JOURNAL OF
FUNCTION SPACES AND APPLICATIONS
Volume 7, Number 3 (2009), 251–288

http://www.jfsa.net

Spaces of Sobolev type with positive smoothness on $\mathbb{R}^n$, embeddings and growth envelopes

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(*Communicated by Hans Triebel*)

2000 Mathematics Subject Classification. 46E35.

Keywords and phrases. Triebel-Lizorkin spaces, Besov spaces, Sobolev spaces, embeddings, growth envelopes.

Abstract. We characterize Triebel-Lizorkin spaces with positive smoothness on $\mathbb{R}^n$, obtained by different approaches. First we present three settings $\mathbf{F}^s_{p,q}(\mathbb{R}^n)$, $F^s_{p,q}(\mathbb{R}^n)$, $\mathfrak{F}^s_{p,q}(\mathbb{R}^n)$ associated to definitions by differences, Fourier-analytical methods and subatomic decompositions. We study their connections and diversity, as well as embeddings between these spaces and into Lorentz spaces. Secondly, relying on previous results obtained for Besov spaces $\mathfrak{B}^s_{p,q}(\mathbb{R}^n)$, we determine their growth envelopes $\mathfrak{E}_{\mathsf{G}}\left(\mathfrak{F}^s_{p,q}(\mathbb{R}^n)\right)$ for $0 < p < \infty$, $0 < q \leq \infty$, $s > 0$, and finally discuss some applications.

1. Introduction

In this article Triebel-Lizorkin spaces of positive smoothness on $\mathbb{R}^n$ are investigated. They were introduced independently by Triebel and Lizorkin in the early 1970s. For a detailed treatment together with historical remarks we refer to Triebel [18, 19]. The idea for this paper originates from its forerunner [12], where we studied corresponding problems for Besov spaces. Since the substantial theory of the Triebel-Lizorkin spaces is strongly linked with the theory of Besov spaces – in the sequel briefly denoted as F-spaces and B-spaces, respectively – the question came up whether those previous results could be carried over to the F-space setting. This paper aims at providing a rather final answer to this question. According to the well-known Besov spaces, Triebel-Lizorkin spaces inherited different characterizations, creating the task of comparing and – in the optimal case – identifying the resulting spaces. In the case, $0 < s \leq n(\frac{1}{p} - 1)$, $0 < p < 1$, for a long time, it was only known that, say, two of the most prominent approaches – based on characterizations by differences on the one hand and by Fourier-analytical decompositions on the other hand – necessarily differ, but may otherwise share similar properties. Modern subatomic characterizations now admit new insights into the nature of these spaces.

More precisely, we restrict ourselves to the following three approaches to F-spaces only: the *classical approach*, which introduces $\mathbf{F}^s_{p,q}(\mathbb{R}^n)$ as those subspaces of $L_p(\mathbb{R}^n)$ such that

$$\|f|\mathbf{F}^s_{p,q}(\mathbb{R}^n)\|_r = \|f|L_p(\mathbb{R}^n)\| + \left\| \left(\int_0^1 t^{-sq} d^r_{t,p} f(\cdot)^q \frac{\mathrm{d}t}{t} \right)^{1/q} |L_p(\mathbb{R}^n) \right\|$$

is finite, where $0 < p < \infty$, $0 < q \leq \infty$, $s > 0$, $r \in \mathbb{N}$ with $r > s$ (appropriately modified for $q = \infty$), and $d^r_{t,p} f(\cdot)$ denote the ball means of $f \in L_p(\mathbb{R}^n)$. Secondly, we deal with the *Fourier-analytical approach* leading to spaces $F^s_{p,q}(\mathbb{R}^n)$ as the set of all tempered distributions $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\left\| f \,|F^s_{p,q}(\mathbb{R}^n) \right\| = \left\| \left\| \left\{ 2^{js} \mathcal{F}^{-1} (\varphi_j \mathcal{F} f)(\cdot) \right\}_{j \in \mathbb{N}_0} |\ell_q \right\| |L_p(\mathbb{R}^n) \right\|$$

is finite, where $s \in \mathbb{R}$, $0 < p < \infty$, $0 < q \leq \infty$, and $\{\varphi_j\}_j$ is a smooth dyadic resolution of unity. Finally, the most recent definition $\mathfrak{F}^s_{p,q}(\mathbb{R}^n)$ relies on *subatomic decompositions* and contains those $f \in L_p(\mathbb{R}^n)$ which can be represented as

$$f(x) = \sum_{\beta \in \mathbb{N}_0^n} \sum_{j=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda^\beta_{j,m} k^\beta_{j,m}(x), \quad x \in \mathbb{R}^n,$$

with coefficients $\lambda = \{\lambda^\beta_{j,m} \in \mathbb{C} : \beta \in \mathbb{N}_0^n, j \in \mathbb{N}_0, m \in \mathbb{Z}^n\}$ belonging to some appropriate sequence space $f^{s,\varrho}_{p,q}$, where $s > 0$, $0 < p < \infty$, $0 < q \leq \infty$, $\varrho \geq 0$, and $k^\beta_{j,m}(x)$ are certain standardized building blocks. Recent results by Hedberg, Netrusov [11] on atomic decompositions and by

Triebel [22, Sect. 9.2] on the reproducing formula prove coincidences

$$\mathbf{F}^s_{p,q}(\mathbb{R}^n) = \mathfrak{F}^s_{p,q}(\mathbb{R}^n), \qquad s > n\left(\frac{1}{\min(p,q)} - \frac{1}{p}\right), \quad 0 < p < \infty, \quad 0 < q \leq \infty,$$

and

$$F^s_{p,q}(\mathbb{R}^n) = \mathfrak{F}^s_{p,q}(\mathbb{R}^n), \qquad s > n\left(\frac{1}{\min(p,q,1)} - 1\right), \quad 0 < p < \infty, \quad 0 < q \leq \infty,$$

resulting in

$$F^s_{p,q}(\mathbb{R}^n) = \mathbf{F}^s_{p,q}(\mathbb{R}^n) = \mathfrak{F}^s_{p,q}(\mathbb{R}^n),$$

whenever

$$0 < p < \infty, \quad 0 < q \leq \infty, \quad s > n\left(\frac{1}{\min(p,q)} - \frac{1}{\max(1,p)}\right)$$

(in terms of equivalent (quasi-)norms). We discuss these three approaches in view of embeddings and envelope results. In particular, our first main result, Theorem 2.16, extends *'limiting embeddings'* of type

$$\mathfrak{F}^s_{p,q_1}(\mathbb{R}^n) \hookrightarrow \mathfrak{F}^\sigma_{r,q_2}(\mathbb{R}^n), \qquad s - \frac{n}{p} = \sigma - \frac{n}{r},$$

to all admitted parameters $0 < \sigma < s$, $0 < p < r < \infty$, $0 < q_1, q_2 \leq \infty$, and similarly for

$$\mathfrak{F}^s_{p,q}(\mathbb{R}^n) \hookrightarrow L_{r,p}(\mathbb{R}^n), \qquad s - \frac{n}{p} = -\frac{n}{r},$$

where $0 < s < \frac{n}{p}$, $0 < q \leq \infty$; in both cases $\mathfrak{F}^s_{p,q}(\mathbb{R}^n)$ can be replaced by $\mathbf{F}^s_{p,q}(\mathbb{R}^n)$, whenever $s > n\left(\frac{1}{\min(p,q)} - \frac{1}{p}\right)$; in fact the second embedding holds in general as well. This result is further clarified in Corollary 4.1.

Secondly, the paper is devoted to the study of the 'typical' singularity behaviour in these F-spaces in the sense of *growth envelopes.* This recently introduced concept originates from such classical ideas as the famous Sobolev embedding theorem [16]. Basically, this characterizes the unboundedness of functions that belong to (classical) Sobolev spaces $W^k_p(\mathbb{R}^n)$, $k \in \mathbb{N}_0$, $1 \leq p < \infty$, (and more general scales of spaces). By Sobolev's embedding theorem it is known that for $k \leq \frac{n}{p}$, $1 \leq p < \infty$, there are (essentially) unbounded functions in $W^k_p(\mathbb{R}^n)$, whereas beyond the 'critical line' $k = \frac{n}{p}$, i.e., for $k > \frac{n}{p}$ (or $k = n$ and $p = 1$) we have $W^k_p(\mathbb{R}^n) \hookrightarrow L_\infty(\mathbb{R}^n)$. In the past a lot of work has been done to refine Sobolev type embeddings in terms of wider classes of function spaces. We do not want to report on this elaborate history here; apart from the original papers assertions of this type are indispensable parts in books dealing with Sobolev spaces and related questions, cf. [1], [26], [14], [5]. We study the growth and unboundedness of such functions (distributions) in terms of their *growth envelope* $\mathfrak{E}_{\mathsf{G}}(X) = (\mathcal{E}_{\mathsf{G}}^X(t), u_{\mathsf{G}}^X)$, where $X \subset L_1^{\mathrm{loc}}$ is a function space and

$$\mathcal{E}_{\mathsf{G}}^X(t) \sim \sup\{f^*(t):\ \|f|X\| \le 1\}, \quad t>0,$$

its *growth envelope function*, and $u_{\mathsf{G}}^X \in (0,\infty]$ is some additional index providing a finer description. Here f^* denotes the non-increasing rearrangement of f, as usual. These concepts were introduced in [21], [8, 9], the latter book also contains a recent survey of the present state-of-the-art (concerning extensions and more general approaches) as well as applications and further references.

Our second main result, Theorem 3.11, can now be formulated as

$$\mathfrak{E}_{\mathsf{G}}(\mathfrak{F}^s_{p,q}(\mathbb{R}^n)) = \left(t^{-\frac{1}{p}+\frac{s}{n}}, p\right),$$

where $0<p<\infty,\ 0<q\le\infty,\ 0<s<\frac{n}{p}$. Moreover, globally we obtain

$$\mathcal{E}_{\mathsf{G}}^{\mathfrak{F}^s_{p,q}}(t) \ \sim\ t^{-\frac{1}{p}} \qquad \text{for} \quad t\to\infty.$$

Similarly for the spaces $\mathbf{F}^s_{p,q}(\mathbb{R}^n)$. This naturally extends results for $F^s_{p,q}(\mathbb{R}^n)$ below the line $s = n\max(\frac{1}{p}-1, 0)$ which – though indispensable for spaces $F^s_{p,q}(\mathbb{R}^n)$ in order to admit an interpretation of $f \in F^s_{p,q}(\mathbb{R}^n)$ as a regular distribution – is not necessary for the approaches $\mathfrak{F}^s_{p,q}(\mathbb{R}^n)$ and $\mathbf{F}^s_{p,q}(\mathbb{R}^n)$, respectively. Moreover, since the scale of F-spaces contains the (*fractional*) *Sobolev spaces* as a special case, i.e.,

$$F^s_{p,2}(\mathbb{R}^n) = H^s_p(\mathbb{R}^n), \qquad s\in\mathbb{R}, \qquad 1<p<\infty,$$

our results admit new insights into the nature of these classical function spaces as well, cf. Remark 2.12 and Corollary 3.12.

The paper is organized as follows. In Section 2 we first present three different approaches to Triebel-Lizorkin spaces of positive smoothness and briefly discuss these concepts. We also extend well-known embedding results to all admitted values of positive smoothness. In Section 3 we recall the concepts of growth envelopes, collect some fundamentals needed below including basic examples. The main results in this context are contained in Section 3.2. Finally Section 4 contains two interesting applications of our results in terms of Hardy-type inequalities and criteria of sharp embeddings.

2. Triebel-Lizorkin spaces with positive smoothness on $\mathbb{R}^n$

We use standard notation. Let $\mathbb{N}$ be the collection of all natural numbers and let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Let $\mathbb{R}^n$ be euclidean n-space, $n \in \mathbb{N}$, $\mathbb{C}$ the complex plane. The set of multi-indices $\beta = (\beta_1, \ldots, \beta_n)$, $\beta_i \in \mathbb{N}_0$, $i = 1, \ldots, n$, is denoted by $\mathbb{N}_0^n$, with $|\beta| = \beta_1 + \cdots + \beta_n$, as usual. Moreover, if $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{N}_0^n$ we put $x^\beta = x_1^{\beta_1} \cdots x_n^{\beta_n}$. We use the equivalence '$\sim$' in

$$a_k \sim b_k \quad \text{or} \quad \varphi(x) \sim \psi(x)$$

always to mean that there are two positive numbers c_1 and c_2 such that

$$c_1\, a_k \le b_k \le c_2\, a_k \quad \text{or} \quad c_1\, \varphi(x) \le \psi(x) \le c_2\, \varphi(x)$$

for all admitted values of the discrete variable k or the continuous variable x, where $\{a_k\}_k$, $\{b_k\}_k$ are non-negative sequences and φ, ψ are non-negative functions. If $a \in \mathbb{R}$, then $a_+ := \max(a, 0)$ and $[a]$ denotes the integer part of a. Given two (quasi-)Banach spaces X and Y, we write $X \hookrightarrow Y$ if $X \subset Y$ and the natural embedding of X in Y is continuous. All unimportant positive constants will be denoted by c, occasionally with subscripts. Integration with respect to the n-dimensional Lebesgue measure in $\mathbb{R}^n$ is denoted by $\mathrm{d}x$, whereas $|A|$ stands for the Lebesgue measure of a Lebesgue-measurable set A in $\mathbb{R}^n$. As we shall always deal with function spaces on $\mathbb{R}^n$, we may usually omit the '$\mathbb{R}^n$' from their notation for convenience.

2.1 Different approaches. In this section we discuss three different approaches to Triebel-Lizorkin spaces with positive smoothness. We first present these approaches separately before we come to some comparison. At the end we collect and extend some embedding results that will also be needed below.

