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Radial variation in some function spaces

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Abstract. In a previous paper [8] we considered properties of the radial variation of analytic functions in a class of Besov spaces A^s_{pq}, $s > 0$. Here we wish to extend these results to certain related spaces. These are the Lipschitz classes Λ_s and the mean Lipschitz classes $\Lambda_{p,s}$ where $p \geq 1, 0 < s < 1$. We also consider A^0_{pq}, where $s = 0$, although the results obtained for these are not as good as when $s > 0$.

1. Introduction

If f is analytic in the disc, the radial variation function of f is the function defined on the disc by

$$F(r,t) = \int_0^r |f'(ue^{it})|\, du, \quad r < 1, \quad 0 \leq t \leq 2\pi. \tag{1}$$

Since $f(re^{it}) - f(0) = \int_0^r f'(ue^{it})\, du$, it is clear that

$$|f(re^{it})| \leq |f(0)| + F(r,t), \quad r < 1, \quad 0 \leq t \leq 2\pi,$$

and $F(r,t)$ is a majorant for f. The function $F(r,t)$ represents the length of the image of the radius vector $[0, re^{it}]$ under the mapping f. It is clear from the definition, that the boundary function $F(t) = \lim_{r\to 1} F(r,t)$ exists, finite or infinite, for all $t \in [0, 2\pi]$. It is known as the radial or total variation. An immediate property of F is that if $F(t) < \infty$, then $\lim_{r\to 1} f(re^{it})$ exists.

We saw in [8] that the property that $f \in A^s_{pq}$, $0 < s < 1, 1 \le p, q < \infty$, translated into meaningful results for F, in particular that $F(r,t)$ satisfies an analogous condition on the disc. In Section 1 we are led naturally to consider the case $s = 0$ when we ask for a condition under which $F(t)$ is an integrable function on the circle. It follows immediately that $F \in L^1$ if and only if $f \in A^0_{11}$. We then show that $F(r,t)$ satisfies a corresponding condition to that by f in the disc . This result extends to the general case $f \in A^0_{pq}$. In Section 3 we suppose that f belongs to a Lipschitz space or a mean Lipschitz space and show that both $F(r,t)$ and $F(t)$ exhibit the expected behaviour.

1.1 Preliminaries. Let D denote the unit disc, T the unit circle in the complex plane and $L^p = L^p(T)$ the usual Lebesgue space when $0 < p < \infty$. For $p \ge 1$ we denote the norm of a function $f \in L^p$ by $||f||_p$. For convenience we shall let m denote normalised Lebesgue measure on the circle T.

Let $\Delta_t f(e^{ix}) = f(e^{i(x+t)}) - f(e^{ix})$ and $\Delta^m_t = \Delta_t(\Delta^{m-1}_t)$. For $0 < s \le 1$, the Lipschitz class Λ_s is the space of 2π-periodic functions on $[-\pi, \pi]$ for which $|\Delta_t f(e^{ix})| = O(|t|^s)$ uniformly in x. A generalization is the mean Lipschitz class $\Lambda(p,s)$ consisting of all functions f for which $||\Delta_t f||_p = O(|t|^s)$ for $t > 0$; $\Lambda(p,s)$ reduces to Λ_s when $p = \infty$. Suppose now that f is analytic in D. If $0 \le r < 1$, let

$$M_p(f,r) = \left(\int_{-\pi}^{\pi} |f(re^{it})|^p \, dm\right)^{1/p}, \quad (0 < p < \infty),$$

denote the integral mean of f of order p. It is well known that $M_p(f,r)$ is an increasing function of r on $[0,1)$ and that the class of functions f for which $\sup_{r<1} M_p(f,r) < \infty$, is the familiar Hardy space H^p [2]. For $1 \le p, q < \infty, s > 0$, and an arbitrary integer $m > s$, we define the Besov space B^s_{pq} by

$$B^s_{pq} = \left\{ f \in L^p : \int_{-\pi}^{\pi} \frac{||\Delta^m_t f||^q_p}{|t|^{1+sq}} \, dm(t) < \infty \right\}.$$

It is well known that the definition is independent of m. For a discussion of these spaces see [1], [3], [4], [6], [7]. When s passes through a positive integer value, the working definition of the Besov space B^s_{pq} may require a change as indicated above.

