

Extended Cesáro operators between generalized Besov spaces and Bloch type spaces in the unit ball

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(Communicated by Hans G. Feichtinger)

2000 Mathematics Subject Classification. Primary: 47B38; Secondary: 46E15, 32A37.

Keywords and phrases. Generalized Besov space, Bloch-type space, extended Cesáro operators, boundedness, compactness.

Abstract. Let g be a holomorphic of the unit ball B in the n -dimensional complex space, and denote by T_g the extended Cesáro operator with symbol g . Let $0 < p < +\infty$, $-n - 1 < q < +\infty$, $q > -1$ and $\alpha > 0$, starting with a brief introduction to well known results about Cesáro operator, we investigate the boundedness and compactness of T_g between generalized Besov space $B(p, q)$ and α - Bloch space B^α in the unit ball, and also present some necessary and sufficient conditions.

1. Introduction

For any $z = (z_1, \dots, z_n)$, $w = (w_1, \dots, w_n) \in C^n$, the inner product is defined by $\langle z, w \rangle = \sum_{k=1}^n z_k \bar{w}_k$. Let B be the unit ball of C^n , the class of all holomorphic functions on B is defined by $H(B)$. For $f \in H(B)$, we write

$$\nabla f(z) = \left(\frac{\partial f}{\partial z_1}(z), \dots, \frac{\partial f}{\partial z_n}(z) \right)$$

and

$$Rf(z) = \langle \nabla f(z), \bar{z} \rangle = \sum_{j=1}^n z_j \frac{\partial f(z)}{\partial z_j}.$$

For $\alpha \geq 0$, f is said to be in the *Bloch* space \mathcal{B}^α provided that $f \in H(B)$ and

$$\|f\|_\alpha = \sup_{z \in B} (1 - |z|^2)^\alpha |\nabla f(z)| < +\infty.$$

As we all know, \mathcal{B}^α is a *Banach* space when $\alpha \geq 1$ under the norm $\|f\|_{\mathcal{B}^\alpha} = |f(0)| + \|f\|_\alpha$. The spaces \mathcal{B}^1 and \mathcal{B}^α ($0 < \alpha < 1$) are just the *Bloch* space and the *Lipschitz* spaces $L_{1-\alpha}$ respectively. From [26] we know that a holomorphic function $f \in \mathcal{B}^\alpha$ if and only if $\sup_{z \in B} (1 - |z|^2)^\alpha |Rf(z)| < +\infty$.

Furthermore, by the Norm Equivalent Theorem we have

$$\|f\|_{\mathcal{B}^\alpha} \approx |f(0)| + \sup_{z \in B} (1 - |z|^2)^\alpha |Rf(z)|,$$

where $M \approx N$ means that the two quantities M and N are comparable, that is, there exist two positive constants C_1 and C_2 such that $C_1 M \leq N \leq C_2 M$.

Let dv be the *Lebesgue* measure on the unit ball B of C^n normalized so that $v(B) = 1$. For $a \in B$, let $g(z, a) = \log |\varphi_a(z)|^{-1}$ be the *Green's* function on B with logarithmic singularity at a , where φ_a is the *Möbius* transformation of B with $\varphi_a(0) = a$, $\varphi_a(a) = 0$, $\varphi_a = \varphi_a^{-1}$.

Let $0 < p, s < +\infty$, $-n - 1 < q < +\infty$ and $q + s > -1$. We say $f \in F(p, q, s)$ provided that $f \in H(B)$ and

$$\|f\|_{F(p, q, s)} = |f(0)| + \left\{ \sup_{a \in B} \int_B |\nabla f(z)|^p (1 - |z|^2)^q g^s(z, a) dv(z) \right\}^{\frac{1}{p}} < +\infty.$$

The space $F(p, q, s)$ was first defined in [31] (see also [27]).

Let $0 < p < +\infty$, $-n - 1 < q < +\infty$ and $q > -1$. We say that $f \in B(p, q)$ if $f \in H(B)$ and

$$\|f\|_{(p, q)} = \left\{ \int_B |\nabla f(z)|^p (1 - |z|^2)^q dv(z) \right\}^{\frac{1}{p}} < +\infty.$$

It is obvious that $B(p, q)$ is a classical Besov space if we take special parameters p, q . It is not hard to show that $B(p, q)$ is a Banach space under the norm $\|f\|_{B(p, q)} = |f(0)| + \|f\|_{(p, q)}$, we refer the reader to see Zhu's book [35]. From Exercises 2.2 in [35], we know that a holomorphic function $f \in B(p, q)$ if and only if $\int_B |Rf(z)|^p (1 - |z|^2)^q < +\infty$.

