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Research Article

On Subclasses of Uniformly Spiral-like Functions Associated with Generalized Bessel Functions

B. A. Frasin [1] and Ibtisam Aldawish [2]

[1] *Faculty of Science, Department of Mathematics, Al al-Bayt University, Mafraq, Jordan*
[2] *Department of Mathematics and Statistics, College of Science, IMSIU (Imam Mohammed Ibn Saud Islamic University), P.O. Box 90950, Riyadh 11623, Saudi Arabia*

Correspondence should be addressed to B. A. Frasin; bafrasin@yahoo.com

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The main object of this paper is to find necessary and sufficient conditions for generalized Bessel functions of first kind $zu_p(z)$ to be in the classes $\mathcal{SP}_p(\alpha,\beta)$ and $\mathcal{UCSP}(\alpha,\beta)$ of uniformly spiral-like functions and also give necessary and sufficient conditions for $z(2-u_p(z))$ to be in the above classes. Furthermore, we give necessary and sufficient conditions for $\mathcal{I}(\kappa,c)f$ to be in $\mathcal{UCSPT}(\alpha,\beta)$ provided that the function f is in the class $\mathcal{R}^{\tau}(A,B)$. Finally, we give conditions for the integral operator $\mathcal{G}(\kappa,c,z)=\int_0^z(2-u_p(t))dt$ to be in the class $\mathcal{UCSPT}(\alpha,\beta)$. Several corollaries and consequences of the main results are also considered.

1. Introduction and Definitions

Let $\mathcal{A}$ denote the class of the normalized functions of the form

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n, \tag{1}$$

which are analytic in the open unit disk $\mathbb{U}=\{z\in\mathbb{C}:|z|<1\}$. Further, let $\mathcal{T}$ be a subclass of $\mathcal{A}$ consisting of functions of the form,

$$f(z)=z-\sum_{n=2}^{\infty}|a_n|z^n,\quad z\in\mathbb{U}. \tag{2}$$

A function $f\in\mathcal{A}$ is spiral-like if

$$\Re\left(e^{-i\alpha}\frac{zf'(z)}{f(z)}\right)>0, \tag{3}$$

for some α with $|\alpha|<\pi/2$ and for all $z\in\mathbb{U}$. Also $f(z)$ is convex spiral-like if $zf'(z)$ is spiral-like.

In [1], Selvaraj and Geetha introduced the following subclasses of uniformly spiral-like and convex spiral-like functions.

Definition 1. A function f of the form (1) is said to be in the class $\mathcal{SP}_p(\alpha,\beta)$ if it satisfies the following condition:

$$\Re\left\{e^{-i\alpha}\left(\frac{zf'(z)}{f(z)}\right)\right\}>\left|\frac{zf'(z)}{f'(z)}-1\right|+\beta \qquad \left(z\in\mathbb{U};|\alpha|<\frac{\pi}{2};0\le\beta<1\right) \tag{4}$$

and $f\in\mathcal{UCSP}(\alpha,\beta)$ if and only if $zf'(z)\in\mathcal{SP}_p(\alpha,\beta)$.

We write

$$\mathcal{SP}_p\mathcal{T}(\alpha,\beta)=\mathcal{SP}_p(\alpha,\beta)\cap\mathcal{T},$$
$$\mathcal{UCSPT}(\alpha,\beta)=\mathcal{UCSP}(\alpha,\beta)\cap\mathcal{T}. \tag{5}$$

In particular, we note that $\mathcal{SP}_p(\alpha,0)=\mathcal{SP}_p(\alpha)$ and $\mathcal{UCSP}(\alpha,0)=\mathcal{UCSP}(\alpha)$, the classes of uniformly spiral-like and uniformly convex spiral-like were introduced by Ravichandran et al. [2]. For $\alpha=0$, the classes $\mathcal{UCSP}(\alpha)$ and $\mathcal{SP}_p(\alpha)$, respectively, reduce to the classes $\mathcal{UCV}$ and $\mathcal{SP}$ introduced and studied by Ronning [3].

For more interesting developments of some related subclasses of uniformly spiral-like and uniformly convex spiral-like, the readers may be referred to the works of Frasin [4, 5], Goodman [6, 7], Tariq Al-Hawary and Frasin [8], Kanas and Wisniowska [9, 10] and Ronning [3, 11].

