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On Bellman-Golubov theorems for the Riemann-Liouville operators

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Abstract. Superposition of Fourier transform with the Riemann - Liouville operators is studied.

1. Introduction

Let $\mathbb{R} := (-\infty, +\infty)$. We denote $\|f\|_p := \left(\int_{\mathbb{R}} |f(x)|^p dx\right)^{1/p}$ for $1 \le p < \infty$ and $\|f\|_\infty := \operatorname{esssup}_{x\in\mathbb{R}} |f(x)|$. By $L^p(\mathbb{R})$ we denote the Lebesgue space of all measurable functions on $\mathbb{R}$ such that $\|f\|_p < \infty$. Similar notations are applied for $\mathbb{R}_+ := [0, +\infty)$.

For $f \in L^1(\mathbb{R})$, the Fourier transform Ff is defined by

$$Ff(x) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{ixt} dt.$$

In particular cases, when f is even or odd, the Fourier transforms are

$$F_c f(x) := \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(t) \cos xt dt$$

and

$$iF_s f(x) := i\sqrt{\frac{2}{\pi}} \int_0^\infty f(t) \sin xt dt,$$

respectively. $F_c f$ and $F_s f$ are called the cosine and sine Fourier transforms and may be independently defined for a function on $\mathbb{R}_+$. It is well known [6, Theorem 74] that for $f \in L^p(\mathbb{R}), 1 < p \le 2,$ there exist $Ff \in L^{p'}(\mathbb{R})$ such that $\lim_{a\to\infty} \|Ff - Ff_a\|_{p'} = 0,$ where $p' = \frac{p}{p-1}$ and

$$Ff_a(x) := \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} e^{ixt} f(t) dt, \ \ a > 0.$$

Moreover, the Planscherel-Titchmarsh inequality

$$\|Ff\|_{p'} \le C(p) \|f\|_p \tag{1}$$

holds. The similar results are valid for the sine and cosine Fourier transforms.

R. Bellman [1] stated and B. I. Golubov [3] proved the following equalities:

$$PF_c f = F_c Q f, \tag{2}$$

if $f \in L^p(\mathbb{R}), 1 \le p \le 2$ and

$$QF_c f = F_c P f, \tag{3}$$

if $f \in L^p(\mathbb{R}), 1 < p \le 2,$ where

$$Pf(x) := \frac{1}{x} \int_0^x f(s) ds$$

and

$$Qf(x) := \int_x^\infty \frac{f(s)}{s} ds$$

are the Hardy operators.

The aim of the paper is to prove the equalities similar to (2) and (3), where the Hardy operators P and Q are replaced by the Riemann-Liouville operators.

By C, we denote constants, which may be different in different occurences.

2. Main results

Let $\alpha > 0$. The Riemann-Liouville operators are defined for a function on the semiaxis $\mathbb{R}_+$ as follows:

$$B_\alpha f(x) := \frac{1}{x^\alpha} \int_0^x (x-t)^{\alpha-1} f(t) dt$$

and

$$H_\alpha f(x) := \int_x^\infty \frac{(t-x)^{\alpha-1}}{t^\alpha} f(t) dt.$$

It is known [4, Theorem 329] that

$$\|B_\alpha f\|_p \le C_{\alpha,p} \|f\|_p \tag{4}$$

for $1 < p \le \infty$ and

$$\|H_\alpha f\|_p \le C_{\alpha,p} \|f\|_p \tag{5}$$

for $1 \le p < \infty$.

