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Research Article

Fixed Point Theorems for Cyclic Weakly Contraction Mappings in Dislocated Quasi Extended b-Metric Space

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In this paper, we establish and prove some theorems about existence and uniqueness of fixed point for cyclic weakly contraction mappings in dislocated quasi extended b-metric space.

1. Introduction

One of the famous generalizations of metric space which was introduced by Bakhtin in 1989 [1] is b-metric space. Many authors utilized the space for fixed point results on contraction mapping or weakly contraction mapping, such as Saluja et al. [2], Mostefaoui et al. [3], Chaudhury et al. [4] and Ansari et al. [5]. In 2012, Shah et al. [6] introduced quasi b-metric space which removed symmetric conditions in b-metric and for utilizing in common fixed point results on contraction mapping. Some authors such as Zhu et al. [7] and Cvetkovic et al. [8] gave some results in that space. In 2013, Hussain et al. [9] introduced dislocated b-metric which weakened first condition in b-metric for fixed point results, and Rasham et al. [10] utilized the space for multivalued fixed point results. In 2016, Rahman et al. [11] generalized the dislocated b-metric to be dislocated quasi b-metric. Several papers has published in dislocated quasi b-metric for containing fixed point results on generalized Banach contraction mappings, such as Klin-eam et al. [12], Suanom et al. [13], and Tiwari et al. [14]. Recently, in 2017, Kamran et al. [15] generalized triangular inequality condition on b-metric such that to be extended b-metric and utilized the space for fixed point results. Samreen et al. [16] yielded some theorems for fixed point results on nonlinear contraction mappings in the space and Alqahtani et al. [17, 18] utilized the space for common fixed point results on two self-mappings and on K-contraction mapping.

Inspired by the extended b-metric space of Samreen et al. [16]. In this work, we introduced a concept of dislocated quasi extended b-metric space as a generalization of dislocated quasi b-metric space [11]. We establish and prove some fixed point theorems in the dislocated quasi extended b-metric space, by utilizing weakly contraction mapping which was introduced by Rhoades [19] and cyclic contraction which was introduced by Zoto et al. [20]. In addition, we also provide some examples to clarify the theorems.

2. Preliminaries

In the following section, we need some definitions to govern and prove our theorems.

Definition 1 (see [1]). Let X be a non-empty set and a real number $k \geq 1$. Let $d : X \times X \longrightarrow [0, \infty)$ be a function. The pair (X, d) is called *b-metric space* if the following conditions are satisfied:

(1) $d(x, y) = 0$ if and only if $x = y$,

(2) $d(x, y) = d(y, x)$,

(3) $d(x, y) \leq k(d(x, z) + d(z, y))$,

for all $x, y, z \in X$.

Example 2 (see [15]). Let $X = l_p(R)$ with $0 < p < 1$, where $l_p(R) = \{\{a_k\} \subseteq R \mid \sum_{k=1}^{\infty} a_k < \infty\}$. Let $d : X \times X \longrightarrow [0, \infty)$ be a function, which is defined as $d(x, y) = \sum_{k=1}^{\infty} |a_k - b_k|^{1/p}$, where $x = \{a_k\}$ and $y = \{b_k\}$. Then d is a b-metric with parameter $b = 2^{1/p}$.

Definition 3 (see [11]). Let X be a nonempty set and a real number $k \geq 1$. Let $d : X \times X \longrightarrow [0, \infty)$ be a function. The pair (X, d) is called a *dislocated quasi b-metric space* (*in short dqb- metric space*) if the following conditions are satisfied:

(1) $d(x, y) = 0$ then $x = y$,

(2) $dd(x, y) \leq k(d(x, z) + d(z, y))$,

for all $x, y, z \in X$.

Example 4 (see [11]). Let $X = R$ and define $d(x, y) = |2x - y|^2 + |2x + y|^2$. It is easy to show that (X, d) is a dislocated quasi b-metric space with $k = 2$.

Definition 5 (see [15]). Let X be a non-empty set and $k : X \times X \longrightarrow [1, \infty)$ be a function. Let $d : X \times X \longrightarrow [0, \infty)$ be a function. The pair (X, d) is called an *extended b-metric space* if the following conditions are satisfied:

(1) $d(x, y) = 0$ if and only if $x = y$,

(2) $d(x, y) = d(y, x)$,

(3) $d(x, y) \leq k(x, y)(d(x, z) + d(z, y))$,

for all $x, y, z \in X$.

Example 6 (see [16]). Let $X = \{1, 2, 3, \ldots\}$. Define $k : X \times X \longrightarrow [1, \infty)$ and $d : X \times X \longrightarrow [0, \infty)$ as follows:

$$k(x, y) = \begin{cases} |x - y|^2 & \text{if } x \neq y \\ 1 & if\ x = y \end{cases} \quad (1)$$

$$\text{and } d(x, y) = (x - y)^4.$$

It is easy to show that (X, d) is a dislocated extended b-metric space.

Definition 7. Let X be a non-empty set and $k : X \times X \longrightarrow [1, \infty)$. Let $d_k : X \times X \longrightarrow [0, \infty)$ be a function. The pair (X, d_k) is called a *quasi extended b-metric space* (*in short qeb-metric space*) if the following conditions are satisfied:

$$\begin{aligned} &(1)\ \ d_k(x, y) = 0 \text{ if and only if } x = y, \\ &(2)\ \ d_k(x, y) \leq k(x, y)(d_k(x, z) + d_k(z, y)), \end{aligned} \quad (2)$$

for all $x, y, z \in X$.

Example 8. Let $X = [0, 1]$ and $d(x, y) = |2^{x-y} - 1|$ for $x, y \in [0, 1]$. Let $k(x, y) = 2^{1-(x+y)/2}$ for $x, y \in [0, 1]$.

It is obvious that for first condition and $d(x, y)$ is not symmetric. For second condition, consider that

$$\begin{aligned} &2^{1-(x+y)/2}(d(x, z) + d(z, y)) \\ &\quad = 2^{1-(x+y)/2}(|2^{x-z} - 1| + |2^{z-y} - 1|) \end{aligned} \quad (3)$$

Since $\min_{z\in[0,1]}|2^{x-z} - 1| + |2^{z-y} - 1| = |2^{x-(x+y)/2} - 1| + |2^{(x+y)/2-y} - 1|$, we get

$$\begin{aligned} &2^{1-(x+y)/2}(|2^{x-z} - 1| + |2^{z-y} - 1|) \\ &\quad \geq 2^{1-(x+y)/2}\left(\left|2^{x-(x+y)/2} - 1\right| + \left|2^{(x+y)/2-y} - 1\right|\right) \\ &\quad = 2^{1-\lfloor(x+y)/2\rfloor}\left(\left|2^{(x-y)/2} - 1\right| + \left|2^{(x-y)/2} - 1\right|\right) \\ &\quad = 2^{2-(x+y)/2}\left(\left|2^{(x-y)/2} - 1\right|\right). \end{aligned} \quad (4)$$

If $x \leq y$ then we have $(x + y)/2 \geq x$, and $2^{2-(x+y)/2} \leq 2^{2-x}$. Therefore, we get

$$\begin{aligned} &2^{1-(x+y)/2}(|2^{x-z} - 1| + |2^{z-y} - 1|) \\ &\quad \geq \left(\left|2^{2-y} - 2^{2-(x+y)/2}\right|\right) \geq \left(\left|2^{2-y} - 2^{2-x}\right|\right) \\ &\quad = 2^2(|2^{-y} - 2^{-x}|) \end{aligned} \quad (5)$$