A function $f \in \mathscr{A}$ is said to be in the class $\mathscr{R}^{\tau}(A,B)$, $\tau \in \mathbb{C} \setminus \{0\}$, $-1 \leq B < A \leq 1$, if it satisfies the inequality

$$\left| \frac{f'(z) - 1}{(A-B)\tau - B[f'(z) - 1]} \right| < 1, \quad z \in \mathbb{U}. \tag{6}$$

This class was introduced by Dixit and Pal [12].

The generalized Bessel function w_p (see, [13]) is defined as a particular solution of the linear differential equation

$$zw''(z) + bzw'(z) + \left[cz^2 - p^2 + (1-b)p\right] w(z) = 0, \tag{7}$$

where $b, p, c \in \mathbb{C}$. The analytic function w_p has the form

$$w_p(z) = \sum_{n=0}^{\infty} \frac{(-1)^n (c)^n}{n! \Gamma(p + n + (b+1)/2)} \cdot \left(\frac{z}{2}\right)^{2n+p}, \quad z \in \mathbb{C}. \tag{8}$$

Now, the generalized and normalized Bessel function u_p is defined with the transformation

$$\begin{aligned} u_p(z) &= 2^p \Gamma\left(p + n + \frac{b+1}{2}\right) z^{-p/2} w_p\left(z^{1/2}\right) \\ &= \sum_{n=0}^{\infty} \frac{(-c/4)^n}{(\kappa)_n n!} z^n, \end{aligned} \tag{9}$$

where $\kappa = p + (b+1)/2 \neq 0, -1, -2, \ldots$ and $(a)_n$ is the well-known Pochhammer (or Appell) symbol, defined in terms of the Euler Gamma function for $a \neq 0, -1, -2, \ldots$ by

$$\begin{aligned} (a)_n &= \frac{\Gamma(a+n)}{\Gamma(a)} \\ &= \begin{cases} 1, & \text{if } n = 0 \\ a(a+1)(a+2)\ldots(a+n-1), & \text{if } n \in \mathbb{N}. \end{cases} \end{aligned} \tag{10}$$

The function u_p is analytic on $\mathbb{C}$ and satisfies the second-order linear differential equation

$$4z^2 u''(z) + 2(2p + b + 1) z u'(z) + czu(z) = 0. \tag{11}$$

Using the Hadamard product, we now considered a linear operator $\mathscr{I}(\kappa, c) : \mathscr{A} \longrightarrow \mathscr{A}$ defined by

$$\begin{aligned} \mathscr{I}(\kappa, c) f &= zu_p(z) * f(z) \\ &= z + \sum_{n=2}^{\infty} \frac{(-c/4)^{n-1}}{(\kappa)_{n-1}(n-1)!} a_n z^n, \end{aligned} \tag{12}$$

where $*$ denote the convolution or Hadamard product of two series.

The study of the generalized Bessel function is a recent interesting topic in geometric function theory. We refer, in this connection, to the works of [13–15] and others.

Motivated by results on connections between various subclasses of analytic univalent functions by using hypergeometric functions (see, for example, [16–20])), and the work done in [21–24], we determine necessary and sufficient conditions for $zu_p(z)$ to be in $\mathcal{SP}_p(\alpha, \beta)$ and $\mathcal{UCSP}(\alpha, \beta)$ and also give necessary and sufficient conditions for $z(2 - u_p(z))$ to be in the function classes $\mathcal{SP}_p\mathcal{T}(\alpha, \beta)$ and $\mathcal{UCSPT}(\alpha, \beta)$. Furthermore, we give necessary and sufficient conditions for $\mathscr{I}(\kappa, c) f$ to be in $\mathcal{UCSPT}(\alpha, \beta)$ provided that the function f is in the class $\mathscr{R}^{\tau}(A, B)$. Finally, we give conditions for the integral operator $\mathscr{G}(\kappa, c, z) = \int_0^z (2 - u_p(t)) dt$ to be in the class $\mathcal{UCSPT}(\alpha, \beta)$.

To establish our main results, we need the following Lemmas.

