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Research Article

A New Coupled Fixed Point Result in Extended Metric Spaces with an Application to Study the Stability of Set-Valued Functional Equations

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In this paper, we introduce a new coupled fixed point theorem in a generalized metric space and utilize the same to study the stability for a system of set-valued functional equations.

1. Introduction and Preliminaries

Discussing the stability of functional equations, we pursue the inquiry put forward in 1940 by Ulam [1] which states that the solution of an equation varying marginally from a given solution, should of need be near the solution for the given equation. Three popular techniques to establish the stability from a perspective Hyers–Ulam of functional equations are the direct technique [2], the technique of invariant means [3, 4] and the fixed point technique (see [5]). In the last technique, different definitely known fixed point theorems are utilized, just as some new have been demonstrated and utilized in a specific circumstance. In 1991, Baker [6] studied the stability of functional equations via the banach fixed point theorem. Since fixed point technique of Baker, Radu [7] gave the stability of an equation of functional by the option of fixed point technique which was presented by Diaz and Margolis [8]. The fixed point technique has given a great deal of impact in the advancement of the stability of functional equations. We allude to numerous papers of stability of equations of functional utilizing the fixed point technique in references on the stability of equations of functional (see [8–10]).

In 2008, Park and An [11] used the fixed point technique to study the stability of functional equations due to Cauchy–Jensen. In 2009, Gao et al. [12] defined the generalized Cauchy–Jensen equation as follows:

Let S be an abelian group and n-divisible, where $n \in \mathbb{N}$, the set of all natural numbers, and X be a normed space with the norm $\|,\|_X$. For any function $f : S \to X$, the equation

$$nf\left(\frac{x+y}{n}+z\right) = f(x) + f(y) + nf(z), \tag{1}$$

for each $x, y, z \in S$ and $n \in \mathbb{N}$ is said to be the generalized equation of Cauchy–Jensen. In special case, when $n = 2$, the equation is called the Cauchy–Jensen equation.

Recently, in 2018, Kaskasem et al. [13] introduced the stability by Hyers–Ulam–Rassias of the generalized set-valued functional equations of Cauchy-Jensen given by

$$\alpha f\left(\frac{x+u}{\alpha}+z\right) = f(x) \oplus f(y) \oplus \alpha f(z), \tag{2}$$

for each $x, y, z \in X$ $\alpha \geq 2$.

The objective of our paper is basically two fold. The first goal is introduce a new fixed point technique dealing with coupled fixed point results for nonlinear contractive mappings on the generalized metric space due to Diaz et al. [8]. The second goal is to apply our new coupled fixed point results to study the stability for the following coupled system of the generalized set-valued functional equations of Cauchy–Jensen:

$$\alpha f\left(\frac{x+u}{\alpha}+z, \frac{y+v}{\alpha}+w\right) = f(x, y) \oplus f(u, v) \oplus \alpha f(z, w), \tag{3}$$

$$\alpha g\left(\frac{x+u}{\alpha}+z,\frac{y+v}{\alpha}+w\right)=g(x,y)\oplus g(u,v)\oplus \alpha g(z,w), \tag{4}$$

for all $x,y,z,u,v,w\in X,\alpha\geq 2$ and $f,g:G\to X$.

Next, we recall some preliminaries that will be used in the main results of this paper.

Let Y be a Banach space. We defined the following:

(i) 2^Y = the set of power sets of Y;
(ii) $C_b(Y)$ = all bounded and closed subsets of Y;
(iii) $C_c(Y)$ = all convex and closed subsets of Y;
(iv) $C_{cb}(Y)$ = all convex closed and bounded subsets of Y.

Definition 1 [12]. On 2^Y, they consider the addition and the scalar multiplication as follows:

$$C+C'=\{x+x\prime x\in C,x\prime\in C\prime\}\text{and}\lambda C=\{\lambda x x\in C\}, \tag{5}$$

where $C,C'\in 2^Y$ and $\lambda\in\mathbf{R}$, the set of all real numbers. Also, we define the following:

$$C\oplus C'=\overline{C+C'}. \tag{6}$$

Then

$$\lambda C+\lambda C'=\lambda(C+C')\text{ and }(\lambda+\mu)C\subseteq\lambda C+\mu C. \tag{7}$$