Since, for $x \in [0, 1]$, $2 - x \geq 1$. Thus we get

$$\begin{aligned} &2^{1-(x+y)/2}(|2^{x-z} - 1| + |2^{z-y} - 1| \\ &\quad \geq 2^2(2^{-x})(|2^{x-y} - 1|) = 2^{2-x}|2^{x-y} - 1| \\ &\quad \geq |2^{x-y} - 1| = d(x, y) \end{aligned} \quad (6)$$

If $x \geq y$ then we have $2^{2-y} \geq 2^{2-x}, (x + y)/2 \leq y$, and $2^{2-(x+y)/2} \leq 2^{2-y}$. Thus we get

$$\begin{aligned} &2^{1-(x+y)/2}(|2^{x-z} - 1| + |2^{z-y} - 1| \\ &\quad \geq \left(\left|2^{2-y} - 2^{2-(x+y)/2}\right|\right) \geq \left(\left|2^{2-x} - 2^{2-y}\right|\right) \\ &\quad = 2^2(|2^{-x} - 2^{-y}|) = 2^2(|2^{-y} - 2^{-x}|) \\ &\quad = 2^2(2^{-x})(|2^{x-y} - 1|) = 2^{2-x}|2^{x-y} - 1| \\ &\quad \geq |2^{x-y} - 1| = d(x, y). \end{aligned} \quad (7)$$

Hence, we have

$$d(x, y) \leq 2^{1-(x+y)/2}(d(x, z) + d(z, y)). \quad (8)$$

Thus d is a quasi b-metric in $X = [0, 1]$.

Definition 9. Let X be a non-empty set and $k : X \times X \longrightarrow [1, \infty)$ and let $d_k : X \times X \longrightarrow [0, \infty)$ be a function. The pair (X, d_k) is called a *dislocated quasi extended b-metric space* (*in short dqeb- metric space*) if the following conditions are satisfied:

$$\begin{aligned} &(1)\ \ d_k(x, y) = 0 \text{ then } x = y, \\ &(2)\ \ d_k(x, y) \leq k(x, y)(d_k(x, z) + d_k(z, y)), \end{aligned} \quad (9)$$

for all $x, y, z \in X$.

Remark 10. If $k(x, y) = k \geq 1$, then *dqeb* is *dqb*.

Example 11. Let $X = [-1, 1]$ and $d_k(x, y) = (|x| + |y|) + |x|^2/m + |y|^2/n, m \neq n$, for $x, y \in [-1, 1]$.

Let $k(x, y) = (2 + |xy|)/2$ for $x \in [-1, 1]$.

In fact, it is clear that if $d(x, y) = 0$, then $x = y = 0$, which is satisfied for first condition. For second condition, we consider,

$$\begin{aligned} k(x, y)(d_k(x, z) + d_k(z, y)) &= \frac{2 + |xy|}{2}\left((|x| + |z|) + \frac{|x|^2}{m} + \frac{|z|^2}{n} + (|z| + |y|) + \frac{|z|^2}{m} + \frac{|y|^2}{n}\right) \\ &\geq \frac{2 + |xy|}{2}\left((|x| + |y|) + \frac{|x|^2}{m} + \frac{|z|^2}{n} + \frac{|z|^2}{m} + \frac{|y|^2}{n}\right) \geq \left((|x| + |y|) + \frac{|x|^2}{m} + \frac{|y|^2}{n}\right) \\ &= d_k(x, y). \end{aligned} \tag{10}$$

Thus (X, d_k) is a dislocated quasi extended b-metric space, with $k(x, y) = (2 + |xy|)/2 \geq 1$.

Definition 12 (see [12, 13]). Let (X, d_k) be a dislocated quasi extended b-metric space and let $\{x_n\}$ be a sequence in X.

(i) $\{x_n\}$ convergent sequence to $x \in X$, if $\lim_{n\longrightarrow\infty} d_k(x_n, x) = \lim_{n\longrightarrow\infty} d_k(x, x_n) = 0$.

(ii) $\{x_n\}$ is called Cauchy in X, if $\lim_{n,m\longrightarrow\infty} d_k(x_n, x_m) = \lim_{n,m\longrightarrow\infty} d_k(x_m, x_n) = 0$.

(iii) (X, d_k) is called complete if every Cauchy sequence in X is convergent in X.

Definition 13 (see [6]). Let X be nonempty set, G and H are subsets of X. A function $T : G \cup H \longrightarrow G \cup H$ is called a *cyclic map* if $T(G) \subseteq H$ and $T(H) \subseteq G$.

Definition 14 (see [17]). Let (X, d_k) be a dislocated quasi extended b-metric space, G and H be subsets of X. A function $T : G \cup H \longrightarrow G \cup H$ is called *dqeb-cyclic weakly contraction* if there exists continuous and non-decreasing function $\varphi : [0, \infty) \longrightarrow [0, \infty)$ such that for every $x \in G$, $y \in H$,

$$k(x, y)\, d_k(Tx, Ty) \leq d_k(x, y) - \varphi(d_k(x, y)), \tag{11}$$

where $\varphi(t) = 0$ if and only if $t = 0$.

Definition 15 (see [20]). Let (X, d_k) be a dislocated quasi extended b-metric space, G and H be subsets of X. A function $T : G \cup H \longrightarrow G \cup H$ is called a *cyclic φ contraction* if T is a cyclic and there exists a continuous and non-decreasing function $\varphi : [0, \infty) \longrightarrow [0, \infty)$ such that for every $x \in G$, $y \in H$

$$d_k(Tx, Ty) \leq \varphi(d_k(x, y)). \tag{12}$$

3. Main Results

In this section, we show some theorems and examples of the existence and uniqueness of fixed point for generalized *dqeb*-cyclic weakly contraction mapping in complete dislocated quasi extended b-metric space.

Theorem 16. *Let (X, d_k) be a complete dislocated quasi extended b-metric space, G and H be closed subsets of X. If $T : G \cup H \longrightarrow G \cup H$ is a cyclic map that satisfies the condition of dqeb-cyclic weakly contraction and $\lim_{n,m\longrightarrow\infty} k(x_n, x_m) = L > 0$, then T has a unique fixed point in $G \cap H$.*

Proof. Since T is a cyclic map, if taking $x_0 \in G$, then $Tx_0 \in H$ and $T^2x_0 \in G$. Define a sequence $\{x_n\}$, where $x_n = Tx_{n-1} = T^n x_0$. So we have $x_{2n} \in G$ and $x_{2n-1} \in H$ for $n = 1, 2, 3 \ldots$

Since $k(x, y) \geq 1$ for all $x, y \in X$, then for all $n \in N$ we have

$$\begin{aligned} d_k(x_{n+1}, x_n) &\leq k(x_{n+1}, x_n)\, d_k(x_{n+1}, x_n) \\ &= k(x_{n+1}, x_n)\, d_k(Tx_n, Tx_{n-1}) \\ &\leq d_k(x_n, x_{n-1}) - \varphi(d_k(x_n, x_{n-1})) \\ &\leq d_k(x_n, x_{n-1}). \end{aligned} \tag{13}$$

Thus we have $\{d_k(x_{n+1}, x_n)\}$ is a nonincreasing sequence of non-negative real numbers.

Claim that $\lim_{n\longrightarrow\infty} d_k(x_{n+1}, x_n) = 0$. Suppose $\lim_{n\longrightarrow\infty} d_k(x_{n+1}, x_n) = \beta$.

Since φ is nondecreasing and $k(x_{n+1}, x_n) \geq 1$, we have

$$\begin{aligned} d_k(x_{n+1}, x_n) &\leq k(x_{n+1}, x_n)\, d_k(x_{n+1}, x_n) \\ &\leq d_k(x_n, x_{n-1}) - \varphi(d_k(x_n, x_{n-1}). \end{aligned} \tag{14}$$

Since φ is continuous then for $n \longrightarrow \infty$, we have $\beta \leq \beta - \varphi(\beta)$. Since $\varphi \geq 0$, thus we get $\varphi(\beta) = 0$. Hence we have $\beta = 0$. Similarly we have $\lim_{n\longrightarrow\infty} d_k(x_n, x_{n+1}) = 0$.