Lemma 2 (see [1]). *(i) A sufficient condition for a function f of the form (1) to be in the class $\mathcal{SP}_p(\alpha, \beta)$ is that*

$$\sum_{n=2}^{\infty} (2n - \cos\alpha - \beta) |a_n| \leq \cos\alpha - \beta \quad (|\alpha| < \pi/2; 0 \leq \beta < 1) \tag{13}$$

and a necessary and sufficient condition for a function f of the form (2) to be in the class $\mathcal{SP}_p\mathcal{T}(\alpha, \beta)$ is that condition (13) is satisfied. In particular, when $\beta = 0$, we obtain a sufficient condition for a function f of the form (1) to be in the class $\mathcal{SP}_p(\alpha)$ is that

$$\sum_{n=2}^{\infty} (2n - \cos\alpha) |a_n| \leq \cos\alpha \quad \left(|\alpha| < \frac{\pi}{2}\right) \tag{14}$$

and a necessary and sufficient condition for a function f of the form (2) to be in the class $\mathcal{SP}_p\mathcal{T}(\alpha)$ is that condition (14) is satisfied.

(ii) A sufficient condition for a function f of the form (1) to be in the class $\mathcal{UCSP}(\alpha, \beta)$ is that

$$\sum_{n=2}^{\infty} n(2n - \cos\alpha - \beta) |a_n| \leq \cos\alpha - \beta \quad \left(|\alpha| < \frac{\pi}{2}; 0 \leq \beta < 1\right) \tag{15}$$

and a necessary and sufficient condition for a function f of the form (2) to be in the class $\mathcal{UCSPT}(\alpha, \beta)$ is that condition (15) is satisfied. In particular, when $\beta = 0$, we obtain a sufficient condition for a function f of the form (1) to be in the class $\mathcal{UCSP}(\alpha)$ is that

$$\sum_{n=2}^{\infty} n(2n - \cos\alpha) |a_n| \leq \cos\alpha \quad \left(|\alpha| < \frac{\pi}{2}\right) \tag{16}$$

and a necessary and sufficient condition for a function f of the form (2) to be in the class $\mathcal{UCSPT}(\alpha)$ is that condition (16) is satisfied.

Lemma 3 (see [12]). *If $f \in \mathscr{R}^{\tau}(A, B)$ is of the form (1), then*

$$|a_n| \leq (A - B) \frac{|\tau|}{n}, \quad n \in \mathbb{N} - \{1\}. \tag{17}$$

The result is sharp for the function

$$f(z) = \int_0^z \left(1 + (A - B)\frac{\tau t^{n-1}}{1 + Bt^{n-1}}\right) dt, \quad (z \in \mathbb{U}; n \in \mathbb{N} - \{1\}). \tag{18}$$

Lemma 4 (see [15]). *If $b, p, c \in \mathbb{C}$ and $\kappa \neq 0, -1, -2, \ldots$, then the function u_p satisfies the recursive relations*

$$u_p'(z) = \frac{(-c/4)}{\kappa} u_{p+1}(z),$$
$$u_p''(z) = \frac{(-c/4)^2}{\kappa(\kappa+1)} u_{p+2}(z), \tag{19}$$

for all $z \in \mathbb{C}$.

2. The Necessary and Sufficient Conditions

Unless otherwise mentioned, we shall assume in this paper that $|\alpha| < \pi/2$ and $0 \leq \beta < 1$.

First we obtain the necessary condition for zu_p to be in $\mathcal{SP}_p(\alpha, \beta)$.

Theorem 5. *If $c < 0$, $\kappa > 0$ ($\kappa \neq 0, -1, -2, \ldots$), then zu_p is in $\mathcal{SP}_p(\alpha, \beta)$ if*

$$2u_p'(1) + (2 - \cos\alpha - \beta)(u_p(1) - 1) \leq \cos\alpha - \beta. \tag{20}$$

Proof. Since

$$zu_P(z) = z + \sum_{n=2}^{\infty} \frac{(-c/4)^{n-1}}{(\kappa)_{n-1}(n-1)!} z^n, \tag{21}$$

according to (13), we must show that

$$\sum_{n=2}^{\infty} (2n - \cos\alpha - \beta) \frac{(-c/4)^{n-1}}{(\kappa)_{n-1}(n-1)!} \leq \cos\alpha - \beta. \tag{22}$$