Let for $0 < p, q \le \infty$ the numbers σ_p and σ_{pq} be given by

$$\sigma_p = n\left(\frac{1}{p} - 1\right)_+ \qquad \text{and} \qquad \sigma_{pq} = n\left(\frac{1}{\min(p,q)} - 1\right)_+ . \tag{2.1}$$

The classical approach: Triebel-Lizorkin spaces $\mathbf{F}^s_{p,q}(\mathbb{R}^n)$. If f is an arbitrary function on $\mathbb{R}^n$, $h \in \mathbb{R}^n$ and $r \in \mathbb{N}$, then

$$(\Delta_h^1 f)(x) = f(x+h) - f(x) \quad \text{and} \quad (\Delta_h^{r+1} f)(x) = \Delta_h^1(\Delta_h^r f)(x), \quad x \in \mathbb{R}^n.$$

For convenience we may write Δ_h instead of Δ_h^1. Furthermore, for a function $f \in L_p(\mathbb{R}^n)$, $0 < p < \infty$, $r \in \mathbb{N}$, the *ball means* are denoted by

$$d_{t,p}^r f(x) = \left(t^{-n} \int_{|h|\le t} |(\Delta_h^r f)(x)|^p \mathrm{d}h \right)^{1/p}, \quad x \in \mathbb{R}^n, \quad t > 0. \tag{2.2}$$

Definition 2.1. Let $0 < p < \infty$, $0 < q \le \infty$, $s > 0$, and $r \in \mathbb{N}$ such that $r > s$. Then $\mathbf{F}_{p,q}^s(\mathbb{R}^n)$ is the collection of all $f \in L_p(\mathbb{R}^n)$ such that

$$\|f|\mathbf{F}_{p,q}^s(\mathbb{R}^n)\|_r = \|f|L_p(\mathbb{R}^n)\| + \left\| \left(\int_0^1 t^{-sq} d_{t,p}^r f(\cdot)^q \frac{\mathrm{d}t}{t} \right)^{1/q} |L_p(\mathbb{R}^n) \right\| \tag{2.3}$$

(with the usual modification if $q = \infty$) is finite.

Remark 2.2. The approach by differences for the spaces $\mathbf{F}_{p,q}^s(\mathbb{R}^n)$ has been described in detail in [18] for those spaces which can also be considered as subspaces of $S'(\mathbb{R}^n)$. Otherwise one finds in [22], Section 9.2.2, pp. 386-390, the necessary explanations and references to the relevant literature. In particular, the spaces in Definition 2.1 are independent of r, meaning that different values of $r > s$ result in (quasi-)norms which are equivalent. Furthermore, the spaces are (quasi-)Banach spaces (Banach spaces, if $1 \le p < \infty$, $1 \le q \le \infty$). Recall that we deal with subspaces of $L_p(\mathbb{R}^n)$, in particular, we have the embedding

$$\mathbf{F}_{p,q}^s(\mathbb{R}^n) \hookrightarrow L_p(\mathbb{R}^n), \qquad s > 0, \quad 0 < p < \infty, \quad 0 < q \le \infty.$$

There is a corresponding approach by differences for the classical Besov spaces $\mathbf{B}_{p,q}^s$. The *k-th modulus of smoothness* of a function $f \in L_p(\mathbb{R}^n)$, $0 < p \le \infty$, $k \in \mathbb{N}$, is defined by

$$\omega_k(f,t)_p = \sup_{|h|\le t} \|\Delta_h^k f \mid L_p(\mathbb{R}^n)\|, \quad t > 0. \tag{2.4}$$

Let $0 < p, q \le \infty$, $s > 0$, and $r \in \mathbb{N}$ such that $r > s$. Then the Besov space $\mathbf{B}_{p,q}^s(\mathbb{R}^n)$ contains all $f \in L_p(\mathbb{R}^n)$ such that

$$\|f|\mathbf{B}_{p,q}^s(\mathbb{R}^n)\|_r = \|f|L_p(\mathbb{R}^n)\| + \left(\int_0^1 t^{-sq} \omega_r(f,t)_p^q \, \frac{\mathrm{d}t}{t} \right)^{1/q} \tag{2.5}$$

(with the usual modification if $q = \infty$) is finite. Further information on the classical approach for B- and F-spaces – treated in a more general context – may be found in [11].

The Fourier-analytical approach: Triebel-Lizorkin spaces $F^s_{p,q}(\mathbb{R}^n)$. The Schwartz space $\mathcal{S}(\mathbb{R}^n)$ and its dual $\mathcal{S}'(\mathbb{R}^n)$ of all complex-valued tempered distributions have their usual meaning here. Let $\varphi_0 = \varphi \in \mathcal{S}(\mathbb{R}^n)$ be such that

$$\operatorname{supp}\varphi \subset \{y \in \mathbb{R}^n : |y| < 2\} \quad \text{and} \quad \varphi(x) = 1 \quad \text{if} \quad |x| \le 1\,, \tag{2.6}$$

and for each $j \in \mathbb{N}$ let $\varphi_j(x) = \varphi(2^{-j}x) - \varphi(2^{-j+1}x)$. Then $\{\varphi_j\}_{j=0}^{\infty}$ forms a *smooth dyadic resolution of unity*. Given any $f \in \mathcal{S}'(\mathbb{R}^n)$, we denote by $\mathcal{F}f$ and $\mathcal{F}^{-1}f$ its Fourier transform and its inverse Fourier transform, respectively. Let $f \in \mathcal{S}'(\mathbb{R}^n)$, then the compact support of $\varphi_j\mathcal{F}f$ implies by the Paley-Wiener-Schwartz theorem that $\mathcal{F}^{-1}(\varphi_j\mathcal{F}f)$ is an entire analytic function on $\mathbb{R}^n$.

Definition 2.3. Let $s \in \mathbb{R}$, $0 < p < \infty$, $0 < q \le \infty$, and $\{\varphi_j\}_j$ a smooth dyadic resolution of unity. The space $F^s_{p,q}(\mathbb{R}^n)$ is the set of all distributions $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\big\|f\,|F^s_{p,q}(\mathbb{R}^n)\big\| = \Big\|\big\|\left\{2^{js}\mathcal{F}^{-1}(\varphi_j\mathcal{F}f)(\cdot)\right\}_{j\in\mathbb{N}_0}|\ell_q\big\||L_p(\mathbb{R}^n)\Big\| \tag{2.7}$$

is finite.

Remark 2.4. The spaces $F^s_{p,q}(\mathbb{R}^n)$ are independent of the particular choice of the smooth dyadic resolution of unity $\{\varphi_j\}_j$ appearing in their definition. They are (quasi-)Banach spaces (Banach spaces for $p,q \ge 1$), and $\mathcal{S}(\mathbb{R}^n) \hookrightarrow F^s_{p,q}(\mathbb{R}^n) \hookrightarrow \mathcal{S}'(\mathbb{R}^n)$, where the first embedding is dense if $q < \infty$. An extension of Definition 2.3 to $p = \infty$ does not make sense if $0 < q < \infty$ (in particular, a corresponding space is not independent of the choice $\{\varphi_j\}_j$). The case $p = q = \infty$ yields the Besov spaces $B^s_{\infty,\infty}(\mathbb{R}^n)$.
In general, the Fourier-analytical Besov spaces $B^s_{p,q}(\mathbb{R}^n)$ are defined correspondingly to the spaces $F^s_{p,q}(\mathbb{R}^n)$ by interchanging the order in which the (quasi-)norms are taken, i.e., first using the L_p-norm and afterwards applying the ℓ_q-norm – in view of (2.7). These B-spaces are closely linked with the Triebel-Lizorkin spaces $F^s_{p,q}(\mathbb{R}^n)$ via

$$B^s_{p,\min(p,q)}(\mathbb{R}^n) \hookrightarrow F^s_{p,q}(\mathbb{R}^n) \hookrightarrow B^s_{p,\max(p,q)}(\mathbb{R}^n). \tag{2.8}$$

We shall later on return to this embedding. The theory of the spaces $F^s_{p,q}(\mathbb{R}^n)$ (and $B^s_{p,q}(\mathbb{R}^n)$) has been developed in detail in [18] and [19] (and continued and extended in the more recent monographs [21], [22]), but has a longer history already including many contributors; we do not further want to discuss this here. Note that the spaces $F^s_{p,q}(\mathbb{R}^n)$ contain tempered distributions which can only be interpreted as regular

distributions (functions) for sufficiently high smoothness. More precisely, we have
(2.9)

$$F^s_{p,q}(\mathbb{R}^n) \subset L_1^{\mathrm{loc}}(\mathbb{R}^n) \quad \text{if, and only if,} \quad \begin{cases} s \geq \sigma_p, & \text{for } 0 < p < 1,\ 0 < q \leq \infty, \\ s > \sigma_p, & \text{for } 1 \leq p < \infty,\ 0 < q \leq \infty, \\ s = \sigma_p, & \text{for } 1 \leq p < \infty,\ 0 < q \leq 2, \end{cases}$$

cf. [17, Thm. 3.3.2]. In particular, for $s < \sigma_p$ one cannot interpret $f \in F^s_{p,q}(\mathbb{R}^n)$ as a regular distribution in general. The scale $F^s_{p,q}(\mathbb{R}^n)$ contains many well-known function spaces. We list a few special cases.

Let $1 < p < \infty$, then

$$F^s_{p,2}(\mathbb{R}^n) = H^s_p(\mathbb{R}^n), \qquad s \in \mathbb{R},$$

are the *(fractional) Sobolev spaces* containing all $f \in S'(\mathbb{R}^n)$ with

$$\mathcal{F}^{-1}(1+|\xi|^2)^{s/2}\mathcal{F}f \in L_p(\mathbb{R}^n).$$

In particular, for $k \in \mathbb{N}_0$, we obtain the *classical Sobolev spaces*

$$F^k_{p,2}(\mathbb{R}^n) = W^k_p(\mathbb{R}^n), \qquad \text{i.e.,} \qquad F^0_{p,2}(\mathbb{R}^n) = L_p(\mathbb{R}^n),$$

usually normed by

$$\|f|W^k_p(\mathbb{R}^n)\| = \left(\sum_{|\alpha| \leq k} \|\mathrm{D}^\alpha f|L_p(\mathbb{R}^n)\|^p \right)^{1/p}.$$

Furthermore,

$$F^0_{p,2}(\mathbb{R}^n) = h_p(\mathbb{R}^n), \qquad 0 < p < \infty,$$

the latter being the *Hardy spaces.*

The subatomic approach: Triebel-Lizorkin spaces $\mathfrak{F}^s_{p,q}(\mathbb{R}^n)$. In this subsection we simultaneously give definitions for the Besov spaces $\mathfrak{B}^s_{p,q}(\mathbb{R}^n)$ and the spaces $\mathfrak{F}^s_{p,q}(\mathbb{R}^n)$. The reason for this is that later on we want to use results previously obtained for the spaces $\mathfrak{B}^s_{p,q}(\mathbb{R}^n)$ in [12], in order to now establish corresponding results for the spaces $\mathfrak{F}^s_{p,q}(\mathbb{R}^n)$.

We complement our notation. Let $Q_{j,m}$ with $j \in \mathbb{N}_0$ and $m \in \mathbb{Z}^n$ denote a cube in $\mathbb{R}^n$ with sides parallel to the axes of coordinates, centered at $2^{-j}m$, and with side length 2^{-j+1}. Besides, if Q is a cube in $\mathbb{R}^n$ and $r > 0$, then rQ is the cube in $\mathbb{R}^n$ concentric with Q and r times the side length of Q.

Let $\chi_{j,m}$ be the characteristic function of $Q_{j,m}$ and

$$\mathbb{R}^n_{++} := \{y \in \mathbb{R}^n : y = (y_1, \dots, y_n),\ y_j > 0\}.$$

The subatomic approach provides a constructive definition for Besov and Triebel-Lizorkin spaces, expanding functions f via building blocks with suitable coefficients, where the latter belong to certain sequence spaces $b^{s,\varrho}_{p,q}$ and $f^{s,\varrho}_{p,q}$, respectively.

Definition 2.5. Let k be a non-negative C^∞-function in $\mathbb{R}^n$ with

$$\operatorname{supp} k \subset \left\{y \in \mathbb{R}^n : |y| < 2^{J-\varepsilon}\right\} \cap \mathbb{R}^n_{++} \tag{2.10}$$

for some fixed $\varepsilon > 0$ and some fixed $J \in \mathbb{N}$, satisfying

$$\sum_{m \in \mathbb{Z}^n} k(x-m) = 1, \ \ x \in \mathbb{R}^n. \tag{2.11}$$

Let $\beta \in \mathbb{N}^n_0$, $j \in \mathbb{N}_0$, $m \in \mathbb{Z}^n$, and set $k^\beta(x) = (2^{-J}x)^\beta k(x)$. Then

$$k^\beta_{j,m}(x) = k^\beta(2^j x - m) \tag{2.12}$$

denote the building blocks related to $Q_{j,m}$.

Remark 2.6. Let $\beta \in \mathbb{N}^n_0$, $j \in \mathbb{N}_0$, $m \in \mathbb{Z}^n$, with $\varepsilon > 0$ and $J \in \mathbb{N}$ as in Definition 2.5. The above definition implies that the building blocks are bounded by

$$0 \leq k^\beta_{j,m}(x) \leq 2^{-\varepsilon|\beta|}, \quad x \in \mathbb{R}^n, \tag{2.13}$$

uniformly in $j \in \mathbb{N}_0$, $m \in \mathbb{Z}^n$, and for their supports we observe that

$$\operatorname{supp} k^\beta_{j,m} \subset 2^{J-\varepsilon} Q_{j,m} \tag{2.14}$$

uniformly in $\beta \in \mathbb{N}^n_0$.

Definition 2.7. Let $\varrho \geq 0$, $s \in \mathbb{R}$, $0 < p, q \leq \infty$ and

$$\lambda = \left\{\lambda^\beta_{j,m} \in \mathbb{C} : \beta \in \mathbb{N}^n_0, m \in \mathbb{Z}^n, j \in \mathbb{N}_0\right\}.$$

(i) Then the sequence space $b^{s,\varrho}_{p,q}$ is defined as

$$b^{s,\varrho}_{p,q} := \left\{\lambda : \|\lambda | b^{s,\varrho}_{p,q}\| < \infty\right\}, \tag{2.15}$$

where

$$\|\lambda|b^{s,\varrho}_{p,q}\| = \sup_{\beta\in\mathbb{N}_0^n} 2^{\varrho|\beta|}\left(\sum_{j=0}^{\infty} 2^{j(s-n/p)q}\left(\sum_{m\in\mathbb{Z}^n}|\lambda^{\beta}_{j,m}|^p\right)^{q/p}\right)^{1/q} \tag{2.16}$$

(with the usual modification if $p=\infty$ and/or $q=\infty$).