Now, we have to prove that $\{x_n\}$ is a Cauchy sequence in X.

Suppose $\{x_n\}$ is not a Cauchy sequence. Then there exists $\varepsilon > 0$ such that every n, there exists n_k, $m_k > n$ such that $d_k(n_k, m_k) \geq \varepsilon$ and $d_k(n_{k-1}, m_k) < \varepsilon$.

From (11) we have

$$\begin{aligned} & k(n_{k-1}, m_{k-1})\, d_k(n_k, m_k) \\ &= k(n_{k-1}, m_{k-1})\, d_k(Tn_{k-1}, Tm_{k-1}) \\ &\leq d_k(n_{k-1}, m_{k-1}) - \varphi(d_k(n_{k-1}, m_{k-1})) \\ &\leq d_k(n_{k-1}, m_{k-1}). \end{aligned} \tag{15}$$

This implies

$$d_k(n_{k-1}, m_{k-1}) \geq \varepsilon k(n_{k-1}, m_{k-1}). \tag{16}$$

We also have that

$$\begin{aligned} & d_k(n_{k-1}, m_{k-1}) \\ &\leq k(n_{k-1}, m_{k-1})(d_k(n_{k-1}, m_k) + d_k(m_k, m_{k-1})) \\ &< \varepsilon k(n_{k-1}, m_{k-1}) + k(n_{k-1}, m_{k-1})\, d_k(m_k, m_{k-1}). \end{aligned} \tag{17}$$

It implies $\lim_{k\longrightarrow\infty} d_k(n_{k-1}, m_{k-1}) \leq \varepsilon \lim_{k\longrightarrow\infty} k(n_{k-1}, m_{k-1})$. Therefore we have $\varepsilon \lim_{k\longrightarrow\infty} k(n_{k-1}, m_{k-1}) \leq \lim_{k\longrightarrow\infty} d_k(n_{k-1}, m_{k-1}) \leq \varepsilon \lim_{k\longrightarrow\infty} k(n_{k-1}, m_{k-1})$, so we have $\varepsilon L \leq \lim_{k\longrightarrow\infty} d_k(n_{k-1}, m_{k-1}) \leq \varepsilon L$, thus we obtain $\lim_{k\longrightarrow\infty} d_k(n_{k-1}, m_{k-1}) = \varepsilon L$. From (15) and (16) we have

$$\begin{aligned} &k(n_{k-1}, m_{k-1}) d_k(n_k, m_k) \\ &\quad \leq d_k(n_{k-1}, m_{k-1}) - \varphi(d_k(n_{k-1}, m_{k-1})) \\ &\quad \leq d_k(n_{k-1}, m_{k-1}) \\ &\varepsilon k(n_{k-1}, m_{k-1}) \\ &\quad \leq d_k(n_{k-1}, m_{k-1}) - \varphi(d_k(n_{k-1}, m_{k-1})) \\ &\quad \leq d_k(n_{k-1}, m_{k-1}). \end{aligned} \tag{18}$$

For $k \longrightarrow \infty$ and using continuity of φ, we get

$$\varepsilon L \leq \varepsilon L - \varphi(\varepsilon L) \leq \varepsilon L. \tag{19}$$

Since $k(x, y) \geq 1$ and $\lim_{n,m\longrightarrow\infty} k(x_n, x_m) = L$, we have $L \geq 1$. Thus we have $\varphi(\varepsilon L) = 0$, this implies $\varepsilon L = 0$. Since $\varepsilon > 0$ then we obtain $L = 0$, which is a contradiction.

Hence $\{x_n\}$ is a Cauchy sequence in X. Since X complete, there exists $x^* \in X$ such that $d_k(x_n, x^*) \longrightarrow 0$ for $n \longrightarrow \infty$. Similarly we can have $d_k(x_n, x^*) \longrightarrow 0$.

Since the sequence $\{x_{2n}\} \in G$, $\{x_{2n-1}\} \in H$ and G, H be closed, then we have $x^* \in G \cap H$.

Now we prove that x^* is a fixed point of T. Using (2) and (11) we have

$$\begin{aligned} &d_k(Tx^*, x^*) \leq k(Tx^*, x^*) \\ &\quad \cdot (d(Tx^*, Tx_{n-1}) + d(Tx_{n-1}, x^*)) \leq k(Tx^*, x^*) \\ &\quad \cdot (d(x^*, x_{n-1}) - \varphi(d(x^*, x_{n-1})) + d(x_n, x^*)). \end{aligned} \tag{20}$$

Using continuity of φ and for $n \longrightarrow \infty$, we have $d_k(Tx^*, x^*) \leq -k(Tx^*, x^*)\varphi(0) \leq 0$.

Thus $d_k(Tx^*, x^*) = 0$, hence $Tx^* = x^*$.

Now we have to show that T has unique fixed point in X. Suppose that u is an another fixed point of T,

$$\begin{aligned} d_k(x^*, u) &= d_k(Tx^*, Tu) \leq k(x^*, u) d_k(Tx^*, Tu) \\ &\leq d_k(x^*, u) - \varphi(d_k(x^*, u)). \end{aligned} \tag{21}$$

Thus we get $\varphi(d_k(x^*, u)) \leq 0$. Since $\varphi \geq 0$, we have $\varphi(d_k(x^*, u)) = 0$. Which implies that $d_k(x^*, u) = 0$, so we have $x^* = u$. □

Example 17. Let $X = [-1, 1]$ and (X, d_k) be a dislocated quasi extended b-metric space which in Example 8. Let $T : G \cup H \longrightarrow G \cup H$ be a function defined by $Tx = -x/2$, where $G = [-1, 0]$, $H = [0, 1]$. Let $\varphi : [0, \infty) \longrightarrow [0, \infty)$ be a function which is defined by $\varphi(t) = t/4$

In fact, It is clear that T is a cyclic map, indeed $T(G) \subseteq H$ and $T(H) \subseteq G$.

Now, we have to show that

$$\begin{aligned} &k(x, y) d_k(Tx, Ty) \leq d_k(x, y) - \varphi(d_k(x, y)) \\ &k(x, y) d_k(Tx, Ty) = \frac{2 + |xy|}{2} d_k\left(\frac{-x}{2}, \frac{-y}{2}\right) \\ &\quad = \frac{2 + |xy|}{2}\left[\left(\left|\frac{-x}{2}\right| + \left|\frac{-y}{2}\right|\right) + \frac{|-x/2|^2}{5}\right. \\ &\quad \left. + \frac{|-y/2|^2}{6}\right] = \frac{2 + |xy|}{4}\left[|x| + |y| + \frac{|x|^2}{10} + \frac{|y|^2}{12}\right] \\ &\quad = \frac{1}{4}\left[\left(2(|x| + |y|) + \frac{|x|^2}{10} + \frac{|y|^2}{12}\right)\right. \\ &\quad \left. + |xy|\left((|x| + |y|) + \frac{|x|^2}{10} + \frac{|y|^2}{12}\right)\right] \\ &\quad \leq \frac{1}{4}\left[\left(2(|x| + |y|) + \frac{|x|^2}{10} + \frac{|y|^2}{12}\right)\right. \\ &\quad \left. + \left((|x| + |y|) + \frac{|x|^2}{10} + \frac{|y|^2}{12}\right)\right] \\ &\quad \leq \frac{1}{4}\left[3(|x| + |y|) + \frac{|x|^2}{5} + \frac{|y|^2}{6}\right)\right] \leq \frac{3}{4}\left(|x|\right. \\ &\quad \left. + |y| + \frac{|x|^2}{10} + \frac{|y|^2}{12}\right) \leq \frac{3}{4}\left(|x| + |y| + \frac{|x|^2}{5}\right. \\ &\quad \left. + \frac{|y|^2}{6}\right) = \left(|x| + |y| + \frac{|x|^2}{5} + \frac{|y|^2}{6}\right) - \frac{1}{4}\left(|x|\right. \\ &\quad \left. + |y| + \frac{|x|^2}{5} + \frac{|y|^2}{6}\right) = d_k(x, y) - \varphi(d_k(x, y)). \end{aligned} \tag{22}$$

Hence, T has a *deqb*-weak contraction property of Theorem 16 and $x = 0$ is the unique fixed point of T.

