

The Jensen functional equation in non-Archimedean normed spaces

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Abstract. We investigate the Hyers–Ulam–Rassias stability of the Jensen functional equation in non-Archimedean normed spaces and study its asymptotic behavior in two directions: bounded and unbounded Jensen differences. In particular, we show that a mapping f between non-Archimedean spaces with $f(0) = 0$ is additive if and only if

$$\|f(\frac{x+y}{2}) - \frac{f(x)+f(y)}{2}\| \rightarrow 0$$

as $\max\{\|x\|, \|y\|\} \rightarrow \infty$.

1. Introduction and preliminaries

The history of the stability theory of functional equations started with a problem concerning group homomorphisms posed by S.M. Ulam [30] in 1940 and its solution given by H.D. Hyers [7] in 1941. Hyers' theorem was generalized by T. Aoki [1] for additive mappings and by Th.M. Rassias [24] for linear mappings by considering an unbounded Cauchy difference. The paper [24] of Th.M. Rassias has provided a lot of influence in the development of what we now call Hyers–Ulam–Rassias stability of functional

equations. During the last decades many stability problems for various functional equations have been studied by numerous mathematicians. We refer the reader to [4, 8, 13, 25, 26] and references therein. The first result on the stability of the classical Jensen equation $f(\frac{x+y}{2}) = \frac{f(x)+f(y)}{2}$ was given by Z. Kominek [16]. The first author, who investigated the stability problem on a restricted domain was F. Skof [29]. The stability of the Jensen equation and its generalizations were studied by numerous researchers, cf. [5, 12, 17, 22]

By a *non-Archimedean field* we mean a field K equipped with a function (valuation) $|\cdot|$ from K into $[0, \infty)$ such that $|r| = 0$ if and only if $r = 0$, $|rs| = |r||s|$, and $|r+s| \leq \max\{|r|, |s|\}$ for all $r, s \in K$. Clearly $|1| = |-1| = 1$ and $|n| \leq 1$ for all $n \in \mathbb{N}$.

Let X be a vector space over a field \mathbb{K} with a non-Archimedean non-trivial valuation $|\cdot|$. A function $\|\cdot\| : X \rightarrow [0, \infty)$ is called a *non-Archimedean norm* if it satisfies the following conditions:

- (i) $\|x\| = 0$ if and only if $x = 0$;
- (ii) $\|rx\| = |r|\|x\|$ ($r \in \mathbb{K}, x \in X$);
- (iii) the strong triangle inequality (ultrametric); namely,

$$\|x + y\| \leq \max\{\|x\|, \|y\|\} \quad (x, y \in X).$$

Then $(X, \|\cdot\|)$ is called a non-Archimedean normed space; cf. [28, 10, 18]. Due to the fact that

$$\|x_n - x_m\| \leq \max\{\|x_{j+1} - x_j\| : m \leq j \leq n-1\} \quad (n > m)$$

a sequence $\{x_n\}$ is Cauchy if and only if $\{x_{n+1} - x_n\}$ converges to zero in a non-Archimedean normed space. By a complete non-Archimedean normed space we mean one in which every Cauchy sequence is convergent. Theory of non-Archimedean normed spaces is not trivial, for instance there may not be any unit vector. Although many results in classical normed space theory have a non-Archimedean counterpart, but their proofs are essentially different and require an entirely new kind of intuition, cf. [20, 21, 23].

In 1897, Hensel [6] discovered the p -adic numbers as a number theoretical analogue of power series in complex analysis. Fix a prime number p . For any nonzero rational number x , there exists a unique integer $n_x \in \mathbb{Z}$ such that $x = \frac{a}{b}p^{n_x}$, where a and b are integers not divisible by p . Then $|x|_p := p^{-n_x}$ defines a non-Archimedean norm on \mathbb{Q} . The completion of \mathbb{Q} with respect to the metric $d(x, y) = |x - y|_p$ is denoted by \mathbb{Q}_p , which is called the p -adic number field; cf. [27, 3]. During the last three decades p -adic numbers have gained the interest of physicists for their research in

particular in problems coming from quantum physics, p -adic strings and superstrings; cf. [15].

