Hindawi
Journal of Function Spaces
Volume 2019, Article ID 2426978, 5 pages
https://doi.org/10.1155/2019/2426978

Research Article

Continuous $*$-K-G-Frame in Hilbert C^*-Modules

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Received 11 May 2019; Accepted 21 July 2019; Published 31 July 2019

Academic Editor: Gestur Ólafsson

Frame theory is exciting and dynamic with applications to a wide variety of areas in mathematics and engineering. In this paper, we introduce the concept of Continuous $*$-K-g-frame in Hilbert C^*-Modules and we give some properties.

1. Introduction and Preliminaries

The concept of frames in Hilbert spaces has been introduced by Duffin and Schaeffer [1] in 1952 to study some deep problems in nonharmonic Fourier series, after the fundamental paper [2] by Daubechies, Grossman and Meyer, frame theory began to be widely used, particularly in the more specialized context of wavelet frames and Gabor frames [3].

Traditionally, frames have been used in signal processing, image processing, data compression, and sampling theory. A discreet frame is a countable family of elements in a separable Hilbert space which allows for a stable, not necessarily unique, decomposition of an arbitrary element into an expansion of the frame elements. The concept of a generalization of frames to a family indexed by some locally compact space endowed with a Radon measure was proposed by G. Kaiser [4] and independently by Ali, Antoine, and Gazeau [5]. These frames are known as continuous frames. Gabardo and Han in [6] called these frames associated with measurable spaces, Askari-Hemmat, Dehghan, and Radjabalipour in [7] called them generalized frames and in mathematical physics they are referred to as coherent states [5].

In this paper, we introduce the notion of Continuous $*$-K-g-Frame which are generalization of $*$-K-g-Frame in Hilbert C^*-Modules introduced by M. Rossafi and S. Kabbaj [8] and we establish some new results.

The paper is organized as follows: we continue this introductory section we briefly recall the definitions and basic properties of C^*-algebra, Hilbert C^*-modules. In Section 2, we introduce the Continuous $*$-K-g-Frame, the Continuous pre-$*$-K-g-frame operator, and the Continuous $*$-K-g-frame operator; also we establish here properties.

In the following we briefly recall the definitions and basic properties of C^*-algebra, Hilbert $\mathscr{A}$-modules. Our reference for C^*-algebras is [9, 10]. For a C^*-algebra $\mathscr{A}$ if $a \in \mathscr{A}$ is positive we write $a \geq 0$ and $\mathscr{A}^+$ denotes the set of positive elements of $\mathscr{A}$.

Definition 1 (see [11]). Let $\mathscr{A}$ be a unital C^*-algebra and $\mathscr{H}$ a left $\mathscr{A}$-module, such that the linear structures of $\mathscr{A}$ and $\mathscr{H}$ are compatible. $\mathscr{H}$ is a pre-Hilbert $\mathscr{A}$-module if $\mathscr{H}$ is equipped with an $\mathscr{A}$-valued inner product $\langle .,. \rangle_{\mathscr{A}} : \mathscr{H} \times \mathscr{H} \longrightarrow \mathscr{A}$, such that is sesquilinear, positive definite, and respects the module action. In the other words,

(i) $\langle x, x \rangle_{\mathscr{A}} \geq 0$ for all $x \in \mathscr{H}$ and $\langle x, x \rangle_{\mathscr{A}} = 0$ if and only if $x = 0$.

(ii) $\langle ax + y, z \rangle_{\mathscr{A}} = a\langle x, y \rangle_{\mathscr{A}} + \langle y, z \rangle_{\mathscr{A}}$ for all $a \in \mathscr{A}$ and $x, y, z \in \mathscr{H}$.

(iii) $\langle x, y \rangle_{\mathscr{A}} = \langle y, x \rangle_{\mathscr{A}}^*$ for all $x, y \in \mathscr{H}$.

For $x \in \mathscr{H}$, we define $\|x\| = \|\langle x, x \rangle_{\mathscr{A}}\|^{1/2}$. If $\mathscr{H}$ is complete with $\|.\|$, it is called a Hilbert $\mathscr{A}$-module or a Hilbert C^*-module over $\mathscr{A}$. For every a in C^*-algebra $\mathscr{A}$, we have $|a| = (a^* a)^{1/2}$ and the $\mathscr{A}$-valued norm on $\mathscr{H}$ is defined by $|x| = \langle x, x \rangle_{\mathscr{A}}^{1/2}$ for $x \in \mathscr{H}$.