(ii) Furthermore, the sequence space $f^{s,\varrho}_{p,q}$ consists of all sequences λ such that

$$f^{s,\varrho}_{p,q} := \left\{\lambda : \|\lambda|f^{s,\varrho}_{p,q}\| < \infty\right\}, \tag{2.17}$$

where

$$\|\lambda|f^{s,\varrho}_{p,q}\| = \sup_{\beta\in\mathbb{N}_0^n} 2^{\varrho|\beta|}\left\|\left(\sum_{j=0}^{\infty}\sum_{m\in\mathbb{Z}^n} 2^{jsq}|\lambda^{\beta}_{j,m}|^q\chi_{j,m}(\cdot)\right)^{1/q}|L_p(\mathbb{R}^n)\right\| \tag{2.18}$$

(with the usual modification if $p=\infty$ and/or $q=\infty$).

We now define the related function spaces.

Definition 2.8. Let $s>0$, $0<q\le\infty$, $\varrho\ge 0$.

(i) Let $0<p\le\infty$. Then $\mathfrak{B}^s_{p,q}(\mathbb{R}^n)$ contains all $f\in L_p(\mathbb{R}^n)$ which can be represented as

$$f(x) = \sum_{\beta\in\mathbb{N}_0^n}\sum_{j=0}^{\infty}\sum_{m\in\mathbb{Z}^n}\lambda^{\beta}_{j,m}k^{\beta}_{j,m}(x), \quad x\in\mathbb{R}^n, \tag{2.19}$$

with coefficients $\lambda = \left\{\lambda^{\beta}_{j,m}\right\}_{\beta\in\mathbb{N}_0^n, j\in\mathbb{N}_0, m\in\mathbb{Z}^n} \in b^{s,\varrho}_{p,q}$. Then

$$\left\|f|\mathfrak{B}^s_{p,q}(\mathbb{R}^n)\right\| = \inf\left\|\lambda|b^{s,\varrho}_{p,q}\right\|, \tag{2.20}$$

where the infimum is taken over all possible representations (2.19).

(ii) Let $0<p<\infty$. Then $\mathfrak{F}^s_{p,q}(\mathbb{R}^n)$ contains all $f\in L_p(\mathbb{R}^n)$ which can be represented as

$$f(x) = \sum_{\beta\in\mathbb{N}_0^n}\sum_{j=0}^{\infty}\sum_{m\in\mathbb{Z}^n}\lambda^{\beta}_{j,m}k^{\beta}_{j,m}(x), \quad x\in\mathbb{R}^n, \tag{2.21}$$

with coefficients $\lambda = \left\{\lambda^{\beta}_{j,m}\right\}_{\beta\in\mathbb{N}_0^n, j\in\mathbb{N}_0, m\in\mathbb{Z}^n} \in f^{s,\varrho}_{p,q}$. Then

$$\left\|f|\mathfrak{F}^s_{p,q}(\mathbb{R}^n)\right\| = \inf\left\|\lambda|f^{s,\varrho}_{p,q}\right\|, \tag{2.22}$$

where the infimum is taken over all possible representations (2.21).

Remark 2.9. The definitions given above follow closely [22, Sect. 9.2]. The spaces $\mathfrak{F}^s_{p,q}(\mathbb{R}^n)$ as well as $\mathfrak{B}^s_{p,q}(\mathbb{R}^n)$ are (quasi-)Banach spaces (Banach spaces for $p, q \geq 1$) and independent of k and ϱ (in terms of equivalent (quasi-)norms). Furthermore, for all admitted parameters p, q, s, we have

$$\mathfrak{F}^s_{p,q}(\mathbb{R}^n) \hookrightarrow L_p(\mathbb{R}^n) \qquad \text{and} \qquad \mathfrak{B}^s_{p,q}(\mathbb{R}^n) \hookrightarrow L_p(\mathbb{R}^n),$$

as well as the following embedding for B- and F-spaces,

$$\mathfrak{B}^s_{p,\min(p,q)}(\mathbb{R}^n) \hookrightarrow \mathfrak{F}^s_{p,q}(\mathbb{R}^n) \hookrightarrow \mathfrak{B}^s_{p,\max(p,q)}(\mathbb{R}^n). \tag{2.23}$$

Proofs of the above assertions can be found in [22, Th. 9.8]. In particular, the Besov and Triebel-Lizorkin spaces coincide if $p = q$, i.e.,

$$\mathfrak{B}^s_{p,p}(\mathbb{R}^n) = \mathfrak{F}^s_{p,p}(\mathbb{R}^n), \qquad 0 < p < \infty.$$

Concerning the convergence of (2.19) and (2.21) one obtains as a consequence of $\lambda \in b^{s,\varrho}_{p,q}$ and $\lambda \in f^{s,\varrho}_{p,q}$, respectively, that the series on the right-hand sides converge absolutely in $L_p(\mathbb{R}^n)$ if $p < \infty$, and in $L_\infty(\mathbb{R}^n, w_\sigma)$, where $w_\sigma(x) = (1 + |x|^2)^{\sigma/2}$ with $\sigma < 0$ if $p = \infty$. Since this implies unconditional convergence we may simplify (2.19), (2.21) and write in the sequel

$$f = \sum_{\beta,j,m} \lambda^\beta_{jm} k^\beta_{jm}.$$

We now discuss the coincidence and diversity of the above presented concepts of F-spaces and may restrict ourselves to positive smoothness $s > 0$. In view of our Remarks 2.2, 2.4 and 2.9 concerning the different nature of these spaces, it is obvious that there cannot be established a complete coincidence of all approaches when $s < \sigma_p$.
It has been shown that such a characterization is also impossible if $\sigma_p < s < \sigma_{pq}$ (in particular, when $0 < q < p$), cf. [22, Rem. 9.15], based on [3] – so the situation is even more complicated. Nevertheless, under certain restrictions on the smoothness parameter s, the above approaches result in the same F-space.

Theorem 2.10. *Let $s > 0$, $0 < p < \infty$ and $0 < q \leq \infty$.*

(i) *Then*

$$\mathbf{F}^s_{p,q}(\mathbb{R}^n) = \mathfrak{F}^s_{p,q}(\mathbb{R}^n), \qquad s > n\left(\frac{1}{\min(p,q)} - \frac{1}{p}\right), \tag{2.24}$$

and

$$F^s_{p,q}(\mathbb{R}^n) = \mathfrak{F}^s_{p,q}(\mathbb{R}^n), \qquad s > \sigma_{pq} \tag{2.25}$$

(in the sense of equivalent (quasi-)norms).

(ii) *Furthermore,*

$$F^s_{p,q}(\mathbb{R}^n) = \mathbf{F}^s_{p,q}(\mathbb{R}^n) = \mathfrak{F}^s_{p,q}(\mathbb{R}^n), \qquad s > n\left(\frac{1}{\min(p,q)} - \frac{1}{\max(1,p)}\right) \tag{2.26}$$

(in the sense of equivalent (quasi-)norms).

Remark 2.11. The first equality in (2.26) is longer known, see [18, Section 2.5.11], [19, Thm. 3.5.3], whereas the second equality in (2.26) is a consequence of the recently proved coincidence (2.24), see [22, Prop. 9.14] (with forerunners in [20, Sect. 13.8], [21, Thm. 2.9]). In the figures aside and below we have indicated the situation in the usual $(\frac{1}{p}, s)$-diagram for different values of q.

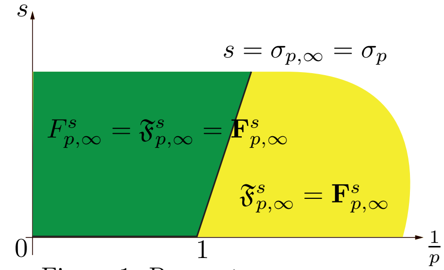

Figure 1: Parameter $q = \infty$

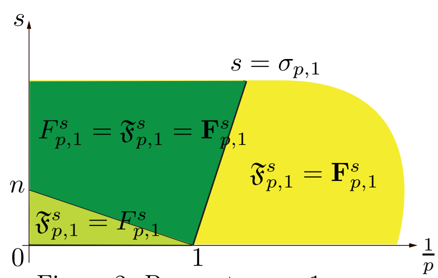

Figure 2: Parameter $q = 1$

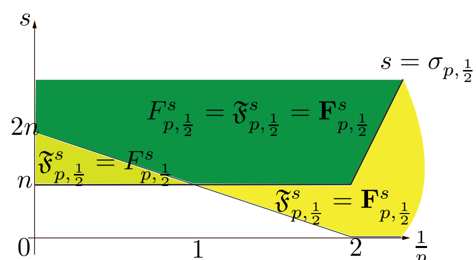

Figure 3: Parameter $q = \frac{1}{2}$

Remark 2.12. In view of the results stated in Theorem 2.10 and Remark 2.4, where we noted that the (fractional) Sobolev spaces are contained in the F-scale as a special case, i.e.,

$$F^s_{p,2}(\mathbb{R}^n) = H^s_p(\mathbb{R}^n), \qquad 1 < p < \infty, \qquad s \in \mathbb{R},$$

it makes sense to introduce new Sobolev-type spaces

(2.27)
$$\mathbf{H}^s_p(\mathbb{R}^n) = \mathbf{F}^s_{p,2}(\mathbb{R}^n), \qquad \mathfrak{H}^s_p(\mathbb{R}^n) = \mathfrak{F}^s_{p,2}(\mathbb{R}^n), \qquad 0 < p < \infty, \quad s > 0.$$

In particular, for $1 < p \leq 2$, these spaces coincide with the (fractional) Sobolev spaces, i.e.,

$$H^s_p(\mathbb{R}^n) = \mathbf{H}^s_p(\mathbb{R}^n) = \mathfrak{H}^s_p(\mathbb{R}^n).$$

The figure aside illustrates the general situation.

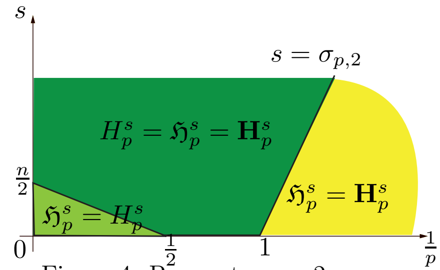

Figure 4: Parameter $q = 2$

Remark 2.13. Let us briefly mention the important feature of *duality* that clearly distinguishes between the spaces $\mathfrak{F}^s_{p,q}(\mathbb{R}^n)$ and $F^s_{p,q}(\mathbb{R}^n)$ when $0 < p < 1$ and $s < \sigma_p$. Then in the usual $(\mathcal{S}(\mathbb{R}^n), \mathcal{S}'(\mathbb{R}^n))$ understanding,

$$\left(F^s_{p,q}(\mathbb{R}^n)\right)' = B^{-s+\sigma_p}_{\infty,\infty}(\mathbb{R}^n), \quad s \in \mathbb{R},\ 0 < p < 1,\ 0 < q < \infty,$$

see [18, Thm. 2.11.3], whereas using (2.23),

(2.28)
$$\left(\mathfrak{F}^s_{p,q}(\mathbb{R}^n)\right)' \subset \left(\mathfrak{B}^s_{p,\min(p,q)}(\mathbb{R}^n)\right)' = \{0\}, \quad 0 < s < \sigma_p,\ 0 < p < 1,\ 0 < q < \infty,$$

complemented by the well-known counterpart for L_p spaces,

$$(L_p(\mathbb{R}^n))' = \{0\}, \quad 0 < p < 1, \tag{2.29}$$

see [23, Thm. 6.37]. This immediately implies that for $0 < p < 1$ neither $L_p(\mathbb{R}^n)$ nor $\mathfrak{F}^s_{p,q}(\mathbb{R}^n)$ or $\mathfrak{B}^s_{p,q}(\mathbb{R}^n)$ with $0 < s < \sigma_p$ admit wavelet frames or bases (in contrast to $F^s_{p,q}(\mathbb{R}^n)$ and $B^s_{p,q}(\mathbb{R}^n)$), cf. [23, Cor. 6.38]. On the other hand, there are atomic characterizations for all spaces covered by (2.24), see [15], [11, Thm. 1.1.14].

2.2 Embeddings. We now come to embedding results and recall what is already known, in particular, for spaces of type $F^s_{p,q}$.

