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Distances from Bloch functions to some Möbius invariant function spaces in the unit ball of $\mathbb{C}^n$

Wen Xu

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Abstract. Distance formulae from Bloch functions to some Möbius invariant function spaces in the unit ball of $\mathbb{C}^n$ such as Q_s spaces, little Bloch space $\mathcal{B}_0$ and Besov spaces B_p are given.

1. Introduction

Let B be the unit ball of $\mathbb{C}^n$ with boundary S, let $d\nu$ denote the Lebesgue measure on B such that $\nu(B) = 1$ and let $d\sigma$ be the rotation invariant positive normalized measure on S, i.e. $\sigma(S) = 1$. Let $d\lambda(z) = (1 - |z|^2)^{-(n+1)} d\nu(z)$. Then $d\lambda$ is Möbius invariant. For $\alpha > -1$, the weighted Lebesgue measure $d\nu_\alpha$ is defined by

$$d\nu_\alpha(z) = c_\alpha (1 - |z|^2)^\alpha d\nu(z),$$

where

$$c_\alpha = \frac{\Gamma(n + \alpha + 1)}{n!\Gamma(\alpha + 1)}$$

is a normalizing constant such that $d\nu_\alpha$ is a probability measure on B, i.e. $\nu_\alpha(B)=1$.

Let $H(B)$ be the class of all holomorphic functions in the unit ball B, and let $\mathrm{Aut}(B)$ be the group of biholomorphic automorphisms of B. For $a\in B$, let $\varphi_a\in\mathrm{Aut}(B)$ denote the Möbius transformation of B which satisfies $\varphi_a(0)=a$, $\varphi_a(a)=0$ and $\varphi_a\circ\varphi_a=I$. Further, $1-|\varphi_a(z)|^2=\frac{(1-|a|^2)(1-|z|^2)}{|1-\langle z,a\rangle|^2}$ for all $z\in B$, where $\langle z,a\rangle$ is the usual inner product on $\mathbb{C}^n$.

For $f\in H(B)$, $\nabla f=(\frac{\partial f}{\partial z_1},\cdots,\frac{\partial f}{\partial z_n})$ is called its complex gradient, $\mathcal{R}f(z)=\sum\limits_j z_j\frac{\partial f}{\partial z_j}(z)$ is called its radial derivative, and $\widetilde{\nabla}f(z)=\nabla(f\circ\varphi_z)(0)$ is called its Möbius invariant gradient. It is known that $(1-|z|^2)|Rf(z)|\le(1-|z|^2)|\nabla f(z)|\le|\widetilde{\nabla}f(z)|$ (cf. [8, Lemma 2.14]).

The invariant Green's function $G(z,a)$ of the unit ball B is defined by $G(z,a)=g(\varphi_a(z))$ (cf. [5]), where

$$g(z)=\frac{n+1}{2n}\int_{|z|}^{1}(1-t^2)^{n-1}t^{-2n+1}dt.$$

In [3], the holomorphic function spaces Q_s associated with the Green's function are introduced and studied. For $s>0$, Q_s is defined by

$$Q_s=\left\{f\in H(B):\sup_{a\in B}\int_B|\widetilde{\nabla}f(z)|^2G(z,a)^s d\lambda(z)<\infty\right\},$$

and its subspace $Q_{s,0}$ is defined by

$$Q_{s,0}=\left\{f\in H(B):\lim_{|a|\to 1}\int_B|\widetilde{\nabla}f(z)|^2G(z,a)^s d\lambda(z)=0\right\}.$$

The Bloch space on B, denoted by $\mathcal{B}$, is the class of all functions $f\in H(B)$, which satisfy

$$\|f\|_1=\sup_{z\in B}(1-|z|^2)|\mathcal{R}f(z)|<\infty.$$

The little Bloch space $\mathcal{B}_0$ on B is a subspace of $\mathcal{B}$, and $f\in\mathcal{B}_0$ if and only if

$$\lim_{|z|\to 1}(1-|z|^2)|\mathcal{R}f(z)|=0.$$

Let

$$\|f\|_2=\sup_{z\in B}(1-|z|^2)|\nabla f(z)|,\quad \|f\|_3=\sup_{z\in B}|\widetilde{\nabla}f(z)|.$$

It is known that the above three semi-norms are equivalent (cf. [8]). Define

$$\|f\|_{\mathcal{B}} = |f(0)| + ||f||_1 = |f(0)| + \sup_{z\in B}(1-|z|^2)|\mathcal{R}f(z)|.$$

Then $\mathcal{B}$ is a Banach space with the norm $\|\cdot\|_{\mathcal{B}}$.

For $\alpha > -1$ and $p > 0$, the weighted Bergman space A^p_α consists of holomorphic functions f satisfying

$$\|f\|_{p,\alpha} = \left(\int_B |f(z)|^p d\nu_\alpha(z)\right)^{\frac{1}{p}} < \infty.$$

For a multi-index $m = (m_1, m_2, \cdots, m_n)$ with non-negative integers, we will employ the notation

$$\frac{\partial^m f}{\partial z^m} = \frac{\partial^{|m|} f}{\partial z_1^{m_1}\cdots \partial z_n^{m_n}},$$

where $|m| = m_1 + m_2 + \cdots + m_n$. For $0 < p < \infty$, the Besov space B_p (see [8]) is the class of holomorphic functions f in B such that the functions

$$(1-|z|^2)^N \frac{\partial^m f}{\partial z^m}(z), \quad |m| = N$$

all belong to $L^p(B, d\lambda)$, where N is any integer satisfying $pN > n$.