Writing

$$n = (n-1) + 1, \tag{23}$$

we have

$$\begin{aligned}
&\sum_{n=2}^{\infty} (2n - \cos\alpha - \beta) \frac{(-c/4)^{n-1}}{(\kappa)_{n-1}(n-1)!} \\
&= 2\sum_{n=2}^{\infty} (n-1) \frac{(-c/4)^{n-1}}{(\kappa)_{n-1}(n-1)!} + \sum_{n=2}^{\infty} (2 - \cos\alpha - \beta) \frac{(-c/4)^{n-1}}{(\kappa)_{n-1}(n-1)!} \\
&= 2\sum_{n=2}^{\infty} \frac{(-c/4)^{n-1}}{(\kappa)_{n-1}(n-2)!} + \sum_{n=2}^{\infty} (2 - \cos\alpha - \beta) \frac{(-c/4)^{n-1}}{(\kappa)_{n-1}(n-1)!} \\
&= 2\sum_{n=0}^{\infty} \frac{(-c/4)^{n+1}}{(\kappa)_{n+1} n!} + (2 - \cos\alpha - \beta) \sum_{n=0}^{\infty} \frac{(-c/4)^{n+1}}{(\kappa)_{n+1}(n+1)!} \\
&= \frac{2(-c/4)}{\kappa} \sum_{n=0}^{\infty} \frac{(-c/4)^n}{(\kappa+1)_n n!} + (2 - \cos\alpha - \beta) \sum_{n=0}^{\infty} \frac{(-c/4)^{n+1}}{(\kappa)_{n+1}(n+1)!} \\
&= \frac{2(-c/4)}{\kappa} u_{p+1}(1) + (2 - \cos\alpha - \beta)(u_p(1) - 1) \\
&= 2u_p'(1) + (2 - \cos\alpha - \beta)(u_p(1) - 1).
\end{aligned} \tag{24}$$

But this last expression is bounded above by $\cos\alpha - \beta$ if (20) holds. □

Corollary 6. *If $c < 0$, $\kappa > 0$ ($\kappa \neq 0, -1, -2, \ldots$), then $z(2 - u_p(z))$ is in $\mathcal{SP}_p\mathcal{T}(\alpha, \beta)$ if and only if the condition (20) is satisfied.*

Proof. Since

$$z(2 - u_p(z)) = z - \sum_{n=2}^{\infty} \frac{(-c/4)^{n-1}}{(\kappa)_{n-1}(n-1)!} z^n. \tag{25}$$

By using the same techniques given in the proof of Theorem 5, we have Corollary 6. □

Theorem 7. *If $c < 0$, $\kappa > 0$ ($\kappa \neq 0, -1, -2, \ldots$), then zu_p is in $\mathcal{SP}_p(\alpha, \beta)$ if*

$$e^{(-c/4\kappa)} \left[\frac{-c}{2\kappa} + (2 - \cos\alpha - \beta)\left(1 - e^{(c/4\kappa)}\right)\right] \leq \cos\alpha - \beta. \tag{26}$$

Proof. We note that $(\kappa)_{n-1} = \kappa(\kappa+1)(\kappa+2)\cdots(\kappa+n-2) \geq \kappa(\kappa+1)^{n-2} \geq \kappa^{n-1}$, $(n \in \mathbb{N})$. From (24), we get

$$\begin{aligned}
&\sum_{n=2}^{\infty} (2n - \cos\alpha - \beta) \frac{(-c/4)^{n-1}}{(\kappa)_{n-1}(n-1)!} \\
&\leq 2\sum_{n=2}^{\infty} (n-1) \frac{\left(-\frac{c}{4\kappa}\right)^{n-1}}{(n-1)!} + (2 - \cos\alpha - \beta) \sum_{n=2}^{\infty} \frac{(-c/4\kappa)^{n-1}}{(n-1)!} \\
&= \left(-\frac{c}{2\kappa}\right) e^{-c/4\kappa} + (2 - \cos\alpha - \beta)\left(e^{-c/4\kappa} - 1\right).
\end{aligned} \tag{27}$$

Therefore, we see that the last expression is bounded above by $\cos\alpha - \beta$ if (26) is satisfied. □

Corollary 8. *If $c < 0$, $\kappa > 0$ $(\kappa \neq 0, -1, -2, \ldots)$, then $z(2 - u_p(z))$ is in $\mathcal{SP}_p\mathcal{T}(\alpha, \beta)$ if and only if the condition (26) is satisfied.*