Let $f(z)$ be a holomorphic function on the unit disc D with *Taylor* expansion $f(z) = \sum_{j=0}^{\infty} a_j z^j$, the classical Cesàro operator acting on f is defined by

$$\mathcal{C}[f](z) = \sum_{j=0}^{\infty} \left(\frac{1}{j+1} \sum_{k=0}^j a_k \right) z^j.$$

Despite the simplicity of the definition of $\mathcal{C}[f](z)$, several problems are encountered when characterizing the boundedness and compactness of Cesàro operator between spaces of holomorphic functions. These problems require profound and interesting analytical machinery. Moreover, the study of Cesàro operator has arguably become a major driving force in the development of modern complex analysis. The papers listed in the bibliography are excellent sources for the recent developments in the theory of Cesàro operators. It is well known that the operator \mathcal{C} is bounded on the usual Hardy spaces $H^p(D)$ for $0 < p < \infty$, Bergman spaces and Dirichlet spaces. See [3, 7, 19, 20, 17, 25].

But the operator \mathcal{C} is not always bounded, in [22], Shi and Ren gave a necessary and sufficient condition for the operator \mathcal{C} to be bounded on mixed norm spaces in the unit disc. It is natural to ask what are the conditions for higher dimensional case.

A little calculation shows $\mathcal{C}[f](z) = \frac{1}{z} \int_0^z f(t) (\log \frac{1}{1-t})' dt$. From this point of view, if $g \in H(B)$, it is natural to consider the extended Cesàro operator (also called Volterra-type operator or Riemann-Stieltjes type operator) T_g on $H(B)$ defined by

$$T_g(f)(z) = \int_0^1 f(tz) Rg(tz) \frac{dt}{t}.$$

It is easy to show that T_g take $H(B)$ into itself. In general, there is no easy way to determine when an extended Cesàro operator is bounded or compact.

The boundedness and compactness of this operator on weighted Bergman, mixed norm, Bloch, and Dirichlet spaces in the unit ball have been studied by Xiao [25], Hu and Zhang [9, 10, 11, 30], Li and Stević [12, 13, 15]. More recently, Li and Stević [14] discuss the compactness of the operator between $F(p, q, s)$ and α -Bloch spaces in the ball, where s must be positive.

In this paper, we will continue this line of research and characterize those g for which T_g is bounded (or compact) between generalized Besov space $B(p, q)$ and α -Bloch space \mathcal{B}^α in the unit ball, and also present some necessary and sufficient conditions. For the proof, we need different method and some complex calculation skills.

For $p > 0, z \in B$, denote the function

$$G_p(z) = \begin{cases} 1, & 0 < p < 1; \\ \log \frac{2}{1-|z|^2}, & p = 1; \\ \left(\frac{1}{1-|z|^2}\right)^{\alpha-1}, & p > 1. \end{cases}$$

The main results of the paper are the following:

Theorem 1. For $0 < p < +\infty, -n - 1 < q < +\infty, q > -1, \alpha \geq 0, g \in H(B), T_g$ is bounded from $B(p, q)$ to \mathcal{B}^α if and only if

$$\sup_{z \in B} (1 - |z|^2)^\alpha G_{\frac{n+1+q}{p}}(z) |Rg(z)| < \infty.$$

Theorem 2. For $0 < p < +\infty, -n - 1 < q < +\infty, q > -1, \alpha \geq 0, g \in H(B), T_g$ is compact from $B(p, q)$ to \mathcal{B}^α if and only if $g \in \mathcal{B}^\alpha$ for $0 < \frac{n+1+q}{p} < 1$, and $\lim_{|z| \rightarrow 1^-} (1 - |z|^2)^\alpha G_{\frac{n+1+q}{p}}(z) |Rg(z)| = 0$ for $\frac{n+1+q}{p} \geq 1$.

2. Some Lemmas

In the following, we will use the symbol c or C to denote a finite positive number which does not depend on variable z and may depend on some norms and parameters p, q, n, α, x, f etc, not necessarily the same at each occurrence. We first give some lemmas.