For two real parameters α and t with the property that neither $n+\alpha$ nor $n+\alpha+t$ is a negative integer, if $f(z) = \sum\limits_{k=0}^{\infty} f_k(z)$ is the homogeneous expansion of f, an invertible operator $\mathcal{R}^{\alpha,t} : H(B) \to H(B)$ is defined in [8] by

$$\mathcal{R}^{\alpha,t} f(z) = \sum_{k=0}^{\infty} \frac{\Gamma(n+1+\alpha)\Gamma(n+1+\alpha+k+t)}{\Gamma(n+1+\alpha+t)\Gamma(n+1+\alpha+k)} f_k(z).$$

For $\zeta \in S$ and $\delta > 0$, let $B(\zeta,\delta) = \{z \in B : |1-\langle z,\zeta\rangle| < \delta\}$. For a positive Borel measure μ on B, if

$$\sup\left\{\frac{\mu(B(\zeta,\delta))}{\delta^{np}} : \zeta \in S, \delta > 0\right\} < \infty,$$

we call μ a p-Carleson measure; if

$$\lim_{\delta\to 0} \frac{\mu(B(\zeta,\delta))}{\delta^{np}} = 0$$

for $\zeta \in S$ uniformly, we call μ a vanishing p-Carleson measure.

The purpose of this paper is to drive distance formulae from Bloch functions to some Möbius invariant function spaces on the unit ball which generalize the results of [7]. R. Zhao, in [7], takes advantage of the second inequality in Lemma 2.5 [2] while we use the third inequality. In fact, the second inequality is not feasible in our multidimensional case.

Throughout this paper, C denotes a positive constant and not necessarily the same at each occurrence.

2. Some Lemmas

We need the following lemmas.

Lemma 1 ([2]). *Let $s>-1$. If $r,t>s+n+1$, then for all $a,w\in B$, we have*

$$\int_B \frac{(1-|z|^2)^s d\nu(z)}{|1-\langle z,w\rangle|^r|1-\langle z,a\rangle|^t} \le \frac{C}{(1-|w|^2)^{r-s-n-1}|1-\langle a,w\rangle|^t}+\frac{C}{(1-|a|^2)^{t-s-n-1}|1-\langle a,w\rangle|^r}.$$

Lemma 2 ([8, Theorem 1.12]). *Suppose c is real and $t>-1$. Then the integrals*

$$I_c(z)=\int_S \frac{d\sigma(\zeta)}{|1-\langle z,\zeta\rangle|^{n+c}},\quad z\in B,$$

and

$$J_{c,t}(z)=\int_B \frac{(1-|w|^2)^t d\nu(w)}{|1-\langle z,w\rangle|^{n+1+t+c}},\quad z\in B,$$

have the following asymptotic properties:

(1) *If $c<0$, then I_c and $J_{c,t}(z)$ are both bounded in B;*
(2) *If $c=0$, then*

$$I_c(z)\sim J_{c,t}(z)\sim \log\frac{1}{1-|z|^2}\quad \text{as } |z|\to 1;$$

(3) *If $c>0$, then*

$$I_c(z)\sim J_{c,t}(z)\sim (1-|z|^2)^{-c}\quad \text{as } |z|\to 1.$$

Lemma 3 ([1,6]). *μ is an s-Carleson measure if and only if*

$$\sup_{a\in B}\int_B \left(\frac{1-|a|^2}{|1-\langle w,a\rangle|^2}\right)^{ns} d\mu(w)<\infty;$$

and μ is a vanishing s-Carleson measure if and only if

$$\lim_{|a|\to 1}\int_B \left(\frac{1-|a|^2}{|1-\langle w,a\rangle|^2}\right)^{ns} d\mu(w)=0.$$

Lemma 4 ([1, Theorem 1,2])**.** *Suppose $n>1$ and $\frac{n-1}{n}<s\le 1$, then $f\in Q_s$ if and only if $|\mathcal{R}f(z)|^2(1-|z|^2)^{ns+2}d\lambda(z)$ is an s-Carleson measure; $f\in Q_{s,0}$ if and only if $|\mathcal{R}f(z)|^2(1-|z|^2)^{ns+2}d\lambda(z)$ is a vanishing s-Carleson measure.*

Lemma 5 ([4, Proposition 5.1.2])**.** *The triangle inequality*

$$|1-\langle a,c\rangle|^{\frac12}\le|1-\langle a,b\rangle|^{\frac12}+|1-\langle b,c\rangle|^{\frac12}$$

holds for all $a,b,c\in\overline{B}$.

