$$\sup_{t\in\Gamma}\sup_{R>0}\left(\frac{1}{R^{p_{\Gamma(t,R)}}}\int_{\Gamma(t,R)} w(\tau)^{p(\tau)}|d\tau|\right)\|w(\cdot)^{-p(\cdot)}\chi_{\Gamma(t,R)}(\cdot)\|_{q(\cdot)/p(\cdot)} < \infty,$$

where

$$p_{\Gamma(t,R)} := \left(\frac{1}{|\Gamma(t,R)|} \int_{\Gamma(t,R)} \frac{1}{p(\tau)} \, |d\tau| \right)^{-1}.$$

Let $HD_{p(\cdot)}(\Gamma)$ denote the class of weights $w : \Gamma \to [0, \infty]$ satisfying (2.2). Following the arguments contained in [5, Remark 3.10], one can show that

$$A_{L^{p(\cdot)}}(\Gamma) \supset HD_{p(\cdot)}(\Gamma)$$

whenever $p : \Gamma \to (1, \infty)$ satisfies the Dini-Lipschitz condition (1.2). We conjecture that the Hästö-Diening characterization remains true also for the operator S in the setting of Carleson curves.

Conjecture 2.6. *Let Γ be a simple Carleson curve, $w : \Gamma \to [0, \infty]$ be a weight, and $p : \Gamma \to (1, \infty)$ be a continuous function satisfying the Dini-Lipschitz condition* (1.2)*. The operator S is bounded on $L^{p(\cdot)}(\Gamma, w)$ if and only if $w \in HD_{p(\cdot)}(\Gamma)$.*

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