Let $\mathscr{H}$ and $\mathscr{K}$ be two Hilbert $\mathscr{A}$-modules. A map $T : \mathscr{H} \longrightarrow \mathscr{K}$ is said to be adjointable if there exists a map $T^* : \mathscr{K} \longrightarrow \mathscr{H}$ such that $\langle Tx, y \rangle_{\mathscr{A}} = \langle x, T^* y \rangle_{\mathscr{A}}$ for all $x \in \mathscr{H}$ and $y \in \mathscr{K}$.

We reserve the notation $End^*_{\mathscr{A}}(\mathscr{H},\mathscr{K})$ for the set of all adjointable operators from $\mathscr{H}$ to $\mathscr{K}$ and $End^*_{\mathscr{A}}(\mathscr{H},\mathscr{H})$ is abbreviated to $End^*_{\mathscr{A}}(\mathscr{H})$.

The following lemmas will be used to prove our mains results

Lemma 2 (see [11]). *Let $\mathscr{H}$ be Hilbert $\mathscr{A}$-module. If $T \in End^*_{\mathscr{A}}(\mathscr{H})$, then*

$$\langle Tx, Tx\rangle \leq \|T\|^2 \langle x, x\rangle, \quad \forall x \in \mathscr{H}. \tag{1}$$

Lemma 3 (see [12]). *Let $\mathscr{H}$ and $\mathscr{K}$ two Hilbert $\mathscr{A}$-Modules and $T \in End^*(\mathscr{H},\mathscr{K})$. Then the following statements are equivalent:*

(i) *T is surjective.*

(ii) *T^* is bounded below with respect to norm; i.e., there is $m > 0$ such that $\|T^*x\| \geq m\|x\|$ for all $x \in \mathscr{K}$.*

(iii) *T^* is bounded below with respect to the inner product; i.e., there is $m' > 0$ such that $\langle T^*x, T^*x\rangle \geq m'\langle x, x\rangle$ for all $x \in \mathscr{K}$.*

Lemma 4 (see [13]). *Let $\mathscr{H}$ and $\mathscr{K}$ be two Hilbert $\mathscr{A}$-Modules and $T \in End^*(\mathscr{H},\mathscr{K})$. Then,*

(i) *if T is injective and T has closed range, then the adjointable map T^*T is invertible and*

$$\left\|(T^*T)^{-1}\right\|^{-1} I_{\mathscr{H}} \leq T^*T \leq \|T\|^2 I_{\mathscr{H}}. \tag{2}$$

(ii) *If T is surjective, then the adjointable map TT^* is invertible and*

$$\left\|(TT^*)^{-1}\right\|^{-1} I_{\mathscr{K}} \leq TT^* \leq \|T\|^2 I_{\mathscr{K}}. \tag{3}$$

2. Continuous ∗-K-g-Frame in Hilbert C^*-Modules

Let X be a Banach space, (Ω, μ) a measure space, and function $f : \Omega \longrightarrow X$ a measurable function. Integral of the Banach-valued function f has defined Bochner and others. Most properties of this integral are similar to those of the integral of real-valued functions. Because every C^*-algebra and Hilbert C^*-module is a Banach space thus we can use this integral and its properties.

Let (Ω, μ) be a measure space, let U and V be two Hilbert C^*-modules, $\{V_w : w \in \Omega\}$ is a sequence of subspaces of V, and $End^*_{\mathscr{A}}(U, V_w)$ is the collection of all adjointable $\mathscr{A}$-linear maps from U into V_w. We define

$$\bigoplus_{w\in\Omega} V_w = \left\{x = \{x_w\} : x_w \in V_w, \left\|\int_\Omega |x_w|^2 d\mu(w)\right\| < \infty\right\}. \tag{4}$$

For any $x = \{x_w : w \in \Omega\}$ and $y = \{y_w : w \in \Omega\}$, if the $\mathscr{A}$-valued inner product is defined by $\langle x, y\rangle = \int_\Omega \langle x_w, y_w\rangle d\mu(w)$, the norm is defined by $\|x\| = \|\langle x, x\rangle\|^{1/2}$, the $\bigoplus_{w\in\Omega} V_w$ is a Hilbert C^*-module.