Lemma 6. *If $s\ge 0$ and $t-ns\ge 0$, and μ is a vanishing s-Carleson measure on B, then*

$$\lim_{|a|\to 1}\int_B \frac{(1-|a|^2)(1-|z|^2)^{t-ns}d\mu(z)}{|1-\langle z,a\rangle|^{t+1}}=0.$$

Proof. Because a vanishing s-Carleson measure must be an s-Carleson measure, we have $\mu(B(\zeta,r))\le Ar^{ns}$, $A>0$, for $\zeta\in S$ and $r>0$. Meanwhile, for $\varepsilon>0$, there exists an $r_0\in(0,1)$ such that $\mu(B(\zeta,r))\le\varepsilon r^{ns}$ for $\zeta\in S$ and $0<r<r_0$. For $j=1,2,\cdots$, $\zeta\in S$ and $r>0$, denote $E_{j,\zeta}(r)=B(\zeta,2^{j+1}r)\backslash B(\zeta,2^jr)$.

Let $a\in B$, $r=1-|a|<r_0/4$ and $\zeta=a/|a|$. Then,

$$|1-\langle a,\zeta\rangle|=1-|a|=r. \tag{2.1}$$

There exists a positive integer m such that $2^{m+1}r<r_0\le 2^{m+2}r$. It is obvious that $m\ge 1$ and $m\to\infty$ as $|a|\to 1$. Then,

$$\mu(B(\zeta,2r))\le\varepsilon(2r)^{ns}, \tag{2.2}$$

(2.3)
$$\mu(E_{j,\zeta}(r))\le\varepsilon(2^{j+1}r)^{ns}\quad\text{if } j\le m,\quad \mu(E_{j,\zeta}(r))\le A(2^{j+1}r)^{ns}\quad\text{if } j>m,$$

$$(1-|z|^2)\le 2(1-|z|)\le 2|1-\langle z,\zeta\rangle|<4r\quad\text{if } z\in B(\zeta,2r), \tag{2.4}$$

$$(1-|z|^2)\le 2(1-|z|)\le 2|1-\langle z,\zeta\rangle|<2^{j+2}r\quad\text{if } z\in E_{j,\zeta}(r), \tag{2.5}$$

and, by Lemma 5, the definition of $E_{j,\zeta}(r)$ and (2.1),
(2.6)
$$|1-\langle z,a\rangle|^{\frac{1}{2}} \geq |1-\langle z,\zeta\rangle|^{\frac{1}{2}}-|1-\langle a,\zeta\rangle|^{\frac{1}{2}} \geq (2^{\frac{j}{2}}-1)r^{\frac{1}{2}} \geq 2^{\frac{j-4}{2}}r^{\frac{1}{2}} \quad \text{if } z \in E_{j,\zeta}(r).$$

Thus, by (2.2)– (2.6),

$$\begin{aligned}
&\int_B \frac{(1-|a|^2)(1-|z|^2)^{t-ns}}{|1-\langle z,a\rangle|^{t+1}}\,d\mu(z) \\
&\quad = \left(\int_{B(\zeta,2r)} + \sum_{j=1}^{m}\int_{E_{j,\zeta}(r)} + \sum_{j=m+1}^{\infty}\int_{E_{j,\zeta}(r)}\right) \frac{(1-|a|^2)(1-|z|^2)^{t-ns}}{|1-\langle z,a\rangle|^{t+1}}\,d\mu(z) \\
&\quad \leq \frac{2r(4r))^{t-ns}(2r)^{ns}\varepsilon}{r^{t+1}} + \sum_{j=1}^{m}\frac{2r(2^{j+2}r)^{t-ns}(2^{j+1}r)^{ns}\varepsilon}{(2^{j-4}r)^{t+1}} \\
&\qquad\qquad + \sum_{j=m+1}^{\infty}\frac{2r(2^{j+2}r)^{t-ns}(2^{j+1}r)^{ns}A}{(2^{j-4}r)^{t+1}} \\
&\quad \leq C\left(\varepsilon + A\sum_{j=m+1}^{\infty}\frac{1}{2^j}\right) = C\left(\varepsilon + \frac{A}{2^m}\right).
\end{aligned}$$

Since $m \to \infty$ as $|a| \to 1$, we have

$$\int_B \frac{(1-|a|^2)(1-|z|^2)^{t-ns}}{|1-\langle z,a\rangle|^{t+1}}\,d\mu(z) < C\varepsilon$$

if $|a|$ is close to 1 sufficiently. This completes the proof. □

Lemma 7 ([8, Theorem 2.2])**.** *If $\alpha > -1$ and $f \in A^1_\alpha$, then*

$$f(z) = \int_B \frac{f(w)d\nu_\alpha(w)}{(1-\langle z,w\rangle)^{n+1+\alpha}}$$

for all $z \in B$.

Lemma 8 ([8, Lemma 6.3])**.** *Suppose $0 < p < \infty$, $n+\alpha$ is not a negative integer, N is a positive integer satisfying $Np > n$, and f is holomorphic in B. Then $f \in B_p$ if and only if the function*

$$F_N(z) = (1-|z|^2)^N \mathcal{R}^{\alpha,N} f(z)$$

belongs to $L^p(B, d\lambda)$.

