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Research Article

On the Baire Generic Validity of the t-Multifractal Formalism in Besov and Sobolev Spaces

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The $t-$multifractal formalism is a formula introduced by Jaffard and Mélot in order to deduce the $t-$spectrum of a function f from the knowledge of the $(p,t)-$oscillation exponent of f. The $t-$spectrum is the Hausdorff dimension of the set of points where f has a given value of pointwise L^t regularity. The $(p,t)-$oscillation exponent is measured by determining to which oscillation spaces $O^s_{p,t}$ (defined in terms of wavelet coefficients) f belongs. In this paper, we first prove embeddings between oscillation and Besov-Sobolev spaces. We deduce a general lower bound for the $(p,t)-$oscillation exponent. We then show that this lower bound is actually equality generically, in the sense of Baire's categories, in a given Sobolev or Besov space. We finally investigate the Baire generic validity of the $t-$multifractal formalism.

1. Introduction

Multifractal analysis is concerned with the pointwise regularity and the scaling behavior of functions. It gives a powerful classification tool in various domains. In the setting of the classical Hölder pointwise regularity, it has been successfully used theoretically in (see [1–18] and references therein) and practically for signal and image processing (see [18–24] and references therein). Several related generic results, in the sense of Baire's categories [15, 25–34] and in the prevalence sense [10, 11, 35, 36], were proved. Recall that a prevalent result is large in a measure sense, whereas a Baire generic result holds in a residual set (any countable intersection of open dense sets) in a well chosen topological vector space. Both Baire and prevalent generic sets are dense and stable by translation, dilation, and countable intersection. However, prevalence and Baire genericity usually differ widely. In $\mathbb{R}^d$, prevalence coincides with Lebesgue almost everywhere, and there exist subsets of $\mathbb{R}^d$ with vanishing Lebesgue measure, but Baire generic. In infinite-dimensional spaces, there are stronger results of this type in [37, 38]. Nevertheless, in [39], Kolàr proved that the so-called HP-notion of genericity yields both Baire and prevalent results (see also [40]).

The Hölder regularity has some limitations (see [41, 42] and references therein). Hölder regularity is only defined for locally bounded functions. It can not take negative values. It is not stable under some pseudodifferential and integral operators. It is not significant in fractal boundaries where it takes only two values 0 and ∞. For instance, in fully developed turbulence, velocity is not bounded near vorticity filaments and yields negative singularities [20]. The same holds for microcalcifications in mammography [20]. In order to overcome these weaknesses, Hölder regularity was replaced by the pointwise L^t regularity introduced by Calderón and Zygmund in [43] for functions that belong locally to L^t to better study elliptic partial differential equations. This notion has recently been put forward in the mathematical literature in [42, 44–46].

Definition 1. Let $t \geq 1$. Let u be a real number and $x \in \mathbb{R}^d$. A function f in $L^t_{loc}(\mathbb{R}^d)$ belongs to $T_{u,t}(x)$ if there exists $R > 0$

and a polynomial P of degree less than u (with $P=0$ if $u<0$), such that

$$\forall r \leq R$$
$$\|f(y)-P(y-x)\|_{L^t(B(x,r))} \leq Cr^{u+d/t} \tag{1}$$

for some constant C independent of r.

The pointwise L^t regularity of f at x is

$$u_t(x)=\sup\{u; f \in T_{u,t}(x)\}. \tag{2}$$

Definition 1 written for $t=\infty$ corresponds to the Hölder regularity.

Definition 1 can be also extended to the case $t \in (0,1)$, where we consider the Hardy space H^t instead of L^t [47]. In [48] Theorem 1 p. 4, it is proved that $u_t(x) \in [-d/t,\infty]$, $t \longmapsto u_t(x)$ is decreasing, and $\nu \longmapsto u_{1/\nu}(x)$ is concave on $[0,1]$. In [43], it is shown that, contrary to the Hölder regularity, the pointwise L^t regularity for $1 \leq t < \infty$ is invariant under pseudodifferential operators of order 0.

The t-sets of f are given by

$$E_t(h)=\{x; u_t(x)=h\}. \tag{3}$$

The $t-$spectrum of f is defined by the function

$$h \longmapsto d_t(h)=dim\, E_t(h) \tag{4}$$

where dim denotes the Hausdorff dimension. By convention $dim\, \emptyset = -\infty$.

In [11, 49], Fraysse has computed the $t-$spectrum for almost every function, in the prevalence setting, in a given Sobolev or Besov space.

In [45, 46], Jaffard and Mélot have shown that the pointwise L^t regularity is well adapted for fractal interfaces. They have also proved that if f belongs to the Besov space $B_t^{\varepsilon,\infty}(\mathbb{R}^d)$ for an $\varepsilon>0$, then the pointwise L^t regularity is characterized by some conditions bearing on the moduli of the wavelet coefficients [47] (the definition of Besov is recalled in the next section). Note that (see [50])

$$B_t^{\varepsilon,\infty}(\mathbb{R}^d) \hookrightarrow L^t(\mathbb{R}^d) \hookrightarrow B_t^{0,t}(\mathbb{R}^d). \tag{5}$$

