Hindawi
Journal of Function Spaces
Volume 2019, Article ID 2038757, 6 pages
https://doi.org/10.1155/2019/2038757

Research Article

Some Noncontractive Maps on Incomplete Metric Spaces Have Also Fixed Points

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Received 6 June 2019; Accepted 24 July 2019; Published 6 August 2019

Academic Editor: Pasquale Vetro

In this article, we discuss a new version of metric fixed point theory. The application of this newly introduced concept is to find some fixed point results where many well-known results in literature cannot be applied. We give some examples to illustrate the given concepts and obtained results.

1. Introduction

The fixed point theory has a long history. After the Banach contraction principle [1], there has been a huge development in metric fixed point theory. This principle has been generalized and extended by many researchers, either by changing the contraction condition or the underlying space. For more details, see [2–11]. The theory of fixed points in ordered sets was started first by Turinici [12]. In 2004, Ran and Reurings [13] have generalized the Banach contraction principle in the setting of ordered sets. The key feature in Ran-Reurings theorem is that the contractive condition on the nonlinear map is only assumed to hold on the comparable elements instead of the whole space as in Banach contraction principle. In 2005, Nieto, Rodríguez-López [14] proved a fixed point theorem by relaxing some conditions in Ran-Reurings [13]. In 2008, Suzuki [15] proved a fixed point theorem by assuming contraction condition on those elements which satisfy the given condition. Besides all these results, there exist various maps on metric spaces which possess a fixed point. This is because either the underlying metric space is not complete or the contraction condition is not satisfied. In this paper, we have tackled both the problems in setting of ordered metric spaces.

First, we recall some well-known results.

Theorem 1 ((Banach contraction principle) [1]). *Let (X, d) be a complete metric space and $f : X \longrightarrow X$ be a self-mapping such that for all $\varsigma, \mu \in X$,*

$$d\left(f(\varsigma), f(\mu)\right) \leq \alpha d(\varsigma, \mu), \tag{1}$$

where $\alpha \in [0, 1)$. Then f has a unique fixed point in X.

Theorem 2 ((Ran-Reurings theorem) [13]). *Let $(X, d, \leq$ be a partially ordered complete metric space such that every pair $(\varsigma, \mu) \in X^2$ has a lower bound and an upper bound. Suppose $F : X \longrightarrow X$ is a continuous and monotone map (i.e., either order-preserving or order-reversing) such that*

(1) *$d(F(\varsigma), F(\mu)) \leq cd(\varsigma, \mu)$ for all $\varsigma \geq \mu$, where $0 < c < 1$;*

(2) *there exists $\varsigma_0 \in X$ such that $\varsigma_0 \leq F(\varsigma_0)$ or $\varsigma_0 \geq F(\varsigma_0)$.*

Then F has a unique fixed point, say $\bar{\varsigma} \in X$. Moreover, for every $\varsigma \in X$,

$$\lim_{n \longrightarrow \infty} F^n(\varsigma) = \bar{\varsigma}. \tag{2}$$

Theorem 3 ((Suzuki) [15]). *Let (X, d) be a complete metric space and $f : X \longrightarrow X$ be a self-map. Define $\psi : [0, 1) \longrightarrow (1/2, 1]$ by*

$$\psi(r) = \begin{cases} 1, & 0 \le r \le \frac{1}{2}\left(\sqrt{5}-1\right), \\ \frac{1-r}{r^2}, & \frac{1}{2}\left(\sqrt{5}-1\right) < r < \frac{1}{\sqrt{2}}, \\ \frac{1}{1+r}, & \frac{1}{\sqrt{2}} \le r < 1. \end{cases} \quad (3)$$

Assume there exists $\alpha \in [0, 1)$ such that

$$\psi(r)\, d(\varsigma, f(\varsigma)) \le d(\varsigma, \mu) \Longrightarrow d(f(\varsigma), f(\mu)) \le \alpha d(\varsigma, \mu), \quad (4)$$

for all $\varsigma, \mu \in X$. Then f has a unique fixed point in X.

Definition 4 (see [12]). A sequence $\{\varsigma_n\}$ in an ordered set $(X, \preceq)$ is said to be increasing or ascending (resp. strictly increasing) if for $m \le n$, $\varsigma_m \preceq \varsigma_n$ (resp. $\varsigma_m \preceq \varsigma_n$ and $\varsigma_m \ne \varsigma_n$. We denote it by $\varsigma_m \prec \varsigma_n$).