Theorem 9. *If $c < 0$, $\kappa > 0 (\kappa \neq 0, -1, -2, \ldots)$, then $zu_p(z)$ is in $\mathcal{UCSP}(\alpha, \beta)$ if*

$$2u_p''(1) + (6 - \cos\alpha - \beta)\, u_p'(1) + (2 - \cos\alpha - \beta)\left(u_p(1) - 1\right) \leq \cos\alpha - \beta. \quad (28)$$

Proof. In view of (15), we must show that

$$\sum_{n=2}^{\infty} n(2n - \cos\alpha - \beta)\frac{(-c/4)^{n-1}}{(\kappa)_{n-1}(n-1)!} \leq \cos\alpha - \beta. \quad (29)$$

Writing

$$n = (n-1) + 1,$$
$$n^2 = (n-1)(n-2) + 3(n-1) + 1. \quad (30)$$

Thus, we have

$$\begin{aligned}
&\sum_{n=2}^{\infty} n(2n - \cos\alpha - \beta)\frac{(-c/4)^{n-1}}{(\kappa)_{n-1}(n-1)!} \\
&= 2\sum_{n=2}^{\infty}(n-1)(n-2)\frac{(-c/4)^{n-1}}{(\kappa)_{n-1}(n-1)!} \\
&\quad + (6 - \cos\alpha - \beta)\sum_{n=2}^{\infty}(n-1)\frac{(-c/4)^{n-1}}{(\kappa)_{n-1}(n-1)!} \\
&\quad + (2 - \cos\alpha - \beta)\sum_{n=2}^{\infty}\frac{(-c/4)^{n-1}}{(\kappa)_{n-1}(n-1)!}. \\
&= 2\sum_{n=3}^{\infty}\frac{(-c/4)^{n-1}}{(\kappa)_{n-1}(n-3)!} \\
&\quad + (6 - \cos\alpha - \beta)\sum_{n=2}^{\infty}\frac{(-c/4)^{n-1}}{(\kappa)_{n-1}(n-2)!} \\
&\quad + (2 - \cos\alpha - \beta)\sum_{n=2}^{\infty}\frac{(-c/4)^{n-1}}{(\kappa)_{n-1}(n-1)!}. \\
&= \frac{2(-c/4)^2}{\kappa(\kappa+1)}\sum_{n=0}^{\infty}\frac{(-c/4)^n}{(\kappa+2)_n\, n!} \\
&\quad + (6 - \cos\alpha - \beta)\frac{(-c/4)}{\kappa}\sum_{n=0}^{\infty}\frac{(-c/4)^n}{(\kappa+1)_n\, n!} \\
&\quad + (2 - \cos\alpha - \beta)\sum_{n=0}^{\infty}\frac{(-c/4)^{n+1}}{(\kappa)_{n+1}(n+1)!} \\
&= 2u_p''(1) + (6 - \cos\alpha - \beta)\, u_p'(1) \\
&\quad + (2 - \cos\alpha - \beta)\left(u_p(1) - 1\right).
\end{aligned} \quad (31)$$

But this last expression is bounded above by $\cos\alpha - \beta$ if (28) holds. □

By using a similar method as in the proof of Corollary 6, we have the following result.

Corollary 10. *If $c < 0$, $\kappa > 0$ $(\kappa \neq 0, -1, -2, \ldots)$, then $z(2 - u_p(z))$ is in $\mathcal{UCSPT}(\alpha, \beta)$ if and only if the condition (28) is satisfied.*

The proof of Theorem 11 (below) is much akin to that of Theorem 7, and so the details may be omitted.

Theorem 11. *If $c < 0$, $\kappa > 0$ $(\kappa \neq 0, -1, -2, \ldots)$, then $z(2 - u_p(z))$ is in $\mathcal{UCSPT}(\alpha, \beta)$ if and only if*

$$e^{(-c/4\kappa)}\left[\frac{c^2}{8\kappa} + (6 - \cos\alpha - \beta)\left(\frac{-c}{4\kappa}\right) + (2 - \cos\alpha - \beta)\left(1 - e^{(c/4\kappa)}\right)\right] \leq \cos\alpha - \beta. \quad (32)$$

3. Inclusion Properties

Making use of Lemma 3, we will study the action of the Bessel function on the class $\mathcal{UCSPT}(\alpha, \beta)$.