Also, when C is convex, we obtain

$$(\lambda+\mu)C=\lambda C+\mu C, \tag{8}$$

for all $\lambda,\mu\in\mathbf{R}$. For any set $C\in 2^Y$, the distance function $d(.,C)$ and the support function $s(.,C)$ are defined by

$$d(x,C)=\inf_{x\in Y}\{\|x-y\|\,|y\in C\}, \tag{9}$$

$$d(x^*,C)=\sup_{x^*\in Y^*}\{\langle x-y\rangle\,|x\in C\}. \tag{10}$$

For all sets $C,C'\in C_b(Y)$, the Hausdorff distance between C and C' is defined by

$$h(C,C')=\inf\{\lambda>0C\subseteq C'+\lambda B_Y,C'\subseteq C+\lambda B_Y\}, \tag{11}$$

where B_Y is the closed unit ball in Y.

Proposition 1 [13]. For any $C,C',K,K'\in C_{cb}(Y)$ and $\lambda>0$, the following properties hold:

(1) $h(C\oplus C',K\oplus K')\leq h(C\oplus K)+(C'\oplus K')$;
(2) $h(\lambda C,\lambda K)=\lambda h(C,K)$.

Definition 2 [14]. Let X be a set. A distance mapping $d:X\times X\to[0,\infty]$ is said to be a generalized metric on X if the following conditions are hold:

(1) $d(a,b)=0$ for all $a,b\in X$ if and only if $a=b$;
(2) $d(a,b)=d(b,a)$ for all $a,b\in X$;
(3) $d(a,u)\leq d(a,b)+d(b,u)$ for all $a,b,u\in X$;
(4) every d-Cauchy sequence in X is d–convergent, i.e., $\lim_{n,m\to\infty}d(x_n,x_m)=0$ for a sequence $x_n\in X$, $n=1,2,\ldots$ implies the existence of an element $x\in X$ with $\lim_{n\to\infty}d(x,x_n)=0$, ($x$ is unique by (1) and (3)).

By the fact that not every two points in X have necessarily a finite distance. One might call such a space a generalized complete metric space.

Example 1. Let $X=\{a,b,c\}$. Define d on $X\times X$ as follows:

$$\begin{gathered}d(a,b)=d(b,a)=1,\\ d(b,c)=d(c,b)=d(c,a)=d(a,c)=\infty,\\ d(x,x)=0,\quad\forall x\in X.\end{gathered} \tag{12}$$

Then (X,d) is a generalized metric space.

Definition 3 [15]. Let $(X,\_\prec)$ be a partially ordered space and let $F:X\times X\to X$. The function F is said to have the mixed monotone property if $F(a,b)$ is nondecreasing monotone in a and is nonincreasing monotone in b, that is, for each $a,b\in X$,

$$a_1,a_2\in X,\ a_1\_\prec a_2\Rightarrow F(a_1,b)\_\prec F(a_2,b), \tag{13}$$

$$b_1,b_2\in Y,\ b_1\_\prec b_2\Rightarrow F(a,b_1)\_\prec F(a,b_2). \tag{14}$$

Definition 4 [15]. A pair $(a,b)\in X\times X$ is called a coupled fixed point of the function $F:X\times X\to X$ if $a=F(a,b)$ and $b=F(b,a)$.

2. Main Coupled Fixed Point Results

Theorem 1. *Suppose that (X,d) is a complete generalized metric space and the function $F:X\times X\to X$ be a continuous mapping having the mixed monotone property on X. Assume that there exists a $k\in[0,1]$ such that for $a,b,u,v\in X$, the following holds:*

$$d(F(a,b),F(u,v))\leq\frac{k}{2}[d(a,u)+d(b,v)], \tag{15}$$

for all $a\leq u,b\geq v$ and $d(a,b)<\infty$. If there exist $a_0,b_0\in X$ such that $a_0\leq F(a_0,b_0)$ and $b_0\geq F(b_0,a_0)$. Then the following alternative holds: either.