Proposition 2.14. *Let $s \in \mathbb{R}$, $0 < p < \infty$ and $0 < q \leq \infty$.*

(i) *Let $0 < p_0 < p < p_1 \leq \infty$, $s_0, s_1 \in \mathbb{R}$ such that*

$$s_0 - \frac{n}{p_0} = s - \frac{n}{p} = s_1 - \frac{n}{p_1}, \tag{2.30}$$

and $0 < q, u, v \leq \infty$. Then

$$B^{s_0}_{p_0,u}(\mathbb{R}^n) \hookrightarrow F^s_{p,q}(\mathbb{R}^n) \hookrightarrow B^{s_1}_{p_1,v}(\mathbb{R}^n) \tag{2.31}$$

if, and only if,

$$0 < u \le p \le v \le \infty. \tag{2.32}$$

(ii) *Let* $\varepsilon > 0$, $0 < u \le \infty$, *and* $q \le v \le \infty$. *Then*

$$F^{s+\varepsilon}_{p,u}(\mathbb{R}^n) \hookrightarrow F^s_{p,q}(\mathbb{R}^n) \qquad \textit{and} \qquad F^s_{p,q}(\mathbb{R}^n) \hookrightarrow F^s_{p,v}(\mathbb{R}^n).$$

(iii) *Let* $\sigma < s$, *and* $p < r < \infty$ *be such that*

$$s - \frac{n}{p} = \sigma - \frac{n}{r}. \tag{2.33}$$

Then for all $0 < q_1, q_2 \le \infty$,

$$F^s_{p,q_1}(\mathbb{R}^n) \hookrightarrow F^\sigma_{r,q_2}(\mathbb{R}^n). \tag{2.34}$$

(iv) *Let* $\sigma_p < s < \frac{n}{p}$, $p \le u \le \infty$, *and* r *such that*

$$s - \frac{n}{p} = -\frac{n}{r}. \tag{2.35}$$

Then

$$F^s_{p,q}(\mathbb{R}^n) \hookrightarrow L_{r,u}(\mathbb{R}^n).$$

Remark 2.15. For a proof of (i) we refer to [17, Sect. 5.2]. The "if"-part of the right-hand embedding is due to Jawerth [13], whereas the "if"-part of the left-hand embedding was proved by Franke [7]. Both original proofs use interpolation techniques. For F-spaces of type $F^s_{p,q}(\mathbb{R}^n)$ it is known that

$$\left(F^{s_0}_{p,q_0}(\mathbb{R}^n), F^{s_1}_{p,q_1}(\mathbb{R}^n)\right)_{\theta,q} = B^s_{p,q}(\mathbb{R}^n)\ , \tag{2.36}$$

where we have to assume $0 < \theta < 1$, $0 < p < \infty$, $s_0, s_1 \in \mathbb{R}$ with $s_0 \neq s_1$, $0 < q_0, q_1, q \le \infty$, and $s = (1-\theta)s_0 + \theta s_1$; we refer to [18, Thm. 2.4.2]. The other results above can be found in [18, Prop. 2.3.2, Thm. 2.7.1]. Limiting embeddings of type (iii) and (iv) with conditions (2.33) and (2.35) refer to embeddings along 'constant differential dimension'. Concerning (iii) this is essentially due to some Plancherel-Polya-Nikolskij inequality (cf. [18, (1.3.2/5), Rem. 1.4.1/4]), whereas (iv) again is a matter of real interpolation (2.36) together with the embedding $F^s_{p,q}(\mathbb{R}^n) \hookrightarrow F^0_{r,2}(\mathbb{R}^n) = L_r(\mathbb{R}^n)$, where the parameters satisfy (2.35).

We want to prove corresponding results for spaces of type $\mathfrak{F}^s_{p,q}$ and $\mathbf{F}^s_{p,q}$.

Theorem 2.16. *Let* $s > 0$, $0 < p < \infty$ *and* $0 < q \le \infty$.

(i) *Let* $0 < p_0 < p < p_1 \leq \infty$, $s_0, s_1 > 0$ *such that*

$$s_0 - \frac{n}{p_0} = s - \frac{n}{p} = s_1 - \frac{n}{p_1}, \tag{2.37}$$

and $0 < q, u, v \leq \infty$. *If*

$$0 < u \leq p \leq v \leq \infty, \tag{2.38}$$

then

$$\mathfrak{B}^{s_0}_{p_0,u}(\mathbb{R}^n) \hookrightarrow \mathfrak{F}^{s}_{p,q}(\mathbb{R}^n) \hookrightarrow \mathfrak{B}^{s_1}_{p_1,v}(\mathbb{R}^n). \tag{2.39}$$

(ii) *Let* $\varepsilon > 0$, $0 < u \leq \infty$, *and* $q \leq v \leq \infty$, *then*

$$\mathfrak{F}^{s+\varepsilon}_{p,u}(\mathbb{R}^n) \hookrightarrow \mathfrak{F}^{s}_{p,q}(\mathbb{R}^n) \qquad \text{and} \qquad \mathfrak{F}^{s}_{p,q}(\mathbb{R}^n) \hookrightarrow \mathfrak{F}^{s}_{p,v}(\mathbb{R}^n).$$

(iii) *Let* $0 < \sigma < s$, *and* $p < r < \infty$ *be such that*

$$s - \frac{n}{p} = \sigma - \frac{n}{r}. \tag{2.40}$$

Then for all $0 < q_1, q_2 \leq \infty$,

$$\mathfrak{F}^{s}_{p,q_1}(\mathbb{R}^n) \hookrightarrow \mathfrak{F}^{\sigma}_{r,q_2}(\mathbb{R}^n). \tag{2.41}$$

(iv) *Let* $s < \frac{n}{p}$, $p \leq u \leq \infty$, *and* r *such that*

$$s - \frac{n}{p} = -\frac{n}{r}. \tag{2.42}$$

Then

$$\mathfrak{F}^{s}_{p,q}(\mathbb{R}^n) \hookrightarrow L_{r,u}(\mathbb{R}^n).$$

Remark 2.17. Previously, we obtained similar results for the spaces $\mathfrak{B}^s_{p,q}(\mathbb{R}^n)$ (which coincide with the classical Besov spaces $\mathbf{B}^s_{p,q}(\mathbb{R}^n)$ if $s > 0$ and $0 < p, q \leq \infty$) in [12]. In particular, Theorem 2.16 (ii) holds analogously for $\mathfrak{B}^s_{p,q}(\mathbb{R}^n)$, whereas (iii) is only valid if $q_1 \leq q_2$, and (iv) for $u \geq q$. Furthermore, we shall strengthen the above assertions in Corollaries 4.1, 4.4 below.

Proof. **Step 1**. We want to establish (i), the so-called Franke-Jawerth embedding. J. Vybíral proved in [24, Thms. 3.1, 3.2, 3.3] corresponding assertions for the underlying sequence spaces, i.e., for $\varrho \geq 0$ and the above given restrictions on the parameters (2.37) we have

$$b^{s_0,\varrho}_{p_0,u} \hookrightarrow f^{s,\varrho}_{p,q} \hookrightarrow b^{s_1,\varrho}_{p_1,v} \tag{2.43}$$

if, and only if,

$$0 < u \le p \le v \le \infty. \tag{2.44}$$

Using this, we are able to obtain similar embeddings for the function spaces. Let $f \in \mathfrak{F}^s_{p,q}$ with representation

$$f(x) = \sum_{\beta,j,m} \lambda^\beta_{j,m} k^\beta_{j,m}(x)$$

according to (2.21). Then a simple calculation yields

$$\|f|\mathfrak{F}^s_{p,q}\| \le \|\lambda|f^{s,\varrho}_{p,q}\| \le c\|\lambda|b^{s_0,\varrho}_{p_0,u}\|,$$

where the last step follows from (2.43). Thus taking the infimum over all representations yields

$$\|f|\mathfrak{F}^s_{p,q}\| \le c\|f|\mathfrak{B}^{s_0}_{p_0,u}\|,$$

which establishes the first embedding in (2.39). The second embedding is proved in the same way.

Step 2. In order to prove (ii), we use a corresponding embedding obtained for the Besov spaces $\mathfrak{B}^s_{p,q}$ in [12]

$$\mathfrak{B}^{s+\varepsilon}_{p,q_1} \hookrightarrow \mathfrak{B}^s_{p,q_2}$$

with parameters $\varepsilon > 0$, $s > 0$, and $0 < p, q_1, q_2 \le \infty$. Together with (2.23) this immediately yields

$$\mathfrak{F}^{s+\varepsilon}_{p,u} \hookrightarrow \mathfrak{B}^{s+\varepsilon}_{p,\max(p,u)} \hookrightarrow \mathfrak{B}^s_{p,\min(p,q)} \hookrightarrow \mathfrak{F}^s_{p,q},$$

where $0 < u, q \le \infty$, which is the desired result. The second embedding follows immediately from the monotonicity of the ℓ_q sequence spaces, i.e., $\ell_q \hookrightarrow \ell_v$ for $q \le v$.

Step 3. We want to prove (iii). Using (2.39) together with an embedding for Besov spaces proved in [12], namely

$$\mathfrak{B}^{s_1}_{p_1,p} \hookrightarrow \mathfrak{B}^{s_1}_{p_1,r} \qquad \text{if} \qquad p \le r$$

(which follows immediately from the monotonicity of the ℓ_q spaces) we see that

$$\mathfrak{F}^s_{p,q_1} \hookrightarrow \mathfrak{B}^{s_1}_{p_1,p} \hookrightarrow \mathfrak{B}^{s_1}_{p_1,r} \hookrightarrow \mathfrak{F}^\sigma_{r,q_2},$$

where $0 < q_1, q_2 \le \infty$, and s_1, p_1 are chosen such that

$$s > s_1 > \sigma \qquad \text{with} \qquad s - \frac{n}{p} = s_1 - \frac{n}{p_1} = \sigma - \frac{n}{r}.$$

Step 4. In order to establish (iv) we again make use of (2.39) and the following embedding for Besov spaces established in [12]

$$\mathfrak{B}^s_{p,q} \hookrightarrow L_{r,u}, \qquad \text{where } s - \frac{n}{p} = -\frac{n}{r}, \qquad u \ge q.$$

This yields

$$\mathfrak{F}^s_{p,q} \hookrightarrow \mathfrak{B}^{s_1}_{p_1,p} \hookrightarrow L_{r,u},$$

with s_1 and p_1 such that $s > s_1 > 0$ and

$$s - \frac{n}{p} = s_1 - \frac{n}{p_1} = -\frac{n}{r} \qquad \text{and} \qquad u \ge p,$$

which completes the proof. □

Remark 2.18. Clearly the above Theorem is covered by Proposition 2.14 whenever $s > \sigma_{pq}$. This follows from Theorem 2.10(i). However, smoothness parameters $0 < s \le \sigma_{pq}$ were not yet covered by these earlier approaches. In the diagram aside we restricted ourselves to the case when $q_1 \ge 1$, and sketched the maximal area of possible embeddings of a fixed original space $F^{s_1}_{p_1,q_1}$ or $\mathfrak{F}^{s_1}_{p_1,q_1}$ into spaces $F^{s_2}_{p_2,q_2}$, $\mathfrak{F}^{s_2}_{p_2,q_2}$, and $L_{r,u}$, respectively.

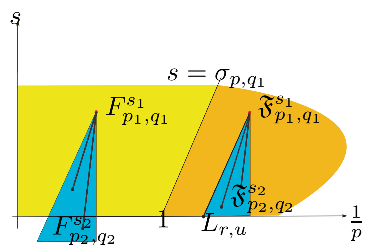

Figure 4: Parameter $q_1 \ge 1$

We finally add what is known for the spaces $\mathbf{F}^s_{p,q}$ in terms of embeddings results.

Proposition 2.19. *Let $s > 0$, $0 < p < \infty$ and $0 < q \le \infty$.*

(i) *Then*

$$\mathbf{B}^s_{p,\min(p,q)}(\mathbb{R}^n) \hookrightarrow \mathbf{F}^s_{p,q}(\mathbb{R}^n) \hookrightarrow \mathbf{B}^s_{p,\max(p,q)}(\mathbb{R}^n). \tag{2.45}$$

(ii) *Let $\varepsilon > 0$, $0 < u \le \infty$, and $q \le v \le \infty$, then*

$$\mathbf{F}^{s+\varepsilon}_{p,u}(\mathbb{R}^n) \hookrightarrow \mathbf{F}^s_{p,q}(\mathbb{R}^n) \qquad \textit{and} \qquad \mathbf{F}^s_{p,q}(\mathbb{R}^n) \hookrightarrow \mathbf{F}^s_{p,v}(\mathbb{R}^n).$$

(iii) *Let* $0 < p < p_1 \leq \infty$, $s_1 > 0$ *such that*

$$s - \frac{n}{p} = s_1 - \frac{n}{p_1}, \tag{2.46}$$

and $0 < q, v \leq \infty$. *If*

$$0 < p \leq v \leq \infty, \tag{2.47}$$

then

$$\mathbf{F}^s_{p,q}(\mathbb{R}^n) \hookrightarrow \mathbf{B}^{s_1}_{p_1,v}(\mathbb{R}^n). \tag{2.48}$$

(iv) *Let* $s < \frac{n}{p}$, $p \leq u \leq \infty$, *and* r *such that*

$$s - \frac{n}{p} = -\frac{n}{r}, \tag{2.49}$$

then

$$\mathbf{F}^s_{p,q}(\mathbb{R}^n) \hookrightarrow L_{r,u}(\mathbb{R}^n).$$

Proof. **Step 1**. We prove (i). Let first $q \geq p$, then we have $\mathbf{B}^s_{p,u} = \mathfrak{B}^s_{p,u}$ and $\mathbf{F}^s_{p,q} = \mathfrak{F}^s_{p,q}$, see Theorem 2.10(i) and [22, Prop. 9.14], such that (2.45) is covered by (2.23). Furthermore (ii), (iii), and (iv) are covered by Theorem 2.16(ii), (i), and (iv), respectively. Hence it remains to deal with the case $q < p$.

Let $r \in \mathbb{N}$ with $r > s$, $0 < u \leq \infty$; then rewriting (2.5) and (2.3) gives

$$\left\| f | \mathbf{B}^s_{p,u} \right\| \sim \| f | L_p \| + \left(\int_0^1 t^{-su} \left(\sup_{|h| \leq t} \| \Delta^r_h f | L_p \| \right)^u \frac{\mathrm{d}t}{t} \right)^{\frac{1}{u}} \tag{2.50}$$

(usual modification for $u = \infty$) and

$$\left\| f | \mathbf{F}^s_{p,q} \right\| \sim \| f | L_p \| + \left(\int_{\mathbb{R}^n} \left(\int_0^1 t^{-(s+\frac{n}{p})q} \left(\int_{|h| \leq t} |\Delta^r_h f(x)|^p \mathrm{d}h \right)^{\frac{q}{p}} \frac{\mathrm{d}t}{t} \right)^{\frac{p}{q}} \mathrm{d}x \right)^{\frac{1}{p}}, \tag{2.51}$$

(usual modification for $q = \infty$).