Lemma 1. If $0 < p < +\infty, -n - 1 < q < +\infty, q > -1$, then $B(p, q) \subset \mathcal{B}^{\frac{n+1+q}{p}}$ and there exists $c > 0$, such that for all $f \in B(p, q)$,

$$\|f\|_{\mathcal{B}^{\frac{n+1+q}{p}}} \leq c \|f\|_{B(p, q)}.$$

Proof. Suppose $f \in B(p, q)$. Fix r_0 with $0 < r_0 < 1$. Since $(Rf) \circ \varphi_a \in H(B), |(Rf) \circ \varphi_a|^p$ is subharmonic in B . That is

$$\begin{aligned} |Rf(a)|^p &= |(Rf) \circ \varphi_a(0)|^p \leq \frac{1}{r_0^{2n}} \int_{r_0 B} |(Rf) \circ \varphi_a(\omega)|^p dv(\omega) \\ &= \frac{1}{r_0^{2n}} \int_{\varphi_a(r_0 B)} |(Rf(z))|^p \frac{(1 - |a|^2)^{n+1}}{|1 - \langle z, a \rangle|^{(2n+2)}} dv(z). \end{aligned}$$

From (5) in [37], we have

$$\frac{1 - r_0}{1 + r_0} (1 - |a|^2) \leq (1 - |z|^2) \leq \frac{1 + r_0}{1 - r_0} (1 - |z|^2)$$

as $z \in \varphi_a(r_0B)$. Thus

$$\frac{(1 - |a|^2)^{n+1}}{|1 - \langle z, a \rangle|^{2n+2}(1 - |z|^2)^q} \leq \frac{4^{n+1}}{(1 - |a|^2)^{n+1+q}} \left(\frac{1 + r_0}{1 - r_0}\right)^{|q|}.$$

Consequently, we have

$$\begin{aligned} |Rf(a)|^p &\leq \frac{1}{r_0^{2n}} \int_{\varphi_a(r_0B)} |Rf(z)|^p \frac{(1 - |a|^2)^{n+1}}{|1 - \langle z, a \rangle|^{2n+2}} dv(z) \\ &= \frac{1}{r_0^{2n}} \int_{\varphi_a(r_0B)} |Rf(z)|^p (1 - |z|^2)^q \frac{(1 - |a|^2)^{n+1}}{|1 - \langle z, a \rangle|^{2n+2}(1 - |z|^2)^q} dv(z) \\ &\leq \frac{4^{n+1} r_0^{-2n}}{(1 - |a|^2)^{n+1+q}} \left(\frac{1 + r_0}{1 - r_0}\right)^{|q|} \|f\|_{B(p,q)}^p. \end{aligned}$$

This shows that $f \in \mathcal{B}^{\frac{n+1+q}{p}}$ and $\|f\|_{\mathcal{B}^{\frac{n+1+q}{p}}} \leq c \|f\|_{B(p,q)}$. \square

Lemma 2. *Let $p > 0$. Then there exists a constant $c > 0$ such that for all $f \in \mathcal{B}^p$ and $z \in B$, the estimate*

$$|f(z)| \leq c G_p(z) \|f\|_{\mathcal{B}^p},$$

holds, where $G_p(z)$ is the function defined in the introduction.

Proof. This Lemma can be easily obtained by some integral estimates. See Lemma 2.2 in [27] for details. \square

Lemma 3. ([30]) *Let $0 < p < 1$, $\{f_j\}$ be any bounded sequence in \mathcal{B}^p and $f_j(z) \rightarrow 0$ on any compact subset of B . Then*

$$\lim_{j \rightarrow \infty} \sup_{z \in B} |f_j(z)| = 0.$$

Lemma 4. *There exists a constant $c > 0$ such that for all $t > -1$ and $z \in B$,*

$$\int_B \left| \log \frac{1}{1 - \langle z, w \rangle} \right|^2 \frac{(1 - |w|^2)^t}{(1 - \langle z, w \rangle)^{n+1+t}} dv(w) \leq C \left(\log \frac{1}{1 - |z|^2} \right)^2.$$

Proof. This Lemma can be proved by Stirling formula and some complex integral estimates. For the convenience of the readers, we will provide the proof here.