Definition 5. Let $K \in End^*_{\mathscr{A}}(U)$; we call $\{\Lambda_w \in End^*_{\mathscr{A}}(U, V_w) : w \in \Omega\}$ a Continuous ∗-K-g-frame for Hilbert C^*-module U with respect to $\{V_w : w \in \Omega\}$ if

(a) for any $x \in U$, the function $\widetilde{x} : \Omega \longrightarrow V_w$ defined by $\widetilde{x}(w) = \Lambda_w x$ is measurable;

(b) there exist two strictly nonzero elements A and B in $\mathscr{A}$ such that

$$A\langle K^*x, K^*x\rangle A^* \leq \int_\Omega \langle \Lambda_w x, \Lambda_w x\rangle d\mu(w) \leq B\langle x, x\rangle B^*, \quad \forall x \in U. \tag{5}$$

The elements A and B are called Continuous ∗-K-g-frame bounds.

If $A = B$ we call this Continuous ∗-K-g-frame a continuous tight ∗-K-g-frame, and if $A = B = 1_{\mathscr{A}}$ it is called a continuous Parseval ∗-K-g-frame. If only the right-hand inequality of (5) is satisfied, we call $\{\Lambda_w : w \in \Omega\}$ a continuous ∗-K-g-Bessel for U with respect to $\{\Lambda_w : w \in \Omega\}$ with Bessel bound B.

Example 6. Let l^∞ be the set of all bounded complex-valued sequences. For any $u = \{u_j\}_{j\in\mathbf{N}}, v = \{v_j\}_{j\in\mathbf{N}} \in l^\infty$, we define

$$uv = \{u_j v_j\}_{j\in\mathbf{N}}, \quad u^* = \{\overline{u_j}\}_{j\in\mathbf{N}}, \quad \|u\| = \sup_{j\in\mathbf{N}} |u_j|. \tag{6}$$

Then $\mathscr{A} = \{l^\infty, \|.\|\}$ is a $\mathbb{C}^*$-algebra.

Let $\mathscr{H} = C_0$ be the set of all sequences converging to zero. For any $u, v \in \mathscr{H}$ we define

$$\langle u, v\rangle = uv^* = \{u_j \overline{u_j}\}_{j\in\mathbf{N}}. \tag{7}$$

Then $\mathscr{H}$ is a Hilbert $\mathscr{A}$-module.

Define $f_j = \{f_i^j\}_{i\in\mathbf{N}^*}$ by $f_i^j = 1/2 + 1/i$ if $i = j$ and $f_i^j = 0$ if $i \neq j$ $\forall j \in \mathbf{N}^*$.

Now define the adjointable operator $\Lambda_j : \mathscr{H} \longrightarrow \mathscr{A}, \Lambda_j x = \langle x, f_j\rangle$.

Then for every $x \in \mathscr{H}$ we have

$$\sum_{j\in\mathbf{N}} \langle \Lambda_j x, \Lambda_j x\rangle = \left\{\frac{1}{2} + \frac{1}{i}\right\}_{i\in\mathbf{N}^*} \langle x, x\rangle \left\{\frac{1}{2} + \frac{1}{i}\right\}_{i\in\mathbf{N}^*}. \tag{8}$$

So $\{\Lambda_j\}_j$ is a $\{1/2 + 1/i\}_{i\in\mathbf{N}^*}$-tight ∗-g-frame.

Let $K : \mathscr{H} \longrightarrow \mathscr{H}$ defined by $Kx = \{x_i/i\}_{i\in\mathbf{N}^*}$.

Then for every $x \in \mathscr{H}$ we have

$$\langle K^*x, K^*x\rangle_{\mathscr{A}} \leq \sum_{j\in\mathbf{N}} \langle \Lambda_j x, \Lambda_j x\rangle = \left\{\frac{1}{2} + \frac{1}{i}\right\}_{i\in\mathbf{N}^*} \langle x, x\rangle \left\{\frac{1}{2} + \frac{1}{i}\right\}_{i\in\mathbf{N}^*}. \tag{9}$$

Now, let (Ω, μ) be a σ-finite measure space with infinite measure and $\{H_w\}_{w\in\Omega}$ be a family of Hilbert $\mathscr{A}$-module ($H_\omega = C_0, \forall w \in \Omega$).

Since Ω is a σ-finite, it can be written as a disjoint union $\Omega = \bigcup \Omega_\omega$ of countably many subsets $\Omega_\omega \subseteq \Omega$, such that $\mu(\Omega_k) < \infty, \forall k \in \mathbb{N}$. Without less of generality, assume that $\mu(\Omega_k) > 0 \forall k \in \mathbb{N}$.