Definition 5 (see [16]). An ordered metric space $(X, d, \preceq)$ is said to be $\overline{O}$-complete, if every increasing Cauchy sequence in X converges in X. In an ordered metric space, completeness implies $\overline{O}$-completeness.

In this paper, we introduce a new contraction condition which is assumed to hold for comparable elements of a subset of whole space. Our result guarantees the existence of a fixed point in such cases where neither Banach contraction principle nor Ran-Reurings and other theorems can be applied. We prove that, under certain conditions, noncontractive maps on incomplete metric spaces have also fixed points. We give examples to illustrate our concepts and obtained results. We also discuss some classes of contraction maps.

2. Main Results

First, we present the following definitions along with some examples.

Definition 6. Let $(X, \preceq)$ be an ordered set and $f : X \longrightarrow X$ be a self-map. A subset $A \subseteq X$ is said to be a t-subset of X with respect to f if

$$\varsigma \preceq f(\varsigma) \quad \text{for all } \varsigma \in A. \quad (5)$$

Example 7. Let $X = \mathbb{R}$ be equipped with the natural ordering $\le$. Define $f : \mathbb{R} \longrightarrow \mathbb{R}$ by $f(\varsigma) = \varsigma^2$. The following are t-subsets of X with respect to f:

(i) $A = [1, \infty)$;
(ii) $A = (-\infty, 0]$.

Example 8. Let $X = C[a, b]$. Equip X with the ordering $\preceq$ defined as $f \preceq g$ iff $f(t) \le g(t)$ for each $t \in [a, b]$ and define $F : X \longrightarrow X$ by $F(f) = 3f + 1$. Then $A = \{f \in X : f(t) \ge 0, \text{for each}, t \in [a, b]\}$, is a t-subset of X with respect to F.

Definition 9. Let $(X, d, \preceq)$ be a partially ordered metric space and $f : X \longrightarrow X$ be a self-map. For any subset $A \subseteq X$ of X, f is said to be a t-contraction with respect to A if for all $\varsigma, \mu \in A$ with $\varsigma \prec \mu$, we have

$$d(\mu, f\mu) \le \alpha d(\varsigma, f\varsigma), \quad (6)$$

where $\alpha \in (0, 1)$.

The following examples illustrate Definition 9.

Example 10. Let us take $X = (-5, 5) \cap \mathbb{Q}$. Endow X with the usual metric of $\mathbb{R}$ and the natural ordering $\le$. Let us consider the subset $A \subseteq X$ defined by $A = \{a_n : a_{n+1} = a_n/3,\ n \ge 0 \text{ with } a_0 = -2\}$. Then $A = \{-2, -2/3, -2/9, \ldots\}$. Define $f : X \longrightarrow X$ by

$$f(\varsigma) = \begin{cases} \frac{\varsigma}{3}, & \varsigma \in A, \\ \varsigma, & \varsigma \in A^c. \end{cases} \quad (7)$$

For any $\varsigma, \mu \in A$ with $\varsigma < \mu$, we have $d(\varsigma, f(\varsigma)) = -2\varsigma/3$, $d(\mu, f(\mu)) = -2\mu/3$ and $(3\mu - \varsigma) \ge 0$. Take $\alpha = 1/3$. For such ς, μ, we get

$$\alpha d(\varsigma, f(\varsigma)) - d(\mu, f(\mu)) = -\frac{2\varsigma}{9} + \frac{2\mu}{3} = \frac{2}{9}(3\mu - \varsigma) \ge 0. \quad (8)$$

Hence f is a t-contraction with respect to A.

Example 11. Let $X = (-\infty, 0) \cap \mathbb{Q}$ be endowed with the usual metric of $\mathbb{R}$ and the natural ordering $\le$. Define $f : X \longrightarrow X$ by $f(\varsigma) = 3\varsigma + 1$ and take $A = \{-41, -14, -5, -2, -1\}$. Clearly, f is a t-contraction with respect to A.

Our first main result is as follows.