3. Results and Proofs

Let A be a subspace of $\mathcal{B}$ and $f \in \mathcal{B}$. We denote the distance in $\mathcal{B}$ of f to A by $d_{\mathcal{B}}(f, A)$. For $f \in H(B)$ and $\varepsilon > 0$, let $\Omega_\varepsilon(f) = \{z \in B : |\mathcal{R}f(z)|(1-|z|^2) \geq \varepsilon\}$ and let $\chi_{\Omega_\varepsilon(f)}$ be the characteristic function of the set $\Omega_\varepsilon(f)$. In these notations, our main result is formulated as follows.

Theorem 1. *Let $n \geq 2$, $\frac{n-1}{n} < s \leq 1$, $0 \leq t < \infty$, and $f \in \mathcal{B}$. Then, the following quantities are equivalent:*

(i) $d_1 = d_{\mathcal{B}}(f, Q_s)$;

(ii) $d_2 = \inf\{\varepsilon : \chi_{\Omega_\varepsilon(f)}(z)(1-|z|^2)^{ns} d\lambda(z)$ *is an s-Carleson measure*$\}$;

(iii) $d_3 = \inf\{\varepsilon : \sup\limits_{a \in B} \int_{\Omega_\varepsilon(f)} |\mathcal{R}f(z)|^t (1-|z|^2)^t (1-|\varphi_a(z)|^2)^{ns} d\lambda(z) < \infty\}$;

(iv) $d_4 = \inf\{\varepsilon : \sup\limits_{a \in B} \int_{\Omega_\varepsilon(f)} |\mathcal{R}f(z)|^t (1-|z|^2)^t G(z,a)^s d\lambda(z) < \infty\}$;

(v) $d_5 = \inf\{\varepsilon : \sup\limits_{a \in B} \int_{\Omega_\varepsilon(f)} |\nabla f(z)|^t (1-|z|^2)^t G(z,a)^s d\lambda(z) < \infty\}$;

(vi) $d_6 = \inf\{\varepsilon : \sup\limits_{a \in B} \int_{\Omega_\varepsilon(f)} |\widetilde{\nabla} f(z)|^t G(z,a)^s d\lambda(z) < \infty\}$;

(vii) $d_7 = \inf\{\varepsilon : \sup\limits_{a \in B} \int_{\Omega_\varepsilon(f)} |\nabla f(z)|^t (1-|z|^2)^t (1-|\varphi_a(z)|^2)^{ns} d\lambda(z) < \infty\}$;

(viii) $d_8 = \inf\{\varepsilon : \sup\limits_{a \in B} \int_{\Omega_\varepsilon(f)} |\widetilde{\nabla} f(z)|^t (1-|\varphi_a(z)|^2)^{ns} d\lambda(z) < \infty\}$.

Proof. First we will prove that $d_1 \leq C d_2$. Let ε be a positive number such that $\chi_{\Omega_\varepsilon(f)}(z)(1-|z|^2)^{ns} d\lambda(z)$ is an s-Carleson measure. Since $f \in \mathcal{B}$, it is easy to see that $\mathcal{R}f(z) \in A^1_\alpha$ for any $\alpha > 0$. Let $\alpha > 0$ be given. According to Lemma 7, we have

$$\mathcal{R}f(z) = \int_B \frac{\mathcal{R}f(w) d\nu_\alpha(w)}{(1-\langle z, w\rangle)^{n+1+\alpha}}, \quad z \in B.$$

Since $\mathcal{R}f(0) = 0$, we have

$$\mathcal{R}f(z) = \int_B \mathcal{R}f(w) \left(\frac{1}{(1-\langle z, w\rangle)^{n+1+\alpha}} - 1 \right) d\nu_\alpha(w), \quad z \in B.$$

It follows that

$$f(z) - f(0) = \int_0^1 \frac{\mathcal{R}f(tz)}{t} dt = \int_B \mathcal{R}f(w) L(z,w) d\nu_\alpha(w),$$

where the kernel

$$L(z,w)=\int_0^1\left(\frac{1}{(1-t\langle z,w\rangle)^{n+1+\alpha}}-1\right)\frac{dt}{t}.$$

Let $f(z)=f_1(z)+f_2(z)$, where

$$f_1(z)=f(0)+\int_{\Omega_\varepsilon(f)}\mathcal{R}f(w)L(z,w)d\nu_\alpha(w)$$

and

$$f_2(z)=\int_{B\setminus\Omega_\varepsilon(f)}\mathcal{R}f(w)L(z,w)d\nu_\alpha(w).$$