For each $\omega \in \Omega$, define the operator: $\Lambda_\omega : H \longrightarrow H_w$ by

$$\Lambda_w(x) = \frac{1}{\mu(\Omega_k)} \langle x, f_k \rangle h_\omega, \quad \forall x \in H \tag{10}$$

where k is such that $w \in \Omega_\omega$ and h_ω is an arbitrary element of H_ω, such that $\|h_\omega\| = 1$.

For each $x \in H$, $\{\Lambda_\omega x\}_{\omega\in\Omega}$ is strongly measurable (since h_ω are fixed) and

$$\int_\Omega \langle \Lambda_\omega x, \Lambda_\omega x \rangle d\mu(\omega) = \sum_{j\in\mathbb{N}} \langle x, f_j \rangle \langle f_j, x \rangle \tag{11}$$

So, therefore

$$\begin{aligned} \langle K^* x, K^* x \rangle &\le \int_\Omega \langle \Lambda_\omega x, \Lambda_\omega x \rangle d\mu(\omega) \\ &= \sum_{j\in\mathbb{N}} \langle x, f_j \rangle \langle f_j, x \rangle \\ &= \left\{ \frac{1}{2} + \frac{1}{i} \right\}_{i\in\mathbb{N}^*} \langle x, x \rangle \left\{ \frac{1}{2} + \frac{1}{i} \right\}_{i\in\mathbb{N}^*} \end{aligned} \tag{12}$$

So $\{\Lambda_\omega\}_{\omega\in\Omega}$ is a continuous $*$-K-g-frame.

Remark 7.

(i) Every continuous $*$-g-frame is a continuous $*$-K-g-frame indeed:

Let $\{\Lambda_w \in End^*_{\mathcal{A}}(U, V_w) : w \in \Omega\}$ be a continuous $*$-g-frame for Hilbert C^*-module U with respect to $\{V_w : w \in \Omega\}$, then

$$A \langle x, x \rangle A^* \le \int_\Omega \langle \Lambda_w x, \Lambda_w x \rangle d\mu(w) \le B \langle x, x \rangle B^*, \quad \forall x \in U. \tag{13}$$

or

$$\langle K^* x, K^* x \rangle \le \|K\|^2 \langle x, x \rangle, \quad \forall x \in U. \tag{14}$$

then

$$\begin{aligned} &\left(\|K\|^{-1} A\right) \langle K^* x, K^* x \rangle \left(\|K\|^{-1} A\right)^* \\ &\quad \le \int_\Omega \langle \Lambda_w x, \Lambda_w x \rangle d\mu(w) \le B \langle x, x \rangle B^* \end{aligned} \tag{15}$$

so let $\{\Lambda_w \in End^*_{\mathcal{A}}(U, V_w) : w \in \Omega\}$ be a continuous $*$-K-g-frame with lower and upper bounds $\|K\|^{-1}A$ and B, respectively.

(ii) If $K \in End^*_{\mathcal{A}}(H)$ is a surjective operator, then every continuous $*$-K-g-frame for H with respect to $\{V_w : w \in \Omega\}$ is a continuous $*$-g-frame.

Indeed,

if K is surjective there exists $m > 0$ such that

$$m \langle x, x \rangle \le \langle K^* x, K^* x \rangle \tag{16}$$

then

$$\left(A\sqrt{m}\right) \langle x, x \rangle \left(A\sqrt{m}\right)^* \le A \langle K^* x, K^* x \rangle A^* \tag{17}$$

or if $\{\Lambda_w \in End^*_{\mathcal{A}}(U, V_w) : w \in \Omega\}$ is a continuous $*$-K-g-frame, we have

$$\begin{aligned} \left(A\sqrt{m}\right) \langle x, x \rangle \left(A\sqrt{m}\right)^* &\le \int_\Omega \langle \Lambda_w x, \Lambda_w x \rangle d\mu(w) \\ &\le B \langle x, x \rangle B^* \end{aligned} \tag{18}$$

hence $\{\Lambda_w \in End^*_{\mathcal{A}}(U, V_w) : w \in \Omega\}$ is a continuous $*$-g-frame for U with lower and upper bounds $A\sqrt{m}$ and B, respectively

Let $K \in End^*_{\mathcal{A}}(U)$, and $\{\Lambda_w \in End^*_{\mathcal{A}}(U, V_w) : w \in \Omega\}$ be a continuous $*$-K-g-frame for Hilbert C^*-module U with respect to $\{V_w : w \in \Omega\}$.