Theorem 12. *Let $(X, d, \preceq)$ be an $\overline{O}$-complete ordered metric space and $f : X \longrightarrow X$ be a self-map. Let $A \subseteq X$ be any nonempty t-subset with respect to f. Suppose that*

(a) *f is a t-contraction with respect to A;*
(b) *f is continuous;*
(c) *$f(A) \subseteq A$.*

Then f has at least one fixed point in X.

Proof. The subset A is nonempty. Let $\varsigma_0 \in A$, so $\varsigma_0 \preceq f(\varsigma_0)$. If $\varsigma_0 = f(\varsigma_0)$, the proof is completed. Otherwise, choose $\varsigma_1 = f(\varsigma_0)$. By assumption (c), we have $\varsigma_1 = f(\varsigma_0) \in A$. By definition of a t-set, $\varsigma_1 \preceq f(\varsigma_1)$. If $\varsigma_1 = f(\varsigma_1)$, the proof is completed. Otherwise, choose $\varsigma_2 = f(\varsigma_1)$. Therefore, $\varsigma_2 = f(\varsigma_1) \in A$. Continuing in this process, we get a strictly increasing sequence $\{\varsigma_n\} \in A$ such that

$$\varsigma_{n+1} = f(\varsigma_n). \quad (9)$$

As $\varsigma_0, \varsigma_1 \in A$ with $\varsigma_0 \prec \varsigma_1$, then by (6) we have

$$d(\varsigma_1, f(\varsigma_1)) \le \alpha d(\varsigma_0, f(\varsigma_0)). \quad (10)$$

Again, as $\varsigma_1, \varsigma_2 \in A$ with $\varsigma_1 \prec \varsigma_2$, we have

$$d(\varsigma_2, f(\varsigma_2)) \leq \alpha d(\varsigma_1, f(\varsigma_1)). \tag{11}$$

Using (10) in (11), we have

$$d(\varsigma_2, f(\varsigma_2)) \leq \alpha^2 d(\varsigma_0, f(\varsigma_0)). \tag{12}$$

Continuing in this way, we get

$$d(\varsigma_n, f(\varsigma_n)) \leq \alpha^n d(\varsigma_0, f(\varsigma_0)). \tag{13}$$

Now, we show that $\{\varsigma_n\}$ is a Cauchy sequence. For $n < m$, by using triangular inequality, (9) and (13), we get

$$\begin{aligned} d(\varsigma_n, \varsigma_m) &\leq d(\varsigma_n, \varsigma_{n+1}) + d(\varsigma_{n+1}, \varsigma_{n+2}) + \cdots \\ &\quad + d(\varsigma_{m-1}, \varsigma_m), \\ &= d(\varsigma_n, f(\varsigma_n)) + d(\varsigma_{n+1}, f(\varsigma_{n+1})) + \cdots \\ &\quad + d(\varsigma_{m-1}, f(\varsigma_{m-1})), \\ &\leq \alpha^n d(\varsigma_0, f(\varsigma_0)) + \alpha^{n+1} d(\varsigma_0, f(\varsigma_0)) + \cdots \\ &\quad + \alpha^{m-1} d(\varsigma_0, f(\varsigma_0)), \\ &\leq \left(\alpha^n + \alpha^{n+1} + \cdots + \alpha^{m-n-1}\right) d(\varsigma_0, f(\varsigma_0)), \\ &\leq \frac{\alpha^n}{1-\alpha} d(\varsigma_0, f(\varsigma_0)). \end{aligned} \tag{14}$$

This shows that $\{\varsigma_n\}$ is an increasing Cauchy sequence in A and hence in X. But X is $\overline{O}$-complete; therefore, there exists $u \in X$ such that

$$\varsigma_n \longrightarrow u \quad \text{as } n \longrightarrow \infty. \tag{15}$$

Since f is continuous, we have

$$f(\varsigma_n) \longrightarrow f(u). \tag{16}$$

Taking $n \longrightarrow \infty$ in (13), we have

$$\lim_{n \longrightarrow \infty} d(\varsigma_n, f(\varsigma_n)) = 0. \tag{17}$$

So $d(\lim_{n \longrightarrow \infty} \varsigma_n, \lim_{n \longrightarrow \infty} f(\varsigma_n)) = 0$. By using (15) and (16), $f(u) = u$. Thus, u is a fixed point of f in X. □

Now, we present an example illustrating Theorem 12, where Banach contraction principle, Ran-Reurings theorem, Suzuki theorem, and other results cannot be applied.