(i) *for all $n\geq 0$, we have*

$$d\left(F^n(a,b),F^{n+1}(a,b)\right)=d\left(F^n(b,a),F^{n+1}(b,a)\right)=\infty,\text{or} \tag{16}$$

(ii) *F has a coupled fixed point in X, that is, there exist $a,b\in X$ such that $a=F(a,b)$ and $b=F(b,a)$.*

Proof. By the given assumptions, there exists $(a_0, b_0) \in X \times X$ such that $a_0 \le F(u_0, v_0)$ and $b_0 \ge F(b_0, a_0)$. Then, we can define $(a_1, b_1) \in X \times X$ such that $a_1 = F(a_0, b_0)$ and $b_1 = F(b_0, a_0)$, then $a_0 \le F(a_0, b_0) = a_1$ and $b_0 \ge F(b_0, a_0) = b_1$. Also there exists $(a_2, b_2) \in X \times X$ such that $a_2 = F(a_1, b_1)$ and $b_2 = F(b_1, a_1)$. Since F has the mixed monotone property, we have,

$$\begin{aligned} a_1 &= F(a_0, b_0) \le F(a_1, b_1) = FF(a_0, b_0) \\ &= F(F(a_0, b_0), F(b_0, a_0)) = a_2, \end{aligned} \tag{17}$$

$$\begin{aligned} b_1 &= F(b_0, a_0) \le F(b_1, a_1) = FF(b_0, a_0) \\ &= F(\mathrm{F}(b_0, a_0), F(a_0, b_0)) = b_2. \end{aligned} \tag{18}$$

Continuing in this way, we construct two sequences $\{a_n\}$ and $\{b_n\}$ in X such that

$$a_{n+1} = F^{n+1}(a_0, b_0) = FF^n(a_0, b_0) = F(F^n(a_0, b_0), F^n(b_0, a_0)) \tag{19}$$

$$b_{n+1} = F^{n_{(n+1)}}(b_0, a_{0^n}) = FF^n(b_0, a_0) = F(F^n(b_0, a_0), F^n(a_0, b_0)) \tag{20}$$

for all $n = 0, 1, 2, \ldots.$

There are two mutually exclusive possibilities: either

(a) for every integer $i = 0, 1, 2 \ldots.$, one has

$$\begin{aligned} &d\big(F^n(a_0, b_0), F^{n+1}(a_0, b_0)\big) = \infty \text{ and} \\ &d\big(F^n(b_0, a_0), F^{n+1}(b_0, a_0)\big) = \infty. \end{aligned} \tag{21}$$

which is exactly the alternative (i) of the conclusion of the theorem, or else

(b) some integer $i = 0, 1, 2 \ldots.$, one has

$$\begin{aligned} &d\big(F^n(a_0, b_0), F^{n+1}(a_0, b_0)\big) < \infty \text{ and} \\ &d\big(F^n(b_0, a_0), F^{n+1}(b_0, a_0)\big) < \infty. \end{aligned} \tag{22}$$

Now, we need to show that (b) implies alternative (ii) of the conclusion of the theorem.

If case (b) holds, let $N = N(a_0, b_0)$ denote a particular one. For definiteness, one could choose the smallest of all integer $n \ge 0$, such that

$$\begin{aligned} &d\big(F^n(a_0, b_0), F^{n+1}(a_0, b_0)\big) < \infty \text{ and} \\ &d\big(F^n(b_0, a_0), F^{n+1}(b_0, a_0)\big) < \infty. \end{aligned} \tag{23}$$

Then, by (15), since $d\big(F^N(a_0, b_0), F^{N+1}(a_0, b_0)\big) < \infty$ and $d\big(F^N(b_0, a_0), F^{N+1}(b_0, a_0)\big) < \infty$, we get

$$\begin{aligned} d\big(F^{N+1}(a_0, b_0), F^{N+2}(a_0, b_0)\big) &= d\big(FF^N(a_0, b_0), FF^{N+1}(a_0, b_0)\big), \\ &= d\big(F\big(F^N(a_0, b_0), F^N(b_0, a_0)\big), \\ &\quad \big(F^{N+1}(a_0, b_0), F^{N+1}(b_0, a_0)\big)\big)\big), \\ &\le \frac{k}{2}\big[d\big(F^N(a_0, b_0), F^{N+1}(a_0, b_0)\big) \\ &\quad + d\big(F^N(b_0, a_0), F^{N+1}(b_0, a_0)\big)\big)\big], \\ &< \infty. \end{aligned} \tag{24}$$

Also,

$$\begin{aligned} d\big(F^{N+1}(b_0, a_0), F^{N+2}(b_0, a_0)\big) &\le \frac{k}{2}\big[d\big(F^N(b_0, a_0), F^{N+1}(b_0, a_0)\big) \\ &\quad + d\big(F^N(a_0, b_0), F^{N+1}(a_0, b_0)\big)\big)\big] \\ &< \infty. \end{aligned} \tag{25}$$