Denote the left term as I_t and let $2\lambda = t + n + 1$. By Taylor expansion

$$\left| \log \frac{1}{1 - \langle z, w \rangle} \right|^2 = \sum_{u,v=1}^{+\infty} \frac{\langle z, w \rangle^u \langle w, z \rangle^v}{uv}$$

and

$$\frac{1}{|1 - \langle z, w \rangle|^{2\lambda}} = \sum_{k,l=0}^{+\infty} \frac{\Gamma(\lambda + k)\Gamma(\lambda + l)}{k!l!\Gamma(\lambda)^2} \langle z, w \rangle^k \langle w, z \rangle^l.$$

Therefore

$$\begin{aligned} I_t &= \int_B \sum_{u,v=1}^{+\infty} \sum_{k,l=0}^{+\infty} \frac{\Gamma(\lambda + k)\Gamma(\lambda + l)}{uvk!l!\Gamma(\lambda)^2} \langle z, w \rangle^{k+u} \langle w, z \rangle^{l+v} (1 - |w|^2)^t dv(w) \\ &= \sum_{u=1}^{+\infty} \sum_{k=0}^{+\infty} \sum_{l=0}^{u+k-1} \frac{\Gamma(\lambda + k)\Gamma(\lambda + l)}{u(u+k-l)k!l!\Gamma(\lambda)^2} \int_B |\langle z, w \rangle|^{2(u+k)} (1 - |w|^2)^t dv(w) \end{aligned}$$

Without any loss of generality, we may assume that $z = |z|e_1$, so that

$$\begin{aligned} &\int_B |\langle z, w \rangle|^{2(u+k)} (1 - |w|^2)^t dv(w) \\ &= \int_B (|z|w_1)^{2(u+k)} (1 - |w|^2)^t dv(w) \\ &= 2n \int_0^1 \int_{\partial B} \rho^{2n-1} |z|^{2(u+k)} |\rho \xi_1|^{2(u+k)} (1 - \rho^2)^t d\rho d\delta_n(\xi) \\ &= 2n|z|^{2(u+k)} \int_0^1 \rho^{2(u+k+n-1)+1} (1 - \rho^2)^t d\rho \int_{\partial B} |\xi_1|^{2(u+k)} d\delta(\xi) \\ &= n|z|^{2(u+k)} \frac{\Gamma(u+k+n)\Gamma(t+1)}{\Gamma(u+k+n+t+1)} \frac{(n-1)!(u+k)!}{(u+k+n-1)!} \\ &= \frac{\Gamma(t+1)\Gamma(u+k+1)n!}{\Gamma(2\lambda+u+k)} |z|^{2(u+k)}, \end{aligned}$$

which gives

$$\begin{aligned} I_t &= \sum_{u=1}^{+\infty} \sum_{k=0}^{+\infty} \sum_{l=0}^{u+k-1} \frac{\Gamma(\lambda + k)\Gamma(\lambda + l)}{u(u+k-l)k!l!\Gamma(\lambda)^2} \frac{\Gamma(t+1)\Gamma(u+k+1)n!}{\Gamma(2\lambda+u+k)} |z|^{2(u+k)} \\ &= \sum_{u=1}^{+\infty} \sum_{k=0}^{+\infty} \frac{n!\Gamma(t+1)\Gamma(\lambda + k)\Gamma(u+k+1)}{uk!\Gamma(\lambda)^2\Gamma(2\lambda+u+k)} \sum_{l=0}^{u+k-1} \frac{\Gamma(\lambda + l)}{(u+k-l)l!} |z|^{2(u+k)} \end{aligned}$$

$$\begin{aligned}
&= \sum_{u=1}^{+\infty} \sum_{k=1}^{+\infty} \frac{n! \Gamma(t+1) \Gamma(\lambda+k) \Gamma(u+k+1)}{uk! \Gamma(\lambda)^2 \Gamma(2\lambda+u+k)} \sum_{l=0}^{u+k-1} \frac{\Gamma(\lambda+l)}{(u+k-l)!} |z|^{2(u+k)} \\
&\quad + \sum_{u=1}^{+\infty} \frac{n! \Gamma(t+1) \Gamma(u+1)}{u \Gamma(\lambda) \Gamma(2\lambda+u)} \sum_{l=0}^{u-1} \frac{\Gamma(\lambda+l)}{(u-l)!} |z|^{2u} \\
&= I_1 + I_2.
\end{aligned}$$