Let us recall the result obtained in [46]; let $(\psi^i)_{i=1,\dots,2^d-1}$ be either the Daubechies [51] compactly supported wavelets in $C^{r_\psi}(\mathbb{R}^d)$ (where r_ψ is the uniform Hölder regularity of ψ) or the Lemarié-Meyer [50, 52] wavelets in the Schwartz class $\mathcal{S}(\mathbb{R}^d)$ of rapidly decreasing C^∞ functions (we will write $r_\psi=\infty$), such that the family $(\psi^i(2^jx-k))$, for $j\in\mathbb{Z}$ and $k\in\mathbb{Z}^d$, form an orthogonal basis of $L^2(\mathbb{R}^d)$ (note that we choose the L^∞ normalization, not L^2). We will omit the letter i and the summation with respect to i. This will not affect the results of this work. The wavelets will be indexed in terms of dyadic cubes

$$\lambda=\lambda_{j,k}=k2^{-j}+\left[0,2^{-j}\right)^d, \tag{6}$$

so that we can write

$$\psi_\lambda(x)=\psi\left(2^jx-k\right). \tag{7}$$

For $j\in\mathbb{N}_0$, set

$$\Lambda_j=\left\{\lambda=\lambda_{j,k} \text{ with } k\in\mathbb{Z}^d\right\}. \tag{8}$$

Put

$$\Lambda=\bigcup_{j\in\mathbb{N}_0}\Lambda_j=\left\{\lambda=\lambda_{j,k} \text{ with } (j,k)\in\mathbb{N}_0\times\mathbb{Z}^d\right\}. \tag{9}$$

Let $\mathcal{S}'(\mathbb{R}^d)$ be the space of tempered distributions (i.e., the dual of $\mathcal{S}(\mathbb{R}^d)$; let $f\in\mathcal{S}'(\mathbb{R}^d)$. Using the notation $\check{\psi}(x)=\overline{\psi}(-x)$, the wavelet coefficient C_λ of f is given by

$$C_\lambda=\left(f\left(\frac{\cdot}{2^j}\right)*\check{\psi}\right)(k)=2^{dj}\langle f,\psi_\lambda\rangle. \tag{10}$$

If $f\in L^2(\mathbb{R}^d)$ then

$$C_\lambda=2^{dj}\int_{\mathbb{R}^d}f(y)\overline{\psi_\lambda}(y)\,dy. \tag{11}$$

And

$$f=\sum_{j\in\mathbb{Z}}\sum_{\lambda\in\Lambda_j}C_\lambda\psi_\lambda. \tag{12}$$

Recall that, for $0<t<\infty$, the $t-$exponent of f is given by

$$\begin{aligned}\eta(t)&=\sup\left\{\tau: f\in B_t^{\tau/t,\infty}\left(\mathbb{R}^d\right)\right\}\\&=d+\liminf_{j\longrightarrow\infty}\frac{\log\left(\sum_{\lambda\in\Lambda_j}|C_\lambda|^t\right)}{\log\left(2^{-j}\right)}.\end{aligned} \tag{13}$$

Remark 2. In [48] Section 3.4, it is proved that if $t\geq1$, then $\eta(t)$ does not depend on the chosen wavelet basis as long as $r_\psi > |h_{\min}|$ (where $h_{\min}=\liminf_{j\longrightarrow\infty}((\log(\sup_{\lambda\in\Lambda_j}|C_\lambda|))/(\log(2^{-j}))))$. When $t\geq1$, if $\eta(t)>0$, then $f\in L^t_{loc}(\mathbb{R}^d)$ and if $\eta(t)<0$ then $f\notin L^t_{loc}(\mathbb{R}^d)$.

For $0<t<\infty$, the $t-$wavelet leader of $f\in L^t_{loc}(\mathbb{R}^d)$ at $\lambda\in\Lambda_j$ was introduced in [46]

$$\ell_{t,\lambda}=\left(\sum_{j'\geq j,\lambda'\subset\lambda}\left|C_{\lambda'}2^{(d/t)(j-j')}\right|^t\right)^{1/t}, \tag{14}$$

where the sum is over all $\lambda'\in\Lambda_{j'}$ such that $\lambda'\subset\lambda$ with $j'\geq j$. The sum (14) is finite if $\eta(t)>0$.

For $x\in\mathbb{R}^d$, denote by $\lambda_j(x)$ the unique cube at the scale j that contains x and $3\lambda_j(x)$ the set formed by the cube $\lambda_j(x)$ and all its 3^d-1 adjacent cubes at scale j. Let $f\in\mathcal{S}'(\mathbb{R}^d)$. Suppose that $1<t<\infty$ and $r_\psi>u_t(x)$ (resp., $1\leq t<\infty$ and $r_\psi=\infty$). If $\eta(t)>0$, then

$$u_t(x)=\liminf_{j\longrightarrow\infty}\frac{\log\left(\ell_{t,3\lambda_j(x)}\right)}{\log\left(2^{-j}\right)}, \tag{15}$$

and see [48], Section 3.3 (resp., [46] Corollary 1 p. 553).

Remark 3. Formula (15) was used in [48] to extend the definition of the pointwise L^t regularity to the case $0<t\leq1$ under the sole condition $f\in B_t^{0,\infty}(\mathbb{R}^d)$ (see [48], Section 3.4). From now on, the pointwise L^t regularity is defined by formula (15) for $t>0$.

Using characterization (15), a t−multifractal formalism associated with pointwise L^t regularities was conjectured by Jaffard and Mélot in [46]. Let us recall it.