However at this point, the triangle property (12 in Definition 2 infers that, at whatever point $n > N$, one has for each $L = 1, 2, \cdots$, that

$$\begin{aligned} d\big(F^n(a_0, b_0), F^{n+1}(a_0, b_0)\big) &\le \sum_{i=1}^{L} d\big(F^{n+i-1}(a_0, b_0), F^{n+i}(a_0, b_0)\big), \\ &\le \sum_{i=1}^{L} \left(\frac{k^{n+1-i-N}}{2^{n+1-i-N}}\right) d\big(F^N(a_0, b_0), F^{N+1}(a_0, b_0)\big), \\ &\le \left(\frac{k}{2}\right)^{n-N} \frac{1-(k/2)^L}{1-k/2} d\big(F^N(a_0, b_0), F^{N+1}(a_0, b_0)\big). \end{aligned} \tag{26}$$

Since $0 < L < 1$, then the sequence $\{F^n(a_0, b_0)\}_{n=0}^{\infty}$ and similarly the sequence $\{F^n(b_0, a_0)\}_{n=0}^{\infty}$ are d–Cauchy sequences and by (15) in Definition 2 they are d–convergent. In other words, there exist $a, b \in X$ such that

$$\lim_{n\to\infty} d(F^n(a_0, b_0), a) = 0 \text{ and } \lim_{n\to\infty} d(F^n(b_0, a_0), b) = 0. \tag{27}$$

At last, we guarantee $F(a, b) = a$ and $F(b, a) = b$, since F is continuous at (a, b) then we have

$$\begin{aligned} F(a, b) &= \lim_{n\to\infty} F(F^n(a_0, b_0), F^n(b_0, a_0)), \\ &= \lim_{n\to\infty} F^{n+1}(a_0, b_0), \\ &= a, \end{aligned} \tag{28}$$

$$\begin{aligned} F(b, a) &= \lim_{n\to\infty} F(F^n(b_0, a_0), F^n(a_0, b_0)), \\ &= \lim_{n\to\infty} F^{n+1}(b_0, a_0), \\ &= b. \end{aligned} \tag{29}$$

Remark 1. Let $f : X \to X$ be a mapping from X into itself. If we put $f(a) = F(a, b)$ and $f(b) = F(b, a)$ in Theorem 1, then one can deduce the following theorem.

Theorem 2. *Suppose that (X, d) is a partially ordered complete generalized metric space and the function $f : X \to X$ be a continuous strictly contractive mapping, that is, there exists a number $k < 1$ such that*

$$d(fa, fb) \le kd(a, b), \forall a \ge b. \tag{30}$$

If there exists $a_0 \in X$ with $a_0 \le f(a_0)$.

Then the following alternative holds: either

(I) *for all $n \ge 0$, we have*

$$d\big(f^n(a), f^{n+1}(a)\big) = \infty, \textit{ or} \tag{31}$$

(II) *f has a coupled fixed point in X, that is, there exist $a \in X$ such that $a = f(a)$.*

Proof 2. By the given assumptions, there exists $a_0 \in X$ such that $a_0 \leq f(u_0)$. Then, we can define $a_1 \in X$ such that $a_1 = f(a_0)$, then $a_0 \leq f(a_0) = a_1$. Also there exists $a_2 \in X$ such that $a_2 = f(a_1)$. Since f has the mixed monotone property, we have,

$$a_1 = f(a_0) \leq f(a_1) = ff(a_0) = f(f(a_0)) = a_2. \quad (32)$$

Continuing in this way, we construct two sequences $\{a_n\}$ in X such that

$$a_{n+1} = f^{n+1}(a_0) = ff^n(a_0) = f(f^n(a_0)) \quad (33)$$

for all $n = 0, 1, 2, \ldots$.

There are two mutually exclusive possibilities: either

(A) for every integer $i = 0, 1, 2\ldots.$, one has

$$d(f^n(a_0), f^{n+1}(a_0) = \infty,) \quad (34)$$

which is exactly the alternative (I) of the conclusion of the theorem, or else

(B) some integer $i = 0, 1, 2\ldots.$, one has

$$d(f^n(a_0), f^{n+1}(a_0)) < \infty. \quad (35)$$

Now, we need to show that (B) implies alternative (II) of the conclusion of the theorem.