Example 13. Let $X = (-4, \infty)$ be endowed with the usual metric and the natural ordering $\leq$. Obviously, $(X, d, \leq)$ is an $\overline{O}$-complete ordered metric space. Define $f : X \longrightarrow X$ by

$$f(\varsigma) = \begin{cases} \dfrac{\varsigma}{2}, & \varsigma < 0 \\ \varsigma^2, & \varsigma \geq 0. \end{cases} \tag{18}$$

Thus f is a continuous mapping on X. Considering the subset $A \subseteq X$ as $A = \{a_n : a_{n+1} = a_n/2,\ n \geq 0 \text{ with } a_0 = -3\}$, then $A = \{-3, -3/2, -3/4, -3/8, \ldots\}$. Clearly, A is a t-subset of X with respect to f and $f(A) \subseteq A$. Now, we show that f is a t-contraction with respect to A. For any $\varsigma, \mu \in A$ with $\varsigma < \mu$, we have $d(\varsigma, f(\varsigma)) = -\varsigma/2$, $d(\mu, f(\mu)) = -\mu/2$, and $(2\mu - \varsigma) \geq 0$. Set $\alpha = 1/2$. For such $\varsigma, \mu \in A$, we get

$$\begin{aligned} \alpha d(\varsigma, f(\varsigma)) - d(\mu, f(\mu)) &= -\frac{\varsigma}{4} + \frac{\mu}{2} = \frac{1}{4}(2\mu - \varsigma) \\ &\geq 0. \end{aligned} \tag{19}$$

Hence f is a t-contraction with respect to A. Thus all the conditions of Theorem 12 are satisfied, and f has a fixed point.

Our second main result is as follows.

Theorem 14. *Let $(X, d, \preceq)$ be a $\overline{O}$-complete ordered metric space and $f : X \longrightarrow X$ be a self-map. Let $A \subseteq X$. Suppose that*

(i) *$f(A) \subseteq A$;*

(ii) *f is nondecreasing on A;*

(iii) *f is continuous on X;*

(iv) *there exists $\varsigma_0 \in A$ such that $\varsigma_0 \preceq f(\varsigma_0)$;*

(v) *f is a t-contraction with respect to A.*

Then f has at least one fixed point in X.

Proof. As $\varsigma_0 \in A$ with $\varsigma_0 \preceq f(\varsigma_0)$. If $\varsigma_0 = f(\varsigma_0)$, the proof is completed. Otherwise, choose $f(\varsigma_0) = \varsigma_1 \in A$ such that $\varsigma_0 \prec \varsigma_1$. Continuing this process and using monotonicity of f in A, we get a strictly increasing Cauchy sequence $\{\varsigma_n\}$ in A such that $\varsigma_{n+1} = f(\varsigma_n)$. As $\varsigma_0 \prec \varsigma_1$, using (6), we have

$$d(\varsigma_1, f(\varsigma_1)) \leq \alpha d(\varsigma_0, f(\varsigma_0)). \tag{20}$$

Again as $\varsigma_1, \varsigma_2 \in A$ with $\varsigma_1 \prec \varsigma_2$, we have

$$d(\varsigma_2, f(\varsigma_2)) \leq \alpha d(\varsigma_1, f(\varsigma_1)). \tag{21}$$

Using (20) in (21), we get

$$d(\varsigma_2, f(\varsigma_2)) \leq \alpha^2 d(\varsigma_0, f(\varsigma_0)). \tag{22}$$

Continuing in this way, we get

$$d(\varsigma_n, f(\varsigma_n)) \leq \alpha^n d(\varsigma_0, f(\varsigma_0)). \tag{23}$$

As Theorem 12, $\{\varsigma_n\}$ is an increasing Cauchy sequence in A and hence in X. But X is $\overline{O}$-complete, so there exists $u \in X$ such that

$$\varsigma_n \longrightarrow u, \quad \text{as } n \longrightarrow \infty. \tag{24}$$

Since f is continuous on X,

$$f(\varsigma_n) \longrightarrow f(u). \tag{25}$$

By taking $n \longrightarrow \infty$ in (23) and using continuity of metric d, we get

$$\lim_{n\longrightarrow\infty} d(\varsigma_n, f(\varsigma_n)) = 0. \quad (26)$$

Then

$$d\left(\lim_{n\longrightarrow\infty} \varsigma_n, \lim_{n\longrightarrow\infty} f(\varsigma_n)\right) = 0. \quad (27)$$

By using (24) and (25), we get $f(u) = u$, and so u is a fixed point of f in X. □

The following examples show that Theorem 12 cannot be applied, while the existence of a fixed point can be obtained using Theorem 14.