Theorem 1. *Let $1 < p \le 2$ and $\alpha > 1/p'$ and suppose that $f \in L^p(\mathbb{R}_+)$. Then*

$$B_\alpha[F_c f](x) = F_c[H_\alpha f](x), \quad \text{a.e } x \in \mathbb{R}_+ \tag{6}$$

and

$$B_\alpha[F_s f](x) = F_s[H_\alpha f](x), \quad \text{a.e } x \in \mathbb{R}_+. \tag{7}$$

Proof. We start with the proof of (6). Let $1 < p \le 2$ and $f \in L^p(\mathbb{R}_+)$. Then $F_c f \in L^{p'}(\mathbb{R}_+)$, where $p' \in [2, \infty)$. Applying (4) we find $B_\alpha[F_c f] \in L^{p'}(\mathbb{R}_+)$. Let $a > 0$ and $f_a(x) = f\chi_{(0,a)}(x)$, where $\chi_{(0,a)}(x)$ is the characteristic function (indicator) of an interval $(0, a)$. Then

$$F_c(f_a)(x) = \sqrt{\frac{2}{\pi}} \int_0^a f(t) \cos xt dt.$$

First we show that if $\alpha > 1/p'$, then

$$B_\alpha[F_c f](x) = \lim_{a\to\infty} B_\alpha[F_c(f_a)](x) \quad \text{for all } x \in \mathbb{R}_+. \tag{8}$$

Indeed, by Hölder's and Planscherel-Titchmarsh's inequalities

$$|B_\alpha[F_c f](x) - B_\alpha[F_c(f_a)](x)| \le \frac{1}{x^\alpha} \int_0^x (x-y)^{\alpha-1} |F_c f(y) - F_c(f_a)(y)| dy$$

$$\leq Cx^{-1/p'}\|F_c f - F_c(f_a)\|_{p'} \to 0, \quad a \to \infty$$

and (8) follows. Next we show that

$$B_\alpha[F_c(f_a)](x) = F_c[H_\alpha(f_a)](x). \tag{9}$$

By the Fubini theorem and by change of variables we have

$$\begin{aligned} B_\alpha[F_c(f_a)](x) &= \frac{1}{x^\alpha}\int_0^x (x-y)^{\alpha-1}F_c(f_a)(y)dy \\ &= \sqrt{\frac{2}{\pi}}\frac{1}{x^\alpha}\int_0^x (x-y)^{\alpha-1}dy\int_0^a f(t)\cos ytdt \\ &= \sqrt{\frac{2}{\pi}}\frac{1}{x^\alpha}\int_0^a f(t)dt\int_0^x (x-y)^{\alpha-1}\cos ytdy = \{yt = xu\} \\ &= \sqrt{\frac{2}{\pi}}\int_0^a \frac{f(t)}{t^\alpha}dt\int_0^t (t-u)^{\alpha-1}\cos xudu. \end{aligned}$$

On the other hand

$$\begin{aligned} F_c[H_\alpha(f_a)](x) &= \sqrt{\frac{2}{\pi}}\int_0^\infty H_\alpha(f_a)(y)\cos xydy \\ &= \sqrt{\frac{2}{\pi}}\int_0^\infty \cos xydy\int_y^\infty \frac{(t-y)^{\alpha-1}}{t^\alpha}f_a(t)dt \\ &= \sqrt{\frac{2}{\pi}}\int_0^a \frac{f(t)}{t^\alpha}dt\int_0^t (t-y)^{\alpha-1}\cos xydy \end{aligned}$$

and (9) follows. Now, since $f \in L^p(\mathbb{R}_+)$, $p \in (1,2]$, then by (5) we have $H_\alpha f \in L^p(\mathbb{R}_+)$ and $F_c[H_\alpha f](x) \in L^{p'}(\mathbb{R}_+)$. We show that

$$\lim_{a\to\infty}\|F_c[H_\alpha f] - F_c[H_\alpha(f_a)]\|_{p'} = 0. \tag{10}$$

Write

$$\begin{aligned} F_c[H_\alpha(f_a)](x) &= \sqrt{\frac{2}{\pi}}\int_0^a \cos xydy\int_y^a \frac{(t-y)^{\alpha-1}}{t^\alpha}f(t)dt \\ &= F_c[H_\alpha f]_a(x) - \sqrt{\frac{2}{\pi}}\int_0^a \cos xydy\int_a^\infty \frac{(t-y)^{\alpha-1}}{t^\alpha}f(t)dt. \end{aligned}$$