We define an operator $T : U \longrightarrow \bigoplus_{w\in\Omega} V_w$ by

$$Tx = \{\Lambda_w x : w \in \Omega\} \quad \forall x \in U, \tag{19}$$

then T is called the continuous $*$-K-g-frame transform.

So its adjoint operator is $T^* : \bigoplus_{w\in\Omega} V_w \longrightarrow U$ given by

$$T^*\left(\{x_\omega\}_{\omega\in\Omega}\right) = \int_\Omega \Lambda^*_\omega x_\omega d\mu(w) \tag{20}$$

By composing T and T^*, the frame operator $S = T^*T$ given by

$Sx = \int_\Omega \Lambda^*_\omega \Lambda_\omega x d\mu(w)$, S is called continuous $*$-K-g frame operator

Theorem 8. *The continuous $*$-K-g frame operator S is bounded, positive, self-adjoint, and $\|A^{-1}\|^{-2}\|K\|^2 \le \|S\| \le \|B\|^2$*

Proof. First we show, S is a self-adjoint operator. By definition we have $\forall x, y \in U$

$$\begin{aligned} \langle Sx, y \rangle &= \left\langle \int_\Omega \Lambda^*_w \Lambda_w x d\mu(w), y \right\rangle \\ &= \int_\Omega \langle \Lambda^*_w \Lambda_w x, y \rangle d\mu(w) \\ &= \int_\Omega \langle x, \Lambda^*_w \Lambda_w y \rangle d\mu(w) \\ &= \left\langle x, \int_\Omega \Lambda^*_w \Lambda_w y d\mu(w) \right\rangle = \langle x, Sy \rangle. \end{aligned} \tag{21}$$

Then S is a self-adjoint.

Clearly S is positive.

By definition of a continuous $*$-K-g-frame we have

$$\begin{aligned} A \langle K^* x, K^* x \rangle A^* &\le \int_\Omega \langle \Lambda_w x, \Lambda_w x \rangle d\mu(w) \\ &\le B \langle x, x \rangle B^*. \end{aligned} \tag{22}$$

So

$$A\left\langle K^{*}x,K^{*}x\right\rangle A^{*}\leq\langle Sx,x\rangle\leq B\langle x,x\rangle B^{*}. \quad (23)$$

This gives

$$\left\|A^{-1}\right\|^{-2}\left\|\left\langle KK^{*}x,x\right\rangle\right\|\leq\|\langle Sx,x\rangle\|\leq\|B\|^{2}\|\langle x,x\rangle\|. \quad (24)$$

If we take supremum on all $x\in U$, where $\|x\|\leq 1$, we have

$$\left\|A^{-1}\right\|^{-2}\|K\|^{2}\leq\|S\|\leq\|B\|^{2}. \quad (25)$$

□

Theorem 9. *Let $K\in End_{\mathcal{A}}^{*}(H)$ be surjective and $\{\Lambda_{w}\in End_{\mathcal{A}}^{*}(U,V_{w}):w\in\Omega\}$ a continuous $*$-K-g-frame for U, with lower and upper bounds A and B, respectively, and with the continuous $*$-K-g-frame operator S.*

Let $T\in End_{\mathcal{A}}^{*}(U)$ be invertible; then $\{\Lambda_{w}T:w\in\Omega\}$ is a continuous $*$-K-g-frame for U with continuous $*$-K-g-frame operator $T^{*}ST$.

Proof. We have

$$\begin{aligned}A\left\langle K^{*}Tx,K^{*}Tx\right\rangle A^{*}&\leq\int_{\Omega}\left\langle\Lambda_{w}Tx,\Lambda_{w}Tx\right\rangle d\mu(w)\\&\leq B\langle Tx,Tx\rangle B^{*},\quad\forall x\in U.\end{aligned} \quad (26)$$

Using Lemma 3, we have $\|(T^{*}T)^{-1}\|^{-1}\langle x,x\rangle\leq\langle Tx,Tx\rangle,\forall x\in U$.