We begin with the left-hand embedding in (2.45). Recall that we need only consider the case $q < p$, that is, $\min(p,q) = q$. In view of (2.50), (2.51) it is sufficient to show that

$$\left(\int\limits_{\mathbb{R}^n}\left(\int\limits_0^1 t^{-(s+\frac{n}{p})q}\left(\int\limits_{|h|\le t}|\Delta_h^r f(x)|^p \mathrm{d}h\right)^{\frac{q}{p}}\frac{\mathrm{d}t}{t}\right)^{\frac{p}{q}}\mathrm{d}x\right)^{\frac{1}{p}}$$

$$\le\ c\left(\int\limits_0^1 t^{-sq}\left(\sup_{|h|\le t}\|\Delta_h^r f|L_p\|\right)^q\frac{\mathrm{d}t}{t}\right)^{\frac{1}{q}}. \tag{2.52}$$

We make use of the generalized triangle inequality for integrals,

$$\left(\int\limits_X\left(\int\limits_Y|\varphi(x,y)|\,\mathrm{d}y\right)^r\mathrm{d}x\right)^{\frac{1}{r}}\le\int\limits_Y\left(\int\limits_X|\varphi(x,y)|^r\,\mathrm{d}x\right)^{\frac{1}{r}}\mathrm{d}y \tag{2.53}$$

for $r>1$, cf. [10, Thm. 202, p. 148]. We put $r=\frac{p}{q}$ and

$$\varphi(x,t)=t^{-(s+\frac{n}{p})q-1}\chi_{(0,1)}(t)\left(\int\limits_{|h|\le t}|\Delta_h^r f(x)|^p\mathrm{d}h\right)^{\frac{q}{p}},\quad x\in\mathbb{R}^n,\ t\in\mathbb{R}.$$

Then the left-hand side of (2.52) can be written as

$$\left(\int\limits_{\mathbb{R}^n}\left(\int\limits_{\mathbb{R}}\varphi(x,t)\mathrm{d}t\right)^r\mathrm{d}x\right)^{\frac{1}{rq}}$$

and an application of (2.53) yields

$$\begin{aligned}
&\left(\int\limits_{\mathbb{R}^n}\left(\int\limits_0^1 t^{-(s+\frac{n}{p})q}\left(\int\limits_{|h|\le t}|\Delta_h^r f(x)|^p\ \mathrm{d}h\right)^{\frac{q}{p}}\frac{\mathrm{d}t}{t}\right)^{\frac{p}{q}}\mathrm{d}x\right)^{\frac{1}{p}}\\
&\qquad\le\ c_1\left(\int\limits_{\mathbb{R}}\left(\int\limits_{\mathbb{R}^n}\varphi(x,t)^{\frac{p}{q}}\mathrm{d}x\right)^{\frac{q}{p}}\mathrm{d}t\right)^{\frac{1}{q}}\\
&\qquad=\ c_1\left(\int\limits_0^1 t^{-(s+\frac{n}{p})q}\left(\int\limits_{\mathbb{R}^n}\int\limits_{|h|\le t}|\Delta_h^r f(x)|^p\ \mathrm{d}h\ \mathrm{d}x\right)^{\frac{q}{p}}\frac{\mathrm{d}t}{t}\right)^{\frac{1}{q}}\\
&\qquad\le\ c_2\left(\int\limits_0^1 t^{-(s+\frac{n}{p})q}\left(\sup_{|h|\le t}\|\Delta_h^r f|L_p\|\,t^{\frac{n}{p}}\right)^q\frac{\mathrm{d}t}{t}\right)^{\frac{1}{q}},
\end{aligned}$$

that is, the right-hand side of (2.52).

We deal with the right-hand embedding of (2.45) and $\max(p,q) = p$. First we use an argument from Step 3 of the proof of [18, Thm. 2.5.12] which gives (with (2.50) applied to $2r$ instead of r),

$$\left\|f|\mathbf{B}^s_{p,p}\right\| \le c\,\|f|L_p\| + c_2\left(\int\limits_{|h|\le 1} |h|^{-sp}\,\|\Delta^r_h f|L_p\|^p\,\frac{\mathrm{d}h}{|h|^n}\right)^{\frac{1}{p}}. \tag{2.54}$$

Note that

$$\begin{aligned}\int\limits_{|h|\le 1} |h|^{-sp}\,\|\Delta^r_h f|L_p\|^p\,\frac{\mathrm{d}h}{|h|^n} &\sim \int\limits_{|h|\le 1} \|\Delta^r_h f|L_p\|^p \int\limits_{|h|}^{1} t^{-sp-n}\,\frac{\mathrm{d}t}{t}\,\mathrm{d}h\\ &\le c\int\limits_0^1 t^{-sp-n}\int\limits_{|h|\le t} \|\Delta^r_h f|L_p\|^p\,\mathrm{d}h\,\frac{\mathrm{d}t}{t}.\end{aligned} \tag{2.55}$$

Since $p > q$ we have for any $x\in\mathbb{R}^n$ that

(2.56)
$$\int\limits_0^1 t^{-(s+\frac{n}{p})p}\int\limits_{|h|\le t} |\Delta^r_h f(x)|^p\,\mathrm{d}h\,\frac{\mathrm{d}t}{t} \le c\left(\int\limits_0^1 t^{-(s+\frac{n}{p})q}\left(\int\limits_{|h|\le t} |\Delta^r_h f(x)|^p\,\mathrm{d}h\right)^{\frac{q}{p}}\frac{\mathrm{d}t}{t}\right)^{\frac{p}{q}},$$

and in view of (2.54), (2.55) and (2.56) we can estimate

$$\begin{aligned}\left\|f|\mathbf{B}^s_{p,\max(p,q)}\right\| &\le c\,\|f|L_p\| + c\left(\int\limits_{\mathbb{R}^n}\int\limits_0^1 t^{-sp-n}\int\limits_{|h|\le t} |\Delta^r_h f(x)|^p\,\mathrm{d}h\,\frac{\mathrm{d}t}{t}\,\mathrm{d}x\right)^{\frac{1}{p}}\\ &\le c'\,\|f|L_p\| + c'\left(\int\limits_{\mathbb{R}^n}\left(\int\limits_0^1 t^{-(s+\frac{n}{p})q}\left(\int\limits_{|h|\le t} |\Delta^r_h f(x)|^p\,\mathrm{d}h\right)^{\frac{q}{p}}\frac{\mathrm{d}t}{t}\right)^{\frac{p}{q}}\mathrm{d}x\right)^{\frac{1}{p}}\\ &\le c''\left\|f|\mathbf{F}^s_{p,q}\right\|.\end{aligned}$$

Step 2. We establish (ii). The second assertion follows from

$$\left\|f|\mathbf{F}^s_{p,q}\right\| \sim \|f|L_p\| + \left(\int\limits_{\mathbb{R}^n}\left(\int\limits_0^1 t^{-(s+\frac{n}{p})q}\left(\int\limits_{|h|\le t} |\Delta^r_h f(x)|^p\mathrm{d}h\right)^{\frac{q}{p}}\frac{\mathrm{d}t}{t}\right)^{\frac{p}{q}}\mathrm{d}x\right)^{\frac{1}{p}}$$

$$\sim \|f|L_p\| + \left(\int\limits_{\mathbb{R}^n} \left(\sum_{k=0}^{\infty} 2^{k(s+\frac{n}{p})q} \left(\int\limits_{|h|\leq 2^{-k}} |\Delta_h^r f(x)|^p \mathrm{d}h \right)^{\frac{q}{p}} \right)^{\frac{p}{q}} \mathrm{d}x \right)^{\frac{1}{p}}$$

and the monotonicity of the ℓ_q sequence spaces. The first embedding is clear using (i) and corresponding assertions for the Besov spaces $\mathbf{B}^s_{p,q}$, cf. [12, Th. 1.16(i)]. We see that

$$\mathbf{F}^{s+\varepsilon}_{p,u} \hookrightarrow \mathbf{B}^{s+\varepsilon}_{p,\max(p,u)} \hookrightarrow \mathbf{B}^{s}_{p,\min(p,q)} \hookrightarrow \mathbf{F}^{s}_{p,q},$$

which yields the desired embedding.

Step 3. Now (iii) follows from (ii) and the Franke-Jawerth embedding as stated in Theorem 2.16(i). We obtain

$$\mathbf{F}^s_{p,q} \hookrightarrow \mathbf{F}^s_{p,\infty} = \mathfrak{F}^s_{p,\infty} \hookrightarrow \mathfrak{B}^{s_1}_{p_1,p} = \mathbf{B}^{s_1}_{p_1,p} \hookrightarrow \mathbf{B}^{s_1}_{p_1,v},$$

where the last embedding follows from the monotonicity of the ℓ_q sequence spaces and holds if $p \leq v$.

Step 4. The proof of (iv) follows from (ii), (iii), and the corresponding assertion for Besov spaces, cf. [12, Th. 1.16(iii)], and

$$\mathbf{F}^s_{p,q} \hookrightarrow \mathbf{F}^s_{p,\infty} \hookrightarrow \mathbf{B}^{s_1}_{p_1,p} \hookrightarrow L_{r,u},$$

where

$$s - \frac{n}{p} = s_1 - \frac{n}{p_1} = -\frac{n}{r} \qquad \text{and} \qquad p \leq u.$$

□

Remark 2.20. If $s > \sigma_{pq}$, then the above proposition is well-known since $\mathbf{B}^s_{p,u} = B^s_{p,u}$ and $\mathbf{F}^s_{p,q} = F^s_{p,q}$, cf. [22, Rem. 9.13].
Using Proposition 2.19(i) together with the interpolation results for classical Besov spaces $\mathbf{B}^s_{p,q}$,

$$\left(\mathbf{B}^{s_0}_{p,q_0}(\mathbb{R}^n), \mathbf{B}^{s_1}_{p,q_1}(\mathbb{R}^n)\right)_{\theta,q} = \mathbf{B}^s_{p,q}(\mathbb{R}^n),$$

where $0 < \theta < 1$, $0 < p < \infty$, $0 < s_0, s_1 < \infty$, $s_0 \neq s_1$, $0 < q_0, q_1, q \leq \infty$, and $s = (1-\theta)s_0 + \theta s_1$, cf. [4, Cor. 6.2], we obtain

$$\left(\mathbf{F}^{s_0}_{p,q_0}(\mathbb{R}^n), \mathbf{F}^{s_1}_{p,q_1}(\mathbb{R}^n)\right)_{\theta,q} = \mathbf{B}^s_{p,q}(\mathbb{R}^n)$$

for the same restrictions on the parameters. Furthermore in Proposition 2.19(iii) we only established the right-hand side of the well-known Franke-Jawerth embedding. Until now we were unable to prove that the left-hand side holds in general for the full range of parameters. But when $p \le q$ Theorem 2.10(i) together with Theorem 2.16(i) yields for $0 < p_0 < p < p_1 \le \infty$, $s_0, s_1 > 0$ such that

$$s_0 - \frac{n}{p_0} = s - \frac{n}{p} = s_1 - \frac{n}{p_1}, \tag{2.57}$$

and $0 < q, u, v \le \infty$, that if

$$0 < u \le p \le v \le \infty, \tag{2.58}$$

then

$$\mathbf{B}^{s_0}_{p_0,u}(\mathbb{R}^n) \hookrightarrow \mathbf{F}^{s}_{p,q}(\mathbb{R}^n) \hookrightarrow \mathbf{B}^{s_1}_{p_1,v}(\mathbb{R}^n). \tag{2.59}$$

3. Growth envelopes

3.1 Definitions and basic properties. Let for some measurable function $f : \mathbb{R}^n \to \mathbb{C}$, finite a.e., its decreasing rearrangement f^* be defined as usual, $f^*(t) = \inf\{s \ge 0 : |\{x \in \mathbb{R}^n : |f(x)| > s\}| \le t\}$, $t \ge 0$.

Definition 3.1. Let X be some (quasi-)normed function space on $\mathbb{R}^n$.

(i) The *growth envelope function* $\mathcal{E}^X_{\mathsf{G}} : (0,\infty) \to [0,\infty]$ of X is defined by

$$\mathcal{E}^X_{\mathsf{G}}(t) = \sup_{\|f|X\| \le 1} f^*(t) , \quad t > 0. \tag{3.1}$$

(ii) Assume $X \not\hookrightarrow L_\infty(\mathbb{R}^n)$. Let $\varepsilon \in (0,1)$, $H(t) = -\log \mathcal{E}^X_{\mathsf{G}}(t)$, $t \in (0,\varepsilon]$, and let μ_H be the associated Borel measure. The number u^X_{G}, $0 < u^X_{\mathsf{G}} \le \infty$, is defined as the infimum of all numbers v, $0 < v \le \infty$, such that

$$\left(\int_0^\varepsilon \left(\frac{f^*(t)}{\mathcal{E}^X_{\mathsf{G}}(t)} \right)^v \mu_H(\mathrm{d}t) \right)^{1/v} \le c \, \|f|X\| \tag{3.2}$$

(with the usual modification if $v = \infty$) holds for some $c > 0$ and all $f \in X$. The couple

$$\mathfrak{E}_{\mathsf{G}}(X) = \left(\mathcal{E}^X_{\mathsf{G}}(\cdot), u^X_{\mathsf{G}} \right)$$

is called *(local) growth envelope* for the function space X.

This concept was introduced and first studied in [21, Ch. 2], [8], see also [9]. For convenience we recall some properties. In view of (i) we obtain – strictly speaking – equivalence classes of growth envelope functions when working with equivalent (quasi-)norms in X as we shall usually do. But we do not want to distinguish between representative and equivalence class in what follows and thus stick at the notation introduced in (i). Concerning (ii) we shall assume that we can choose a continuous representative in the equivalence class $[\mathcal{E}_{\mathsf{G}}^{X}]$, for convenience (but in a slight abuse of notation) denoted by $\mathcal{E}_{\mathsf{G}}^{X}$ again. It is obvious that (3.2) holds for $v=\infty$ and any X. Moreover, one verifies that

(3.3)
$$\sup_{0<t\leq\varepsilon} \frac{g(t)}{\mathcal{E}_{\mathsf{G}}^{X}(t)} \leq c_1 \left(\int_0^{\varepsilon} \left(\frac{g(t)}{\mathcal{E}_{\mathsf{G}}^{X}(t)} \right)^{v_1} \mu_H(\mathrm{d}t) \right)^{\frac{1}{v_1}} \leq c_2 \left(\int_0^{\varepsilon} \left(\frac{g(t)}{\mathcal{E}_{\mathsf{G}}^{X}(t)} \right)^{v_0} \mu_H(\mathrm{d}t) \right)^{\frac{1}{v_0}}$$

for $0<v_0<v_1<\infty$ and all non-negative monotonically decreasing functions g on $(0,\varepsilon]$; cf. [21, Prop. 12.2]. So with $g=f^*$ we observe that the left-hand sides in (3.2) are monotonically ordered in v and it is natural to look for the smallest possible one.