In [2], the authors investigated stability of approximate additive mappings $f : \mathbb{Q}_p \rightarrow \mathbb{R}$. The stability of the Cauchy equation in normed spaces over fields with valuation was studied in [14]. In [20], the stability of Cauchy and quadratic functional equations were investigated in the context of non-Archimedean normed spaces. In this paper, using some ideas from [9, 11, 12, 19] we establish the Hyers–Ulam–Rassias stability of the Jensen functional equation in the setting of non-Archimedean normed spaces and study its asymptotic behavior in two directions: bounded and unbounded Jensen differences.

2. Stability of the Jensen equation

In this section, we prove the stability of the Jensen functional equation. Throughout this section we assume that X is a non-Archimedean normed space and Y is a non-Archimedean Banach space over a non-Archimedean field \mathbb{K} with $|3| < 1$.

Theorem 2.1. *Suppose that $\alpha, \beta \geq 0$, $0 \leq p < 1$ and $f : X \rightarrow Y$ is a mapping satisfying $\|f(0)\| \leq \beta$ and*

$$(2.1) \quad \left\| f\left(\frac{x+y}{2}\right) - f(x) - f(y) \right\| \leq \alpha \max\{\|x\|^p, \|y\|^p\} \quad (x, y \in X \setminus \{0\}).$$

Then there exists a unique Jensen mapping $T : X \rightarrow Y$ such that

$$(2.2) \quad \|f(x) - T(x)\| \leq \max\left\{\frac{\alpha}{|3|^p}\|x\|^p, |2|\beta\right\} \quad (x \in X \setminus \{0\}).$$

Proof. Let $x \in X \setminus \{0\}$. Replace x and y by x and $\frac{-x}{3}$ in (2.1), respectively, to get

$$(2.3) \quad \begin{aligned} \left\| 2f\left(\frac{x}{3}\right) - f(x) - f\left(\frac{-x}{3}\right) \right\| &\leq \alpha \max\left\{\|x\|^p, \left\|\frac{-x}{3}\right\|^p\right\} \\ &= \frac{\alpha}{|3|^p}\|x\|^p \quad (\text{by } |3|^p < 1). \end{aligned}$$

Replace x and y by $\frac{x}{3}$ and $\frac{-x}{3}$ in (2.1), respectively, to obtain

$$\|2f(0) - f\left(\frac{x}{3}\right) - f\left(\frac{-x}{3}\right)\| \leq \alpha \max\left\{\left\|\frac{x}{3}\right\|^p, \left\|\frac{-x}{3}\right\|^p\right\} = \frac{\alpha}{|3|^p}\|x\|^p$$

whence

$$(2.4) \quad \left\| f\left(\frac{x}{3}\right) + f\left(\frac{-x}{3}\right) \right\| \leq \max \left\{ \frac{\alpha}{|3|^p} \|x\|^p, |2| \|f(0)\| \right\} \leq \max \left\{ \frac{\alpha}{|3|^p} \|x\|^p, |2|\beta \right\}.$$

It follows from (2.3) and (2.4) that

$$(2.5) \quad \begin{aligned} \left\| 3f\left(\frac{x}{3}\right) - f(x) \right\| &\leq \max \left\{ \left\| 2f(0) - f\left(\frac{x}{3}\right) - f\left(\frac{-x}{3}\right) \right\|, \left\| f\left(\frac{x}{3}\right) + f\left(\frac{-x}{3}\right) \right\| \right\} \\ &\leq \max \left\{ \frac{\alpha}{|3|^p} \|x\|^p, \max \left\{ \frac{\alpha}{|3|^p} \|x\|^p, |2|\beta \right\} \right\} \\ &= \max \left\{ \frac{\alpha}{|3|^p} \|x\|^p, |2|\beta \right\} \quad (x \in X \setminus \{0\}). \end{aligned}$$

Given $x \in X \setminus \{0\}$, replace x by $\frac{x}{3^n}$ in (2.5) and multiple the obtained inequality with $|3|^n$ to get

$$\begin{aligned} \left\| 3^{n+1}f\left(\frac{x}{3^{n+1}}\right) - 3^n f\left(\frac{x}{3^n}\right) \right\| &\leq \max \left\{ \frac{\alpha}{|3|^p} |3|^n \left\| \frac{x}{3^n} \right\|^p, |3|^n |2|\beta \right\} \\ &= \max \left\{ (|3|^{1-p})^n \frac{\alpha}{|3|^p} \|x\|^p, |3|^n |2|\beta \right\} \end{aligned}$$

The right hand side tends to zero as $n \rightarrow \infty$, so the sequence $\{3^n f(\frac{x}{3^n})\}$ is Cauchy. Since Y is complete, we conclude that $\{3^n f(\frac{x}{3^n})\}$ is convergent.