Definition 4. For $s \in \mathbb{R}$, $p, t > 0$, the oscillation space $O^s_{p,t}(\mathbb{R}^d)$ (see [46]) is the space of tempered distributions f that satisfy

$$\sup_{j\in\mathbb{N}_0}\left(2^{(s-d/p)j}\left(\sum_{\lambda\in\Lambda_j}(\ell_{t,\lambda})^p\right)^{1/p}\right)<\infty \quad \text{if } 0<p<\infty$$
$$\sup_{j\in\mathbb{N}_0,\lambda\in\Lambda_j}\left(2^{sj}\ell_{t,\lambda}\right) \quad \text{if } p=\infty. \tag{16}$$

The left-hand term defines the $O^s_{p,t}(\mathbb{R}^d)$-seminorm.

It is independent of the choice of the smooth enough wavelet basis or in the Schwartz class (see [53], Section 3.2).

This space together with global notions (3) and (4) can also be defined locally; let

$$\mathcal{O}=\left\{\Omega\subset\mathbb{R}^d : \Omega \text{ is a nonempty bounded open subset of } \mathbb{R}^d\right\}. \tag{17}$$

For $\Omega\in\mathcal{O}$, set

$$E^\Omega_t(h)=\{x\in\Omega;\ u_t(x)=h\} \tag{18}$$

and

$$d^\Omega_t(h)=\dim E^\Omega_t(h). \tag{19}$$

Clearly

$$d_t(h)=\sup_{\Omega\in\mathcal{O}} d^\Omega_t(h). \tag{20}$$

For $\Omega\in\mathcal{O}$ and $j\in\mathbb{N}_0$, set

$$\Lambda_j(\Omega)=\left\{\lambda\in\Lambda_j : \lambda\cap\Omega\neq\emptyset\right\}. \tag{21}$$

Definition 5. For $s\in\mathbb{R}$, $p,t>0$, the oscillation space $O^s_{p,t}(\Omega)$ is the space of tempered distributions f that satisfy

$$\sup_{j\in\mathbb{N}_0}\left(2^{(s-d/p)j}\left(\sum_{\lambda\in\Lambda_j(\Omega)}(\ell_{t,\lambda})^p\right)^{1/p}\right)<\infty \quad \text{if } 0<p<\infty$$
$$\sup_{j\in\mathbb{N}_0,\lambda\in\Lambda_j(\Omega)}\left(2^{sj}\ell_{t,\lambda}\right) \quad \text{if } p=\infty. \tag{22}$$

Remark 6. For this space, we can assume that functions and wavelets are compactly supported, with support K that contains Ω. Then, there exists $C>0$, such that, at each scale j, there are at most $C2^{dj}$ dyadic cubes $\lambda\in\Lambda_j(\Omega)$ for which $\ell_{t,\lambda}$ does not vanish (see [17] in the proof of Proposition 10 p.36).

For $0<t,p<\infty$, the local (p,t)−oscillation exponent of f on Ω is given by

$$\zeta^\Omega_t(p)=\sup\left\{\tau : f\in O^{\tau/p}_{p,t}(\Omega)\right\} = d+\liminf_{j\longrightarrow\infty}\frac{\log\left(\sum_{\lambda\in\Lambda_j(\Omega)}(\ell_{t,\lambda})^p\right)}{\log\left(2^{-j}\right)}. \tag{23}$$

The (p,t)−oscillation exponent of f is defined as

$$\zeta_t(p)=\inf_{\Omega\in\mathcal{O}}\zeta^\Omega_t(p). \tag{24}$$

The t−multifractal formalism in [46] states that

$$d_t(h)=\inf_{0<p<\infty}\left(d+hp-\zeta_t(p)\right). \tag{25}$$

The local t−multifractal formalism on Ω states that

$$d^\Omega_t(h)=\inf_{0<p<\infty}\left(d+hp-\zeta^\Omega_t(p)\right). \tag{26}$$

These formalisms yield upper bounds valid for any function (see either [46], Theorem 2, p. 561, or [53], Section 3.2); for $t>0$, define the upper t-set $B_t(h)$ of f by

$$B_t(h)=\{x; u_t(x)\le h\} \tag{27}$$

and the upper t−spectrum by

$$h\longmapsto D_t(h)=\dim B_t(h). \tag{28}$$

Define the local upper t-set $B^\Omega_t(h)$ of f by

$$B^\Omega_t(h)=B_t(h)\cap\Omega \tag{29}$$

and the local upper t−spectrum on Ω by

$$h\longmapsto D^\Omega_t(h)=\dim B^\Omega_t(h). \tag{30}$$

Then

$$\forall t>0,\quad \eta(t)>0 \quad \forall h\ d_t(h)\le D_t(h)\le\inf_{0<p<\infty}\left(d+hp-\zeta_t(p)\right) \tag{31}$$

and

$$\forall t>0,\quad \eta^\Omega(t)>0 \quad \forall h\ d^\Omega_t(h)\le D^\Omega_t(h)\le\inf_{0<p<\infty}\left(d+hp-\zeta^\Omega_t(p)\right) \tag{32}$$

where

$$\eta^\Omega(t)=d+\liminf_{j\longrightarrow\infty}\frac{\log\left(\sum_{\lambda\in\Lambda_j(\Omega)}\left|C_\lambda\right|^t\right)}{\log\left(2^{-j}\right)}. \tag{33}$$

As in (26), we state the t−multifractal formalism for upper t-sets as

$$D_t(h)=\inf_{0<p<\infty}\left(d+hp-\zeta_t(p)\right). \tag{34}$$

We also state the local t−multifractal formalism for upper t-sets on Ω as

$$D_t^{\Omega}(h) = \inf_{0<p<\infty} \left(d + hp - \zeta_t^{\Omega}(p)\right). \tag{35}$$

In [53], Leonarduzzi et al. have validated the t−multifractal formalism for some synthetic images and signals that include independent realizations of random processes.