Proposition 3.2. **(i)** *Let X_i, $i=1,2$, be some function spaces on $\mathbb{R}^n$. Then $X_1 \hookrightarrow X_2$ implies that there is some positive constant c such that for all $t>0$,*

$$\mathcal{E}_{\mathsf{G}}^{X_1}(t) \ \leq \ c\, \mathcal{E}_{\mathsf{G}}^{X_2}(t). \tag{3.4}$$

(ii) *We have $X \hookrightarrow L_\infty$ if, and only if, $\mathcal{E}_{\mathsf{G}}^{X}$ is bounded.*

(iii) *Let X_i, $i=1,2$, be some function spaces on $\mathbb{R}^n$ with $X_1 \hookrightarrow X_2$. Assume for their growth envelope functions*

$$\mathcal{E}_{\mathsf{G}}^{X_1}(t) \ \sim \ \mathcal{E}_{\mathsf{G}}^{X_2}(t), \qquad t\in(0,\varepsilon),$$

for some $\varepsilon>0$. Then we get for the corresponding indices $u_{\mathsf{G}}^{X_i}$, $i=1,2$, that

$$u_{\mathsf{G}}^{X_1} \ \leq \ u_{\mathsf{G}}^{X_2}.$$

This result coincides with [9, Props. 3.4, 4.5].

Remark 3.3. For rearrangement-invariant Banach function spaces X with fundamental function φ_X it is proved in [9, Sect. 2.3] that

$$\mathcal{E}_{\mathsf{G}}^{X}(t) \sim \frac{1}{\varphi_X(t)} = \left\|\chi_{A_t}\middle|X\right\|^{-1}, \quad t>0, \tag{3.5}$$

where $A_t \subset \mathbb{R}^n$ with $|A_t|=t$.

In contrast to the local characterization in Definition 3.1(ii) it turned out recently, that sometimes also the *global behaviour* of the envelope function, $\mathcal{E}_{\mathsf{G}}^{X}(t)$ for $t \to \infty$ is of interest.

Example 3.4. Let $L_{p,q}(\mathbb{R}^n)$, $0<p<\infty$, $0<q\le\infty$, denote the well-known Lorentz spaces, consisting of all functions f for which the quantity

$$\|f|L_{p,q}(\mathbb{R}^n)\| = \begin{cases} \left(\int\limits_0^\infty \left[t^{\frac{1}{p}} f^*(t)\right]^q \frac{\mathrm{d}t}{t}\right)^{1/q} & , \text{ if } \quad 0<q<\infty, \\ \sup\limits_{0<t<\infty} t^{\frac{1}{p}} f^*(t) & , \text{ if } \quad q=\infty, \end{cases} \tag{3.6}$$

is finite. They are natural refinements of the scale of Lebesgue spaces; we refer to [2, Ch. 4] for further details. It is proved in [9, Thm. 4.7, Cor. 10.14] that

$$\mathfrak{E}_{\mathsf{G}}(L_{p,q}(\mathbb{R}^n)) = \left(t^{-\frac{1}{p}}, q\right), \tag{3.7}$$

and

$$\mathcal{E}_{\mathsf{G}}^{L_{p,q}}(t) \sim t^{-\frac{1}{p}} \qquad \text{for} \quad t\to\infty. \tag{3.8}$$

3.2 Growth envelopes for F-spaces. We now turn to Triebel-Lizorkin and Besov spaces and first collect what is known. We will make use of our previous results obtained for growth envelopes in Besov spaces $\mathfrak{B}^s_{p,q}(\mathbb{R}^n)$, cf. [12].

As explained, the above concept is interesting only for spaces $X \not\hookrightarrow L_\infty$; in case of F-spaces this reads as follows.

Proposition 3.5. *Let* $0<p<\infty$, $0<q\le\infty$ *and* $s\in\mathbb{R}$.

(i) *Then*

$$F^s_{p,q}(\mathbb{R}^n) \hookrightarrow L_\infty(\mathbb{R}^n) \quad \textit{if, and only if,} \quad \begin{cases} 0<p<\infty, & \textit{if} \quad s>\frac{n}{p}, \\ 0<p\le 1, & \textit{if} \quad s=\frac{n}{p}. \end{cases}$$

(ii) *Furthermore,*

$$\mathfrak{F}^s_{p,q}(\mathbb{R}^n) \hookrightarrow L_\infty(\mathbb{R}^n) \quad \textit{if, and only if,} \quad \begin{cases} 0<p<\infty, & \textit{if} \quad s>\frac{n}{p}, \\ 0<p\le 1, & \textit{if} \quad s=\frac{n}{p}. \end{cases} \tag{3.9}$$

Proof. Part (i) coincides with [6, 2.3.3(iii)], so we are left to prove (ii). Clearly the limiting case $s=\frac{n}{p}$ is of interest only, in view of Theorem

2.16(i) and the corresponding result for the Besov spaces in [12, Prop. 2.5],

$$\mathfrak{B}^s_{p,q}(\mathbb{R}^n) \hookrightarrow L_\infty(\mathbb{R}^n) \quad \text{if, and only if,} \quad \begin{cases} 0 < q \le \infty, & \text{if} \quad s > \frac{n}{p}, \\ 0 < q \le 1, & \text{if} \quad s = \frac{n}{p}. \end{cases} \tag{3.10}$$

But also in the limiting case the argument relies on the result for $\mathfrak{B}$-spaces. Assume

$$\mathfrak{F}^{n/p}_{p,q} \hookrightarrow L_\infty,$$

then for all $0 < r < p$, Theorem 2.16(i) gives

$$\mathfrak{B}^{n/r}_{r,p} \hookrightarrow \mathfrak{F}^{n/p}_{p,q} \hookrightarrow L_\infty,$$

such that (3.10) implies

$$0 < p \le 1.$$

Conversely, if $0 < p \le 1$, we may choose r with $p < r < \infty$, such that (3.10) yields

$$\mathfrak{F}^{n/p}_{p,q} \hookrightarrow \mathfrak{B}^{n/r}_{r,p} \hookrightarrow L_\infty.$$

This completes the proof. □

Remark 3.6. Note that we did not use the identity (2.25), that is

$$F^s_{p,q}(\mathbb{R}^n) = \mathfrak{F}^s_{p,q}(\mathbb{R}^n)$$

in the above proof, which is only clear for $s > \sigma_{pq}$. This implies that for $s = \frac{n}{p}$ these spaces may differ if $q < \min(p,1)$. However, as verified above, they are both embedded into L_∞.

We separately investigate the situation for the spaces $\mathbf{F}^s_{p,q}$.

Proposition 3.7. *Let $0 < p < \infty$, $0 < q \le \infty$, and $s > 0$.*

(i) *Then*

$$\mathbf{F}^s_{p,q}(\mathbb{R}^n) \hookrightarrow L_\infty(\mathbb{R}^n) \quad \textit{if} \quad \begin{cases} 0 < p < \infty, & s > \frac{n}{p}, \\ 0 < p \le 1, & s = \frac{n}{p}. \end{cases}$$

(ii) *Assume $s < \frac{n}{p}$ or $s = \frac{n}{p}$, $1 < p < \infty$ and $q > \min\left(\frac{p}{2}, 1\right)$. Then*

$$\mathbf{F}^s_{p,q}(\mathbb{R}^n) \not\hookrightarrow L_\infty(\mathbb{R}^n). \tag{3.11}$$

Proof. The proof of (i) follows immediately from Theorem 2.10(i) and Proposition 3.5(ii), since for $s > \frac{n}{p}$ or $s = \frac{n}{p}$ and $0 < p \le 1$ we have

$$\mathbf{F}^s_{p,q} \hookrightarrow \mathbf{F}^s_{p,\infty} = \mathfrak{F}^s_{p,\infty} \hookrightarrow L_\infty.$$

Concerning (ii), if $s < \frac{n}{p}$ we proceed indirectly assuming $\mathbf{F}^s_{p,q} \hookrightarrow L_\infty$. Choosing $s < \sigma < \frac{n}{p}$ we see that

$$\mathfrak{F}^\sigma_{p,p} = \mathbf{F}^\sigma_{p,p} \hookrightarrow \mathbf{F}^s_{p,q} \hookrightarrow L_\infty,$$

which gives the desired contradiction according to Proposition 3.5(ii).
If $s = \frac{n}{p}$, $1 < p < \infty$, Theorem 2.10(i) yields

$$\mathbf{F}^{n/p}_{p,q} = \mathfrak{F}^{n/p}_{p,q}, \qquad \text{where} \quad \frac{1}{q} < \frac{s}{n} + \frac{1}{p} = \frac{2}{p},$$

from which we see – again using Proposition 3.5(ii) – that

$$\mathbf{F}^s_{p,q} \not\hookrightarrow L_\infty.$$

On the other hand, if $s = \frac{n}{p}$, $1 < p < \infty$, and $q \le \frac{p}{2}$, then $\mathbf{F}^{n/p}_{p,q} \hookrightarrow L_\infty$ implies $\mathbf{B}^{n/p}_{p,q} \hookrightarrow L_\infty$ leading to $q \le 1$. $\square$

Remark 3.8. In view of Proposition 3.5 we claim that "if" in Proposition 3.7(i) could probably be replaced by "if, and only if", i.e., when $0 < q \le \infty$ Proposition 3.7(ii) can be generalized to

$$\mathbf{F}^{n/p}_{p,q} \not\hookrightarrow L_\infty, \qquad \text{if} \quad 1 < p < \infty \quad \text{and} \quad q \le \min\left(\frac{p}{2}, 1\right).$$

We now focus on growth envelopes for the spaces $\mathfrak{F}^s_{p,q}$. In the diagram aside we have shaded the area corresponding to the remaining cases (assuming that $q \ge 1$) apart from (3.9), where the lower right triangle refers to our new result in Theorem 3.11 below, extending the already known situation repeated below.

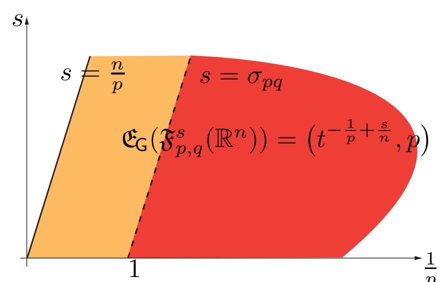

Figure 5: Parameter $q \ge 1$

Recall that by (2.9) and Proposition 3.5(i) only smoothness parameters $0 \le s \le \frac{n}{p}$ are of interest for the local behaviour of $\mathcal{E}_{\mathsf{G}}(t)$.

Proposition 3.9. *Let* $0 < p < \infty$, $0 < q \le \infty$ *and* $s \ge 0$.

(i) *Let* $\sigma_p < s < \frac{n}{p}$, *then*

$$\mathfrak{E}_{\mathsf{G}}(F^s_{p,q}(\mathbb{R}^n)) = \left(t^{-\frac{1}{p}+\frac{s}{n}}, p\right).$$

(ii) *Let* $s = \frac{n}{p}$, $1 < p < \infty$ *and* p' *given by* $\frac{1}{p} + \frac{1}{p'} = 1$. *Then*

$$\mathfrak{E}_{\mathsf{G}}(F^{n/p}_{p,q}(\mathbb{R}^n)) = \left(|\log t|^{1/p'}, p\right).$$

(iii) *Let* $1 \le p < \infty$, $0 < q \le 2$, *then*

$$\mathfrak{E}_{\mathsf{G}}(F^{0}_{p,q}(\mathbb{R}^n)) = \left(t^{-1/p}, p\right).$$

(iv) *Let* $s = \sigma_p$, $0 < p < 1$, *and* $0 < q \le \infty$, *then*

$$\mathfrak{E}_{\mathsf{G}}(F^{s}_{p,q}(\mathbb{R}^n)) = \left(t^{-1}, p\right).$$

(v) *Assume* $0 < p < \infty$, $s > \sigma_p$, *then*

$$\mathcal{E}_{\mathsf{G}}^{F^s_{p,q}}(t) \ \sim \ t^{-\frac{1}{p}} \qquad \textit{for} \quad t \to \infty.$$

Proofs can be found in [9, Thms. 8.1, 8.16, 10.19, Props. 8.12, 8.14], concerning (i) and (ii) also in [21, Thms. 13.2, 15.2]. Furthermore, the assertion on the additional index in (iv) – still an open problem in [9, Proposition 8.14]– was proved recently in [25, Theorem 1.3]. In view of Theorem 2.10 - assuming that $q \ge p$, and $0 < p \le 1$ - we thus have results for the spaces $\mathbf{F}^s_{p,q}$ and $\mathfrak{F}^s_{p,q}$ in case of $\sigma_p < s < \frac{n}{p}$ and want to extend this to $s > 0$.

In [12] we established the following results concerning growth envelopes for Besov spaces $\mathfrak{B}^s_{p,q}(\mathbb{R}^n)$.

Proposition 3.10. *Let* $0 < p < \infty$, $0 < q \le \infty$ *and* $s > 0$.

(i) *Let* $s < \frac{n}{p}$, *then*

$$\mathfrak{E}_{\mathsf{G}}(\mathfrak{B}^{s}_{p,q}(\mathbb{R}^n)) = \left(t^{-\frac{1}{p}+\frac{s}{n}}, q\right).$$

(ii) *Let* $s = \frac{n}{p}$, $1 < q \le \infty$ *and* q' *given by* $\frac{1}{q} + \frac{1}{q'} = 1$. *Then*

$$\mathfrak{E}_{\mathsf{G}}(\mathfrak{B}^{n/p}_{p,q}(\mathbb{R}^n)) = \left(|\log t|^{1/q'}, q\right).$$

(iii) *We have*

$$\mathcal{E}_{\mathsf{G}}^{\mathfrak{B}^s_{p,q}}(t) \ \sim \ t^{-\frac{1}{p}} \qquad \textit{for} \quad t \to \infty.$$

Note that we could replace $\mathfrak{B}^s_{p,q}(\mathbb{R}^n)$ in the above proposition by the classical Besov spaces $\mathbf{B}^s_{p,q}(\mathbb{R}^n)$, since these coincide for $s > 0$.
We are now able to formulate corresponding results for the spaces $\mathfrak{F}^s_{p,q}(\mathbb{R}^n)$.