Applying Minkowskii's inequality

$$\begin{aligned}\|F_c[H_\alpha f]-F_c[H_\alpha(f_a)]\|_{p'} &\le \|F_c[H_\alpha f]-F_c[H_\alpha f]_a\|_{p'}\\ &\quad+\left(\int_0^\infty\left|\int_0^a \cos xydy\int_a^\infty \frac{(t-y)^{\alpha-1}}{t^\alpha}f(t)dt\right|^{p'}dx\right)^{1/p'}.\end{aligned}$$

By Planscherel-Titchmarsh's inequality

$$\begin{aligned}\|F_c[H_\alpha f]-F_c[H_\alpha f]_a\|_{p'} &\le C(p)\|H_\alpha f-[H_\alpha f]_a\|_p\\ &= C(p)\left(\int_a^\infty |H_\alpha f(x)|^p dx\right)^{1/p}\to 0,\quad a\to\infty\end{aligned}$$

and (10) follows if

$$\lim_{a\to\infty}\int_0^\infty\left|\int_0^a \cos xydy\int_a^\infty \frac{(t-y)^{\alpha-1}}{t^\alpha}f(t)dt\right|^{p'}dx=0. \tag{11}$$

By Planscherel-Titchmarsh's and Hölder's inequalities we find

$$\begin{aligned}&\left(\int_0^\infty\left|\int_0^a \cos xydy\int_a^\infty \frac{(t-y)^{\alpha-1}}{t^\alpha}f(t)dt\right|^{p'}dx\right)^{1/p'}\\ &\qquad\le C\left(\int_0^a\left|\int_a^\infty\left(1-\frac{y}{t}\right)^{(\alpha-1)}\frac{f(t)}{t}dt\right|^p dy\right)^{1/p}\\ &\qquad\le C\left(\int_0^a\left(1-\frac{y}{a}\right)^{p(\alpha-1)}dy\right)^{1/p}\left|\int_a^\infty\frac{f(t)}{t}dt\right|\\ &\qquad= C_1a^{1/p}\left|\int_a^\infty\frac{f(t)}{t}dt\right|\\ &\qquad\le C_2\left(\int_a^\infty|f(t)|^pdt\right)^{1/p}\to 0,\quad a\to\infty\end{aligned}$$

since $0<1-\frac{y}{a}\le 1-\frac{y}{t}\le 1,\quad 0\le y\le a\le t<\infty$, and (11) is proved.

Observe that (10) implies the existence of a subsequence $\{a_k\}, a_k\to\infty$ such that

$$F_c[H_\alpha f](x)=\lim_{k\to\infty}F_c[H_\alpha(f_{a_k})](x)\quad \text{a.e } x\in\mathbb{R}_+.$$

Now (6) follows from this, (8) and (9). The proof of (7) is analogous. □

Theorem 2. *Let* $f\in L^p(\mathbb{R}_+)$, *where* $1<p\le 2$ *and let* $\alpha>1/p'$. *Then*

$$H_\alpha[F_cf](x)=F_c[B_\alpha f](x),\quad \text{a.e. } x\in\mathbb{R}_+ \tag{12}$$

and

$$H_\alpha[F_s f](x) = F_s[B_\alpha f](x), \quad \text{a.e. } x \in \mathbb{R}_+. \tag{13}$$

Proof. Applying Planscherel-Titchmarsh's inequality and (4) we have $H_\alpha[F_c f](x) \in L^{p'}(\mathbb{R}_+)$. For $A > x > 0$ by Fubini's theorem we write

$$\int_x^A \frac{(y-x)^{\alpha-1}}{y^\alpha} dy \int_0^a f(t)\cos yt dt = \int_0^a f(t) dt \int_x^A \frac{(y-x)^{\alpha-1}}{y^\alpha} \cos yt dy. \tag{14}$$