Theorem 18. *Let (X, d_k) be a complete dislocated quasi extended b-metric space, G and H be closed subsets of X. If $T : G \cup H \longrightarrow G \cup H$ is a cyclic map, continuous mapping and $\lim_{n,m\longrightarrow\infty} k(x_n, x_m) = L > 0$, such that*

$$k(x, y) d_k(Tx, Ty) \leq d_k(x, y) - \varphi(d_k(Tx, Ty)), \tag{23}$$

where $\varphi : [0, \infty) \longrightarrow [0, \infty)$ is a nondecreasing, continuous mapping and $\varphi(t) = 0$ iff $t = 0$.

Then T has a unique fixed point in $G \cap H$.

Proof. Since T is a cyclic map, if taking $x_0 \in G$, then $Tx_0 \in H$ and $T^2 x_0 \in G$. Define a sequence $\{x_n\}$, where $x_n = Tx_{n-1} = T^n x_0$. So we have $x_{2n} \in G$ and $x_{2n-1} \in H$ for $n = 1, 2, 3 \ldots$

By using (23) and for all $n \in N$, we have

$$\begin{aligned} d_k(x_{n+1}, x_n) &\leq k(x_{n+1}, x_n) d_k(x_{n+1}, x_n) \\ &= k(x_{n+1}, x_n) d_k(Tx_n, Tx_{n-1}) \\ &\leq d_k(x_n, x_{n-1}) - \varphi(d_k(Tx_n, Tx_{n-1})) \\ &\leq d_k(x_n, x_{n-1}). \end{aligned} \quad (24)$$

Thus we have $\{d_k(x_{n+1}, x_n)\}$ is a nonincreasing seq-uence of non-negative real numbers. Claim that $\lim_{n\longrightarrow\infty} d_k(x_{n+1}, x_n) = 0$. Suppose $\lim_{n\longrightarrow\infty} d_k(x_{n+1}, x_n) = \beta$.

Since φ is non-decreasing and $k(x_{n+1}, x_n) \geq 1$, then we have

$$\begin{aligned} d_k(x_{n+1}, x_n) &\leq k(x_{n+1}, x_n) d_k(x_{n+1}, x_n) \\ &\leq d_k(x_n, x_{n-1}) - \varphi(d_k(Tx_n, Tx_{n-1})) \\ &\leq d_k(x_n, x_{n-1}) - \varphi(d_k(Tx_n, Tx_{n-1})) \\ &= d_k(x_n, x_{n-1}) - \varphi(d_k(x_{n+1}, x_n)). \end{aligned} \quad (25)$$

Since φ is a continuous mapping then for $n \longrightarrow \infty$, we have $\beta \leq \beta - \varphi(\beta)$. Since $\varphi \geq 0$, thus we get $\varphi(\beta) = 0$. Hence we have $\beta = 0$. Similarly we have $\lim_{n\longrightarrow\infty} d_k(x_n, x_{n+1}) = 0$. □

Now, we have to prove that $\{x_n\}$ is a Cauchy sequence in X.

Suppose $\{x_n\}$ is not a Cauchy, then there exists $\varepsilon > 0$ such that every n, there exists $n_k, m_k > n$ such that

$$\begin{aligned} &d_k(n_k, m_k) \geq \varepsilon \\ &\text{and } d_k(n_{k-1}, m_k) < \varepsilon. \end{aligned} \quad (26)$$

From (23) we have

$$\begin{aligned} &k(n_{k-1}, m_{k-1}) d_k(n_k, m_k) \\ &\quad = k(n_{k-1}, m_{k-1}) d_k(Tn_{k-1}, Tm_{k-1}) \\ &\quad \leq d_k(n_{k-1}, m_{k-1}) - \varphi(d_k(Tn_{k-1}, Tm_{k-1})) \\ &\quad \leq d_k(n_{k-1}, m_{k-1}). \end{aligned} \quad (27)$$

By using (26) and (27), then we have

$$d_k(n_{k-1}, m_{k-1}) \geq \varepsilon k(n_{k-1}, m_{k-1}). \quad (28)$$

By using (2) and (26) we also have that

$$\begin{aligned} &d_k(n_{k-1}, m_{k-1}) \\ &\quad \leq k(n_{k-1}, m_{k-1}) (d_k(n_{k-1}, m_k) + d_k(m_k, m_{k-1})) \\ &\quad < \varepsilon k(n_{k-1}, m_{k-1}) + k(n_{k-1}, m_{k-1}) d_k(m_k, m_{k-1}). \end{aligned} \quad (29)$$

It implies $\lim_{k\longrightarrow\infty} d_k(n_{k-1}, m_{k-1}) \leq \varepsilon \lim_{k\longrightarrow\infty} k(n_{k-1}, m_{k-1})$. By using (28) and (29), we have $\varepsilon \lim_{k\longrightarrow\infty} k(n_{k-1}, m_{k-1}) \leq \lim_{k\longrightarrow\infty} d_k(n_{k-1}, m_{k-1}) \leq \varepsilon \lim_{k\longrightarrow\infty} k(n_{k-1}, m_{k-1})$, so we get $\varepsilon L \leq \lim_{k\longrightarrow\infty} d_k(n_{k-1}, m_{k-1}) \leq \varepsilon L$, thus we obtain $\lim_{k\longrightarrow\infty} d_k(n_{k-1}, m_{k-1}) = \varepsilon L$.

From (23) and (28) we have

$$\begin{aligned} &k(n_{k-1}, m_{k-1}) d_k(n_k, m_k) \\ &\quad \leq d_k(n_{k-1}, m_{k-1}) - \varphi(d_k(n_k, m_k)) \\ &\quad \leq d_k(n_{k-1}, m_{k-1}). \end{aligned} \quad (30)$$

Since φ is a non-decreasing and $\varphi \geq 0$, we have

$$\begin{aligned} \varepsilon k(n_{k-1}, m_{k-1}) &\leq d_k(n_{k-1}, m_{k-1}) - \varphi(d_k(n_k, m_k)) \\ &\leq d_k(n_{k-1}, m_{k-1}) \\ &\quad - \varphi(\varepsilon k(n_{k-1}, m_{k-1})) \\ &\leq d_k(n_{k-1}, m_{k-1}). \end{aligned} \quad (31)$$

For $k \longrightarrow \infty$ and using continuity of φ, we get

$$\varepsilon L \leq \varepsilon L - \varphi(\varepsilon L) \leq \varepsilon L. \quad (32)$$

Since $k(x, y) \geq 1$ and $\lim_{n,m\longrightarrow\infty} k(x_n, x_m) = L$, thus we have $L \geq 1$. However, from (32) we have $\varphi(\varepsilon L) = 0$, this implies $\varepsilon L = 0$. Since $\varepsilon > 0$ then we obtain $L = 0$ which is a contradiction.

Hence $\{x_n\}$ is a Cauchy sequence in X.