If case (B) holds, let $N = N(a_0)$ denote a particular one. For definiteness, one could choose the smallest of all integer $n \geq 0$, such that

$$d(f^n(a_0), f^{n+1}(a_0)) < \infty. \quad (36)$$

Then, by (30), since $d(f^N(a_0), f^{N+1}(a_0)) < \infty$, we get

$$\begin{aligned} \mathrm{d}(f^{N+1}(a_0), f^{N+2}(a_0)) &= d(ff^N(a_0), ff^{N+1}(a_0)) \\ &= d(f(f^N(a_0), f^N(b_0)), \\ &\quad (f^{N+1}(a_0), f^{N+1}(b_0)))) \\ &\leq k[d(f^N(a_0), f^{N+1}(a_0))] \\ &< \infty. \end{aligned} \quad (37)$$

However at this point, the triangle property (12) in Definition 3 infers that, at whatever point $n > N$, one has for each $L = 1, 2, \ldots$, that

$$\begin{aligned} d(f^n(a_0), f^{n+1}(a_0)) &\leq \sum_{i=1}^{L} d(f^{n+i-1}(a_0), f^{n+i}(a_0)) \\ &\leq \sum_{i=1}^{L} k^{n+1-i-N} d(f^N(a_0), f^{N+1}(a_0)) \\ &\leq (k)^{n-N}(1 - k^L) d(f^N(a_0), f^{N+1}(a_0)). \end{aligned} \quad (38)$$

Since $0 < L < 1$, then the sequence $\{f^n(a_0)\}_{n=0}^{\infty}$ is a d–Cauchy sequence and by (15) in Definition 2 it is d–convergent. In other words, there exists a point $a \in X$ such that

$$\lim_{n\to\infty} d(f^n(a_0), a) = 0. \quad (39)$$

At last, we guarantee $\lim_{n\to\infty} f^n(a) = x$, since f is continuous at a then we have

$$f(a) = \lim_{n\to\infty} f(f^n(a_0)) = \lim_{n\to\infty} f^{n+1}(a_0) = a. \quad (40)$$

Remark 2. We note that the contractive condition in [8, Theorem 1.6] is slightly stronger than the condition (30) of Theorem 2.

3. Stability of the Cauchy–Jensen Functional Equations

Let X be a real normed space and Y be a real banach space.

Definition 5. Let $f, g : X \times X \to C_{cb}(Y)$ be two set-valued mappings.

(1) The coupled generalized Cauchy-Jensen set-valued functional equation is defined by

$$\alpha f\left(\frac{x+u}{\alpha} + z, \frac{y+v}{\alpha} + w\right) = f(x, y) \oplus f(u, v) \oplus \alpha f(z, w), \quad (41)$$

$$\alpha g\left(\frac{x+u}{\alpha} + z, \frac{y+v}{\alpha} + w\right) = g(x, y) \oplus g(u, v) \oplus \alpha g(z, w), \quad (42)$$

for all $x, y, z, u, v, w \in X$ and $\alpha \geq 2$

(2) Every solution of the generalized Cauchy-Jensen set-valued functional equation is called a Cauchy–Jensen set-valued mapping.

Theorem 3. *Let f, g be two set-valued mappings defined on $X \times X$ into $(C_{cb}(Y), \oplus, h)$ such that there exists a function*

$\psi : X \times X \times X \to [0, \infty$ satisfying

$$h\left(\alpha f\left(\frac{x+u}{\alpha} + z, \frac{y+v}{\alpha} + w\right), f(x, y) \oplus f(u, v) \oplus \alpha f(z, w)\right) \leq \psi(x, u, z) + \psi(y, v, w), \quad (43)$$

for all $x, y, z, u, v, w \in X$ and $\alpha \geq 2$. If there exists $L < 1$ such that

$$\psi(x, y, z) \leq \frac{L}{2} \alpha \psi(\alpha x, \alpha y, \alpha z), \quad (44)$$

$$\psi(u, v, w) \leq \frac{L}{2} \alpha \psi(\alpha u, \alpha v, \alpha w), \quad (45)$$

for all $x, y, z, u, v, w \in X$, then there exists unique generalized Cauchy–Jensen set-valued mappings $F, G : X \times X \to (C_{cb}(Y), \oplus, h)$ such that

$$h(f(x, y), F(x, y)) \leq \frac{1}{(1-L)(2+\alpha)} [\psi(x, x, x) + \psi(y, y, y)], \quad (46)$$

$$h(g(y, x), G(y, x)) \leq \frac{1}{(1-L)(2+\alpha)} [\psi(x, x, x) + \psi(y, y, y)], \quad (47)$$

for all $x, y \in X$. Moreover, if there exist positive real number r and M with $r < 1$ such that diam $f(x, y) \leq M\|(x, y)\|_X^r$, $g(x, y) \leq M\|(x, y)\|_X^r$ for all $x, y \in X$, then $F(x, y), G(y, x)$ are singleton sets.