Let us show that if $a \to +\infty$, then the equality

$$\int_x^A \frac{(y-x)^{\alpha-1}}{y^\alpha} F_c(f)(y) dy = \sqrt{\frac{2}{\pi}} \int_0^\infty f(t) dt \int_x^A \frac{(y-x)^{\alpha-1}}{y^\alpha} \cos yt dy \tag{15}$$

holds. Indeed, for the left hand side of (14) we have

$$\sqrt{\frac{2}{\pi}} \int_x^A \frac{(y-x)^{\alpha-1}}{y^\alpha} dy \int_0^a f(t)\cos yt dt = \int_x^A \frac{(y-x)^{\alpha-1}}{y^\alpha} F_c(f_a)(y) dy$$

and by Lebesgue's theorem on dominated convergence

$$\lim_{a\to\infty} \int_x^A \frac{(y-x)^{\alpha-1}}{y^\alpha} F_c(f_a)(y) dy = \int_x^A \frac{(y-x)^{\alpha-1}}{y^\alpha} F_c f(y) dy.$$

It implies that the right hand side of (14) is convergent for $a \to \infty$ and (15) holds with the first integral on the right in the Riemann sense. Now, by Hölder's inequality

$$\begin{aligned} \int_x^\infty \left| \frac{(y-x)^{\alpha-1}}{y^\alpha} F_c f(y) dy \right| &\le \|F_c f\|_{p'} \left(\int_x^\infty \frac{(y-x)^{(\alpha-1)p}}{y^{\alpha p}} dy \right)^{1/p} \\ &= C \|F_c f\|_{p'} x^{-\frac{1}{p'}} < \infty. \end{aligned}$$

Therefore, by Lebesgue's theorem on dominated convergence there exist a finite limit of the left hand side of (15)

$$\lim_{A\to+\infty} \int_x^A \frac{(y-x)^{\alpha-1}}{y^\alpha} F_c f(y) dy = \int_x^\infty \frac{(y-x)^{\alpha-1}}{y^\alpha} F_c f(y) dy.$$

Hence,

$$H_\alpha[F_c f](x) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(t) dt \int_x^\infty \frac{(y-x)^{\alpha-1}}{y^\alpha} \cos yt dy \tag{16}$$

for all $x \in \mathbb{R}_+$ with both the integrals on the right in the Riemann sense. To justify (16) with both the integrals on the right in the Lebesgue sense, we consider the function from the right hand side of (15)

$$h_{x,A}(t) := \int_x^A \frac{(y-x)^{\alpha-1}}{y^\alpha} \cos ytdy,$$

when $t \to +0$ and $t \to +\infty$. If $t \to +0$, then

$$\begin{aligned} h_{x,A}(t) &= \int_{xt}^{At} \frac{(z-xt)^{\alpha-1}}{z^\alpha} \cos zdz \\ &= \int_u^{Au/x} \frac{(z-u)^{\alpha-1}}{z^\alpha} \cos zdz \\ &= \int_0^{(A/x-1)u} \frac{s^{\alpha-1}}{(s+u)^\alpha} \cos(s+u)ds := W(u) \end{aligned}$$

To estimate $W(u)$, we shall use the asymptotic formulae from [2, Chapter 1, Section 4]. Let $\theta \in \mathbb{R}$, $\beta > 0$, $\varphi(t) \in C[0,a]$ and let $\theta + \beta = N$, where N is a non-negative integer. Then

$$\begin{aligned} &\int_0^a t^{\beta-1}(t+\varepsilon)^\theta \varphi(t,\varepsilon)dt \\ &\quad \sim \sum_{n\geq \max[0,-N]}^{N} \binom{n+N}{\theta} \frac{\partial^n \varphi(t,\varepsilon)}{\partial t^n}\bigg|_{t=0} \varepsilon^{n+N} \ln(1/\varepsilon) + \sum_{n=0}^{\infty} b_n \varepsilon^n, \end{aligned} \tag{17}$$

for $\varepsilon \to 0$, if $\varepsilon \in S_\delta := \{0 < |\varepsilon| \leq r, |\arg \varepsilon| \leq \pi - \delta]\} \subset \mathbb{C}$, b_n are constants and $\varphi(t,\varepsilon) \in C^\infty([0,a] \times [\varepsilon : |\varepsilon| < r])$.