By Stirling formula, there exists an absolute constant C_1 such that for all $l, u, k \geq 1$

$$\begin{aligned}
\frac{\Gamma(\lambda+l)}{l!} &\leq C_1 l^{\lambda-1}, \quad \frac{\Gamma(u+k+1)}{\Gamma(2\lambda+u+k)} \leq C_1 (u+k)^{1-2\lambda}, \\
\frac{\Gamma(u+k+1)}{\Gamma(2\lambda+u)} &\leq C_1 u^{1-2\lambda}, \quad \frac{\Gamma(\lambda+k)}{k!} \leq C_1 k^{\lambda-1}
\end{aligned}$$

so that

$$I_1 \leq C_1^3 \sum_{u=1}^{+\infty} \sum_{k=1}^{+\infty} \frac{n! \Gamma(t+1) k^{\lambda-1} (u+k)^{1-2\lambda}}{u \Gamma(\lambda)^2} \sum_{l=1}^{u+k-1} \frac{l^{\lambda-1}}{(u+k-l)} |z|^{2(u+k)}$$

and

$$I_2 \leq C_1^2 \sum_{u=1}^{+\infty} \frac{n! \Gamma(t+1) u^{1-2\lambda}}{u \Gamma(\lambda)} \sum_{l=1}^{u-1} \frac{l^{\lambda-1}}{(u-l)} |z|^{2u}.$$

Note that

$$\sum_{l=1}^{M-1} \frac{l^{\lambda-1}}{M-l} \approx M^{\lambda-2} \log M$$

for any $M \geq 2$. Consequently, there exists a constant C , such that

$$\begin{aligned}
I_1 &\leq C \sum_{u=1}^{+\infty} \sum_{k=1}^{+\infty} \frac{n! \Gamma(t+1) k^{\lambda-1} (u+k)^{1-2\lambda}}{\Gamma(\lambda)^2 u} (u+k)^{\lambda-2} \log(u+k) |z|^{2(u+k)} \\
&= C \sum_{u=1}^{+\infty} \sum_{k=1}^{+\infty} \frac{n! \Gamma(t+1)}{\Gamma(\lambda)^2} \frac{k^\lambda}{(u+k)^\lambda} \frac{\log(u+k)}{u+k} \frac{1}{uk} |z|^{2(u+k)} \\
&\leq C \sum_{u=1}^{+\infty} \sum_{k=1}^{+\infty} \frac{1}{uk} |z|^{2(u+k)} = C \left(\log \frac{1}{1-|z|^2} \right)^2
\end{aligned}$$

and

$$\begin{aligned}
 I_2 &\leq C \sum_{u=1}^{+\infty} \frac{n! \Gamma(t+1) u^{1-2\lambda}}{\Gamma(\lambda) u} u^{\lambda-2} \log u |z|^{2u} \\
 &= C \sum_{u=1}^{+\infty} \frac{n! \Gamma(t+1)}{\Gamma(\lambda)} \frac{1}{u^{\lambda+1}} \frac{\log u}{u} |z|^{2u}.
 \end{aligned}$$

Clearly, I_2 can be controlled by $(\log \frac{1}{1-|z|^2})^2$ and the assertion is proved. □

Lemma 5. *Let g be a holomorphic self-map of B , K is an arbitrary compact subset of B . Then $T_g : B(p, q) \rightarrow \mathcal{B}^\alpha$ is compact if and only if for any uniformly bounded sequence $\{f_j\}(j \in N)$ in $B(p, q)$ which converges to zero uniformly for z on K when $j \rightarrow \infty$, $\|T_g f_j\|_{\mathcal{B}^\alpha} \rightarrow 0$ holds.*

Proof. Assume that T_g is compact and suppose $\{f_j\}$ is a sequence in $B(p, q)$ with $\sup_{j \in N} \|f_j\|_{B(p, q)} < \infty$ and $f_j \rightarrow 0$ uniformly on compact subsets of B . By the compactness of T_g we have that $\{T_g f_j\}$ has a subsequence $\{T_g f_{j_m}\}$ which converges in \mathcal{B}^α , say, to h . By Lemma 2, it follows that for any compact set $K \subset B$, there is a positive constant C_K independent of f such that $|T_g f_j(z) - h(z)| \leq C_K \|T_g f_j - h\|_{\mathcal{B}^\alpha}$ for all $z \in K$. This implies that $T_g f_j(z) - h(z) \rightarrow 0$ uniformly on compact sets of B . Since K is a compact subset of B , by the hypothesis and the definition of T_g , $T_g f_j(z)$ converges to zero uniformly on K . Since K is arbitrary, it follows that the limit function h is equal to 0. Since it is true for arbitrary subsequence of $\{f_j\}$, we see that $T_g f_j \rightarrow 0$ in \mathcal{B}^α .