The previous definition is no longer valid when $s \le 0$; for these cases another description is required. For $n \ge 1$ we let W_n be the polynomial on T whose Fourier coefficients satisfy $\hat{W}_n(2^n) = 1, \hat{W}_n(j) = 0$ for $j \notin (2^{n-1}, 2^{n+1})$ and $\hat{W}_n$ is a linear function on $[2^{n-1}, 2^n]$ and on $[2^n, 2^{n+1}]$. If $n < 0$ we put $W_n = \overline{W}_{-n}$. We put $W_0 = \overline{z}+1+z$. For $s \le 0$, $1 \le p, q < \infty$, B^s_{pq} consists of all distributions f on T for which

$$\sum_{n=-\infty}^{\infty} 2^{|n|s} \|f * W_n\|_p^q < \infty.$$

It is known that this description is equivalent to the previous one for $s > 0$, but for $s = 0$ in particular, only the second definition is valid. See [4] Appendix 2, [1]. In fact when $q > p$ there exist $f \in B^0_{pq}$ such that $f \notin L^p$.

Let A^s_{pq} denote the subspace of B^s_{pq} consisting of analytic functions. The space A^s_{pq} for $s > 0$, may be characterized as follows: for an arbitrary integer $m > s$ the analytic function $f \in A^s_{pq}$ if and only if

$$\|f\|_A = |f(0)| + \left\{ \int_0^1 (1-r^2)^{q(m-s)-1} M_p(f^{(m)}, r)^q r \, dr \right\}^{1/q} < \infty.$$

Once again the definition is independent of m for $m > s$. For $s = 0$ this definition is easily modified. This is because of the property that $f \in A^0_{pq}$ if and only if $If \in A^1_{pq}$ where I is the integration operator. Therefore $f \in A^0_{pq}$ if and only if with $m = 2$,

$$||f||_A = |f(0)| + \left\{ \int_0^1 (1-r^2)^{q-1} M_p(f', r)^q r \, dr \right\}^{1/q} < \infty,$$

and with $m = 3$, if and only if

$$||f||_A = |f(0)| + \left\{ \int_0^1 (1-r^2)^{2q-1} M_p(f^{(2)}, r)^q r \, dr \right\}^{1/q} < \infty.$$

We shall need both of these representations. In particular with $p = q = 1$ we have $f \in A^0_{11}$ if and only if

$$||f||_A = |f(0)| + \int_0^1 \int_0^{2\pi} |f'(re^{it})| \, dmr \, dr < \infty.$$

2. Integrability of F

The function $F(t) = F(1,t)$ is given from (1) by

$$F(t) = \int_0^1 |f'(ue^{it})|\ du, \quad 0 \le t \le 2\pi.$$

We now ask what is a sufficient condition that $F \in L^1$? Since $F \in L^1$ if and only if $\int_0^{2\pi}\int_0^1 |f'(re^{it})| r\ dr\ dm < \infty$, the answer is immediate from the definition:

Proposition 1. *$F \in L^1(T)$ if and only if $f \in A^0_{11}$. Moreover*

$$\|F\|_1 + |f(0)| = \|f\|_A. \tag{2}$$

It may be observed here that if $f \in A^0_{11}$ then its boundary function $f(e^{it})$ exists a.e.; in fact $f \in H^1$. This follows by integrating the obvious inequality $|f(re^{it})| \le |f(0)| + \int_0^r |f'(ue^{it})|\ du$.

We can equally express the relationship in terms of the A-norm of $F(r,t)$. For this purpose we introduce the gradient of F: $\nabla F(r,t) = (\frac{\partial F}{\partial r}, 1/r\frac{\partial F}{\partial t}) = (|f'(re^{it})|, 1/r\frac{\partial F}{\partial t})$. The relationship referred to is

$$f \in A^0_{11} \text{ if and only if } \int_0^1\int_0^{2\pi} |\nabla F(r,t)|\ dmr\ dr < \infty.$$

If the integral is finite then it follows very simply that $f \in A^0_{11}$ and that $\|f\|_A \le |f(0)| + \|F\|_A$. The proof in the other direction has already been done in essence in [8] where we considered only $s > 0$. In fact we can state a more general result which follows from Theorem 1 there, and which works without any changes for our situation.