K is surjective, then there exists m such that

$$m\langle Tx,Tx\rangle\leq\left\langle K^{*}Tx,K^{*}Tx\right\rangle \quad (27)$$

and then

$$m\left\|\left(T^{*}T\right)^{-1}\right\|^{-1}\langle x,x\rangle\leq\left\langle K^{*}Tx,K^{*}Tx\right\rangle \quad (28)$$

so

$$m\left\|\left(T^{*}T\right)^{-1}\right\|^{-1}A\langle x,x\rangle A^{*}\leq A\left\langle K^{*}Tx,K^{*}Tx\right\rangle A^{*} \quad (29)$$

Or $\|T^{-1}\|^{-2}\leq\|(T^{*}T)^{-1}\|^{-1}$, this implies

$$\begin{aligned}&\left(\left\|T^{-1}\right\|^{-1}\sqrt{m}A\right)\langle x,x\rangle\left(\left\|T^{-1}\right\|^{-1}\sqrt{m}A\right)^{*}\\&\quad\leq A\left\langle K^{*}Tx,K^{*}Tx\right\rangle A^{*},\quad\forall x\in U.\end{aligned} \quad (30)$$

And we know that $\langle Tx,Tx\rangle\leq\|T\|^{2}\langle x,x\rangle$, $\forall x\in U$. This implies that

$$B\langle Tx,Tx\rangle B^{*}\leq(\|T\|B)\langle x,x\rangle(\|T\|B)^{*},\quad\forall x\in U. \quad (31)$$

Using (26), (30), (31) we have

$$\begin{aligned}&\left(\left\|T^{-1}\right\|^{-1}\sqrt{m}A\right)\langle x,x\rangle\left(\left\|T^{-1}\right\|^{-1}\sqrt{m}A\right)^{*}\\&\quad\leq\int_{\Omega}\left\langle\Lambda_{w}Tx,\Lambda_{w}Tx\right\rangle d\mu(w)\\&\quad\leq(\|T\|B)\langle x,x\rangle(\|T\|B)^{*}\end{aligned} \quad (32)$$

So $\{\Lambda_{w}T:w\in\Omega\}$ is a continuous $*$-K-g-frame for U. Moreover for every $x\in U$, we have

$$\begin{aligned}T^{*}STx&=T^{*}\int_{\Omega}\Lambda_{w}^{*}\Lambda_{w}Txd\mu(w)\\&=\int_{\Omega}T^{*}\Lambda_{w}^{*}\Lambda_{w}Txd\mu(w)\\&=\int_{\Omega}\left(\Lambda_{w}T\right)^{*}\left(\Lambda_{w}T\right)xd\mu(w).\end{aligned} \quad (33)$$

This completes the proof. □

Corollary 10. *Let $\{\Lambda_{w}\in End_{\mathcal{A}}^{*}(U,V_{w}):w\in\Omega\}$ be a continuous $*$-K-g-frame for U and let $K\in End_{\mathcal{A}}^{*}(U)$ be surjective, with continuous $*$-K-g-frame operator S. Then $\{\Lambda_{w}S^{-1}:w\in\Omega\}$ is a continuous $*$-K-g-frame for U.*

Proof. Result from the last theorem by taking $T=S^{-1}$ □

The following theorem characterizes a continuous $*$-K-g-frame by its frame operator.

Theorem 11. *Let $\{\Lambda_{\omega}\}_{\omega\in\Omega}$ be a continuous $*$-g-Bessel for H with respect to $\{H_{\omega}\}_{\omega\in\Omega}$, then $\{\Lambda_{\omega}\}_{\omega\in\Omega}$ is a continuous $*$-K-g-frame for H with respect to $\{H_{\omega}\}_{\omega\in\Omega}$ if and only if there exists a constant $A>0$ such that $S\geq AKK^{*}$ where S is the frame operator for $\{\Lambda_{\omega}\}_{\omega\in\Omega}$.*

Proof. We know $\{\Lambda_{\omega}\}_{\omega\in\Omega}$ is a continuous $*$-K-g-frame for H with bounded A and B if and only if

$$\begin{aligned}A\left\langle K^{*}x,K^{*}x\right\rangle A^{*}&\leq\int_{\Omega}\left\langle\Lambda_{w}x,\Lambda_{w}x\right\rangle d\mu(w)\\&\leq B\langle x,x\rangle B^{*}\end{aligned} \quad (34)$$

If and only if

$$\begin{aligned}A\left\langle KK^{*}x,x\right\rangle A^{*}&\leq\int_{\Omega}\left\langle\Lambda_{w}^{*}\Lambda_{w}x,x\right\rangle d\mu(w)\\&\leq B\langle x,x\rangle B^{*}\end{aligned} \quad (35)$$

If and only if

$$A\left\langle KK^{*}x,x\right\rangle A^{*}\leq\langle Sx,x\rangle\leq B\langle x,x\rangle B^{*} \quad (36)$$

where S is the continuous $*$-K-g frame operator for $\{\Lambda_{\omega}\}_{\omega\in\Omega}$.

Therefore, the conclusion holds. □

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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