Since

$$\mathcal{R}L(z,w)=\int_0^1\frac{(n+1+\alpha)\langle z,w\rangle dt}{(1-t\langle z,w\rangle)^{n+1+\alpha+1}}=\frac{1}{(1-\langle z,w\rangle)^{n+1+\alpha}}-1,$$

we have

$$\begin{aligned}|\mathcal{R}f_1(z)| &= \left|\int_{\Omega_\varepsilon(f)}\mathcal{R}f(w)\mathcal{R}L(z,w)d\nu_\alpha(w)\right|\\ &\le \int_{\Omega_\varepsilon(f)}|\mathcal{R}f(w)|\left(\frac{1}{|1-\langle z,w\rangle|^{n+1+\alpha}}+1\right)d\nu_\alpha(w)\\ &\le C\left(\int_B\frac{(1-|w|^2)^{\alpha-1}d\nu(w)}{|1-\langle z,w\rangle|^{n+1+\alpha}}+1\right),\end{aligned}$$

and

$$\begin{aligned}|\mathcal{R}f_2(z)| &= \left|\int_{B\setminus\Omega_\varepsilon(f)}\mathcal{R}f(w)\mathcal{R}L(z,w)d\nu_\alpha(w)\right|\\ &\le C\varepsilon\left(\int_B\frac{(1-|w|^2)^{\alpha-1}d\nu(w)}{|1-\langle z,w\rangle|^{n+1+\alpha}}+1\right).\end{aligned}$$

Thus, using Lemma 2 with $c=1$ we get

$$|\mathcal{R}f_1(z)|\le\frac{C}{1-|z|^2},\quad |\mathcal{R}f_2(z)|\le\frac{C\varepsilon}{1-|z|^2},$$

and consequently, $f_1\in\mathcal{B}$ and $||f_2||_{\mathcal{B}}=||f_2||_1\le C\varepsilon$, since $f_2(0)=0$. Further we want to prove that $f_1\in Q_s$.

We have

$$\begin{aligned}
I(f_1,a) &= \int_B |\mathcal{R}f_1(z)|^2(1-|z|^2)^2(1-|\varphi_a(z)|^2)^{ns}d\lambda(z)\\
&\le ||f_1||_1\int_B |\mathcal{R}f_1(z)|(1-|z|^2)(1-|\varphi_a(z)|^2)^{ns}d\lambda(z)\\
&\le ||f_1||_1\int_B\left(\int_{\Omega_\varepsilon(f)}|\mathcal{R}f(w)|\left(\frac{1}{|1-\langle z,w\rangle|^{n+1+\alpha}}+1\right)d\nu_\alpha(w)\right)\\
&\qquad\qquad \times(1-|z|^2)(1-|\varphi_a(z)|^2)^{ns}d\lambda(z)\\
&\le ||f_1||_1||f||_1\int_{\Omega_\varepsilon(f)}d\nu_{\alpha-1}(w)\int_B\frac{(1-|z|^2)^{ns-n}(1-|a|^2)^{ns}d\nu(z)}{|1-\langle z,w\rangle|^{n+1+\alpha}|1-\langle z,a\rangle|^{2ns}}\\
&\qquad +||f_1||_1||f||_{1,\alpha}\int_B\frac{(1-|a|^2)^{ns}(1-|z|^2)^{ns-n}d\nu(z)}{|1-\langle z,a\rangle|^{2ns}}\\
&= ||f_1||_1||f||_1 I_1+||f_1||_1||f||_{1,\alpha}I_2.
\end{aligned}$$

Since $\alpha>0$, $n\ge 2$ and $(n-1)/n<s\le 1$, we have $ns-n>-1$, $ns-1>0$, $n+1+\alpha>ns-n+n+1$ and $2ns>ns-n+n+1$. Thus, by Lemma 2 and Lemma 1,

$$I_2=\int_B\frac{(1-|a|^2)^{ns}(1-|z|^2)^{ns-n}d\nu(z)}{|1-\langle z,a\rangle|^{n+1+(ns-n)+ns-1}}\le C(1-|a|^2),$$

and

$$\begin{aligned}
I_1 &\le C\int_{\Omega_\varepsilon(f)}\frac{(1-|a|^2)^{ns}(1-|w|^2)^{ns}d\lambda(w)}{|1-\langle a,w\rangle|^{2ns}}\\
&\qquad +C(1-|a|^2)\int_{\Omega_\varepsilon(f)}\frac{(1-|w|^2)^{\alpha-1}d\nu(w)}{|1-\langle a,w\rangle|^{n+1+\alpha}}.
\end{aligned}$$

Because $\chi_{\Omega_\varepsilon(f)}(w)(1-|w|^2)^{ns}d\lambda(w)$ is an s-Carleson measure, by Lemma 3, we have

$$\sup_{a\in B}\int_{\Omega_\varepsilon(f)}\left(\frac{1-|a|^2}{|1-\langle a,w\rangle|^2}\right)^{ns}(1-|w|^2)^{ns}d\lambda(w)<\infty.$$

Using Lemma 2 we get

$$(1-|a|^2)\int_{\Omega_\varepsilon(f)}\frac{(1-|w|^2)^{\alpha-1}d\nu(w)}{|1-\langle a,w\rangle|^{n+1+\alpha}}\le C.$$

So $\sup_{a\in B}I(f_1,a)<\infty$ and, by Lemma 3, $|\mathcal{R}f_1(z)|^2(1-|z|^2)^{ns+2}d\lambda(z)$ is an s-Carleson measure. Consequently, by Lemma 4, $f_1\in Q_s$. We have proved above that $||f_2||_{\mathcal{B}} \leq C\varepsilon$. Thus, $d_{\mathcal{B}}(f, Q_s) \leq ||f - f_1||_{\mathcal{B}} = ||f_2||_{\mathcal{B}} \leq C\varepsilon$. This shows that $d_1 \leq Cd_2$.