Conversely, let $\{f_j\} \in K_r = B_{B(p, q)}(0, r)$, where $B_{B(p, q)}(0, r)$ is a ball in $B(p, q)$. Then by Lemma 2, $\{f_j\}$ is uniformly bounded in arbitrary compact subset M of B . By *Montel's Lemma*, $\{f_j\}$ is a normal family, therefore there is a subsequence $\{f_{j_m}\}$ which converges uniformly to $f \in H(B)$ on compact subsets of B . It follows that $\nabla f_{j_m} \rightarrow \nabla f$ uniformly on compact subsets of B .

Denote $B_k = B(0, 1 - \frac{1}{k}) \subset C^n$, then

$$\begin{aligned}
 &\int_B |\nabla f|^p (1 - |z|^2)^q dv(z) \\
 &= \lim_{k \rightarrow +\infty} \int_{B_k} \lim_{m \rightarrow +\infty} |\nabla f_{j_m}|^p (1 - |z|^2)^q dv(z) \\
 &\leq \lim_{k \rightarrow +\infty} \lim_{m \rightarrow +\infty} \int_{B_k} |\nabla f_{j_m}|^p (1 - |z|^2)^q dv(z).
 \end{aligned}$$

But $\{f_{j_m}\} \subset B_{B(p,q)}(0, r)$. Therefore

$$\int_{B_k} |\nabla f_{j_m}|^p (1 - |z|^2)^q dv(z) < r^p,$$

so that

$$\int_B |\nabla f|^p (1 - |z|^2)^q dv(z) \leq r^p.$$

Consequently, $\|f\|_{B(p,q)} \leq r$, and $f \in B(p, q)$. Hence $\|f_{j_m} - f\| \leq 2r < \infty$ and the sequence $\{f_{j_m} - f\}$ converges to 0 on compact subsets of B , by the hypothesis of this lemma, it follows that $T_g f_{j_m} \rightarrow T_g f$ in \mathcal{B}^α . Thus the set $T_g(K_r)$ is relatively compact and the proof is complete. \square

Lemma 6. *Let $g \in H(B)$, then for any $f \in H(B)$ and $z \in B$*

$$R[T_g f](z) = f(z)Rg(z).$$

Proof. Suppose the holomorphic function fRg has the *Taylor* expansion

$$(fRg)(z) = \sum_{|\alpha| \geq 1} a_\alpha z^\alpha.$$

Then we have

$$\begin{aligned} R(T_g f)(z) &= R \int_0^1 f(tz)R(tz) \frac{dt}{t} = R \int_0^1 \sum_{|\alpha| \geq 1} a_\alpha (tz)^\alpha \frac{dt}{t} \\ &= R \left[\sum_{|\alpha| \geq 1} \frac{a_\alpha z^\alpha}{|\alpha|} \right] = \sum_{|\alpha| \geq 1} a_\alpha z^\alpha = (fRg)(z). \end{aligned}$$

\square

3. Proof of Theorem 1

Suppose $\sup_{z \in B} (1 - |z|^2)^\alpha G_{\frac{n+1+q}{p}}(z) |Rg(z)| < \infty$. By Lemmas 1, 2 and 6, it follows that for all $f \in H(B)$

$$\begin{aligned} (1 - |z|^2)^\alpha |R[T_g f](z)| &= (1 - |z|^2)^\alpha |f(z)| |Rg(z)| \\ &\leq c (1 - |z|^2)^\alpha G_{\frac{n+1+q}{p}}(z) |Rg(z)| \\ &\leq c \|f\|_{B(p,q)} (1 - |z|^2)^\alpha G_{\frac{n+1+q}{p}}(z) |Rg(z)| \\ &\leq c \|f\|_{B(p,q)} \end{aligned}$$

which implies that T_g is bounded. On the other hand, suppose T_g is bounded, with $\|T_g f\|_{\mathcal{B}^\alpha} \leq c\|f\|_{B(p,q)}$.