Proof 3. First, we consider the set $S = \{g : X \times X \to C_{cb}(Y) | g(0, 0) = 0\}$ and introduce the generalized metric on X as follows:

$$d(g(x, y), f(x, y)) = \inf\{M \in [0, \infty) h(g(x, y), f(x, y)) \leq M\eta\}, \quad (48)$$

where $\eta = \psi(x, x, x) + \psi(y, y, y)$ and $\inf\psi = +\infty$. Then (S, d) is a complete generalized metric space (see[[16], Theorem (3)]). Now, we consider a linear mapping $T : S \times S \to S$ such that

$$T(f(x, y), g(x, y)) = \frac{1}{\beta} f(\beta x, \beta y), \forall x, y \in X, \quad (49)$$

$$T(g(x, y), f(x, y)) = \frac{1}{\beta} g(\beta x, \beta y), \forall x, y \in X, \quad (50)$$

where $\beta = (2/\alpha) + 1$.

Next, we show that T is a strictly contractive mapping with Lipschitz constant L. Let $g, f \in S$ with $d(f(x, y), u(x, y)) = K$ and $d(g(x, y), v(x, y)) = K'$ for some $K, K' \in R_+$. It follows from (48) that

$$h(f(x, y), u(x, y)) \leq K[\psi(x, x, x) + \psi(y, y, y), \quad (51)$$

$$h(g(x, y), v(x, y)) \leq K'[\psi(x, x, x) + \psi(y, y, y), \quad (52)$$

for all $x, y \in X$. From Proposition 1, (44), (45) and (52) we obtain that

$$\begin{aligned} h(T(f, g), T(u, v)) &= h\left(\frac{1}{\beta} f(\beta x, \beta y), \frac{1}{\beta} u(\beta x, \beta y)\right) \\ &= \frac{1}{\beta} h(f(\beta x, \beta y), u(\beta x, \beta y)) \\ &\leq \frac{L}{2}(K + K')[\psi(x, x, x) + \psi(y, y, y)]\}, \end{aligned} \quad (53)$$

for all $x \in X$. Hence, $d(T(f, g), T(u, v)) \leq L/2(K + K')$, that is, $d(T(f, g), T(u, v)) \leq L/2[d(f, u) + d(g, v)]$. Therefore, we suppose that $x = u$, $z = (\beta - 2/\beta)x$, $y = v$, $w = (\beta - 2/\beta)y$ and in (43) since $f(x, y)$ is convex, we have

$$h(\beta f(\beta x, \beta y), \beta^2 f(x, y)) \leq \psi(x, x, x) + \psi(y, y, y), \quad (54)$$

for all $x, y \in X$. Then, we have

$$\beta^2 h\left(\frac{1}{\beta} f(\beta x, \beta y), f(x, y)\right) \leq \psi(x, x, x) + \psi(y, y, y), \quad (55)$$

for all $x, y \in X$. Thus, by (2), we have

$$h(T(f, g), f) \leq \frac{1}{\beta^2}[\psi(x, x, x) + \psi(y, y, y)], \quad (56)$$

for all $x, y \in X$.

Similarly, one can deduce that

$$h(T(g, f), g) \leq \frac{1}{\beta^2}[\psi(x, x, x) + \psi(y, y, y)], \quad (57)$$

and so

$$d(T(f, g), f) \leq \frac{1}{\beta^2} < \infty, \quad (58)$$

$$d(T(g, f), g) \leq \frac{1}{\beta^2} < \infty. \quad (59)$$

for all $x, y \in X$.