By Lemma 3, $\chi_{\Omega_\varepsilon(f)}(z)(1-|z|^2)^{ns}d\lambda(z)$ is an s-Carleson measure if and only if

$$\sup_{a\in B}\int_{\Omega_\varepsilon(f)}(1-|\varphi_a(z)|^2)^{ns}d\lambda(z) = \sup_{a\in B}\int_{\Omega_\varepsilon(f)}\frac{(1-|a|^2)^{ns}(1-|z|^2)^{ns}}{|1-\langle z,a\rangle|^{2ns}}d\lambda(z) < \infty.$$

Since

$$\varepsilon \leq |\mathcal{R}f(z)|(1-|z|^2) \leq \|f\|_{\mathcal{B}}, \quad z \in \Omega_\varepsilon(f),$$

we obtain $d_2 = d_3$. It is obvious that $d_3 \leq d_4$, since $(1 - |\varphi_a(z)|^2)^n \leq CG(z,a)$. It follows from $(1-|z|^2)|\mathcal{R}f(z)| \leq (1-|z|^2)|\nabla f(z)| \leq |\widetilde{\nabla} f(z)|$ that $d_4 \leq d_5 \leq d_6$.

Now, we are going to prove that $d_6 \leq d_1$. For $\varepsilon > d_1$, there exists a function $f_\varepsilon \in Q_s$ such that $\|f - f_\varepsilon\|_{\mathcal{B}} < (d_1+\varepsilon)/2$. Then,

$$\begin{aligned}
|\widetilde{\nabla} f_\varepsilon(z)| &\geq |\mathcal{R}f_\varepsilon(z)|(1-|z|^2)\\
&\geq |\mathcal{R}f(z)|(1-|z|^2) - |\mathcal{R}(f-f_\varepsilon)(z)|(1-|z|^2)\\
&\geq \varepsilon - (d_1+\varepsilon)/2 = (\varepsilon - d_1)/2, \quad z \in \Omega_\varepsilon(f).
\end{aligned}$$

Thus,

$$\begin{aligned}
&\sup_{a\in B}\int_{\Omega_\varepsilon(f)}|\widetilde{\nabla} f(z)|^t G(z,a)^s d\lambda(z)\\
&\qquad\leq \|f\|_3^t \sup_{a\in B}\int_{\Omega_\varepsilon(f)} G(z,a)^s d\lambda(z)\\
&\qquad\leq \frac{4\|f\|_3^t}{(\varepsilon-d_1)^2}\sup_{a\in B}\int_{\Omega_\varepsilon(f)}|\widetilde{\nabla} f_\varepsilon(z)|^2 G(z,a)^s d\lambda(z) < \infty.
\end{aligned}$$

This shows that $d_6 \leq \varepsilon$. Since ε may be close to d_1 arbitrarily, we obtain $d_6 \leq d_1$.

We have proved that $d_1, d_2, \cdots, d_6$ are equivalent. It follows from

$$(1-|z|^2)|\mathcal{R}f(z)| \leq (1-|z|^2)|\nabla f(z)| \leq |\widetilde{\nabla} f(z)|$$

and

$$(1-|\varphi_a(z)|^2)^n \leq CG(z,a)$$

that $d_3 \leq d_7 \leq d_8 \leq d_6$. Thus, all of $d_1, d_2, \cdots, d_8$ are equivalent. The proof is complete. □

Corollary 1. *Let $n \geq 2$, $\frac{n-1}{n} < s \leq 1$ and $0 \leq t < \infty$. Let $f \in \mathcal{B}$. Then the following conditions are equivalent:*

(i) *f is in the closure of Q_s in $\mathcal{B}$;*

(ii) *$\chi_{\Omega_\varepsilon(f)}(1-|z|^2)^{ns}d\lambda(z)$ is an s-Carleson measure for every $\varepsilon>0$;*

(iii) $\sup\limits_{a\in B}\int_{\Omega_\varepsilon(f)}|\mathcal{R}f(z)|^t(1-|z|^2)^t(1-|\varphi_a(z)|^2)^{ns}d\lambda(z)<\infty$ *for every* $\varepsilon>0$*;*

(iv) $\sup\limits_{a\in B}\int_{\Omega_\varepsilon(f)}|\mathcal{R}f(z)|^t(1-|z|^2)^tG(z,a)^sd\lambda(z)<\infty$ *for every* $\varepsilon>0$*;*

(v) $\sup\limits_{a\in B}\int_{\Omega_\varepsilon(f)}|\nabla f(z)|^t(1-|z|^2)^tG(z,a)^sd\lambda(z)<\infty$ *for every* $\varepsilon>0$*;*

(vi) $\sup\limits_{a\in B}\int_{\Omega_\varepsilon(f)}|\widetilde{\nabla}f(z)|^tG(z,a)^sd\lambda(z)<\infty$ *for every* $\varepsilon>0$*;*

(vii) $\sup\limits_{a\in B}\int_{\Omega_\varepsilon(f)}|\nabla f(z)|^t(1-|z|^2)^t(1-|\varphi_a(z)|^2)^{ns}d\lambda(z)<\infty$ *for every* $\varepsilon>0$*;*

(viii) $\sup\limits_{a\in B}\int_{\Omega_\varepsilon(f)}|\widetilde{\nabla}f(z)|^t(1-|\varphi_a(z)|^2)^{ns}d\lambda(z)<\infty$ *for every* $\varepsilon>0$.