This paper is devoted to the study of the Baire generic validity of both local and global t−multifractal formalisms in a given Sobolev or Besov space. Recall that Besov and Sobolev spaces are complete metrizable spaces [54]. Note that, if $s > d/p$ then $B_p^{s,q}(\mathbb{R}^d) \subset C^{s-d/p}(\mathbb{R}^d) \subset L^\infty(\mathbb{R}^d)$ for all q. On the contrary, if $s = d/p$ and $q > 1$ (resp., $s < d/p$), Jaffard and Meyer [55] (resp., [28]) have proved that a function in $B_p^{s,q}(\mathbb{R}^d)$ can be infinite on a dense set and thus nowhere Hölder regular. Similarly, if $s \le d/p$, there exist functions in the usual Sobolev space

$$L^{p,s}\left(\mathbb{R}^d\right) = \left\{f \in L^p\left(\mathbb{R}^d\right);\ (-\Delta)^{s/2} f \in L^p\left(\mathbb{R}^d\right)\right\} \tag{36}$$

which are everywhere locally unbounded [55].

In this paper, we are interested in the Baire generic validity of the t−multifractal formalisms in Besov spaces $B_{p_0}^{s_0,q_0}(\mathbb{R}^d)$ for $s_0, q_0, p_0 > 0$ and Sobolev spaces $L^{p_0,s_0}(\mathbb{R}^d)$ for $s_0 > 0$ and $1 \le p_0 < \infty$, under the condition $s_0 - d/p_0 > -d/t$. In the next section, we recall some tools from the theory of Besov spaces. We also add some embeddings between oscillation spaces and a relationship with the space BMO of functions of bounded mean oscillation. In the third section, we prove embeddings between Besov and oscillation spaces. We deduce a general lower bound for the (p,t)−oscillation exponent. In the fourth section, we show that the obtained lower bound is actually equality generically, in the sense of Baire categories, in a given Sobolev or Besov space. In the fifth section, we investigate the Baire generic validity of the t−multifractal formalisms. Finally, in the sixth section, we deduce a conclusion on the range of Baire validity of both the t−multifractal formalism and the t−multifractal formalism for upper t−sets.

All generic results are studied locally on $\Omega_L = L + (0,1)^d$ ($L \in \mathbb{Z}^d$) and also globally on $\mathbb{R}^d$.

Remark 7. Since only t−wavelet leaders for $j \ge 0$ are needed in the values of pointwise L^t regularity and the (p,t)−exponent, then, from now on, we will identify functions that have the same wavelet coefficients C_λ for $j \ge 0$.

2. Besov, Sobolev, and Oscillation Spaces

2.1. Besov and Sobolev Spaces. Let us first recall some properties of Besov spaces (for details, see, for example, [54, 56, 57]). Given a function $f \in \mathcal{S}(\mathbb{R}^d)$, its Fourier transform is denoted by $\hat{f}$. Let $\Psi_0 \in \mathcal{S}(\mathbb{R}^d)$ with

$$\begin{aligned} \widehat{\Psi_0}(\xi) &= 1 \quad \text{if } |\xi| \le 1, \\ \widehat{\Psi_0}(\xi) &= 0 \quad \text{if } |\xi| \ge 2. \end{aligned} \tag{37}$$

For $j \in \mathbb{N}$, let

$$\widehat{\Psi_j}(\xi) = \widehat{\Psi_0}\left(2^{-j}\xi\right) - \widehat{\Psi_0}\left(2^{-(j-1)}\xi\right). \tag{38}$$

Then the support of $\widehat{\Psi_j}$ is included in the annulus $\{\xi; 2^{j-1} < |\xi| \le 2^{j+1}\}$ and

$$\sum_{j=0}^{\infty} \widehat{\Psi_j} = 1 \tag{39}$$

is a partition of the unity. The Littlewood Paley definition of Besov spaces is the following.

Definition 8. Let $0 < p, q \le \infty$, $s \in \mathbb{R}$. Then

$$B_p^{s,q}\left(\mathbb{R}^d\right) = \left\{f \in \mathcal{S}'\left(\mathbb{R}^d\right);\ \|f\|_{B_p^{s,q}(\mathbb{R}^d)} < \infty\right\} \tag{40}$$

where

$$\|f\|_{B_p^{s,q}(\mathbb{R}^d)} = \begin{cases} \left(\sum_{j=0}^{\infty} 2^{jsq} \left\|f * \Psi_j\right\|_{L^p(\mathbb{R}^d)}^q\right)^{1/q} & \text{if } 0 < q < \infty \\ \sup_{j\in\mathbb{N}_0} 2^{js} \left\|f * \Psi_j\right\|_{L^p(\mathbb{R}^d)} & \text{if } q = \infty. \end{cases} \tag{41}$$

Note that $\|.\|_{B_p^{s,q}(\mathbb{R}^d)}$ is a norm (quasinorm when $p < 1$ or $q < 1$) on $B_p^{s,q}(\mathbb{R}^d)$.

Besov spaces do not depend on the choice of Ψ_0. They are separable when both p and q are finite.