Case 1. If $0 < \frac{n+1+q}{p} < 1$, it is clear that $T_g f \in \mathcal{B}^\alpha$ for $f = 1$, i.e.,

$$\sup_{z \in B} (1 - |z|^2)^\alpha |RT_g f(z)| = \sup_{z \in B} (1 - |z|^2)^\alpha |Rg(z)| < \infty.$$

Case 2. If $\frac{n+1+q}{p} > 1$, we need to prove that

$$\sup_{z \in B} (1 - |z|^2)^\alpha \left(\frac{1}{1 - |z|^2} \right)^{\frac{n+1+q}{p} - 1} |Rg(z)| < \infty.$$

For $w \in B$, consider the test function

$$f_w(z) = \frac{1 - |w|^2}{(1 - \langle z, w \rangle)^{\frac{n+1+q}{p}}}.$$

It is easy to see that

$$\int_B (1 - |z|^2)^q |\nabla f_w(z)|^p dv(z) \leq c(1 - |w|^2)^p \int_B \frac{(1 - |z|^2)^q}{|1 - \langle z, w \rangle|^{n+1+q+p}} dv(z) \leq c.$$

The last inequality follows from [35], so that $f_w \in B(p, q)$ for any $w \in B$. With the boundedness of T_g , we get

$$\begin{aligned} (1 - |z|^2)^\alpha \left(\frac{1}{1 - |z|^2} \right)^{\frac{n+1+q}{p} - 1} |Rg(z)| &= (1 - |z|^2)^\alpha |f_z(z)| |Rg(z)| \\ &= (1 - |z|^2)^\alpha |R(T_g f_z)(z)| \\ &\leq \|T_g f_z\|_{\mathcal{B}^\alpha} \leq c\|T_g\| < \infty. \end{aligned}$$

Case 3. If $\frac{n+1+q}{p} = 1$, namely $p = n + 1 + q$, we need to prove

$$\sup_{z \in B} (1 - |z|^2)^\alpha \log \frac{2}{1 - |z|^2} |Rg(z)| < \infty.$$

For $w \in B$, consider the test function

$$f_w(z) = \left(\log \frac{1}{1 - |w|^2} \right)^{-\frac{2}{p}} \left(\log \frac{1}{1 - \langle z, w \rangle} \right)^{1 + \frac{2}{p}}.$$

It is easy to show that $f_w \in B(p, q)$ from Lemma 4. The result follows by the same discussion as that of Case 2, the detail is omitted. So, the proof of Theorem 1 is completed.

4. Proof of Theorem 2

Case 1. $0 < \frac{n+1+q}{p} < 1$. If T_g is compact, it is obvious that T_g is bounded, in the proof of Theorem 1, we have got that $g \in \mathcal{B}^\alpha$.

Conversely, let $\{f_j\}$ be an uniformly bounded sequence in $B(p, q)$ which converges to zero uniformly on any compact subset of B when $j \rightarrow \infty$. From Lemma 1, we know that $\|f_j\|_{\mathcal{B}^{\frac{n+1+q}{p}}} \leq c\|f_j\|_{B(p, q)}$, thus $\{f_j\}$ is uniformly bounded in $\mathcal{B}^{\frac{n+1+q}{p}}$. Then by the hypothesis and Lemma 3, we get that

$$\lim_{j \rightarrow \infty} \sup_{z \in B} |f_j(z)| = 0.$$

From which we have

$$\|T_g f_j\|_{\mathcal{B}^\alpha} \leq c \sup_{z \in B} (1 - |z|^2)^\alpha |f_j(z) Rg(z)| \leq c \|g\|_{\mathcal{B}^\alpha} \sup_{z \in B} |f_j(z)| \rightarrow 0,$$

when $j \rightarrow \infty$. So T_g is compact from Lemma 5.

Case 2. If $\frac{n+1+q}{p} = 1$, it is sufficient to prove

$$\lim_{|z| \rightarrow 1} (1 - |z|^2)^\alpha |Rg(z)| \log \frac{1}{1 - |z|^2} = 0.$$

Suppose that $\lim_{|z| \rightarrow 1} (1 - |z|^2)^\alpha |Rg(z)| \log \frac{1}{1 - |z|^2} \neq 0$. Then there exists $\varepsilon_0 > 0$, $\{z^j\} \in B$, such that

$$(1 - |z^j|^2)^\alpha |Rg(z^j)| \log \frac{1}{1 - |z^j|^2} \geq \varepsilon_0.$$