Since X complete, there exists $x^* \in X$ such that $d_k(x_n, x^*) \longrightarrow 0$ and $d_k(x^*, x_n) \longrightarrow 0$ for $n \longrightarrow \infty$.

Since the sequence $\{x_{2n}\} \in G$,$\{x_{2n-1}\} \in H$ and G, H closed, we have $x^* \in G \cap H$.

Now we have to prove that x^* is a fixed point of T. By using (2) and (23), we have

$$\begin{aligned} &d_k(Tx^*, x^*) \leq k(Tx^*, x^*) \\ &\quad \cdot (d(Tx^*, Tx_{n-1}) + d(Tx_{n-1}, x^*)) = k(Tx^*, x^*) \\ &\quad \cdot (d(Tx^*, Tx_{n-1}) + d(x_n, x^*)) \leq k(Tx^*, x^*) \\ &\quad \cdot (k(x^*, x_{n-1}) d(Tx^*, Tx_{n-1}) + d(x_n, x^*)) \\ &\quad \leq k(Tx^*, x^*) \\ &\quad \cdot (d(x^*, x_{n-1}) - \varphi(d(Tx^*, Tx_{n-1})) + d(x_n, x^*)) \\ &\quad \leq k(Tx^*, x^*) (d(x^*, x_{n-1}) + d(x_n, x^*)). \end{aligned} \quad (33)$$

Thus for $n \longrightarrow \infty$, we have $d_k(Tx^*, x^*) = 0$, hence $Tx^* = x^*$.

Now we have to show that T has unique fixed point in X. Suppose that u is an another fixed point T,

$$\begin{aligned} d_k(x^*, u) = d_k(Tx^*, Tu) &\leq k(x^*, u) d_k(Tx^*, Tu) \\ &\leq d_k(x^*, u) - \varphi(d_k(Tx^*, Tu)) \\ &= d_k(x^*, u) - \varphi(d_k(x^*, u)). \end{aligned} \quad (34)$$

Thus we get $\varphi(d_k(x^*, u)) \leq 0$. Since $\varphi \geq 0$, we have $\varphi(d_k(x^*, u)) = 0$. Which implies that $d_k(x^*, u) = 0$, so we have $x^* = u$.

Example 19. Let $X=[-1,1]$ and (X,d_k) be a dislocated quasi extended b-metric space which in Example 8. Let $T: G\cup H\longrightarrow G\cup H$ be a function defined by $Tx=-x/2$, where $G=[-1,0], H=[0,1]$. Let $\varphi:[0,\infty)\longrightarrow[0,\infty)$ be a function and defined as, $\varphi(t)=t/8$.

In fact, it is clear that T is cyclic map, indeed $T(G)\subseteq H$ and $T(H)\subseteq G$.

Now, for all $x,y\in X$ we have to show that

$$k(x,y)\,d_k(Tx,Ty)\le d_k(x,y)-\varphi(d_k(Tx,Ty)). \quad (35)$$

$$\begin{aligned}
k(x,y)\,d_k(Tx,Ty) &= \frac{2+|xy|}{2}d_k\left(\frac{-x}{2},\frac{-y}{2}\right) \\
&= \frac{2+|xy|}{2}\left[\left(\left|\frac{-x}{2}\right|+\left|\frac{-y}{2}\right|\right)+\frac{|-x/2|^2}{5}\right. \\
&\quad \left.+\frac{|-y/2|^2}{6}\right] = \frac{2+|xy|}{4}\left[|x|+|y|+\frac{|x|^2}{10}+\frac{|y|^2}{12}\right] \\
&= \frac{1}{4}\left[\left(2(|x|+|y|)+\frac{|x|^2}{10}+\frac{|y|^2}{12}\right)\right. \\
&\quad \left.+|xy|\left((|x|+|y|)+\frac{|x|^2}{10}+\frac{|y|^2}{12}\right)\right] \\
&\le \frac{1}{4}\left[\left(2(|x|+|y|)+\frac{|x|^2}{10}+\frac{|y|^2}{12}\right)\right. \\
&\quad \left.+\left((|x|+|y|)+\frac{|x|^2}{10}+\frac{|y|^2}{12}\right)\right] = \frac{1}{4}\left[3(|x|+|y|)\right. \\
&\quad \left.+\frac{|x|^2}{5}+\frac{|y|^2}{6}\right)\Bigg] \le \frac{3}{4}\left(|x|+|y|+\frac{|x|^2}{5}+\frac{|y|^2}{6}\right) \\
&= \left(|x|+|y|+\frac{|x|^2}{5}+\frac{|y|^2}{6}\right)-\frac{1}{4}\left(|x|+|y|\right. \\
&\quad \left.+\frac{|x|^2}{5}+\frac{|y|^2}{6}\right) \le \left(|x|+|y|+\frac{|x|^2}{5}+\frac{|y|^2}{6}\right) \\
&\quad -\frac{1}{16}\left(|x|+|y|+\frac{|x|^2}{5}+\frac{|y|^2}{6}\right) = \left(|x|+|y|\right. \\
&\quad \left.+\frac{|x|^2}{5}+\frac{|y|^2}{6}\right)-\frac{1}{8}\left(\frac{|x|}{2}+\frac{|y|}{2}+\frac{|x|^2}{10}+\frac{|y|^2}{12}\right) \\
&\le \left(|x|+|y|+\frac{|x|^2}{5}+\frac{|y|^2}{6}\right)-\frac{1}{8}\left(\frac{|x|}{2}+\frac{|y|}{2}\right. \\
&\quad \left.+\frac{|x|^2}{20}+\frac{|y|^2}{24}\right) = \left(|x|+|y|+\frac{|x|^2}{5}+\frac{|y|^2}{6}\right) \\
&\quad -\frac{1}{8}\left(\left|\frac{-x}{2}\right|+\left|\frac{-y}{2}\right|+\frac{|-x/2|^2}{5}+\frac{|-y/2|^2}{6}\right) \\
&= d_k(x,y)-\varphi\left(d_k\left(\frac{-x}{2},\frac{-y}{2}\right)\right) = d_k(x,y) \\
&\quad -\varphi(d_k(Tx,Ty)).
\end{aligned} \quad (36)$$

Hence, T has a *deqb*-weak contraction property of Theorem 18 and $x=0$ is the unique fixed point of T.

Theorem 20. *Let (X,d_k) be a complete dislocated quasi extended b-metric space, G and H be closed subsets of X. If $T: G\cup H\longrightarrow G\cup H$ is a cyclic, continuous mapping and $\lim_{n,m\longrightarrow\infty}k(x_n,x_m)=L>0$, such that*

$$k(Tx,Ty)\,d_k(Tx,Ty)\le d_k(x,y)-\varphi(d_k(x,y)), \quad (37)$$

where $\varphi:[0,\infty)\longrightarrow[0,\infty)$ be a nondecreasing, continuous function and $\varphi(t)=0$ iff $t=0$.

Then T has a unique fixed point in $G\cap H$.

Proof. Since T is a cyclic map, taking $x_0\in G$, then $Tx_0\in H$ and $T^2x_0\in G$. Define a sequence $\{x_n\}$, where $x_n=Tx_{n-1}=T^nx_0$. So we have $x_{2n}\in G$ and $x_{2n-1}\in H$ for $n=1,2,3\ldots$

From (37), then for all $n\in N$ we have

$$\begin{aligned}
d_k(x_{n+1},x_n) &\le k(x_{n+1},x_n)\,d_k(x_{n+1},x_n) \\
&= k(Tx_n,Tx_{n-1})\,d_k(Tx_n,Tx_{n-1}) \\
&\le d_k(x_n,x_{n-1})-\varphi(d_k(x_n,x_{n-1})) \\
&\le d_k(x_n,x_{n-1}).
\end{aligned} \quad (38)$$

Thus we have $\{d_k(x_{n+1},x_n)\}$ be a nonincreasing seq-uence of non-negative real numbers. Claim that $\lim_{n\longrightarrow\infty}d_k(x_{n+1},x_n)=0$. Suppose $\lim_{n\longrightarrow\infty}d_k(x_{n+1},x_n)=\beta$.