Theorem 12. *Let $c < 0$, $\kappa > 0$ $(\kappa \neq 0, -1, -2, \ldots)$. If $f \in \mathcal{R}^\tau(A, B)$, then $\mathcal{I}(\kappa, c)f$ is in $\mathcal{UCSPT}(\alpha, \beta)$ if and only if*

$$(A - B)|\tau|\left[2u_p'(1) + (2 - \cos\alpha - \beta)\left(u_p(1) - 1\right)\right] \leq \cos\alpha - \beta. \quad (33)$$

Proof. In view of (15), it suffices to show that

$$\sum_{n=2}^{\infty} n(2n - \cos\alpha - \beta)\frac{(-c/4)^{n-1}}{(\kappa)_{n-1}(n-1)!}\left|a_n\right| \leq \cos\alpha - \beta. \quad (34)$$

Since $f \in \mathcal{R}^\tau(A, B)$, then by Lemma 3, we get

$$\left|a_n\right| \leq \frac{(A - B)|\tau|}{n}. \quad (35)$$

Thus, we must show that

$$\begin{aligned}
&\sum_{n=2}^{\infty} n(2n - \cos\alpha - \beta)\frac{(-c/4)^{n-1}}{(\kappa)_{n-1}(n-1)!}\left|a_n\right| \\
&\leq (A - B)|\tau|\left[\sum_{n=2}^{\infty}(2n - \cos\alpha - \beta)\frac{(-c/4)^{n-1}}{(\kappa)_{n-1}(n-1)!}\right].
\end{aligned} \quad (36)$$

The remaining part of the proof of Theorem 12 is similar to that of Theorem 5, and so we omit the details. □

4. An Integral Operator

In this section, we obtain the necessary and sufficient conditions for the integral operator $\mathcal{G}(\kappa, c, z)$ defined by

$$\mathcal{G}(\kappa, c, z) = \int_0^z \left(2 - u_p(t)\right) dt \quad (37)$$

to be in $\mathcal{UCSPT}(\alpha, \beta)$.

Theorem 13. *If* < 0, $\kappa > 0$ $(\kappa \neq 0, -1, -2, \ldots)$, *then the integral operator* $\mathcal{G}(\kappa, c, z)$ *is in* $\mathcal{UCSPT}(\alpha, \beta)$ *if and only if the condition (20) is satisfied.*

Proof. Since

$$\mathcal{G}(\kappa, c, z) = z - \sum_{n=2}^{\infty} \frac{(-c/4)^{n-1}}{(\kappa)_{n-1}} \frac{z^n}{n!} \tag{38}$$

then, in view of (15), we need only to show that

$$\sum_{n=2}^{\infty} n (2n - \cos\alpha - \beta) \frac{(-c/4)^{n-1}}{(\kappa)_{n-1}\, n!} \leq \cos\alpha - \beta \tag{39}$$

or equivalently

$$\sum_{n=2}^{\infty} (2n - \cos\alpha - \beta) \frac{(-c/4)^{n-1}}{(\kappa)_{n-1} (n-1)!} \leq \cos\alpha - \beta. \tag{40}$$

The remaining part of the proof is similar to that of Theorem 5, and so we omit the details. □

The proofs of Theorems 14 and 15 are much akin to that of Theorem 7, and so the details may be omitted.

Theorem 14. *Let* $c < 0$, $\kappa > 0$ $(\kappa \neq 0, -1, -2, \ldots)$. *If* $f \in \mathcal{R}^{\tau}(A, B)$, *then* $\mathcal{I}(\kappa, c) f$ *is in* $\mathcal{UCSPT}(\alpha, \beta)$ *if and only if*

$$\begin{aligned}(A - B)|\tau| e^{(-c/4\kappa)} \left[\frac{-c}{2\kappa} + (2 - \cos\alpha - \beta)\left(1 - e^{(c/4\kappa)}\right)\right] \\ \leq \cos\alpha - \beta.\end{aligned} \tag{41}$$

Theorem 15. *If* < 0, $\kappa > 0$ $(\kappa \neq 0, -1, -2, \ldots)$, *then the integral operator* $\mathcal{G}(\kappa, c, z)$ *is in* $\mathcal{UCSPT}(\alpha, \beta)$ *if and only if the condition (32) is satisfied.*

5. Corollaries and Consequences

In this section, we apply our main results in order to deduce each of the following corollaries and consequences.