If we take $\varphi(s,u) := \cos(s+u)$, $\beta := \alpha$, $\theta := -\alpha$, so that $\theta + \beta = N = 0$, then by (17), we find

$$W(u) \sim \cos u \ln(1/u), \qquad u \to +0.$$

Hence,

$$h_{x,A}(t) = O(\ln(1/xt)), \qquad t \to +0. \tag{18}$$

For the case $t \to +\infty$ we write

$$\begin{aligned} h_{x,A}(t) &= \int_x^A \frac{(y-x)^{\alpha-1}}{y^\alpha} \cos ytdy = \int_0^{A-x} \frac{y^{\alpha-1}}{(y+x)^\alpha} \cos t(y+x)dy \\ &= \Re e^{ixt} \int_0^{A-x} g(y)d\left(-\int_y^\infty u^{\alpha-1} e^{iut}du\right) =: \Re e^{ixt}\Phi_\alpha(t), \end{aligned}$$

where $g(y)=(y+x)^{-\alpha}$. Integrating by parts, we obtain

$$\Phi_\alpha(t)=g(0)\int_0^\infty u^{\alpha-1}e^{iut}du-g(A-x)\int_{A-x}^\infty u^{\alpha-1}e^{iut}du$$

$$+\int_0^{A-x}\left(\int_y^\infty u^{\alpha-1}e^{iut}du\right)g'(y)dy.$$

Put $u-y=\rho e^{i\frac{\pi}{2}}$ and $u=re^{i\theta}$, then $\rho\le r, 0\le\theta\le\frac{\pi}{2}$. If $|u|\ge y$, then $|u|^{\alpha-1}\le y^{\alpha-1}$, so that

$$\left|\int_y^\infty u^{\alpha-1}e^{iut}du\right|\le y^{\alpha-1}\int_0^\infty e^{-t\rho}d\rho=\Gamma(1)y^{\alpha-1}t^{-1}.$$

Hence,

$$\int_0^{A-x}\left(\int_y^\infty u^{\alpha-1}e^{iut}du\right)g'(y)dy=O(t^{-1}).$$

Analogously,

$$\int_{A-x}^\infty u^{\alpha-1}e^{iut}du=O(t^{-1})$$

and also it is known that

$$\int_0^\infty u^{\alpha-1}e^{iut}du=e^{-\pi i\alpha/2}t^{-\alpha}\Gamma(\alpha).$$

Therefore,

$$h_{x,A}(t)=O(t^{-\alpha}),\quad t\to+\infty. \tag{19}$$

Thus, it follows from (18) and (19) that there exist a function

$$G(t)=|f(t)|\{\chi_{[0,1/2]}(t)|\ln(1/xt)|+\chi_{[1/2,\infty)}(t)t^{-\alpha}\}\in L^1(0,\infty)$$

such that

$$|f(t)h_{x,A}(t)|\le G(t).$$

By Lebesgue's theorem on dominated convergence

$$\int_0^\infty f(t)dt\int_x^\infty\frac{(y-x)^{\alpha-1}}{y^\alpha}\cos yt dy=\lim_{A\to\infty}\int_0^\infty f(t)h_{x,A}(t)dt.$$

It implies

$$H_\alpha[F_cf](x)=\sqrt{\frac{2}{\pi}}\int_0^\infty f(t)h_{x,\infty}(t)dt. \tag{20}$$

By change of variables we find

$$h_{x,\infty}(t) = \int_x^\infty \frac{(y-x)^{\alpha-1}}{y^\alpha} \cos yt dy = \{yt \to v\}$$

$$= \int_{xt}^\infty \frac{(v-xt)^{\alpha-1}}{v^\alpha} \cos v dv = \{v \to xy\}$$

$$= \int_t^\infty \frac{(y-t)^{\alpha-1}}{y^\alpha} \cos xy dy = h_{t,\infty}(x).$$