Set $T(x) := \lim_{n \rightarrow \infty} 3^n f(\frac{x}{3^n})$. Assume that

$$(2.6) \quad \left\| 3^n f\left(\frac{x}{3^n}\right) - f(x) \right\| \leq \max \left\{ \frac{\alpha}{|3|^p} \|x\|^p, |2|\beta \right\}$$

for some $n \in \mathbb{N}$. Then

$$\begin{aligned} \left\| 3^{n+1}f\left(\frac{x}{3^{n+1}}\right) - f(x) \right\| &\leq \max \left\{ \left\| 3^{n+1}f\left(\frac{x}{3^{n+1}}\right) - 3^n f\left(\frac{x}{3^n}\right) \right\|, \left\| 3^n f\left(\frac{x}{3^n}\right) - f(x) \right\| \right\} \\ &\leq \max \left\{ \max \left\{ (|3|^{1-p})^n \frac{\alpha}{|3|^p} \|x\|^p, |3|^n |2|\beta \right\}, \right. \\ &\quad \left. \max \left\{ \frac{\alpha}{|3|^p} \|x\|^p, |2|\beta \right\} \right\} \\ &= \max \left\{ \frac{\alpha}{|3|^p} \|x\|^p, |2|\beta \right\} \end{aligned}$$

Hence (2.6) holds for all positive integer n . Letting n approach to infinity in (2.6) we get

$$\|T(x) - f(x)\| \leq \max \left\{ \frac{\alpha}{|3|^p} \|x\|^p, |2|\beta \right\}.$$

Replacing x and y by $\frac{x}{3^n}$ and $\frac{y}{3^n}$, respectively, in (2.1) we get

$$\|3^n 2f\left(\frac{x+y}{2 \cdot 3^n}\right) - 3^n f\left(\frac{x}{3^n}\right) - 3^n f\left(\frac{y}{3^n}\right)\| \leq \alpha(|3|^{1-p})^n \max \{\|x\|^p, \|y\|^p\}.$$

Taking the limit as $n \rightarrow \infty$ we obtain

$$2T\left(\frac{x+y}{2}\right) = T(x) + T(y) \quad (x, y \in X \setminus \{0\}).$$

If T' is another Jensen mapping satisfying (2.2), then

$$\begin{aligned} \|T(x) - T'(x)\| &= \lim_{k \rightarrow \infty} |3|^k \|T\left(\frac{x}{3^k}\right) - T'\left(\frac{x}{3^k}\right)\| \\ &\leq \lim_{k \rightarrow \infty} |3|^k \max \left\{ \|T\left(\frac{x}{3^k}\right) - f\left(\frac{x}{3^k}\right)\|, \|f\left(\frac{x}{3^k}\right) - T'\left(\frac{x}{3^k}\right)\| \right\} \\ &\leq \lim_{k \rightarrow \infty} |3|^k \max \left\{ \frac{\alpha}{|3|^p} \left\| \frac{x}{3^k} \right\|^p, |2|\beta \right\} \\ &= \lim_{k \rightarrow \infty} \max \left\{ \frac{\alpha}{|3|^p} (|3|^{1-p})^k \|x\|^p, |3|^k |2|\beta \right\} = 0, \quad x \in X \setminus \{0\}. \end{aligned}$$

Therefore $T = T'$. This completes the proof of the uniqueness of T . \square

Corollary 2.2. *Suppose that $\alpha \geq 0$, $0 \leq p < 1$ and $f : X \rightarrow Y$ is a mapping satisfying $f(0) = 0$ and*

$$\left\| f\left(\frac{x+y}{2}\right) - f(x) - f(y) \right\| \leq \alpha \max \{\|x\|^p, \|y\|^p\} \quad (x, y \in X).$$

Then there exists a unique additive mapping $T : X \rightarrow Y$ such that

$$(2.7) \quad \|f(x) - T(x)\| \leq \frac{\alpha}{|3|^p} \|x\|^p \quad (x \in X).$$

Proof. It follows from Theorem 2.1 that there is a unique Jensen mapping T satisfying (2.2). Since $f(0) = 0$, we have $T(0) = 0$. Hence T is clearly additive and satisfies (2.7). \square

3. Asymptotic aspect of a bounded Jensen difference

In this section, we deal with the asymptotic behavior of the Jensen functional equation. Throughout this section we assume that X is a non-Archimedean normed space over \mathbb{K} with $\{\|x\| : x \in \mathcal{X}\} = \{|r| : r \in \mathcal{K}\}$ and Y is a non-Archimedean Banach space over a non-Archimedean field \mathbb{K} with $|\beta| < 1$. Utilizing the strategy of Theorem 3 of [12] we get the following result.