Corollary 16. *If* $c < 0$, $\kappa > 0$ $(\kappa \neq 0, -1, -2, \ldots)$,*then* zu_p *is in* $\mathcal{SP}_p(\alpha)$ *if*

$$2u_p'(1) + (2 - \cos\alpha)\left(u_p(1) - 1\right) \leq \cos\alpha. \tag{42}$$

Corollary 17. *If* $c < 0$, $\kappa > 0$ $(\kappa \neq 0, -1, -2, \ldots)$, *then* $z(2 - u_p(z))$ *is in* $\mathcal{SP}_p\mathcal{T}(\alpha)$ *if and only if the condition (42) is satisfied.*

Corollary 18. *If* $c < 0$, $\kappa > 0$ $(\kappa \neq 0, -1, -2, \ldots)$, *then* zu_p *is in* $\mathcal{SP}_p(\alpha)$ *if*

$$e^{(-c/4\kappa)} \left[\frac{-c}{2\kappa} + (2 - \cos\alpha)\left(1 - e^{(c/4\kappa)}\right)\right] \leq \cos\alpha. \tag{43}$$

Corollary 19. *If* $c < 0$, $\kappa > 0$ $(\kappa \neq 0, -1, -2, \ldots)$, *then* $z(2 - u_p(z))$ *is in* $\mathcal{SP}_p\mathcal{T}(\alpha)$ *if and only if the condition (43) is satisfied.*

Corollary 20. *If* $c < 0$, $\kappa > 0$ $(\kappa \neq 0, -1, -2, \ldots)$, *then* $zu_p(z)$ *is in* $\mathcal{UCSP}(\alpha)$ *if*

$$\begin{aligned}2u_p''(1) + (6 - \cos\alpha) u_p'(1) + (2 - \cos\alpha)\left(u_p(1) - 1\right) \\ \leq \cos\alpha.\end{aligned} \tag{44}$$

Corollary 21. *If* $c < 0$, $\kappa > 0$ $(\kappa \neq 0, -1, -2, \ldots)$, *then* $z(2 - u_p(z))$ *is in* $\mathcal{UCSPT}(\alpha)$ *if and only if*

$$\begin{aligned}e^{(-c/4\kappa)} \Bigg[\frac{c^2}{8\kappa} + (6 - \cos\alpha)\left(\frac{-c}{4\kappa}\right) \\ + (2 - \cos\alpha)\left(1 - e^{(c/4\kappa)}\right)\Bigg] \leq \cos\alpha.\end{aligned} \tag{45}$$

Corollary 22. *Let* $c < 0$, $\kappa > 0$ $(\kappa \neq 0, -1, -2, \ldots)$. *If* $f \in \mathcal{R}^{\tau}(A, B)$, *then* $\mathcal{I}(\kappa, c) f$ *is in* $\mathcal{UCSPT}(\alpha)$ *if and only if*

$$\begin{aligned}(A - B)|\tau| \left[2u_p'(1) + (2 - \cos\alpha)\left(u_p(1) - 1\right)\right] \\ \leq \cos\alpha.\end{aligned} \tag{46}$$

Corollary 23. *If* < 0, $\kappa > 0$ $(\kappa \neq 0, -1, -2, \ldots)$, *then the integral operator* $\mathcal{G}(\kappa, c, z)$ *is in* $\mathcal{UCSPT}(\alpha)$ *if and only if the condition (42) is satisfied.*

Corollary 24. *Let* $c < 0$, $\kappa > 0$ $(\kappa \neq 0, -1, -2, \ldots)$. *If* $f \in \mathcal{R}^{\tau}(A, B)$, *then* $\mathcal{I}(\kappa, c) f$ *is in* $\mathcal{UCSPT}(\alpha)$ *if and only if*

$$\begin{aligned}(A - B)|\tau| e^{(-c/4\kappa)} \left[\frac{-c}{2\kappa} + (2 - \cos\alpha)\left(1 - e^{(c/4\kappa)}\right)\right] \\ \leq \cos\alpha.\end{aligned} \tag{47}$$