Example 15. Let $X = \mathbb{R}$ be endowed with the usual metric and the natural ordering $\leq$. Then $(X, d, \preceq)$ is an $\overline{O}$-complete metric space. Let us define $f : X \longrightarrow X$ by $f(\varsigma) = \varsigma^3$. Clearly, f is a continuous function. Note that neither Banach contraction principle, nor Ran-Reurings theorem, nor Suzuki result can be applied. Now, we show that Theorem 14 can work in this case. Choose the subset $A \subseteq X$ such that

$$A = \left\{a_n : a_{n+1} = a_n^3, \text{ for } n \geq 0, \text{ with } a_0 = -2\right\} \cup \{-1\}. \quad (28)$$

Then $A = \{\ldots, -512, -8, -2, -1\}$. Clearly, A is not a t-subset with respect to f. Thus, Theorem 12 cannot be applied. Now, $f(A) \subseteq A$ and f is nondecreasing in A. So, it remains to prove that (6) is satisfied. For any $\varsigma, \mu \in A$ with $\varsigma < \mu$, we have $d(\varsigma, f(\varsigma)) = \varsigma - \varsigma^3$ and $d(\mu, f(\mu)) = \mu - \mu^3$. Choose $\alpha = 1/3$. We have

$$\begin{aligned} \alpha d(\varsigma, f(\varsigma)) - d(\mu, f(\mu)) &= \frac{1}{3}\left(\varsigma - \varsigma^3\right) - \left(\mu - \mu^3\right) \\ &= \frac{1}{3}\left[\left(3\mu^3 - \varsigma^3\right) - (3\mu - \varsigma)\right] \geq 0. \end{aligned} \quad (29)$$

Thus, all the conditions of Theorem 14 are satisfied, and so f has a fixed point in X.

Example 16. Let $X = \mathbb{R}$ be endowed with the usual metric and the natural ordering $\leq$. Then $(X, d, \preceq)$ is an $\overline{O}$-complete metric space. Let us define $f : X \longrightarrow X$ by

$$f(\varsigma) = \begin{cases} -\varsigma^2 - \varsigma, & \varsigma \leq -2 \\ \varsigma^3 + 6, & \varsigma > -2. \end{cases} \quad (30)$$

Mention that f is a continuous function on X. Choose the subset $A \subseteq X$ such that

$$A = \left\{a_n : a_{n+1} = -a_n^2 - a_n, \text{ for } n \geq 0, \text{ with } a_0 = -3\right\} \cup \{-2\}. \quad (31)$$

Then $A = \{\ldots, -30, -6, -3, -2\}$. Clearly, $f(A) \subseteq A$ and f is nondecreasing in A. Now, it remains to prove that (6) is satisfied. For any $\varsigma, \mu \in A$ with $\varsigma < \mu$, we have $d(\varsigma, f(\varsigma)) = \varsigma^2 + 2\varsigma$ and $d(\mu, f(\mu)) = \mu^2 + 2\mu$. Choose $\alpha = 1/2$. We have

$$\begin{aligned} \alpha d(\varsigma, f(\varsigma)) - d(\mu, f(\mu)) &= \frac{1}{2}\left(\varsigma^2 + 2\varsigma\right) - \left(\mu^2 + 2\mu\right) \\ &= \frac{1}{2}\left[\left(\varsigma^2 - 2\mu^2\right) - (4\mu - 2\varsigma)\right] \geq 0. \end{aligned} \quad (32)$$

Thus, all the conditions of Theorem 14 are satisfied, so f has a fixed point in X.