Theorem 3.1. *Suppose that $\alpha, \beta > 0$ and $f : X \rightarrow Y$ is a mapping satisfying $f(0) = 0$ and*

$$(3.1) \quad \left\| f\left(\frac{x+y}{2}\right) - f(x) - f(y) \right\| \leq \alpha$$

for all $x, y \in X$ with $\max\{\|x\|, \|y\|\} \geq |\beta|$. Then there exists a unique additive mapping $T : X \rightarrow Y$ such that

$$\|f(x) - T(x)\| \leq \alpha \quad (x \in X).$$

Proof. Assume that $\max\{\|x\|, \|y\|\} < |\beta|$. For $x = y = 0$ take $z \in X$ to be an element of X with $\|z\| = |\beta|$. Without loss of generality, assume that $\|y\| \leq \|x\| < |\beta|$. Let $\gamma \in K$ with $|\gamma| = \|x\|$. Set $z := x + \frac{\beta}{3^n \gamma} x$ for large enough n such that $x \neq 0$ or $y \neq 0$,

$$(3.2) \quad \begin{aligned} \max\{\|x - z\|, \|y + z\|\} &\geq |\beta| \\ \max\{\|2z\|, \|x - z\|\} &\geq |\beta| \\ \max\{\|y\|, \|2z\|\} &\geq |\beta| \\ \max\{\|y + z\|, \|z\|\} &\geq |\beta| \\ \max\{\|x\|, \|z\|\} &\geq |\beta|. \end{aligned}$$

It follows from (3.1) and (3.2), we get

$$\begin{aligned} \left\| f\left(\frac{x+y}{2}\right) - \frac{f(x) + f(y)}{2} \right\| &\leq \max \left\{ \left\| f\left(\frac{x+y}{2}\right) - \frac{f(x-z) + f(y+z)}{2} \right\|, \right. \\ &\quad \left\| \frac{f(2z) + f(x-z)}{2} - f\left(\frac{x+z}{2}\right) \right\|, \\ &\quad \left\| f\left(\frac{y+2z}{2}\right) - \frac{f(y) + f(2z)}{2} \right\|, \\ &\quad \left\| \frac{f(y+z) + f(z)}{2} - f\left(\frac{y+2z}{2}\right) \right\|, \\ &\quad \left. \left\| f\left(\frac{x+z}{2}\right) - \frac{f(x) + f(z)}{2} \right\| \right\}. \end{aligned}$$

Thus

$$\|f(\frac{x+y}{2}) - \frac{f(x)+f(y)}{2}\| \leq \alpha$$

for all $x, y \in X$. Now the result is deduced from Corollary 2.2. \square

Theorem 3.2. *Suppose that $f : X \rightarrow Y$ is a mapping satisfying $f(0) = 0$. Then f is additive if and only if*

$$(3.3) \quad \|f(\frac{x+y}{2}) - \frac{f(x)+f(y)}{2}\| \rightarrow 0$$

as $\max\{\|x\|, \|y\|\} \rightarrow \infty$.

Proof. If f is additive, then (3.3) evidently holds. Conversely, use the limit (3.3) to get for each $n \in \mathbb{N}$ a real number $\beta_n > |\beta_n|$ (replace β_n by a real number of the form $k(n)\beta_n$ where $k(n)$ is an integer, if necessary, to get $k(n)\beta_n > |\beta_n| \geq |k(n)\beta_n|$, since $|k(n)| \leq 1$) such that

$$\|f(\frac{x+y}{2}) - \frac{f(x)+f(y)}{2}\| \leq 1/n$$

for all $x, y \in X$ with

$$\max\{\|x\|, \|y\|\} \geq \beta_n > |\beta_n|.$$

Next use Theorem 3.1 to conclude a unique additive mapping T_n such that

$$(3.4) \quad \|f(x) - T_n(x)\| \leq 1/n$$

for all $x \in X$. Thus $\|f(x) - T_1(x)\| \leq 1$ and $\|f(x) - T_n(x)\| \leq 1/n \leq 1$ for each n . By the uniqueness of T_1 we conclude that $T_n = T_1$ for all n . Tending n to infinity in (3.4) we deduce that $f = T_1$ is additive. \square

4. Asymptotic aspect of an unbounded Jensen difference

In this section, we deal with the asymptotic behavior of an unbounded Jensen difference. Throughout this section we assume that Let X is a non-Archimedean normed space and Y is a non-Archimedean Banach space over a non-Archimedean field \mathbb{K} with $|2| < 1$.