Since φ is a nondecreasing and $k(x_{n+1},x_n)\ge 1$, then we have

$$\begin{aligned}
d_k(x_{n+1},x_n) &\le k(x_{n+1},x_n)\,d_k(x_{n+1},x_n) \\
&\le d_k(x_n,x_{n-1})-\varphi(d_k(x_n,x_{n-1})).
\end{aligned} \quad (39)$$

Since φ is continuous then for $\longrightarrow\infty$, we have $\beta\le\beta-\varphi(\beta)$. Since $\varphi\ge 0$, thus we get $\varphi(\beta)=0$. Hence we have $\beta=0$. Similarly we have $\lim_{n\longrightarrow\infty}d_k(x_n,x_{n+1})=0$.

Now, we have to prove that $\{x_n\}$ is a Cauchy sequence in X.

Suppose $\{x_n\}$ is not a Cauchy, then there exists $\varepsilon>0$ such that every n, there exists $n_k,m_k>n$ such that $d_k(n_k,m_k)\ge\varepsilon$ and $d_k(n_{k-1},m_k)<\varepsilon$.

By using (37) we have

$$\begin{aligned}
&k(n_k,m_k)\,d_k(n_k,m_k) \\
&\quad = k(Tn_{k-1},Tm_{k-1})\,d_k(Tn_{k-1},Tm_{k-1}) \\
&\quad \le d_k(n_{k-1},m_{k-1})-\varphi(d_k(n_{k-1},m_{k-1})) \\
&\quad \le d_k(n_{k-1},m_{k-1}).
\end{aligned} \quad (40)$$

Since $d_k(n_k, m_k) \geq \varepsilon$, we get

$$d_k(n_{k-1}, m_{k-1}) \geq \varepsilon k(n_k, m_k). \quad (41)$$

By using (2), we also have that

$$\begin{aligned} & d_k(n_{k-1}, m_{k-1}) \\ & \quad \leq k(n_{k-1}, m_{k-1})(d_k(n_{k-1}, m_k) + d_k(m_k, m_{k-1})) \\ & \quad \leq \varepsilon k(n_{k-1}, m_{k-1}) + k(n_{k-1}, m_{k-1}) d_k(m_k, m_{k-1}). \end{aligned} \quad (42)$$

It implies that $\lim_{k\longrightarrow\infty} d_k(n_{k-1}, m_{k-1}) \leq \varepsilon \lim_{k\longrightarrow\infty} k(n_{k-1}, m_{k-1})$.

Therefore, from (40) and (41), we have

$$\begin{aligned} \varepsilon \lim_{k\longrightarrow\infty} k(n_k, m_k) & \leq \lim_{k\longrightarrow\infty} d_k(n_{k-1}, m_{k-1}) \\ & \leq \varepsilon \lim_{k\longrightarrow\infty} k(n_{k-1}, m_{k-1}). \end{aligned} \quad (43)$$

Thus we have $\varepsilon L \leq \lim_{k\longrightarrow\infty} d_k(n_{k-1}, m_{k-1}) \leq \varepsilon L$, and we obtain

$$\lim_{k\longrightarrow\infty} d_k(n_{k-1}, m_{k-1}) = \varepsilon L. \quad (44)$$

From (40) and (44) and $d_k(n_k, m_k) \geq \varepsilon$, we have

$$\begin{aligned} \varepsilon k(n_k, m_k) & \leq k(n_k, m_k) d_k(n_k, m_k) \\ & \leq d_k(n_{k-1}, m_{k-1}) - \varphi(d_k(n_{k-1}, m_{k-1})) \\ & \leq d_k(n_{k-1}, m_{k-1}). \end{aligned} \quad (45)$$

By using (45) and continuity of φ, then for $k \longrightarrow \infty$ we get

$$\varepsilon L \leq \varepsilon L - \varphi(\varepsilon L) \leq \varepsilon L. \quad (46)$$

Since $k(x, y) \geq 1$ and $\lim_{n,m\longrightarrow\infty} k(x_n, x_m) = L$, thus we have $L \geq 1$. However, from (46) we have $\varphi(\varepsilon L) = 0$, this implies $\varepsilon L = 0$ and since $\varepsilon > 0$ then we obtain $L = 0$ which is a contradiction.

Hence $\{x_n\}$ is a Cauchy sequence in X.

Since X complete, there exists $x^* \in X$ such that $d_k(x_n, x^*) \longrightarrow 0$ for $n \longrightarrow \infty$. Since the sequence $\{x_{2n}\} \in G$, $\{x_{2n-1}\} \in H$ and G, H closed, it implies that $x^* \in G \cap H$.

Now, we have to prove that x^* is a fixed point of T.

$$\begin{aligned} & d_k(Tx^*, x^*) \leq k(Tx^*, x^*) \\ & \quad \cdot (d(Tx^*, Tx_{n-1}) + d(Tx_{n-1}, x^*)) = k(Tx^*, x^*) \\ & \quad \cdot (d(Tx^*, Tx_{n-1}) + d(x_n, x^*)) \leq k(Tx^*, x^*) \\ & \quad \cdot (k(Tx^*, Tx_{n-1}) d(Tx^*, Tx_{n-1}) + d(x_n, x^*)) \\ & \quad \leq k(Tx^*, x^*) \\ & \quad \cdot (d(x^*, x_{n-1}) - \varphi(d(x^*, x_{n-1})) + d(x_n, x^*)) \\ & \quad \leq k(Tx^*, x^*)(d(x^*, x_{n-1}) + d(x_n, x^*)). \end{aligned} \quad (47)$$

Thus for $n \longrightarrow \infty$, we have $d_k(Tx^*, x^*) = 0$, hence $Tx^* = x^*$.

Now we have to show that T has unique fixed point in X. Suppose u is an another fixed point of T,

$$\begin{aligned} d_k(x^*, u) & = d_k(Tx^*, Tu) \\ & \leq k(Tx^*, Tu) d_k(Tx^*, Tu) \\ & \leq d_k(x^*, u) - \varphi(d_k(x^*, u)). \end{aligned} \quad (48)$$

Thus we get $\varphi(d_k(x^*, u)) \leq 0$. Since $\varphi \geq 0$, we have $\varphi(d_k(x^*, u)) = 0$. Which implies that $d_k(x^*, u) = 0$, so we have $x^* = u$. □

Example 21. Let $X = [-1, 1]$ and $(\mathrm{X}, \mathrm{d_k})$ be a dislocated quasi extended b-metric space which in Example 8. Let $T : G \cup H \longrightarrow G \cup H$ be a function defined by $Tx = -x/2$, where $G = [-1, 0]$, $H = [0, 1]$, and let $\varphi : [0, \infty) \longrightarrow [0, \infty)$ be a function and defined as, $\varphi(t) = (7/16)t$.

In fact it clear T is cyclic, since $T(G) \subseteq H$ and $T(H) \subseteq G$.