Let us recall their wavelet characterizations; for a given sequence of scalar numbers $(a_\lambda)_{\lambda\in\Lambda}$ (where Λ is as in (9)), let

$$\left\|(a_\lambda)_{\lambda\in\Lambda}\right\|_{b_p^{s,q}} = \left(\sum_{j=0}^{\infty}\left(2^{(s-d/p)j}\left(\sum_{\lambda\in\Lambda_j}\left|a_\lambda\right|^p\right)^{1/p}\right)^q\right)^{1/q}, \tag{42}$$

with the usual modification when $p = \infty$ or $q = \infty$ (i.e., $2^{(s-d/\infty)j} = 2^{sj}$ and the sums of pth or qth powers replaced by suprema over the same sets of indices), and

$$b_p^{s,q} = \left\{(a_\lambda)_{\lambda\in\Lambda} : \left\|(a_\lambda)_{\lambda\in\Lambda}\right\|_{b_p^{s,q}} < \infty\right\}. \tag{43}$$

Then, for the above wavelets if either $r_\psi = \infty$ or r_ψ is large enough ($r_\psi > \max\{s, 2d/p + d/2 - s\}$ in [57], Theorem 1.64, p. 48, and ($r_\psi > |s|$ if $p \ge 1$, and $r_\psi > \max\{s, 2d/p - d - s\}$ if $0 < p < 1$) in [50, 58]), then

$$I : f \longrightarrow \left(C_\lambda(f)\right)_{\lambda\in\Lambda} \tag{44}$$

is an isomorphic map from $B_p^{s,q}(\mathbb{R}^d)$ onto $b_p^{s,q}$.

Under the same condition on r_ψ, it is also proved that, for $1 \le p < \infty$ and $s > 0$, the Sobolev space $L^{p,s}(\mathbb{R}^d)$ given in (36) is characterized by

$$\|f\|_{L^{p,s}(\mathbb{R}^d)} := \left\|\left(\sum_{j=0}^{\infty} 2^{2sj} \sum_{\lambda\in\Lambda_j} \left|C_\lambda\right|^2 \chi_\lambda\right)^{1/2}\right\|_{L^p(\mathbb{R}^d)} < \infty, \tag{45}$$

where χ_λ denotes the characteristic function of the cube λ.

Note that characterizations (44) and (45) do not depend on choice of the wavelet basis (see [50]).

Besov and usual Sobolev spaces are closely related (see [12, 59]):

$$\forall p \geq 1\ \forall s > 0 \quad B_p^{s,1}\left(\mathbb{R}^d\right) \hookrightarrow L^{p,s}\left(\mathbb{R}^d\right) \hookrightarrow B_p^{s,\infty}\left(\mathbb{R}^d\right) \tag{46}$$

and

$$\forall p > 1\ \forall s > \varepsilon > 0\ \forall q > 0 \quad B_p^{s-\varepsilon,q}\left(\mathbb{R}^d\right) \hookrightarrow L^{p,s}\left(\mathbb{R}^d\right) \hookrightarrow B_p^{s+\varepsilon,q}\left(\mathbb{R}^d\right). \tag{47}$$

The following embeddings hold (for example, see [60], Proposition 2.6, p. 245, Proposition 2.8, p. 245, and Theorem 2.14, p. 248, respectively):

$$\forall 0 < q_2 \leq q_1 \quad b_p^{s,q_2} \hookrightarrow b_p^{s,q_1}, \tag{48}$$

$$\forall s_1 \leq s_2 \quad b_p^{s_2,q_2} \hookrightarrow b_p^{s_1,q_1}, \tag{49}$$

$$\forall 0 < p_2 \leq p_1 \quad b_{p_2}^{s,q} \hookrightarrow b_{p_1}^{s+d/p_1-d/p_2,q}. \tag{50}$$

The following interpolation property holds:

$$\forall 0 \leq \theta \leq 1 \quad b_{p_1}^{s_1,q} \cap b_{p_2}^{s_2,q} \hookrightarrow b_p^{\theta s_1+(1-\theta)s_2,q}, \quad \text{where } \frac{1}{p} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2}. \tag{51}$$

Using the isomorphic (44), we obtain similar embeddings between Besov spaces.

2.2. Oscillation Spaces

Proposition 9. *The following embeddings between oscillation spaces hold:*

$$\forall p,t > 0\ \forall s_1 \leq s_2 \quad O_{p,t}^{s_2}\left(\mathbb{R}^d\right) \hookrightarrow O_{p,t}^{s_1}\left(\mathbb{R}^d\right), \tag{52}$$

$$\forall t > 0\ \forall 0 < p_2 \leq p_1 \quad O_{p_2,t}^{s}\left(\mathbb{R}^d\right) \hookrightarrow O_{p_1,t}^{s+d/p_1-d/p_2}\left(\mathbb{R}^d\right), \tag{53}$$

and

$$\forall 0 \leq \theta \leq 1 \quad O_{p_1,t}^{s_1}\left(\mathbb{R}^d\right) \cap O_{p_2,t}^{s_2}\left(\mathbb{R}^d\right) \hookrightarrow O_{p,t}^{\theta s_1+(1-\theta)s_2}\left(\mathbb{R}^d\right), \quad \text{where } \frac{1}{p} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2}. \tag{54}$$

We also have

$$\forall 0 < \theta < 1 \quad O_{p,t_1}^{s_1}\left(\mathbb{R}^d\right) \cap O_{p,t_2}^{s_2}\left(\mathbb{R}^d\right) \hookrightarrow O_{p,\theta t_1+(1-\theta t_2)}^{(\theta t_1/(\theta t_1+(1-\theta)t_2))s_1+((1-\theta)t_2/(\theta t_1+(1-\theta)t_2))s_2}\left(\mathbb{R}^d\right). \tag{55}$$