Corollary 25. *If* < 0, $\kappa > 0$ $(\kappa \neq 0, -1, -2, \ldots)$, *then the integral operator* $\mathcal{G}(\kappa, c, z)$ *is in* $\mathcal{UCSPT}(\alpha)$ *if and only if the condition (45) is satisfied.*

Remark 26. If we put $\alpha = 0$ in Corollary 6, we obtain Theorem 5 in [22] for $\lambda = 1$ and $\beta = 1$.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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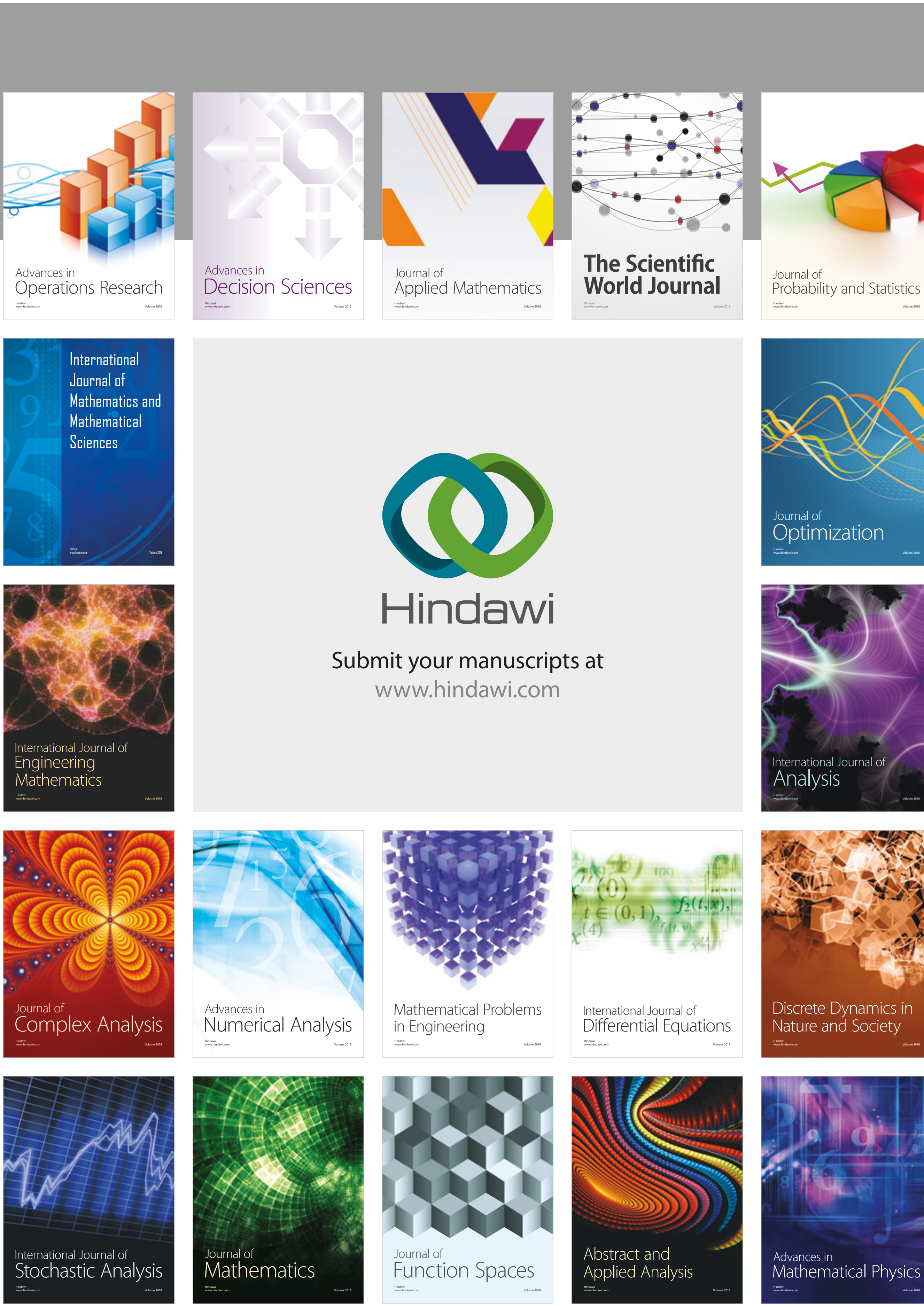

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