Now, we have to show that

$$k(Tx, Ty) d_k(Tx, Ty) \leq d_k(x, y) - \varphi(d_k(x, y)).$$

$$\begin{aligned} & k(Tx, Ty) d_k(Tx, Ty) = k\left(\frac{-x}{2}, \frac{-y}{2}\right) d_k\left(\frac{-x}{2}, \frac{-y}{2}\right) \\ & \quad = \frac{8 + |xy|}{8}\left[\left(\left|\frac{-x}{2}\right| + \left|\frac{-y}{2}\right|\right) + \frac{|-x/2|^2}{5}\right. \\ & \quad \left. + \frac{|-y/2|^2}{6}\right] = \frac{8 + |xy|}{16}\left[|x| + |y| + \frac{|x|^2}{10} + \frac{|y|^2}{12}\right] \\ & \quad = \frac{1}{16}\left[\left(8(|x| + |y|) + \frac{|x|^2}{10} + \frac{|y|^2}{12}\right)\right. \\ & \quad \left. + |xy|\left((|x| + |y|) + \frac{|x|^2}{10} + \frac{|y|^2}{12}\right)\right] \\ & \quad \leq \frac{1}{16}\left[\left(8(|x| + |y|) + \frac{|x|^2}{10} + \frac{|y|^2}{12}\right)\right. \\ & \quad \left. + \left((|x| + |y|) + \frac{|x|^2}{10} + \frac{|y|^2}{12}\right)\right] \\ & \quad = \frac{1}{16}\left(9(|x| + |y|) + \frac{|x|^2}{5} + \frac{|y|^2}{6}\right) \leq \frac{9}{16}\left(|x|\right. \\ & \quad \left. + |y| + \frac{|x|^2}{5} + \frac{|y|^2}{6}\right) = \left(|x| + |y| + \frac{|x|^2}{5} + \frac{|y|^2}{6}\right) \\ & \quad - \frac{7}{16}\left(|x| + |y| + \frac{|x|^2}{5} + \frac{|y|^2}{6}\right) = d_k(x, y) \\ & \quad - \varphi(d_k(x, y)). \end{aligned} \quad (49)$$

Hence, T has a *deqb*- weak contraction property of Theorem 20 and $x = 0$ is the unique fixed point of T.

Theorem 22. *Let (X, d_k) be a complete dislocated quasi extended b-metric space, G and H be closed subsets of X and let $0 < \lambda < 1$. If $T : G \cup H \longrightarrow G \cup H$ is a cyclic, continuous function which satisfy the conditions*

$$k(Tx, Ty)\, d_k(Tx, Ty) \leq \lambda\varphi(d_k(x, y)), \quad (50)$$

where $\varphi : [0, \infty) \longrightarrow [0, \infty)$ be a φ nondecreasing and continuous function, $\varphi(t) = 0$ if only if $t = 0$ and $\varphi(\lambda t) \leq \lambda\varphi(t)$, $\varphi^{n+1}(t) \leq \varphi^n(t)$, $\varphi^{n+1}(t) = \varphi(\varphi^n(t))$, for $n = 1, 2, 3, \ldots$, and $\lim_{n,m\longrightarrow\infty} k(x_n, x_m) < 1/\lambda$.

Then T has unique fixed point in $G \cap H$.

Proof. Since T is a cyclic map, for $x_0 \in G$, then $Tx_0 \in H$ and $T^2x_0 \in G$. Define a sequence $\{x_n\}$, where $x_n = Tx_{n-1} = T^n x_0$. So we have $x_{2n} \in G$ and $x_{2n-1} \in H$ for $n = 1, 2, 3\ldots$

Since $k(x, y) \geq 1$ and $0 < \lambda < 1$ then for all $n \in N$, we have

$$\begin{aligned} k(x_n, x_{n+1})\, d_k(x_n, x_{n+1}) &= k(Tx_{n-1}, Tx_n)\, d_k(Tx_{n-1}, Tx_n) \\ &\leq \lambda\varphi(d_k(x_{n-1}, x_n)) \leq \lambda\varphi(\lambda\varphi(d_k(x_{n-2}, x_{n-1}))) \\ &= \lambda^2\varphi^2((d_k(x_{n-2}, x_{n-1}))) \leq \lambda^n\varphi^n((d_k(x_0, x_1))). \end{aligned} \quad (51)$$

We have

$$d_k(x_n, x_{n+1}) \leq k(x_n, x_{n+1})\, d_k(x_n, x_{n+1}) \leq \lambda^n\varphi^n(t_0), \quad (52)$$

where $t_0 = d_k(x_0, x_1)$.

By using (2) and (52), we have

$$\begin{aligned} d_k(x_m, x_m) &\leq k(x_n, x_m)(d_k(x_n, x_{n+1}) \\ &\quad + d_k(x_{n+1}, x_m)) \leq k(x_n, x_m) \\ &\quad \cdot (d_k(x_n, x_{n+1}) + d_k(x_{n+1}, x_m))) \leq k(x_n, x_m) \\ &\quad \cdot (\lambda^n\varphi^n(t_0) + d_k(x_{n+1}, x_m)) \leq k(x_n, x_m)(\lambda^n \\ &\quad \cdot \varphi^n(t_0) + k(x_{n+1}, x_m)(d_k(x_{n+1}, x_{n+2}) \\ &\quad + d_k(x_{n+2}, x_m)) \leq k(x_n, x_m)(\lambda^n\varphi^n(t_0) \\ &\quad + k(x_{n+1}, x_m)\, d_k(x_{n+1}, x_{n+2}) + d_k(x_{n+2}, x_m)) \\ &\leq k(x_n, x_m)(\lambda^n\varphi^n(t_0) + k(x_{n+1}, x_m) \\ &\quad \cdot \left(\lambda^{n+1}\varphi^{n+1}(t_0) + d_k(x_{n+2}, x_m)\right) \leq k(x_n, x_m) \\ &\quad \cdot \left(\lambda^n\varphi^n(t_0) + k(x_{n+1}, x_m)\, \lambda^{n+1}\varphi^{n+1}(t_0)\right. \\ &\quad \left. + k(x_{n+1}, x_m)\, d_k(x_{n+2}, x_m)\right) \leq k(x_n, x_m)(\lambda^n \\ &\quad \cdot \varphi^n(t_0) + k(x_{n+1}, x_m)\, \lambda^{n+1}\varphi^{n+1}(t_0) \\ &\quad + k(x_{n+1}, x_m)\, k(x_{n+2}, x_m)(d_k(x_{n+2}, x_{n+3}) \\ &\quad + d_k(x_{n+3}, x_m)) \leq k(x_n, x_m)(\lambda^n\varphi^n(t_0) \\ &\quad + k(x_{n+1}, x_m)\, \lambda^{n+1}\varphi^{n+1}(t_0) + k(x_{n+1}, x_m) \\ &\quad \cdot k(x_{n+2}, x_m)\left(\lambda^{n+2}\varphi^{n+2}(t_0) + d_k(x_{n+3}, x_m)\right) \\ &\leq k(x_n, x_m)(\lambda^n\varphi^n(t_0) + k(x_{n+1}, x_m) \\ &\quad \cdot \lambda^{n+1}\varphi^{n+1}(t_0) + k(x_{n+1}, x_m)\, k(x_{n+2}, x_m) \\ &\quad \cdot \lambda^{n+1}\varphi^{n+2}(t_0) + \cdots + k(x_{n+1}, x_m)\, k(x_{n+2}, x_m) \\ &\quad \cdots k(x_{m-1}, x_m)\, \lambda^{m-1}\varphi^{m-1}(t_0) \\ &= \sum_{i=0}^{m-n-1} \lambda^{n+i}\varphi^{n+i}(t_0) \prod_{j=0}^{i} k\left(x_{n+j}, x_m\right) \\ &= \sum_{i=n}^{m-1} \lambda^i\varphi^i(t_0) \prod_{j=0}^{i} k\left(x_{n+j}, x_m\right). \end{aligned} \quad (53)$$

We have

$$d_k(x_n, x_m) \leq \sum_{i=n}^{m-1} \lambda^i\varphi^i(t_0) \prod_{j=0}^{i} k\left(x_{n+j}, x_m\right). \quad (54)$$

Let $a_i = \lambda^i\varphi^i(t_0)\prod_{j=0}^{i} k(x_{n+j}, x_m)$.