And

$$\forall p > 0\ \forall s \quad O_{p,t}^{s}\left(\mathbb{R}^d\right) \hookrightarrow B_p^{s,\infty}\left(\mathbb{R}^d\right) \tag{56}$$

and

$$O_{\infty,2}^{0}\left(\mathbb{R}^d\right) = BMO. \tag{57}$$

If $\mathcal{O}$ is as in (17), then the following local embedding holds:

$$\forall \Omega \in \mathcal{O}\ \forall t > 0\ \forall 0 < p_1 \leq p_2 \quad O_{p_2,t}^{s}(\Omega) \hookrightarrow O_{p_1,t}^{s}(\Omega). \tag{58}$$

Proof. Embeddings (52) (resp., (53)) follows directly from (49) (resp., (50)). Embedding (54) follows from (51) and the equivalence

$$f \in O_{p,t}^{s}\left(\mathbb{R}^d\right) \Longleftrightarrow \left(\ell_{t,\lambda}\right)_{\lambda\in\Lambda} \in b_p^{s,\infty}. \tag{59}$$

Let us now prove embedding (55). Write the t−wavelet leader of f at a cube $\lambda \in \Lambda_j$ (given in (14)) as

$$\ell_{t,\lambda} = 2^{(d/t)j} a_{t,\lambda}^{1/t} \tag{60}$$

where

$$a_{t,\lambda} = \sum_{j'=j}^{\infty} \left(2^{-dj'} \sum_{\lambda'\subset\lambda} \left|C_{\lambda'}\right|^t\right). \tag{61}$$

If $\theta \in (0,1)$, then

$$a_{\theta t_1+(1-\theta)t_2,\lambda} = \sum_{j'=j}^{\infty} \sum_{\lambda'\subset\lambda} \left(\left|C_{\lambda'}\right|^{t_1} 2^{-dj'}\right)^{\theta} \left(\left|C_{\lambda'}\right|^{t_2} 2^{-dj'}\right)^{1-\theta}. \tag{62}$$

We know that, by Cauchy-Schwartz inequality, if $P \geq 1$ and $1/P + 1/Q = 1$, then

$$\forall A_i, B_i \geq 0\ \forall \quad \sum A_i B_i \leq \left(\sum A_i^P\right)^{1/P} \left(\sum B_i^Q\right)^{1/Q}. \tag{63}$$

If $P = 1/\theta$ then $Q = 1/(1-\theta)$; then relation (63) applied to (62) with $A_i = (|C_{\lambda'}|^{t_1} 2^{-dj'})^{\theta}$ and $B_i = (|C_{\lambda'}|^{t_2} 2^{-dj'})^{1-\theta}$ (with $i = (j', \lambda')$) yields

$$a_{\theta t_1+(1-\theta)t_2,\lambda} \leq \left(a_{t_1,\lambda}\right)^{\theta} \left(a_{t_2,\lambda}\right)^{1-\theta}. \tag{64}$$

Then

$$2^{dj}a_{\theta t_1+(1-\theta)t_2,\lambda} \le \left(2^{dj}a_{t_1,\lambda}\right)^{\theta}\left(2^{dj}a_{t_2,\lambda}\right)^{1-\theta}. \quad (65)$$

It follows from (60) that

$$\begin{aligned}&\ell_{\theta t_1+(1-\theta)t_2,\lambda}\\ &\quad\le \left(\ell_{t_1,\lambda}\right)^{\theta t_1/(\theta t_1+(1-\theta)t_2)}\left(\ell_{t_2,\lambda}\right)^{(1-\theta)t_2/(\theta t_1+(1-\theta)t_2)}.\end{aligned} \quad (66)$$

Put

$$S_t(p,j) := 2^{-dj}\sum_{\lambda\in\Lambda_j(\Omega)}\left(\ell_{t,\lambda}\right)^p = 2^{d(p/t-1)j}\sum_{\lambda\in\Lambda_j(\Omega)} a_{t,\lambda}^{p/t}. \quad (67)$$

Then, as in (64), relation (66) yields

$$\begin{aligned}S_{\theta t_1+(1-\theta)t_2}(p,j) &\le \left(S_{t_1}(p,j)\right)^{\theta t_1/(\theta t_1+(1-\theta)t_2)}\\ &\quad\cdot\left(S_{t_2}(p,j)\right)^{(1-\theta)t_2/(\theta t_1+(1-\theta)t_2)}.\end{aligned} \quad (68)$$

This achieves the proof of embedding (55).

Embedding (56) follows from the fact that $|C_\lambda| \le \ell_{t,\lambda}$.

With regard to (57), for $p=\infty$, $s=0$, and $t=2$,

$$\begin{aligned}\|f\|_{O^0_{\infty,2}(\mathbb{R}^d)} &= \sup_{j\in\mathbb{N}_0,\lambda\in\Lambda_j}\left(\ell_{2,\lambda}\right)\\ &= \sup_{j\in\mathbb{N}_0,\lambda\in\Lambda_j}\left(\sum_{\lambda'\subset\lambda}\left|C_{\lambda'}2^{(d/2)(j-j')}\right|^2\right)^{1/2}.\end{aligned} \quad (69)$$

It follows that

$$\begin{aligned}&f\in O^0_{\infty,2}\left(\mathbb{R}^d\right)\Longleftrightarrow\\ &\exists C>0:\forall j\in\mathbb{N}_0\ \forall\lambda\in\Lambda_j\\ &\sum_{\lambda'\subset\lambda}\left|C_{\lambda'}\right|^2 2^{-dj'} \le C2^{-dj}.\end{aligned} \quad (70)$$

Thanks to our L^∞ (resp., Meyer L^2) normalization (7) for wavelets and (10), the right-hand term in (70) coincides with the Carleson condition for the wavelet characterization of the *BMO* space given in Theorem 4, p. 150-151 in [50].