It follows from this and (20) that

$$H_\alpha[F_c f](x) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(t)dt \int_t^\infty \frac{(y-t)^{\alpha-1}}{y^\alpha} \cos xy dy.$$

Now we show that

$$H_\alpha[F_c f](x) = \lim_{a\to\infty} \sqrt{\frac{2}{\pi}} \int_0^a f(t)dt \int_t^a \frac{(y-t)^{\alpha-1}}{y^\alpha} \cos yx dy. \tag{21}$$

We have

$$\left|\int_a^\infty \frac{(y-t)^{\alpha-1}}{y^\alpha} \cos xy dy\right|$$
$$= \frac{1}{x}\left|\frac{(y-t)^{\alpha-1}}{y^\alpha} \sin xy\Big|_a^\infty - \int_a^\infty \sin xy\, d\left(\frac{(y-t)^{\alpha-1}}{y^\alpha}\right)\right|$$
$$\le \frac{(a-t)^{\alpha-1}}{a^\alpha x} + \frac{1}{x}\int_a^\infty \left|d\left(\frac{(y-t)^{\alpha-1}}{y^\alpha}\right)\right| \le \frac{2(a-t)^{\alpha-1}}{a^\alpha x},$$

then

$$\left|\int_0^a f(t)dt \int_t^\infty \frac{(y-t)^{\alpha-1}}{y^\alpha} \cos yx dy - \int_0^a f(t)dt \int_t^a \frac{(y-t)^{\alpha-1}}{y^\alpha} \cos yx dy\right|$$

$$= \left|\int_0^a f(t)dt \int_a^\infty \frac{(y-t)^{\alpha-1}}{y^\alpha} \cos yx dy\right| \le \int_0^a \frac{2(a-t)^{\alpha-1}}{a^\alpha x} |f(t)| dt$$

and by Hölder's inequality

$$\lim_{a\to\infty} \int_0^a \frac{2(a-t)^{\alpha-1}}{a^\alpha x} |f(t)| dt \le \lim_{a\to\infty} 2C(\alpha,p) \frac{a^{\alpha-1/p}}{a^\alpha x} \|f\|_{L^p} = 0.$$

Again applying Hölder's inequality

$$\left|\int_0^\infty f(t)dt \int_t^\infty \frac{(y-t)^{\alpha-1}}{y^\alpha} \cos yx dy - \int_0^a f(t)dt \int_t^\infty \frac{(y-t)^{\alpha-1}}{y^\alpha} \cos yx dy\right|$$

$$= \left|\int_a^\infty f(t)h_{x,A}(t)dt\right| \le C\left(\int_a^\infty |f|^p dt\right)^{1/p}\left(\int_a^\infty t^{-\alpha p'}dt\right)^{1/p'} \to 0,\ \ a\to\infty$$

and (21) follows. Changing the order of integrals on the right hand side of (21), we find

$$H_\alpha[F_c f](x) = \lim_{a\to\infty} F_c[B_\alpha f]_a(x) \tag{22}$$

Since

$$\lim_{a\to\infty} \|F_c[B_\alpha f] - F_c[B_\alpha f]_a\|_{p'} = 0, \tag{23}$$

then there exist a sequence a_k such that $a_k\to\infty,\ k\to\infty$ and

$$F_c[B_\alpha f] = \lim_{k\to\infty} F_c[B_\alpha f]_{a_k}(x),\ \ a.e.\ x\in\mathbb{R}_+ \tag{24}$$

and (12) follows from (22), (23) and (24). The proof of (13) is analogous. □

For the case $p=1$ we have the following analog of Theorems 1 and 3.

Theorem 3. *If $f\in L^1(\mathbb{R}_+)$ and $\alpha>0$, then for all $x\in\mathbb{R}_+$ the equalities*

$$B_\alpha[F_c f](x) = F_c[H_\alpha f](x),\ \ B_\alpha[F_s f](x) = F_s[H_\alpha f](x) \tag{25}$$

hold.