Since $\varphi^{n+1}(t) \leq \varphi^n(t)$ we have

$$\begin{aligned} \frac{a_{i+1}}{a_i} &= \frac{\varphi^{n+i+1}(t_0)}{\varphi^{n+i}(t_0)} \lambda k(x_{n+i+1}, x_m) \\ &\leq \lambda k(x_{n+i+1}, x_m). \\ \lim_{i\longrightarrow\infty} \frac{a_{i+1}}{a_i} &= \lim_{i\longrightarrow\infty} \frac{\varphi^{n+i+1}(t_0)}{\varphi^{n+i}(t_0)} \lambda k(x_{n+i+1}, x_m) \\ &\leq \lim_{i\longrightarrow\infty} \lambda k(x_{n+i+1}, x_m) < 1. \end{aligned} \quad (55)$$

By using the ratio test criteria, we get $\sum_{i=0}^{\infty} \lambda^{n+i}\varphi^{n+i}(t_0)\prod_{j=0}^{i} k(x_{n+j}, x_m) = \sum_{i=0}^{\infty} a_i$ convergence.

Let $S_p = \sum_{i=0}^{p} a_i$, then from (54), we get

$$\begin{aligned} d_k(x_n, x_m) &\leq \sum_{i=0}^{m-n-1} \lambda^{n+i}\varphi^{n+i}(t_0) \prod_{j=0}^{i} k\left(x_{n+j}, x_m\right) \\ &= \sum_{i=n}^{m-1} a_i = S_{m-1} - S_{n-1} \leq \left|S_{m-1} - S_{n-1}\right|. \end{aligned} \quad (56)$$

Thus for $n, m \longrightarrow \infty$ we get $d_k(x_n, x_m) \longrightarrow 0$. Hence $\{x_n\}$ is a Cauchy sequence in X.

Since X complete, there exists $x^* \in X$ such that $d_k(x_n, x^*) \longrightarrow 0$ for $n \longrightarrow \infty$.

Similarly, we can have $d_k(x^*, x_n) \longrightarrow 0$.

Since the sequence $\{x_{2n}\} \in G$, $\{x_{2n-1}\} \in H$ and G, H closed, thus we have $x^* \in G \cap H$.

Now we prove that x^* is a fixed point of T. Using (2) and (11) we have

$$\begin{aligned} d_k(Tx^*, x^*) &\le k(Tx^*, x^*) \\ &\cdot (d(Tx^*, Tx_{n-1}) + d(Tx_{n-1}, x^*)) \le k(Tx^*, x^*) \\ &\cdot (k(Tx^*, Tx_{n-1}) d(Tx^*, Tx_{n-1}) + d(Tx_{n-1}, x^*)) \\ &\le k(Tx^*, x^*)(\lambda\varphi(d(x^*, x_{n-1})) + d(x_n, x^*)). \end{aligned} \tag{57}$$

Using continuity of φ, and $\varphi(0) = 0$, then for $n \longrightarrow \infty$, we have $d_k(Tx^*, x^*) \le \lambda k(Tx^*, x^*)\varphi(0) \le 0$.

Thus $d_k(Tx^*, x^*) = 0$, hence $Tx^* = x^*$.

Now we have to show that T has unique fixed point in X. Suppose u is an another fixed point of T,

$$\begin{aligned} d_k(x^*, u) &= d_k(Tx^*, Tu) \\ &\le k(Tx^*, Tu) d_k(Tx^*, Tu) \\ &\le \lambda\varphi(d_k(x^*, u)). \end{aligned} \tag{58}$$

We have

$$(1-\lambda)\varphi d_k(x^*, u) \le 0. \tag{59}$$

Since $1-\lambda > 0$ thus we get $\varphi(d_k(x^*, u)) \le 0$. Since $\varphi \ge 0$, then $\varphi(d_k(x^*, u)) = 0$. Which implies that $d_k(x^*, u) = 0$, so we have $x^* = u$. □

Example 23. Let $X = [-1, 1]$ and (X, d_k) be a dislocated quasi extended b-metric space which in Example 8. Let $T : G \cup H \longrightarrow G \cup H$ be a function defined by $Tx = -x^3/8$, where $G = [-1, 0]$, $H = [0, 1]$. Let $\varphi : [0, \infty) \longrightarrow [0, \infty)$ be a function and defined as, $\varphi(t) = (3/4)t^2$ and $\lambda = 1/4$.

In fact, it clear that $\varphi(\lambda t) \le \lambda\varphi(t)$, $\varphi^{n+1}(t) \le \varphi^n(t)$ and T is cyclic, since $T(G) \subseteq H$ and $T(H) \subseteq G$.

Since $x_n, x_m \in X = [-1, 1]$ and $k(x_n, x_m) = (2 + x_n x_m)/2$, it is easy to show that $\lim_{n,m\longrightarrow\infty} k(x_n, x_m) < 1/\lambda$.

Now, we have to show that

$$\begin{aligned} &k(Tx, Ty) d_k(Tx, Ty) \le \lambda\varphi(d_k(x, y)). \\ &k(Tx, Ty) d_k(Tx, Ty) \\ &\quad = k\left(\frac{-x^3}{8}, \frac{-y^3}{8}\right) d_k\left(\frac{-x^3}{8}, \frac{-y^3}{8}\right) \\ &\quad = \frac{2 + \left|(-x^3/8)(-y^3/8)\right|}{2} d_k\left(\frac{-x^3}{8}, \frac{-y^3}{8}\right) \\ &\quad \le \frac{2 + \left|x^3 y^3\right|}{2} d_k\left(\frac{-x^3}{8}, \frac{-y^3}{8}\right) \\ &\quad \le \frac{2 + \left|x^3 y^3\right|}{2}\left[\left(\left|\frac{-x^3}{8}\right| + \left|\frac{-y^3}{8}\right|\right) + \frac{\left|-x^3/8\right|^2}{5}\right. \\ &\quad \left. + \frac{\left|-y^3/8\right|^2}{6}\right] = \frac{2 + \left|x^3 y^3\right|}{16}\left[\left|x^3\right| + \left|y^3\right| + \frac{\left|x^3\right|^2}{40}\right. \\ &\quad \left. + \frac{\left|y^3\right|^2}{48}\right] = \frac{3}{16}\left[\left|x^3\right| + \left|y^3\right| + \frac{\left|x^3\right|^2}{40} + \frac{\left|y^3\right|^2}{48}\right] \\ &\quad \le \frac{3}{16}\left(\left|x^2\right| + \left|y^2\right| + \frac{\left|x^2\right|^2}{25} + \frac{\left|y^2\right|^2}{36}\right) \le \frac{3}{16}\left((|x|\right. \\ &\quad \left. + |y| + \frac{|x|^2}{5} + \frac{|y|^2}{6}\right)^2 = \frac{1}{4}\varphi(d_k(x, y)). \end{aligned} \tag{60}$$

Hence, T has a *deqb*-weak contraction property of Theorem 22 and $x = 0$ is the unique fixed point of T.

4. Conclusion

In this article, we considered and proved the fixed point theorems for cyclic weakly contraction mapping in complete dislocated quasi extended b-metric space. These results generalize the recent results of Samreen [14] and Rahman [9], which was in our results more general in the sense by utilizing dislocated quasi extended b-metric and cyclic weakly contraction. Furthermore, In Theorems 16, 18, 20, and 22 one can derive several consequences in dislocated quasi b-metric by letting $k(x, y) = K \ge 1$ and in dislocated quasi metric by letting $k(x, y) = 1$.

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares that there are no conflicts of interest.

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