Finally result (58) follows from Remark 6 and the Hölder inequality:

$$\begin{aligned}&\sum_{\lambda\in\Lambda_j(\Omega)}\left(\ell_{t,\lambda}\right)^{p_1}\\ &\quad\le C2^{dj(1-p_1/p_2)}\left(\sum_{\lambda\in\Lambda_j(\Omega)}\left(\ell_{t,\lambda}\right)^{p_2}\right)^{p_1/p_2}.\end{aligned} \quad (71)$$

□

Oscillation spaces $O^s_{p,t}(\mathbb{R}^d)$ were defined in terms of wavelet coefficients. They can be characterized by some Littlewood Paley conditions; moreover $t-$wavelet leaders can also be replaced by L^t local norms [61].

When $t=\infty$, the $t-$wavelet leaders of $f\in L^\infty_{loc}(\mathbb{R}^d)$ given in (14) boil down to the classical wavelet leaders

$$\ell_{\infty,\lambda} = \sup_{\lambda'\subset\lambda}\left|C_{\lambda'}\right| \quad (72)$$

used for the characterization of the Hölder exponent. The corresponding oscillation spaces $O^s_{p,\infty}(\mathbb{R}^d)$ have been studied in [17, 62, 63] and denoted by $O^s_p(\mathbb{R}^d)$. Their characterizations by differences were obtained in [62]. Their independence of the chosen wavelet basis in the Schwartz class was obtained in [17]. For either $s\ge 0$ or $s\le -d/p$, it is proved in [63] that these spaces are a variation on the definition of Besov spaces. On the contrary, the spaces $O^s_p(\mathbb{R}^d)$ for $-d/p<s<0$ cannot be sharply imbedded between Besov spaces and thus are new spaces of really different nature.

Generalized oscillation spaces $O^{s,s'}_{p,t}(\mathbb{R}^d)$ were also introduced in [46]; if $s'>0$ (resp., $s'<0$), then f belongs to $O^{s,s'}_{p,t}(\mathbb{R}^d)$ if its fractional derivative (resp., primitive) or order s' which we denote by $(-\Delta)^{s'/2}f$ (i.e., such that $\widehat{(-\Delta)^{s'/2}}f(\xi) = |\xi|^{s'}\widehat{f}(\xi)$) belongs to $O^s_{p,t}(\mathbb{R}^d)$. Spaces $O^{s,s'}_{p,\infty}(\mathbb{R}^d)$ allow the computation of fractal dimension of graphs and yield a multifractal formalism for chirp-type Hölder singularities that behave like $|x-x_0|^h\sin(1/|x-x_0|^\beta)$ (see [62]).

3. General Lower Bound on the $(p,t)-$Oscillation Exponent

The following general lower bounds of both local and global $(p,t)-$oscillation exponents hold.

Theorem 10. *Let f be any function in either the Besov space $B^{s_0,q_0}_{p_0}(\mathbb{R}^d)$ for $s_0,q_0,p_0>0$ or the Sobolev space $L^{p_0,s_0}(\mathbb{R}^d)$ for $s_0>0$ and $p_0\ge 1$. Then, for all $t>0$ such that $s_0-d/p_0>-d/t$,*

$$\forall\Omega\in\mathcal{O}$$

$$\zeta^\Omega_t(p) \ge \begin{cases} p\left(s_0-\dfrac{d}{p_0}\right)+d & \forall p\ge p_0\\ ps_0 & \forall p\le p_0\end{cases} \quad (73)$$

and

$$\zeta_t(p) \ge \begin{cases} p\left(s_0-\dfrac{d}{p_0}\right)+d & \forall p\ge p_0\\ ps_0 & \forall p\le p_0.\end{cases} \quad (74)$$

Proof. For $\Omega\in\mathcal{O}$, define the local Besov space $B^{s,q}_p(\Omega)$ (resp., Sobolev space $L^s_p(\Omega)$) as in (44) (resp., (45)) with Λ replaced by $\bigcup_{j\in\mathbb{N}_0}\Lambda_j(\Omega)$ (resp., Λ_j replaced by $\Lambda_j(\Omega)$).

Result (73) will follow from (23) and the following proposition. Result (74) will be deduced from (24).

Proposition 11. *Let $\Omega\in\mathcal{O}$. The following embeddings hold for all $s_0,q_0,p_0,t>0$ such that $s_0-d/p_0>-d/t$:*

$$\begin{aligned}&\mathit{If}\ t\ge p_0\\ &\mathit{then}\ B^{s_0,q_0}_{p_0}(\Omega)\hookrightarrow O^{s_0-d/p_0+d/p}_{p,t}(\Omega)\quad \mathit{if}\ p\ge p_0\\ &B^{s_0,q_0}_{p_0}(\Omega)\hookrightarrow O^{s_0}_{p,t}(\Omega)\quad\forall p\le p_0\end{aligned} \quad (75)$$

and

$$\text{if } t < p_0 \text{ then} \forall \varepsilon > 0$$

$$B_{p_0}^{s_0,q_0}(\Omega) \hookrightarrow O_{p,t}^{s_0-d/p_0+d/p-\varepsilon}(\Omega) \quad \forall p \geq p_0 \tag{76}$$

$$B_{p_0}^{s_0,q_0}(\Omega) \hookrightarrow O_{p,t}^{s_0-\varepsilon}(\Omega) \quad \forall p \leq p_0.$$

Embeddings between similar local Sobolev and oscillation spaces also hold (using (46) and (47)).