Corollary 17. *If all the conditions of Theorem 12 are satisfied, then the fixed point of f exists and lies in $\overline{A}$ (closure of A).*

Proof. Theorem 12 guarantees the existence of a fixed point in X. From (15), $\{\varsigma_n\}$ is a sequence in A converging to $u \in X$; therefore, $u \in \overline{A}$. □

Theorem 18. *Let $(X, d, \preceq)$ be an ordered metric space not necessarily complete and $f : X \longrightarrow X$ be a continuous self-map such that $f(X)$ is $\overline{O}$-complete and $\varsigma \preceq f(\varsigma)$ for all $\varsigma \in f(X)$. Suppose that*

$$d(\mu, f(\mu)) \leq \alpha d(\varsigma, f(\varsigma)), \quad \forall \varsigma, \mu \in f(X) \text{ with } \varsigma \prec \mu, \quad (33)$$

where $\alpha \in (0, 1)$. Then f has at least one fixed point in $f(X)$. Also, every strict upper bound of fixed points of f in $f(X)$ is also a fixed point of f.

Proof. Let $\varsigma_0 \in f(X)$. By assumption, $\varsigma_0 \preceq f(\varsigma_0)$. If $\varsigma_0 = f(\varsigma_0)$, the proof is completed. Otherwise, let $\varsigma_1 = f(\varsigma_0)$. Then $\varsigma_1 \in f(X)$ and we have $\varsigma_1 \preceq f(\varsigma_1)$. If $\varsigma_1 = f(\varsigma_1)$, again the proof is completed. Otherwise, continuing in the process, we get an increasing sequence $\{\varsigma_n\}$ in $f(X)$ such that $\varsigma_{n+1} = f(\varsigma_n)$. Now, $\varsigma_0, \varsigma_1 \in f(X)$ with $\varsigma_0 \prec \varsigma_1$; then, by (33), we have

$$d(\varsigma_1, f(\varsigma_1)) \leq \alpha d(\varsigma_0, f(\varsigma_0)). \quad (34)$$

Again, as $\varsigma_1, \varsigma_2 \in f(X)$ with $\varsigma_1 \prec \varsigma_2$, we have

$$d(\varsigma_2, f(\varsigma_2)) \leq \alpha d(\varsigma_1, f(\varsigma_1)). \quad (35)$$

Using (34), we get

$$d(\varsigma_2, f(\varsigma_2)) \leq \alpha^2 d(\varsigma_0, f(\varsigma_0)). \quad (36)$$

Continuing in this way, we get

$$d(\varsigma_n, f(\varsigma_n)) \leq \alpha^n d(\varsigma_0, f(\varsigma_0)). \quad (37)$$

As Theorem 12, $\{\varsigma_n\}$ is an increasing Cauchy sequence in $f(X)$, which is $\overline{O}$-complete, so there exists $z \in f(X)$ such that

$$\varsigma_n \longrightarrow z \quad \text{as } n \longrightarrow \infty. \quad (38)$$

The continuity of f implies that

$$f(\varsigma_n) \longrightarrow f(z) \quad \text{as } n \longrightarrow \infty. \quad (39)$$

Letting $n \longrightarrow \infty$ in (37) and using (38) and (39), we get

$$\lim_{n \longrightarrow \infty} d(\varsigma_n, f(\varsigma_n)) = 0. \quad (40)$$

Hence $d(\lim_{n\longrightarrow\infty}\varsigma_n, \lim_{n\longrightarrow\infty} f(\varsigma_n)) = 0$. We deduce that $f(z) = z$. Thus z is a fixed point of f in $f(X)$.

Now, let k be any strict upper bound of z in $f(X)$, *i.e.*, $k, z \in f(X)$ such that $z \prec k$. By (33), we have

$$d(k, f(k)) \leq \alpha d(z, f(z)) = 0 \quad (41)$$

This implies that $k = f(k)$, so k is also a fixed point of f in $f(X)$. □

Example 19. Let $X = (-1, \infty)$ be equipped with the natural ordering $\leq$ and the usual metric d. Let $f : X \longrightarrow X$ be a self-map defined by

$$f(\varsigma) = \begin{cases} \varsigma^2, & -1 < \varsigma < 0 \\ \varsigma, & \varsigma \geq 0. \end{cases} \quad (42)$$

Clearly, f is continuous and $f(X) = [0, \infty)$ is an $\overline{O}$-complete metric space. It can be seen that f satisfies all the assumptions of Theorem 18, and so there exists a fixed point of f in $f(X)$. It can be seen that the existing known classical fixed point results in literature cannot be applied.