Theorem 3.11. *Let* $0 < p < \infty$, $0 < q \le \infty$ *and* $s > 0$.

(i) *Let* $s < \frac{n}{p}$, *then*

$$\mathfrak{E}_{\mathsf{G}}(\mathfrak{F}^s_{p,q}(\mathbb{R}^n)) = \left(t^{-\frac{1}{p}+\frac{s}{n}}, p\right).$$

(ii) *Let* $s = \frac{n}{p}$, $1 < p < \infty$ *and* p' *given by* $\frac{1}{p} + \frac{1}{p'} = 1$. *Then*

$$\mathfrak{E}_{\mathsf{G}}(\mathfrak{F}^{n/p}_{p,q}(\mathbb{R}^n)) = \left(|\log t|^{1/p'}, p\right).$$

(iii) *We have*

$$\mathcal{E}_{\mathsf{G}}^{\mathfrak{F}^s_{p,q}}(t) \ \sim \ t^{-\frac{1}{p}} \qquad \textit{for} \quad t \to \infty.$$

Proof. **Step 1**. We show that $\mathcal{E}_{\mathsf{G}}^{\mathfrak{F}^s_{p,q}}(t) \sim t^{-\frac{1}{p}+\frac{s}{n}}$, $\quad 0 < t < 1$. Using (2.23)

$$\mathfrak{B}^s_{p,\min(p,q)} \hookrightarrow \mathfrak{F}^s_{p,q} \hookrightarrow \mathfrak{B}^s_{p,\max(p,q)},$$

together with Proposition 3.2(i), and the results for Besov spaces from Proposition 3.10(i), we see that

$$c_1 t^{-\frac{1}{r}} \leq c_2 \mathcal{E}_{\mathsf{G}}^{\mathfrak{B}^s_{p,\min(p,q)}}(t) \leq \mathcal{E}_{\mathsf{G}}^{\mathfrak{F}^s_{p,q}}(t) \leq c_3 \mathcal{E}_{\mathsf{G}}^{\mathfrak{B}^s_{p,\max(p,q)}}(t) \leq c_4 t^{-\frac{1}{r}}, \qquad 0 < t < 1,$$

where $-\frac{1}{r} = -\frac{1}{p} + \frac{s}{n}$.

Step 2. In order to show that for the additional index $u_{\mathsf{G}}^{\mathfrak{F}^s_{p,q}} = p$, we use the Franke-Jawerth embedding from Theorem 2.16(i),

$$\mathfrak{B}^{s_0}_{p_0,p} \hookrightarrow \mathfrak{F}^s_{p,q} \hookrightarrow \mathfrak{B}^{s_1}_{p_1,p},$$

where we may choose s_0, s_1 and p_0, p_1 such that

$$s_0 > s > s_1 > 0, \qquad \text{and} \qquad s_0 - \frac{n}{p_0} = s - \frac{n}{p} = s_1 - \frac{n}{p_1}.$$

But then the growth envelope functions for all these spaces turn out to be equivalent. This observation together with Proposition 3.2(iii) and the fact that $u_{\mathsf{G}}^{\mathfrak{B}^{s_i}_{p_i,p}} = p$, $i = 0, 1$, yields $u_{\mathsf{G}}^{\mathfrak{F}^s_{p,q}} = p$.

Step 3. We prove (ii). Choosing $r_1 < p < r_2$ according to Theorem 2.16(i) such that

$$\mathfrak{B}^{n/r_1}_{r_1,p} \hookrightarrow \mathfrak{F}^{n/p}_{p,q} \hookrightarrow \mathfrak{B}^{n/r_2}_{r_2,p},$$

we see from Propostion 3.10(ii) that

$$\mathfrak{E}_{\mathsf{G}}(\mathfrak{B}^{n/r_i}_{r_i,p}(\mathbb{R}^n)) = \left(|\log t|^{1/p'}, p\right), \qquad i = 1, 2.$$

But then Proposition 3.2(i),(iii) yields

$$\mathfrak{E}_{\mathsf{G}}(\mathfrak{F}^{n/p}_{p,q}(\mathbb{R}^n)) = \left(|\log t|^{1/p'}, p\right).$$

Step 4. We establish $\mathcal{E}_{\mathsf{G}}^{\mathfrak{F}^s_{p,q}}(t) \sim t^{-\frac{1}{p}}$ for $t \to \infty$.
Concerning the global behaviour – using the same argumentation as in Step 1 together with Proposition 3.10(iii) – we see that

$$c_1 t^{-\frac{1}{p}} \le c_2 \mathcal{E}_{\mathsf{G}}^{\mathfrak{B}^s_{p,\min(p,q)}}(t) \le \mathcal{E}_{\mathsf{G}}^{\mathfrak{F}^s_{p,q}}(t) \le c_3 \mathcal{E}_{\mathsf{G}}^{\mathfrak{B}^s_{p,\max(p,q)}}(t) \le c_4 t^{-\frac{1}{p}}, \qquad t \to \infty,$$

which completes the proof. □

In terms of the Sobolev-type spaces introduced in Remark 2.12 the results read as follows.

Corollary 3.12. *Let* $0 < p < \infty$ *and* $s > 0$.

(i) *Let* $s < \frac{n}{p}$, *then*

$$\mathfrak{E}_{\mathsf{G}}(\mathfrak{H}^s_p(\mathbb{R}^n)) = \left(t^{-\frac{1}{p}+\frac{s}{n}}, p\right).$$

(ii) *Let* $s = \frac{n}{p}$, $1 < p < \infty$ *and* p' *given by* $\frac{1}{p} + \frac{1}{p'} = 1$. *Then*

$$\mathfrak{E}_{\mathsf{G}}(\mathfrak{H}^{n/p}_p(\mathbb{R}^n)) = \left(|\log t|^{1/p'}, p\right).$$

(iii) *We have*

$$\mathcal{E}_{\mathsf{G}}^{\mathfrak{H}^s_p}(t) \ \sim \ t^{-\frac{1}{p}} \qquad \textit{for} \quad t \to \infty.$$

We derive the following results concerning the growth envelopes for the spaces $\mathbf{F}^s_{p,q}$.

Proposition 3.13. *Let* $0 < p < \infty$, $0 < q \le \infty$ *and* $s > 0$.

(i) *Let* $s < \frac{n}{p}$, *then*

$$\mathfrak{E}_{\mathsf{G}}(\mathbf{F}^s_{p,q}(\mathbb{R}^n)) = \left(t^{-\frac{1}{p}+\frac{s}{n}}, u_{\mathsf{G}}^{\mathbf{F}^s_{p,q}}\right) \quad \textit{with} \quad \begin{cases} u_{\mathsf{G}}^{\mathbf{F}^s_{p,q}} = p & \textit{if} \ \frac{1}{q} < \frac{s}{n} + \frac{1}{p}, \\ q \le u_{\mathsf{G}}^{\mathbf{F}^s_{p,q}} \le p & \textit{if} \ \frac{1}{q} \ge \frac{s}{n} + \frac{1}{p}. \end{cases}$$

(ii) *Let* $s = \frac{n}{p}$, $1 < p < \infty$, p' *given by* $\frac{1}{p} + \frac{1}{p'} = 1$, *and* $q > \frac{p}{2}$. *Then*

$$\mathfrak{E}_{\mathsf{G}}(\mathbf{F}^{n/p}_{p,q}(\mathbb{R}^n)) = \left(|\log t|^{1/p'}, p\right).$$

(iii) *We have*

$$\mathcal{E}_{\mathsf{G}}^{\mathbf{F}^s_{p,q}}(t) \ \sim \ t^{-\frac{1}{p}} \qquad \textit{for} \quad t \to \infty.$$

Proof. Note that (ii) is simply a consequence of Theorem 2.10(i) and Proposition 3.11(ii).
The results for the growth envelope functions follow immediately from Proposition 2.19(i),

$$\mathbf{B}^s_{p,\min(p,q)} \hookrightarrow \mathbf{F}^s_{p,q} \hookrightarrow \mathbf{B}^s_{p,\max(p,q)}, \tag{3.12}$$

together with Proposition 3.2(i) and previous results for the Besov spaces $\mathbf{B}^s_{p,q}$ as stated in Proposition 3.10(i),(iii).
As for the additional index note that for $q \geq p$ or $q < p$ with $\frac{1}{q} < \frac{s}{n} + \frac{1}{p}$ we have by Theorem 2.10(i)

$$\mathbf{F}^s_{p,q} = \mathfrak{F}^s_{p,q},$$

which gives $u_{\mathsf{G}}^{\mathbf{F}^s_{p,q}} = p$, cf. Theorem 3.11(i). On the other hand, when $q < p$ and $\frac{1}{q} \geq \frac{s}{n} + \frac{1}{p}$, we see from (3.12) that

$$\mathbf{B}^s_{p,q} \hookrightarrow \mathbf{F}^s_{p,q} \hookrightarrow \mathbf{B}^s_{p,p},$$

leading to $q \leq u_{\mathsf{G}}^{\mathbf{F}^s_{p,q}} \leq p$ in terms of Theorem 3.10(i) and Proposition 3.2(iii). □

Remark 3.14. Compared to the other approaches associated to the spaces $F^s_{p,q}$ and $\mathfrak{F}^s_{p,q}$, our results for the spaces $\mathbf{F}^s_{p,q}$ are not as complete. The reason for this is mainly the lacking left-hand side of the Franke-Jawerth embedding, as already mentioned in Remark 2.20. If (2.59) was true in general, we immediately obtained for the additional index that

$$u_{\mathsf{G}}^{\mathbf{F}^s_{p,q}} \geq p, \qquad \text{if } 0 < q < p < \infty, \quad 0 < s < \frac{n}{p},$$

as well as the embedding

$$\mathbf{F}^{s_1}_{p_1,q_1} \hookrightarrow \mathbf{F}^{s_2}_{p_2,q_2}$$

for parameters

$$s_1 - \frac{n}{p_1} = s_2 - \frac{n}{p_2}, \quad s_1 > s_2, \quad 0 < q_1, q_2 \leq \infty.$$

Furthermore we had similar assertions as in Proposition 3.5 for the spaces $\mathbf{F}^s_{p,q}$. But this is not yet verified by our arguments.

4. Applications

We briefly present two typical applications of the preceding envelope results: sharp embedding criteria and Hardy-type inequalities.

4.1 Sharp embeddings.

Corollary 4.1. **(i):** *Let* $s > \sigma > 0$, $0 < p, r < \infty$ *and* $0 < q_1, q_2 \leq \infty$. *Then*

$$\mathfrak{F}^s_{p,q_1}(\mathbb{R}^n) \hookrightarrow \mathfrak{F}^\sigma_{r,q_2}(\mathbb{R}^n)$$

if, and only if,

$$s - \frac{n}{p} \geq \sigma - \frac{n}{r}, \qquad p \leq r.$$

(ii): *Let* $0 < p < \infty$, $0 < s < \frac{n}{p}$, $0 < q, u \leq \infty$ *with* $s - \frac{n}{p} = -\frac{n}{r}$. *Then*

$$\mathfrak{F}^s_{p,q}(\mathbb{R}^n) \hookrightarrow L_{r,u}(\mathbb{R}^n)$$

if, and only if,

$$p \leq u.$$

Proof. **Step 1**. We establish (i) and first assume that

$$s - \frac{n}{p} \geq \sigma - \frac{n}{r}, \qquad p \leq r.$$

But then necessity follows immediately from Theorem 2.16(ii), (iii).
In order to show sufficiency we assume

$$\mathfrak{F}^s_{p,q_1} \hookrightarrow \mathfrak{F}^\sigma_{r,q_2}. \tag{4.1}$$

The global behaviour of the growth envelope functions obtained in Theorem 3.11(iii) together with Proposition 3.2(i) yields

$$t^{-\frac{1}{p}+\frac{1}{r}} \leq c \qquad \text{as} \qquad t \to \infty,$$

implying $p \leq r$. Furthermore, applying the Franke-Jawerth embedding (2.39) on both sides of (4.1) we obtain for $s_1 > s > \sigma > \sigma_1$,

$$s_1 - \frac{n}{p_1} = s - \frac{n}{p}, \qquad \text{and} \qquad \sigma_1 - \frac{n}{r_1} = \sigma - \frac{n}{r}$$

the embedding,

$$\mathfrak{B}^{s_1}_{p_1,u} \hookrightarrow \mathfrak{F}^s_{p,q_1} \hookrightarrow \mathfrak{F}^\sigma_{r,q_2} \hookrightarrow \mathfrak{B}^{\sigma_1}_{r_1,v},$$

where in particular, $u \le p \le r \le v$. But then [12, Cor. 3.1(i)] yields

$$s - \frac{n}{p} = s_1 - \frac{n}{p_1} \ge \sigma_1 - \frac{n}{r_1} = \sigma - \frac{n}{r}.$$

This completes the proof of (i).

Step 2. Now we turn our attention towards (ii). Assume that $p \le u$. Using the embedding obtained in Theorem 2.16(iv) yields

$$\mathfrak{F}^s_{p,q} \hookrightarrow L_{r,p} \hookrightarrow L_{r,u}$$

where the second embedding is well-known for Lorentz spaces and can be found in [2, Prop. 4.2.].
In order to prove sufficiency, again we gain from corresponding results for Besov spaces. In [12] we proved that

$$\mathfrak{B}^s_{p,q} \hookrightarrow L_{r,u}$$

if, and only if, q$\le u$.

But then, using this together with (2.39) yields

$$\mathfrak{F}^s_{p,q} \hookrightarrow \mathfrak{B}^{s_1}_{p_1,p} \hookrightarrow L_{r,u},$$

for s_1, p_1 chosen such that

$$s_1 - \frac{n}{p_1} = s - \frac{n}{p}, \qquad 0 < s_1 < s.$$

This finally shows that $p \le u$. $\square$

In terms of the spaces $\mathbf{F}^s_{p,q}$ the sharp embedding results read as follows.