Let

$$f_j(z) = \left(\log \frac{1}{1 - |z^j|^2} \right)^{-\frac{2}{p}} \left(\log \frac{1}{1 - \langle z, z^j \rangle} \right)^{1 + \frac{2}{p}}.$$

We have shown that $f_j \in B(p, q)$ with $\|f_j\|_{B(p, q)} \leq c$, and it is obvious that $f_j \rightarrow 0$ uniformly on any compact subset of B as $j \rightarrow \infty$. While

$$\begin{aligned} & \|T_g f_j\|_{\mathcal{B}^\alpha} \\ & \geq (1 - |z^j|^2)^\alpha |f_j(z^j)| |Rg(z^j)| \\ & = \left\{ (1 - |z^j|^2)^\alpha |Rg(z^j)| \log \frac{1}{1 - |z^j|^2} \right\} |f_j(z^j)| \left(\log \frac{1}{1 - |z^j|^2} \right)^{-1} \\ & \geq \varepsilon_0 |f_j(z^j)| \left(\log \frac{1}{1 - |z^j|^2} \right)^{-1} \\ & = \varepsilon_0, \end{aligned}$$

then $\|T_g f_j\|_{\mathcal{B}^\alpha}$ doesn't tend to 0 when $j \rightarrow \infty$. It's a contraction by Lemma 5. So

$$\lim_{|z| \rightarrow 1} (1 - |z|^2)^\alpha |Rg(z)| \log \frac{1}{1 - |z|^2} = 0.$$

Meanwhile, note that $\lim_{|z| \rightarrow 1} \log \frac{1}{1 - |z|^2} = \infty$, it is easy to see that $\lim_{|z| \rightarrow 1} (1 - |z|^2)^\alpha |Rg(z)| = 0$.

Therefore, we have

$$\lim_{|z| \rightarrow 1} (1 - |z|^2)^\alpha |Rg(z)| \log \frac{2}{1 - |z|^2} = 0.$$

If $\frac{n+1+q}{p} > 1$, then consider

$$f_j(z) = \frac{1 - |z^j|^2}{(1 - \langle z, z^j \rangle)^{\frac{n+1+q}{p}}}$$

and use the same method as in the case of $\frac{n+1+q}{p} = 1$, we can also prove

$$\lim_{|z| \rightarrow 1} (1 - |z|^2)^\alpha |Rg(z)| \log \frac{1}{1 - |z|^2} = 0.$$

Conversely, let $\{f_j\}$ be an uniformly bounded sequence in $B(p, q)$ which converges to zero uniformly on any compact subset of B when $j \rightarrow \infty$. If $\lim_{|z| \rightarrow 1} (1 - |z|^2)^\alpha G_{\frac{n+1+q}{p}}(z) |Rg(z)| = 0$, then for all $\varepsilon > 0$, there exists $r \in (0, 1)$, such that

$$(1 - |z|^2)^\alpha G_{\frac{n+1+q}{p}}(z) |Rg(z)| < \varepsilon, r < |z| < 1.$$

Then

$$\begin{aligned} & \|T_g f_j\|_{\mathcal{B}^\alpha} \\ & \leq c \sup_{|z| \leq r} (1 - |z|^2)^\alpha |f_j(z) Rg(z)| + c \sup_{r < |z| < 1} (1 - |z|^2)^\alpha |f_j(z) Rg(z)| \\ & \leq c \sup_{|z| \leq r} (1 - |z|^2)^\alpha |Rg(z)| |f_j(z)| \\ & \quad + c \sup_{r < |z| < 1} (1 - |z|^2)^\alpha G_{\frac{n+1+q}{p}}(z) |Rg(z)| \|f_j\|_{B(p, q)} \\ & \leq c \sup_{|z| \leq r} (1 - |z|^2)^{q\alpha} |Rg(z)| |f_j(z)| + c\varepsilon \|f_j\|_{B(p, q)} \\ & \leq c\varepsilon \end{aligned}$$

if j is sufficiently large. It follows from Lemma 5 that T_g is compact. So now, the proof of Theorem 2 is completed.

Acknowledgements. The first author is partly supported by the National Natural Science Foundation of China (Grand Nos.10671141, 10371091). The authors would like to express their sincere thanks to the editor and referees for helpful comments and suggestions which improved the manuscript considerably.

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(Received : September 2006)



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