Theorem 1. *Suppose that $1 \le p, q < \infty$. There is a constant $C = C(p,q)$ such that if $f \in A^0_{pq}$ then*

$$\int_0^1 (1-r^2)^{q-1}\left(\int_{-\pi}^{\pi} |\nabla F(r,t)|^p\ dm\right)^{q/p} r\ dr \le C||f||_A^q.$$

Proof. The proof in [8] goes through word for word with $s = 0$. In the case $p = q = 1$ it is simpler since the use of Hölder's inequality is not needed. We do make use of the alternative definitions of A^0_{pq} mentioned above. □

If the double integral for $F(r,t)$ is finite then as noted already it is clear that $f \in A^0_{pq}$. The question when $F \in L^p,\ p > 1$, does not have so neat an answer. A reasonable sufficient condition is given by

Theorem 2. *Suppose that* $1 \le p, q < \infty$. *If* $f \in A^0_{p1}$ *then*

$$\|F\|_p \le ||f||_A.$$

Proof. By Minkowski's Inequality in continuous form

$$\begin{aligned}\left(\int_0^{2\pi} |F(t)|^p\ dm\right)^{1/p} &= \left(\int_0^{2\pi}\left(\int_0^1 |f'(re^{it})|\ dr\right)^p\ dm\right)^{1/p}\\ &\le \int_0^1 \Big(\int_0^{2\pi} |f'(re^{it})|^p\ dm\Big)^{1/p}\ dr\\ &< \infty,\end{aligned}$$

and $||F||_p \le ||f||_A$. □

Remark. The condition $f \in A^0_{p1}$ implies that $f \in H^p$ for all $p \ge 1$. To see this we note that for $r < 1$

$$|f(re^{it})| \le |f(0)| + \int_0^r |f'(ue^{it})|\ du.$$

On using Minkowski's Inequality again we obtain

$$\begin{aligned}M_p(f,r) &\le |f(0)| + \int_0^r M_p(f',u)\ du\\ &\le ||f||_A\end{aligned}$$

and the result is immediate.

In [8] it was shown that if $f \in A^s_{pq},\ 0 < s < 1$, then the boundary function $F \in B^s_{pq}$. We do not know whether this is true for the case $s = 0$ since the proof given there is no longer valid.

3. The Lipschitz spaces

The Lipschitz space $\Lambda_s, 0 < s < 1$, may be regarded as the Besov space $B^s_{\infty\infty}$. It is well known that for an analytic function f on the disc, $f \in \Lambda_s$ if and only if there exists M such that

$$|f'(z)| \le \frac{M}{(1-r)^{1-s}} \tag{3}$$

This property has its counterpart for the function $F(r,t)$.

Theorem 3. *The function* $f \in \Lambda_s$, $0 < s < 1$, *if and only if* $\nabla F(r,t) = O((1-r)^{s-1})$.

Proof. Suppose $f \in \Lambda_s$ and let M be the number noted above. First we show that $F(t)$ is bounded.

$$\begin{aligned} F(r,t) = \int_0^r |f'(ue^{it})|\, du &\le M \int_0^r \frac{1}{(1-u)^{1-s}}\, du \\ &= M\left(1-(1-r)^s\right)/s \le M/s, \end{aligned}$$

for all $r < 1$ and so $F(t)$ is bounded.

Since the first component of $\nabla F(r,t)$ is $|f'(re^{it})|$ we need only consider the second. Now by Lemma 3 of [8], $\frac{\partial F}{\partial t}(r,t) = \int_0^r \frac{\partial |f'|}{\partial t}(ue^{it})\, du$ and

$$\begin{aligned} \left|1/r \frac{\partial F}{\partial t}(r,t)\right| &= \left|1/r \int_0^r \frac{\partial |f'|}{\partial t}(ue^{it})\, du\right| \\ &\le 1/r \int_0^r u|f''(ue^{it})|\, du \\ &\le M \int_0^r \frac{1}{(1-u)^{2-s}}\, du \le M' \frac{1}{(1-r)^{1-s}}. \end{aligned}$$

In the second inequality above we used Theorem 5.5 of [2]. The result follows. □

There is a corresponding result for $F(t)$.