In the little o-spaces $\mathcal{B}_0$ and $Q_{s,0}$ we have the following results.

Theorem 2. *Let $n\geq 2$, $\frac{n-1}{n}<s\leq 1$, $0\leq t<\infty$, and $f\in\mathcal{B}$. Then the following quantities are equivalent:*

(i) $d_1'=d_{\mathcal{B}}(f,\mathcal{B}_0)$;

(ii) $d_1=d_{\mathcal{B}}(f,Q_{s,0})$;

(iii) $d_2=\inf\{\varepsilon:\chi_{\Omega_\varepsilon(f)}(1-|z|^2)^{ns}d\lambda(z)$ *is a vanishing s-Carleson measure*$\}$;

(iv) $d_3=\inf\{\varepsilon:\lim\limits_{|a|\to 1}\int_{\Omega_\varepsilon(f)}|\mathcal{R}f(z)|^t(1-|z|^2)^t(1-|\varphi_a(z)|^2)^{ns}d\lambda(z)=0\}$;

(v) $d_4=\inf\{\varepsilon:\lim\limits_{|a|\to 1}\int_{\Omega_\varepsilon(f)}|\mathcal{R}f(z)|^t(1-|z|^2)^tG(z,a)^sd\lambda(z)=0\}$;

(vi) $d_5=\inf\{\varepsilon:\lim\limits_{|a|\to 1}\int_{\Omega_\varepsilon(f)}|\nabla f(z)|^t(1-|z|^2)^tG(z,a)^sd\lambda(z)=0\}$;

(vii) $d_6=\inf\{\varepsilon:\lim\limits_{|a|\to 1}\int_{\Omega_\varepsilon(f)}|\widetilde{\nabla}f(z)|^tG(z,a)^sd\lambda(z)=0\}$;

(viii) $d_7=\inf\{\varepsilon:\lim\limits_{|a|\to 1}\int_{\Omega_\varepsilon(f)}|\nabla f(z)|^t(1-|z|^2)^t(1-|\varphi_a(z)|^2)^{ns}d\lambda(z)=0\}$;

(ix) $d_8=\inf\{\varepsilon:\lim\limits_{|a|\to 1}\int_{\Omega_\varepsilon(f)}|\widetilde{\nabla}f(z)|^t(1-|\varphi_a(z)|^2)^{ns}d\lambda(z)=0\}$.

Proof. It is known that the closure of polynomials in $\mathcal{B}$ is just $\mathcal{B}_0$ (cf. [8]), and $Q_{s,0}$ contains all polynomials. So, $\mathcal{B}_0\subset\overline{Q_{s,0}}$, where $\overline{Q_{s,0}}$ is the closure of $Q_{s,0}$ with respect to $\mathcal{B}$. On the other hand, $Q_{s,0}\subset\mathcal{B}_0$, and so $\overline{Q_{s,0}}\subset\mathcal{B}_0$. Thus $\overline{Q_{s,0}}=\mathcal{B}_0$, which implies that d_1' and d_1 are equivalent.

We can show the equivalence of $d_1,d_2,\cdots,d_8$ in a similar way as in the proof of Theorem 1. The only point we have to explain is the proof of

$d_1 \le Cd_2$. Note that

$$I(f_1,a) \le C\int_B \frac{(1-|a|^2)^{ns}}{|1-\langle a,w\rangle|^{2ns}}\chi_{\Omega_\varepsilon(f)}(w)(1-|w|^2)^{ns}d\lambda(w)$$
$$+\,C\int_B \frac{(1-|a|^2)(1-|w|^2)^{\alpha+n-ns}}{|1-\langle a,w\rangle|^{n+\alpha+1}}\chi_{\Omega_\varepsilon(f)}(w)(1-|w|^2)^{ns}d\lambda(w) + C(1-|a|^2).$$

Now, since $\chi_{\Omega_\varepsilon(f)}(1-|w|^2)^{ns}d\lambda(w)$ is a vanishing s-Carleson measure, by Lemma 3 and Lemma 6, we see that $I(f_1,a)\to 0$ as $|a|\to 1$. Therefore, by Lemma 3 and Lemma 4, $f_1 \in Q_{s,0}$. The proof is complete. □

Corollary 2. *Let $n\ge 2$, $\frac{n-1}{n} < s \le 1$ and let $f\in\mathcal{B}$. Then $f\in\mathcal{B}_0$ if and only if $\chi_{\Omega_\varepsilon(f)}(1-|z|^2)^{ns}d\lambda(z)$ is a vanishing s-Carleson measure for every $\varepsilon>0$.*

Finally, we will discuss the distance formulae from Bloch functions to Besov spaces on the unit ball.