Corollary 4.2. *Let $s > \sigma > 0$, $0 < p, r < \infty$ and $0 < q_1, q_2 \le \infty$.*

(i): *Let $\left(\frac{\sigma}{n} + \frac{1}{r}\right)^{-1} < q_2 \le \infty$. If*

$$s - \frac{n}{p} \ge \sigma - \frac{n}{r}, \qquad p \le r,$$

then

$$\mathbf{F}^s_{p,q_1}(\mathbb{R}^n) \hookrightarrow \mathbf{F}^\sigma_{r,q_2}(\mathbb{R}^n).$$

(ii): *Let $\left(\frac{s}{n} + \frac{1}{p}\right)^{-1} < q_1 \le \infty$. If*

$$\mathbf{F}^s_{p,q_1}(\mathbb{R}^n) \hookrightarrow \mathbf{F}^\sigma_{r,q_2}(\mathbb{R}^n),$$

then

$$s - \frac{n}{p} \ge \sigma - \frac{n}{r}, \qquad p \le r.$$

Proof. Concerning (i) we use Theorem 2.10(i) and Corollary 4.1, which give

$$\mathbf{F}^s_{p,q_1} \hookrightarrow \mathbf{F}^s_{p,\infty} = \mathfrak{F}^s_{p,\infty} \hookrightarrow \mathfrak{F}^\sigma_{r,q_2} = \mathbf{F}^\sigma_{r,q_2}.$$

The proof of (ii) follows in a similar way from

$$\mathfrak{F}^s_{p,q_1} = \mathbf{F}^s_{p,q_1} \hookrightarrow \mathbf{F}^\sigma_{r,q_2} \hookrightarrow \mathbf{F}^\sigma_{r,\infty} = \mathfrak{F}^\sigma_{r,\infty}$$

and an application of Corollary 4.1. □

Remark 4.3. Observe that (3.7) and Theorem 3.11(i) imply

$$\mathfrak{E}_\mathsf{G}\Big(L_{r,p}\Big) = \Big(t^{-\frac{1}{r}},\ p\Big) = \mathfrak{E}_\mathsf{G}\Big(\mathfrak{F}^s_{p,q}\Big), \quad s - \frac{n}{p} = -\frac{n}{r},$$

where $0 < p < \infty$, $0 < q \leq \infty$, and $s > 0$; that is, we have by (2.42) the embedding $\mathfrak{F}^s_{p,q} \hookrightarrow L_{r,p}$ only, whereas the corresponding envelopes even coincide. This can be interpreted as $L_{r,p}$ being indeed the best possible space within the Lorentz scale in which $\mathfrak{F}^s_{p,q}$ can be embedded continuously. On the other hand this can also be understood in the sense that $L_{r,p}$ is *'as good as'* $\mathfrak{F}^s_{p,q}$ – as far as only the growth of the unbounded functions belonging to the spaces under consideration is concerned, whereas (additional) smoothness features are obviously *'ignored'*.

Corollary 4.4. *Let* $s_0 > s > s_1 > 0$, $0 < p_0 < p < p_1 < \infty$, $0 < u, q, v \leq \infty$ *with*

$$s_0 - \frac{n}{p_0} = s - \frac{n}{p} = s_1 - \frac{n}{p_1}. \tag{4.2}$$

Then

$$\mathfrak{B}^{s_0}_{p_0,u}(\mathbb{R}^n) \hookrightarrow \mathfrak{F}^s_{p,q}(\mathbb{R}^n) \hookrightarrow \mathfrak{B}^{s_1}_{p_1,v}(\mathbb{R}^n) \tag{4.3}$$

if, and only if,

$$0 < u \leq p \leq v \leq \infty. \tag{4.4}$$

Proof. **Step 1**. The necessity, i.e., that (4.4) implies (4.3) is covered by Theorem 2.16(i). It remains to show the converse implication. This is done in two steps: first we use our envelope results for small smoothness parameters, that is, when $0 < s < \frac{n}{p}$; secondly we combine Proposition 2.14(i),(iii) with the identity (2.25) in Theorem 2.10(i).

Step 2. First we assume $0 < s < \frac{n}{p}$. Hence by (4.2) also $0 < s_i < \frac{n}{p_i}$ and all spaces in (4.3) possess non-trivial growth envelopes. Moreover, Proposition 3.10(i) and Theorem 3.11(i) together with (4.2) lead to

$$\mathcal{E}_{\mathsf{G}}^{\mathfrak{B}^{s_0}_{p_0,u}}(t) \sim \mathcal{E}_{\mathsf{G}}^{\mathfrak{F}^{s}_{p,q}}(t) \sim \mathcal{E}_{\mathsf{G}}^{\mathfrak{B}^{s_1}_{p_1,v}}(t) \sim t^{-\frac{1}{p}+\frac{s}{n}}, \qquad 0 < t < 1,$$

such that (4.3) and Proposition 3.2(iii) give

$$u_{\mathsf{G}}^{\mathfrak{B}^{s_0}_{p_0,u}} \leq u_{\mathsf{G}}^{\mathfrak{F}^{s}_{p,q}} \leq u_{\mathsf{G}}^{\mathfrak{B}^{s_1}_{p_1,v}}.$$

In view of Proposition 3.10(i) and Theorem 3.11(i) this is just (4.4).

Step 3. We now assume $s \geq \frac{n}{p}$, hence by (4.2) $s_i \geq \frac{n}{p_i}$, $i = 0, 1$, too. Since $\mathfrak{B}^s_{p,q} = B^s_{p,q}$ whenever $s > \sigma_p$, cf. [22, Prop. 9.14], we immediately obtain

$$\mathfrak{B}^{s_0}_{p_0,u} = B^{s_0}_{p_0,u}, \qquad \text{and} \qquad \mathfrak{B}^{s_1}_{p_1,v} = B^{s_1}_{p_1,v}.$$

One is tempted to use $\mathfrak{F}^s_{p,q} = F^s_{p,q}$ in order to apply Proposition 2.14(i), but this is not always true when $s \geq \frac{n}{p}$, e.g. when $q < \min(p, 1)$, recall Theorem 2.10(i). However, one can circumvent this difficulty in the following way. Let us first deal with the left-hand embedding,

$$\mathfrak{B}^{s_0}_{p_0,u} \hookrightarrow \mathfrak{F}^s_{p,q},$$

which implies

$$B^{s_0}_{p_0,u} = \mathfrak{B}^{s_0}_{p_0,u} \hookrightarrow \mathfrak{F}^s_{p,q} \hookrightarrow \mathfrak{F}^s_{p,\infty} = F^s_{p,\infty}$$

in view of Theorems 2.10(i), 2.16(ii) and [22, Prop. 9.14]. Thus Proposition 2.14(i) yields $u \leq p$ as desired.

As far as the right-hand embedding is concerned, we proceed by contradiction. Let us assume that $v < p$. Hence, there exists some $\varepsilon > 0$ such that $v < p - \varepsilon =: p_\varepsilon < p$ and a number $\sigma_\varepsilon > s$ with

$$\sigma_\varepsilon - \frac{n}{p_\varepsilon} = s - \frac{n}{p}.$$

Using Theorem 2.16(iii), we obtain as a consequence of the right-hand embedding that

$$\mathfrak{B}^{\sigma_\varepsilon}_{p_\varepsilon,p_\varepsilon} = \mathfrak{F}^{\sigma_\varepsilon}_{p_\varepsilon,p_\varepsilon} \hookrightarrow \mathfrak{F}^s_{p,q} \hookrightarrow \mathfrak{B}^{s_1}_{p_1,v}.$$

Finally, we apply [12, Cor. 3.1(i)] to the limiting embedding of $\mathfrak{B}$-spaces leading to $p_\varepsilon \leq v$ in contrast to our assumption. □

4.2 Hardy-type inequalities. Our next application concerns Hardy-type inequalities. This follows immediately from our above results together with the monotonicity (3.3), the properties of f^* and the fact that, given

$\varkappa$ non-negative on $(0,\varepsilon]$,

$$\sup_{0<t\le\varepsilon} \varkappa(t)\, \frac{f^*(t)}{\mathcal{E}^X_{\mathsf{G}}(t)} \;\le\; c$$

holds for some $c>0$ and all $f\in X$, $\|f|X\|\le 1$, if, and only if, $\varkappa$ is bounded, cf. [9, Prop. 3.4(v)].

Corollary 4.5. *Let $s>0$, $0<q\le\infty$, $0<p<\infty$, $\varkappa(t)$ be a positive monotonically decreasing function on $(0,\varepsilon]$ and $0<\nu\le\infty$.*

(i): *Let $s<\frac{n}{p}$. Then*

$$\left(\int_0^\varepsilon \left[\varkappa(t)t^{\frac{1}{p}-\frac{s}{n}}f^*(t)\right]^\nu \frac{\mathrm{d}t}{t}\right)^{\frac{1}{\nu}} \le c\|f|\mathfrak{F}^s_{p,q}(\mathbb{R}^n)\|,$$

for some $c>0$ and all $f\in\mathfrak{F}^s_{p,q}(\mathbb{R}^n)$, if, and only if, $\varkappa$ is bounded and $p\le\nu\le\infty$, with the modification

$$\sup_{t\in(0,\varepsilon)} \varkappa(t)t^{\frac{1}{p}-\frac{s}{n}}f^*(t) \le c\|f|\mathfrak{F}^s_{p,q}(\mathbb{R}^n)\|, \tag{4.5}$$

if $\nu=\infty$. In particular, if $\varkappa$ is an arbitrary non-negative function on $(0,\varepsilon]$, then (4.5) holds if, and only if, $\varkappa$ is bounded.

(ii): *Let $s=\frac{n}{p}$, $1<p<\infty$, and p' given by $\frac{1}{p}+\frac{1}{p'}=1$. Then*

$$\left(\int_0^\varepsilon \left[\varkappa(t)\cdot|\log t|^{-1/p'}f^*(t)\right]^\nu \frac{\mathrm{d}t}{t}\right)^{\frac{1}{\nu}} \le c\|f|\mathfrak{F}^s_{p,q}(\mathbb{R}^n)\|,$$

for some $c>0$ and all $f\in\mathfrak{F}^s_{p,q}(\mathbb{R}^n)$, if, and only if, $\varkappa$ is bounded and $p\le\nu\le\infty$, with the modification

$$\sup_{t\in(0,\varepsilon)} \varkappa(t)\cdot|\log t|^{-1/p'}f^*(t) \le c\|f|\mathfrak{F}^s_{p,q}(\mathbb{R}^n)\|, \tag{4.6}$$

if $\nu=\infty$. In particular, if $\varkappa$ is an arbitrary non-negative function on $(0,\varepsilon]$, then (4.6) holds if, and only if, $\varkappa$ is bounded.

Proof. In view of our preceding remarks this is an immediate consequence of Theorem 3.11 since $\mathfrak{E}_{\mathsf{G}}(\mathfrak{F}^s_{p,q}) = \left(t^{-\frac{1}{p}+\frac{s}{n}},p\right)$. □

Concerning the spaces $\mathbf{F}^s_{p,q}$ the Hardy-type inequalities read as follows.

Corollary 4.6. *Let $s>0$, $0<q\le\infty$, $0<p<\infty$, $\varkappa(t)$ be a positive monotonically decreasing function on $(0,\varepsilon]$ and $0<\nu\le\infty$.*

(i): *Let* $s < \frac{n}{p}$ *and* $\left(\frac{s}{n} + \frac{1}{p}\right)^{-1} < q \le \infty$. *Then*

$$\left(\int_0^\varepsilon \left[\varkappa(t) t^{\frac{1}{p} - \frac{s}{n}} f^*(t) \right]^\nu \frac{\mathrm{d}t}{t} \right)^{\frac{1}{\nu}} \le c \| f | \mathbf{F}^s_{p,q}(\mathbb{R}^n) \|,$$

for some $c > 0$ *and all* $f \in \mathbf{F}^s_{p,q}(\mathbb{R}^n)$, *if, and only if,* $\varkappa$ *is bounded and* $p \le \nu \le \infty$, *with the modification*

$$\sup_{t \in (0,\varepsilon)} \varkappa(t) t^{\frac{1}{p} - \frac{s}{n}} f^*(t) \le c \| f | \mathbf{F}^s_{p,q}(\mathbb{R}^n) \|, \tag{4.7}$$

if $\nu = \infty$. *In particular, if* $\varkappa$ *is an arbitrary non-negative function on* $(0, \varepsilon]$, *then* (4.7) *holds if, and only if,* $\varkappa$ *is bounded.*

(ii): *Let* $s = \frac{n}{p}$, $q > \frac{p}{2}$, $1 < p < \infty$, *and* p' *given by* $\frac{1}{p} + \frac{1}{p'} = 1$. *Then*

$$\left(\int_0^\varepsilon \left[\varkappa(t) \cdot |\log t|^{-1/p'} f^*(t) \right]^\nu \frac{\mathrm{d}t}{t} \right)^{\frac{1}{\nu}} \le c \| f | \mathbf{F}^s_{p,q}(\mathbb{R}^n) \|,$$

for some $c > 0$ *and all* $f \in \mathbf{F}^s_{p,q}(\mathbb{R}^n)$, *if, and only if,* $\varkappa$ *is bounded and* $p \le \nu \le \infty$, *with the modification*

$$\sup_{t \in (0,\varepsilon)} \varkappa(t) \cdot |\log t|^{-1/p'} f^*(t) \le c \| f | \mathbf{F}^s_{p,q}(\mathbb{R}^n) \|, \tag{4.8}$$

if $\nu = \infty$. *In particular, if* $\varkappa$ *is an arbitrary non-negative function on* $(0, \varepsilon]$, *then* (4.8) *holds if, and only if,* $\varkappa$ *is bounded.*

Proof. The proof follows immediately from Theorem 2.10 (i) and Corollary 4.5 above. □

Acknowledgment. The research of the author was supported by DFG Graduiertenkolleg (Research Training Group) 597.

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(*Received : June 2008*)