Proof. By Fubini theorem we write

$$\begin{aligned} B_\alpha[F_c f](x) &= \frac{1}{x^\alpha}\int_0^x (x-y)^{\alpha-1}F_c f(y)dy \\ &= \sqrt{\frac{2}{\pi}}\frac{1}{x^\alpha}\int_0^x (x-y)^{\alpha-1}\left\{\int_0^\infty f(t)\cos yt dt\right\}dy\ . \\ &= \sqrt{\frac{2}{\pi}}\frac{1}{x^\alpha}\int_0^\infty f(t)\left\{\int_0^x (x-y)^{\alpha-1}\cos yt dy\right\}dt, \end{aligned}$$

Changing $ty=xu$, we have $x-y=\frac{xt-xu}{t}$, $dy=\frac{x}{t}du$ and

$$B_\alpha[F_c f](x) = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{f(t)}{t^\alpha}\left\{\int_0^t (t-u)^{\alpha-1}\cos xu du\right\}dt.$$

On the other hand again by Fubini's theorem

$$\begin{aligned} F_c[H_\alpha f](x) &= \sqrt{\frac{2}{\pi}}\int_0^\infty H_\alpha f(y)\cos xy dy \\ &= \sqrt{\frac{2}{\pi}}\int_0^\infty \cos xy \left\{\int_y^\infty \frac{(t-y)^{\alpha-1}}{t^\alpha} f(t)dt\right\} dy \\ &= \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{f(t)}{t^\alpha}\left\{\int_0^t (t-y)^{\alpha-1}\cos xy dy\right\} dt \end{aligned}$$

and for all $x > 0$ the equality

$$B_\alpha[F_c f](x) = F_c[H_\alpha f](x)$$

follows. If $x = 0$ then

$$F_c[H_\alpha f](0) = \sqrt{\frac{2}{\pi}}\frac{1}{\alpha}\int_0^\infty f(t)dt = \frac{1}{\alpha}F_c f(0)$$

and for a continuous f at $x = 0$ we have

$$B_\alpha f(0) := \lim_{x\to 0} B_\alpha f(x) = \lim_{x\to 0}\frac{1}{x^\alpha}\int_0^x (x-t)^{\alpha-1} f(t)dt = \frac{1}{\alpha} f(0).$$

Therefore, if $f \in L^1(\mathbb{R}_+)$, then $F_c f(x)$ is continuous and

$$B_\alpha[F_c f](0) := \lim_{x\to 0} B_\alpha[F_c f](x) = \frac{1}{\alpha}F_c f(0).$$

□

Now we extend the operators B_α and H_α on $\mathbb{R}$ as follows.

$$B_\alpha f(x) := \begin{cases} \dfrac{1}{x^\alpha}\displaystyle\int_0^x (x-t)^{\alpha-1} f(t)dt, & x > 0 \\ \dfrac{1}{|x|^\alpha}\displaystyle\int_x^0 (|x|-|t|)^{\alpha-1} f(t)dt, & x < 0 \end{cases}$$

and

$$H_\alpha f(x) := \begin{cases} \displaystyle\int_x^\infty \frac{(t-x)^{\alpha-1}}{t^\alpha} f(t)dt, & x > 0 \\ \displaystyle\int_{-\infty}^x \frac{(|t|-|x|)^{\alpha-1}}{|t|^\alpha} f(t)dt, & x < 0. \end{cases}$$

It is easy to see, that for even or odd functions $f(x)$, the images $B_\alpha f(x)$ and $H_\alpha f(x)$ are even or odd too. As a consequence of Theorems 1 and 2, we obtain the following result.

Theorem 4. If $1 < p \le 2$ and $\alpha > 1/p'$, then

$$B_\alpha[Ff](x) = F[H_\alpha f](x), \quad \text{a.e } x \in \mathbb{R}$$

and

$$H_\alpha[Ff](x) = F[B_\alpha f](x), \quad \text{a.e } x \in \mathbb{R}$$

for any $f \in L^p(\mathbb{R})$.

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