Theorem 4.1. *Suppose that $\alpha \geq 0$, $M > 0$, $0 \leq p < 1$ and $f : X \rightarrow Y$ is a mapping satisfying $f(0) = 0$ and*

$$(4.1) \quad \|2f(\frac{x+y}{2}) - f(x) - f(y)\| \leq \alpha \max\{\|x\|^p, \|y\|^p\}$$

for all $x, y \in X$ with $\max\{\|x\|, \|y\|\} \geq M$. Then there exists an additive mapping $T : X \rightarrow Y$ such that

$$(4.2) \quad \|f(x) - T(x)\| \leq \alpha \|x\|^p$$

for all $x \in X$ with $\|x\| \geq M$. Furthermore, T is independent of given positive numbers α and M .

Proof. Let $\|x\| \geq M$. Put $y = 0$ in (4.1) to get

$$(4.3) \quad \|2f(\frac{x}{2}) - f(x)\| \leq \alpha \|x\|^p$$

Using (4.3) and the fact that $\|x/2^n\| \geq M$ for large enough n , we can follow the same argument as in the proof of Theorem 2.1 to get a mapping $\tau : \{x \in X : \|x\| \geq M\} \rightarrow Y$ defined by $\lim_{n \rightarrow \infty} 2^n f(x/2^n)$ satisfying

$$\|\tau(x) - f(x)\| \leq \alpha \|x\|^p \quad (\|x\| \geq M).$$

and

$$\tau(\frac{x+y}{2}) = \frac{1}{2}(\tau(x) + \tau(y))$$

for all $x, y \in X$ with $\max\{\|x\|, \|y\|\} \geq M$. For each $x \in X$ with $\|x\| \geq M$ we have $\|x/2\| \geq \|x\| \geq M$, whence

$$(4.4) \quad \tau(x/2) = \lim_{n \rightarrow \infty} 2^n f(x/2^{n+1}) = \frac{1}{2} \lim_{n \rightarrow \infty} 2^{n+1} f(x/2^{n+1}) = \frac{1}{2} \tau(x)$$

Given any $x \in X$ with $0 < \|x\| < M$, let $k = k(x)$ denotes the least positive integer such that $\|x/2^k\| \geq M$ and define

$$T(x) = \begin{cases} 0 & x = 0 \\ 2^k \tau(\frac{x}{2^k}) & \|x\| < M \\ \tau(x) & \|x\| \geq M \end{cases}$$

Now we show that

$$(4.5) \quad T(x) = \lim_{n \rightarrow \infty} 2^n f(\frac{x}{2^n}) \quad (x \in X).$$

To see this, take any $x \in X$ with $0 < \|x\| < M$. Let k be the least positive integer satisfying $\|x/2^{k-1}\| \geq M$. Then $k-1$ is the least positive integer satisfying $\|\frac{x}{2^{k-2}}\| \geq M$. Therefore

$$T(x/2) = 2^{k-1} \tau(\frac{x}{2^{k-1}}) = \frac{1}{2} 2^k \tau(\frac{x}{2^k}) = \frac{1}{2} T(x).$$

By (4.4) and the definition of T , we therefore conclude that

$$(4.6) \quad T(x/2) = \frac{1}{2}T(x)$$

for all $x \in X$. Let $x \in X \setminus \{0\}$. There is a positive integer k_0 such that $\|2^{-k_0}x\| \geq M$. We have

$$\begin{aligned} T(x) &= 2^{k_0}T(2^{-k_0}x) = 2^{k_0}\tau(2^{-k_0}x) = 2^{k_0} \lim_{n \rightarrow \infty} f\left(\frac{2^{-k_0}x}{2^n}\right) \\ &= \lim_{n \rightarrow \infty} 2^{n+k_0}f\left(\frac{x}{2^{n+k_0}}\right) = \lim_{n \rightarrow \infty} 2^n f\left(\frac{x}{2^n}\right). \end{aligned}$$