Corollary 20. *If, in Theorem 18, we replace the $\overline{O}$-completeness of $f(X)$ by the fact that $\overline{A}$ (closure of A) is complete, then f has a fixed point.*

Proof. From (15), we have proved that $\{\varsigma_n\}$ is a Cauchy sequence in A and hence in $\overline{A}$. Since $\overline{A}$ is complete, $\{\varsigma_n\}$ converges to $u \in \overline{A}$. We have proved in Theorem 12 that u is a fixed point. □

Remarks. Let $(X, d, \preceq)$ be a partially ordered metric space. Consider

$$B_t(f, X) = \{f : X \longrightarrow X : d(f\varsigma, fy) \leq \alpha d(\varsigma, \mu);\ \varsigma, \mu \in X,\ \alpha \in (0, 1)\}, \quad (43)$$

and

$$R_t(f, X) = \{f : X \longrightarrow X : d(f\varsigma, f\mu) \leq \alpha d(\varsigma, \mu);\ \varsigma, \mu \in X,\ \varsigma \preceq \mu,\ \alpha \in (0, 1)\}. \quad (44)$$

We have $B_t(f, X) \subseteq R_t(f, X)$. Further, let A be a t-subset of X such that $f(A) \subseteq A$. Define

$$Q(f, A, X) = \{f : X \longrightarrow X : d(\mu, f\mu) \leq \alpha d(\varsigma, f\varsigma);\ \varsigma, \mu \in A,\ \varsigma \prec \mu,\ \alpha \in (0, 1)\}. \quad (45)$$

We shall show that $R_t(f, X) \subseteq Q(f, A, X)$ provided that A exists.

For this, let $g \in R_t(f, X)$. Suppose there exists $A \subseteq X$ such that $g(A) \subseteq A$ and A is a t-subset with respect to g. Then $d(g\varsigma, g\mu) \leq \alpha d(\varsigma, \mu)$ for all $\varsigma, \mu \in X$ with $\varsigma \preceq \mu$. In particular,

$$d(g\varsigma, g\mu) \leq \alpha d(\varsigma, \mu), \quad \forall \varsigma, \mu \in A \text{ with } \varsigma \prec \mu. \quad (46)$$

As $\varsigma, \mu \in A$, $\varsigma \preceq g\varsigma$ and $\mu \preceq g\mu$. Also, $g(A) \subseteq A$ implies that $g\varsigma, g\mu \in A$. Since (46) holds for all $\mu \in A$ such that $\varsigma \prec \mu$, we can replace μ by $g\varsigma$. From (46), we get

$$d(\mu, g\mu) \leq \alpha d(\varsigma, g\varsigma), \quad \forall \varsigma, \mu \in A,\ \varsigma \prec \mu. \quad (47)$$

This shows that $g \in Q(f, A, X)$. Hence $R_t(f, X) \subseteq Q(f, A, X)$. It should be noted that the converse is not true in general.

3. Conclusion

In this article, we have introduced new contraction type mappings assumed to hold only on comparable elements of a subset of whole space. By using this concept, we have guaranteed the existence of a fixed point in such cases where Banach contraction principle, Ran-Reurings theorem, Suzuki theorem, and others remain silent. The field for applying this result is not restricted to only contractive maps. This result can be applied to expansive and nonexpansive maps too. Further, it is clear that the fixed point is unique if the t-subset of X with respect to f is connected and either bounded above or bounded below, but not both. However, proof for uniqueness needs further explorations.

Inclosing, we want to bring to the reader attention the following open questions.

Question 1. Is it possible to replace the continuity hypothesis in Theorems 12 and 18, by a weaker condition?

Question 2. In Theorems 12 and 18, under what condition we will have uniqueness of the fixed point? If such condition exists, does it give us uniqueness in an orbit or in the whole space?

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

Acknowledgments

The third author would like to thank Prince Sultan University for funding this work through research group Nonlinear Analysis Methods in Applied Mathematics (NAMAM) group number RG-DES-2017-01-17.

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