Theorem 3. *Let $1\le p<\infty$ and $f\in\mathcal{B}$. Then the following quantities are equivalent:*

(i) $d_1 = d_{\mathcal{B}}(f,\mathcal{B}_0)$;

(ii) $d_2 = d_{\mathcal{B}}(f,B_p)$;

(iii) $d_3 = \inf\{\varepsilon : \lambda(\Omega_\varepsilon(f)) < \infty\}$, *where* $\lambda(\Omega_\varepsilon(f)) = \int_{\Omega_\varepsilon(f)} d\nu(z)/(1-|z|^2)^{n+1}$.

Proof. Because $B_p \subset \mathcal{B}_0$, in a similar way to the proof of Theorem 2, it is easy to get that d_1 is equivalent to d_2.

In order to prove $d_2 \le Cd_3$, let $f_1(z)$, $f_2(z)$ be the same as in the proof of Theorem 1. What we need to do is to show $f_1\in B_p$ for $1\le p<\infty$. We have

$$f_1(z) = f(0) + \int_{\Omega_\varepsilon(f)} \mathcal{R}f(w)L(z,w)d\nu_\alpha(w).$$

Then

$$\mathcal{R}^{\alpha,n+1}f_1(z) = \int_{\Omega_\varepsilon(f)} \mathcal{R}f(w)\mathcal{R}^{\alpha,n+1}L(z,w)d\nu_\alpha(w),$$

where

$$\mathcal{R}^{\alpha,n+1}L(z,w) = \int_0^1 \left(\frac{1}{(1-t\langle z,w\rangle)^{n+1+\alpha+n+1}} - 1\right)\frac{dt}{t}$$

satisfying

$$|\mathcal{R}^{\alpha,n+1}L(z,w)| \le \frac{C}{|1-\langle z,w\rangle|^{n+1+\alpha+n}}$$

for all z and w in B.

Thus we have

$$\begin{aligned}
&\int_B (1-|z|^2)^{n+1}|\mathcal{R}^{\alpha,n+1}f_1(z)|d\lambda(z)\\
&\le \int_B\int_{\Omega_\varepsilon(f)}|\mathcal{R}f(w)||\mathcal{R}^{\alpha,n+1}L(z,w)|d\nu_\alpha(w)d\nu(z)\\
&\le C\int_{\Omega_\varepsilon(f)}d\nu_{\alpha-1}(w)\int_B\frac{C}{|1-\langle z,w\rangle|^{n+1+\alpha+n}}d\nu(z)\\
&\le C\int_{\Omega_\varepsilon(f)}\frac{d\nu_{\alpha-1}(w)}{(1-|w|^2)^{n+\alpha}}\\
&= C\int_{\Omega_\varepsilon(f)}\frac{d\nu(w)}{(1-|w|^2)^{n+1}}.
\end{aligned}$$

Using Lemma 8 with $p=1$ and $N=n+1$ we get $f_1\in B_1\subset B_p$ if $\lambda(\Omega_\varepsilon(f))<\infty$.

The last matter is to prove that $d_3\le d_2$. Since $B_{p_1}\subset B_{p_2}$, where $1\le p_1<p_2<\infty$, we may assume $n<p<\infty$. For $\varepsilon>d_2$, there exists a function $f_\varepsilon\in B_p$ such that $\|f-f_\varepsilon\|_{\mathcal{B}}<(d_2+\varepsilon)/2$. Then

$$\begin{aligned}
|\mathcal{R}f_\varepsilon(z)|(1-|z|^2) &\ge |\mathcal{R}f(z)|(1-|z|^2)-|\mathcal{R}(f-f_\varepsilon)(z)|(1-|z|^2)\\
&\ge \varepsilon-(d_2+\varepsilon)/2=(\varepsilon-d_2)/2,\quad z\in\Omega_\varepsilon(f).
\end{aligned}$$

By [8, Exercise 6.8] and $f_\varepsilon\in B_p$,

$$\int_B (1-|z|^2)^p|\mathcal{R}f_\varepsilon(z)|^p d\lambda(z)<\infty,\quad n<p<\infty.$$

Thus

$$\begin{aligned}
\lambda(\Omega_\varepsilon(f)) &= \int_{\Omega_\varepsilon(f)}\frac{d\nu(z)}{(1-|z|^2)^{n+1}}\\
&\le \frac{2^p}{(\varepsilon-d_2)^p}\int_{\Omega_\varepsilon(f)}(1-|z|^2)^p|\mathcal{R}f_\varepsilon(z)|^p d\lambda(z)<\infty,
\end{aligned}$$

which shows that $d_3\le\varepsilon$. Since ε may be close to d_2 arbitrarily, we have $d_3\le d_2$. The proof is complete. □

Corollary 3. *Let $f\in\mathcal{B}$. Then $f\in\mathcal{B}_0$ if and only if $\lambda(\Omega_\varepsilon(f))<\infty$ for every $\varepsilon>0$.*

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Department of Physics and Mathematics
University of Joensuu
P. O. Box 111
FI-80101 Joensuu
Finland
(*E-mail : wxu@cc.joensuu.fi*)

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