Trivially $T(0) = 0 = \lim_{n \rightarrow \infty} 2^n f(\frac{0}{2^n})$. Hence (4.5) holds for all $x \in X$. If $x = 0$ or $y = 0$, by taking (4.6) into account, we get $T(\frac{x+y}{2}) = \frac{1}{2}(T(x) + T(y))$. So we may assume that $x \neq 0$ and $y \neq 0$. Choose $n \in \mathbb{N}$ to be large enough such that

$$\min \left\{ \left\| \frac{x}{2^n} \right\|, \left\| \frac{y}{2^n} \right\| \right\} \geq M.$$

Utilizing (4.1) we obtain

$$\left\| 2^n f\left(\frac{(x+y)/2}{2^n}\right) - \frac{1}{2}2^n f\left(\frac{x}{2^n}\right) - \frac{1}{2}2^n f\left(\frac{y}{2^n}\right) \right\| \leq (|2|^{1-p})^n \max \{ \|x\|^p, \|y\|^p \}.$$

Letting n approach to infinity we get

$$T\left(\frac{x+y}{2}\right) = \frac{1}{2}(T(x) + T(y)).$$

Hence T is additive.

Suppose that T' is another additive mapping satisfying (4.2) with α and M replaced by α' and M' , respectively. For $x \in X$ choose $n \in \mathbb{N}$ large enough so that $\|2^{-n}x\| \geq \max \{M, M'\}$. Then

$$\begin{aligned} \|T(x) - T'(x)\| &= \lim_{k \rightarrow \infty} |2|^k \|T\left(\frac{x}{2^k}\right) - T'\left(\frac{x}{2^k}\right)\| \\ &\leq \lim_{k \rightarrow \infty} |2|^k \max \left\{ \left\| T\left(\frac{x}{2^k}\right) - f\left(\frac{x}{2^k}\right) \right\|, \left\| f\left(\frac{x}{2^k}\right) - T'\left(\frac{x}{2^k}\right) \right\| \right\} \\ &\leq \lim_{k \rightarrow \infty} |2|^k \max \left\{ \alpha \left\| \frac{x}{2^k} \right\|^p, \alpha' \left\| \frac{x}{2^k} \right\|^p \right\} \\ &= \lim_{k \rightarrow \infty} (|2|^{1-p})^k \|x\|^p \max \{ \alpha, \alpha' \} \\ &= 0. \end{aligned}$$

Hence $T(x) = T'(x)$. □

Let $f : X \rightarrow Y$ be a mapping. Following [9]

- (i) f is called *p-asymptotic close to an additive mapping T* if
$$\lim_{\|x\| \rightarrow \infty} \frac{\|f(x) - T(x)\|}{\|x\|^p} = 0.$$
- (ii) f is said to satisfy *p-asymptotically the Jensen equation* if for each $\alpha > 0$ there exists $M > 0$ such that

$$\|f(\frac{x+y}{2}) - f(x) - f(y)\| \leq \alpha \max\{\|x\|^p, \|y\|^p\}$$

for all $x, y \in X$ with $\max\{\|x\|, \|y\|\} \geq M$.

Applying Theorem 4.1 and the uniqueness of obtained additive mapping we infer the following corollary.

Corollary 4.2. *If $0 < p < 1$ and a mapping $f : X \rightarrow Y$ with $f(0) = 0$ satisfies p-asymptotically the Jensen equation, then it is p-asymptotic close to an additive mapping.*

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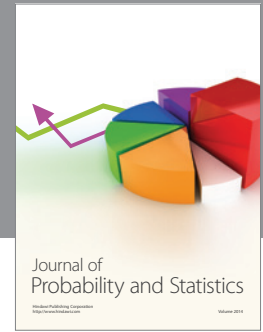
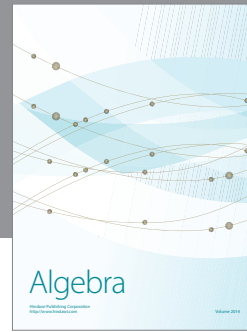
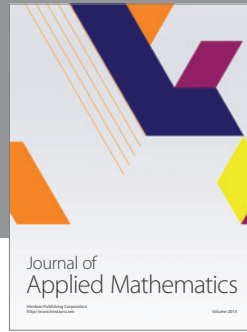
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