By Theorem 1, there exist two mappings $F, G : X \times X \to (C_{cb}(Y), h)$ such that the following conditions hold;

(a) (F, G) is a coupled fixed point of T, that is, $F(x, y) = T(F(x, y), G(x, y))$ and $G(x, y) = T(G(x, y), F(x, y))$, for all $x, y \in X$. Then we have

$$\begin{aligned} F(x, y) &= T(F(x, y), G(x, y)) = \frac{1}{\beta} F(\beta x, \beta y) \\ &\Rightarrow F(\beta x, \beta y) = \beta F(x, y), \end{aligned} \quad (60)$$

$$\begin{aligned} G(x, y) &= T(G(x, y), F(x, y)) = \frac{1}{\beta} G(\beta x, \beta y) \\ &\Rightarrow G(\beta x, \beta y) = \beta G(x, y). \end{aligned} \quad (61)$$

(b) The sequences $\{T_n(f, g)\}$ and $\{T_n(g, f)\}$ converge to F, G respectively. This implies the following equality:

$$F(x, y) = \lim_{n\to\infty} \frac{1}{\beta^n} f(\beta^n x, \beta^n y) \forall x, y \in X, \quad (62)$$

$$G(x, y) = \lim_{n\to\infty} \frac{1}{\beta^n} g(\beta^n x, \beta^n y) \forall x, y \in X. \quad (63)$$

(c) We obtain that $d(f, F) \leq 1/(1 - L)d(f, T(f, g))$ and $d(g, G) \leq 1/(1 - L)d(g, T(g, f))$ which implies to the following inequality:

$$d(f, F) \leq \frac{1}{(1 - L)\beta} \text{ and } d(g, G) \leq \frac{1}{(1 - L)\beta}. \quad (64)$$

Thus the inequalities (46) hold.

It follows from (41) and (42) that

$$\begin{aligned} &h\left(\alpha F\left(\frac{x + u}{\alpha} + z, \frac{y + v}{\alpha} + w\right), (x, u) \oplus (y, v) \oplus \alpha(z, w)\right) \\ &= \lim_{n\to\infty} \frac{1}{\beta^n} h\left(\alpha f\left(\frac{\beta^n x + \beta^n u}{\alpha} + \beta^n z, \frac{\beta^n y + \beta^n u}{\alpha} + \beta^n w,\right),\right. \\ &\quad f(\beta^n x, \beta^n y) \oplus f(\beta^n u, \beta^n v) \oplus f(\beta^n z, \beta^n w)) \\ &\leq \lim_{n\to\infty} \frac{1}{\beta^n} \psi(\beta^n x, \beta^n u, \beta_n z) + \psi(\beta^n y, \beta^n v, \beta_n w) = 0, \end{aligned} \quad (65)$$

for all $x, y, z, u, v, w \in X$. Thus, we have

$$h\left(\alpha F\left(\frac{x + u}{\alpha} + z, \frac{y + v}{\alpha} + w\right), F(x, y) \oplus F(u, v) \oplus \alpha F(z, w)\right) = 0. \quad (66)$$

So, we have

$$\alpha F\left(\frac{x + u}{\alpha} + z, \frac{y + v}{\alpha} + w\right) = F(x, u) \oplus F(y, v) \oplus \alpha F(z, w), \quad (67)$$

and similarly, one can get that

$$\alpha G\left(\frac{x+u}{\alpha}+z,\frac{y+v}{\alpha}+w\right)=G(x,u)\oplus G(y,v)\oplus\alpha G(z,w),\tag{68}$$

for all $x, y, z, u, v, w \in X$. Moreover, let r and M be positive real numbers with $r < 1$ and $diam f(x, y) \leq MM\|(x, y)\|^r$ for all $x, y \in X$. Then, we have

$$diam\left(\left(\frac{1}{\beta}\right)^n f(\beta^n x,\beta^n y)\right)\leq\left(\left(\frac{1}{\beta}\right)^n f(\beta^n x,\beta^n y)\right)\tag{69}$$

for all $x, y \in X$. Since $1/\beta^{n-rn} < 1$, we have $\lim_{n\to\infty}(1/\beta)^n \|(\beta^n x,\beta^n y)\|_X^r = 0$.

This implies that $F(x) = \lim_{n\to\infty}(1/\beta)^n\|(\beta^n x,\beta^n y)\|_X^r$ is a singleton set. This completes the proof.

Open Problem 13. Can our results in this paper be extended in generalized b–metric spaces as in Aydi and Czerwik [17] and Karapinar et al. [18].

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

The authors read and approved the final manuscript.

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