Theorem 4. *If* $f \in \Lambda_s$, $0 < s < 1$, *then* $F(t) \in \Lambda_s$.

Proof. We have shown that F is bounded. We write

$$F(x) - F(t) = F(x) - F(r,x) + F(r,x) - F(r,t) + F(r,t) - F(t).$$

But

$$\begin{aligned} F(x) - F(r,x) = \int_r^1 |f'(ue^{ix})|\, du &\le M \int_r^1 \frac{1}{(1-r)^{1-s}}\, du \\ &\le M(1-r)^s/s \end{aligned}$$

and the same holds for $F(r,t) - F(t)$. Moreover $F(r,x) - F(r,t) = \int_t^x \frac{\partial F}{\partial v}(r,v)\, dv$. Consequently

$$|F(r,x)-F(r,t)| \le \left|\int_t^x |\frac{\partial F}{\partial v}(r,v)|\ dv\right| \le M'\left|\int_t^x \frac{1}{(1-r)^{1-s}}\ dv\right| = M'\frac{1}{(1-r)^{1-s}}|t-x|,$$

on using the previous theorem. If we now choose $1-r=|x-t|$ we get $|F(r,x)-F(r,t)| \le M''|t-x|^s$ and $F(t) \in \Lambda_s$. □

The mean Lipschitz classes $\Lambda_{p,s}(T)$, $1 \le p$, $0 < s < 1$, are indentical with the Besov spaces $B^s_{p\infty}$. They satisfy the condition: A function $g \in L^p(T)$ belongs to $\Lambda_{p,s}$ if

$$\|g\|_{p,s} = \left(\int_0^{2\pi} |g(x+t)\ - g(x)|^p\ dx\right)^{1/p} = O(|t|^s)$$

for small t. It is known (Theorem 5.4 of [2]) that an analytic function f is in $\Lambda_{p,s}$ if and only if $M_p(f',r) = O\big(\frac{1}{(1-r)^{1-s}}\big)$ $\quad 0<r<1$. With the aid of this, similar results to those of the last two theorems can be shown to hold and the proofs are straightforward.

Theorem 5. *If $f \in \Lambda_{p,s}$, $1 \le p$, $0 < s < 1$, then there exists $C = C(p,s)$ such that*

(a) $\left(\int_{-\pi}^{\pi} |\nabla F(r,t)|^p\ dm\right)^{1/p} \le C\|f\|_{p,s}(1-r)^{s-1}$*;*

(b) $F(t) \in \Lambda_{p,s}$ *and* $\|F\|_{p,s} \le C\|f\|_{p,s}$.

Whether a particular type of continuity for f implies the same holds for F is uncertain. The boundary function $f(e^{it})$ is absolutely continuous if and only if $f' \in H^1$. We dont know that this implies that $F(t)$ is absolutely continuous but it does imply that F is continuous.

Proposition 2. *If $f(e^{it})$ is absolutely continuous then $F(t)$ is continuous.*

Proof. We have $F(t+x)-F(t) = \int_0^1 (|f'(re^{i(t+x)})| - |f'(re^{it})|)\ dr$ and therefore

$$|F(t+x)-F(t)| \le \int_0^1 |f'(re^{i(t+x)}) - f'(re^{it})|\ dr = \int_0^1 |f'_x(re^{it}) - f'(re^{it})|\ dr$$

where $g_x(t) = g(t+x)$ is a translate of g. The Fejer-Riesz inequality allows us to conclude

$$|F(t+x) - F(t)| + |F(t+x+\pi) - F(t+\pi)| \\ \leq \int_{-1}^{1} |f'_x(re^{it}) - f'(re^{it})| \, dr \leq \frac{1}{2} \int_0^{2\pi} |f'_x(re^{it}) - f'(re^{it})| \, dx \to 0$$

as $x \to 0$ uniformly in t, because the translation map $x \to g_x$ is uniformly continuous from T to L^1. The proof is complete. $\square$

In [8] it was seen that if we assume slightly more, namely if $f \in A^1_{11}$, then $F \in B^1_{11}$ which implies that F is absolutely continuous. However mere continuity of f on the circle does not even imply that F is bounded. In fact Walter Rudin [5] has shown that there exists an analytic function f continuous in the closed disc, such that $F(t) = \infty$ almost everywhere.

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