Proof. Let $f \in B_{p_0}^{s_0,q_0}(\Omega)$ for $s_0, q_0, p_0 > 0$ and $t > 0$ such that $s_0 - d/p_0 > -d/t$.

(1) Assume that $t \geq p_0$. If $a_{t,\lambda}$ is as in (61), then using the property

$$\forall x_n \geq 0 \ \forall p \geq 1$$
$$\sum x_n^p \leq \left(\sum x_n\right)^p \tag{77}$$

we get

$$a_{t,\lambda}^{1/t} \leq \left(\sum_{j'=j}^{\infty}\left(2^{-d(p_0/t)j'}\sum_{\lambda'\subset\lambda}\left|C_{\lambda'}\right|^{p_0}\right)\right)^{1/p_0}. \tag{78}$$

(a) If $p \geq p_0$, then by Remark 6, the quantity $S_t(p,j)$ given in (67) satisfies

$$S_t(p,j) \leq C2^{d(p/t-1)j}\left(\sum_{\lambda\in\Lambda_j(\Omega)}\sum_{j'=j}^{\infty}\sum_{\lambda'\subset\lambda}\left|C_{\lambda'}\right|^{p_0} \cdot 2^{-d(p_0/t)j'}\right)^{p/p_0}. \tag{79}$$

Clearly

$$\begin{aligned}\sum_{\lambda\in\Lambda_j(\Omega)}\sum_{j'=j}^{\infty}\sum_{\lambda'\subset\lambda}\left|C_{\lambda'}\right|^{p_0}2^{-d(p_0/t)j'} \\ = \sum_{j'=j}^{\infty}\sum_{\lambda'\in\Lambda_{j'}(\Omega)}\left|C_{\lambda'}\right|^{p_0}2^{-d(p_0/t)j'}.\end{aligned} \tag{80}$$

Since $f \in B_{p_0}^{s_0,q_0}(\Omega)$, then $f \in B_{p_0}^{s_0,\infty}(\Omega)$. There exists j_0 such that

$$\forall j' \geq j_0$$
$$2^{-dj'}\sum_{\lambda'\in\Lambda_{j'}(\Omega)}\left|C_{\lambda'}\right|^{p_0} \leq C2^{-s_0p_0j'}. \tag{81}$$

It follows that for all $j \geq j_0$

$$S_t(p,j) \leq C2^{d(p/t-1)j}\left(\sum_{j'=j}^{\infty}2^{d(1-p_0/t)j'}2^{-s_0p_0j'}\right)^{p/p_0}. \tag{82}$$

Since $s_0 - d/p_0 > -d/t$, then

$$S_t(p,j) \leq C2^{(-d+(d-s_0p_0)(p/p_0))j}. \tag{83}$$

Thus

$$f \in O_{p,t}^{s_0-d/p_0+d/p}(\Omega). \tag{84}$$

(b) Let now $p \leq p_0$.

$$\forall 0 \leq r \leq 1$$
$$\sum_{\lambda\in\Lambda_j(\Omega)}\left(\ell_{t,\lambda}\right)^{rp} \geq C2^{dj(1-r)}\left(\sum_{\lambda\in\Lambda_j(\Omega)}\left(\ell_{t,\lambda}\right)^p\right)^r. \tag{85}$$

It follows that

$$S_t(rp,j) \geq C\left(S_t(p,j)\right)^r. \tag{86}$$

From 1.(a) since $s_0 > 0$, then $f \in O_{t,p_0}^{s_0}(\Omega)$. Then

$$S_t(p_0,j) \leq C2^{-s_0p_0j}. \tag{87}$$

By taking $r = p_0/p$, we deduce that

$$S_t(p,j) \leq C2^{-s_0pj}. \tag{88}$$

Thus

$$f \in O_{p,t}^{s_0}(\Omega). \tag{89}$$

(2) Assume that $t < p_0$. Since $s_0 - d/p_0 > -d/t$, then there exists $t' > t$ such that $s_0 - d/p_0 = -d/t'$. It follows from the fact that $s_0 > 0$ that $t' > p_0$. Let $t_1 \in (p_0, t')$ and $t_2 < t$. Since $t < p_0$ write $t = \theta t_1 + (1-\theta)t_2$.

Since $t_1 < p'$, then $s_0 - d/p_0 > -d/t_1$. Since $t_1 > p_0$, then from above 1.(a) yields

$$\begin{aligned} &f \in O_{p,t_1}^{s_0-d/p_0+d/p}(\Omega) \quad \text{if } p \geq p_0 \\ &f \in O_{p,t_1}^{s_0}(\Omega) \quad \text{if } p < p_0. \end{aligned} \tag{90}$$

On the other hand, since $t_2 < p_0$ then $f \in O_{p,t_2}^{s_0-d/p_0}(\Omega)$. In fact

$$\begin{aligned} a_{t_2,\lambda} &\leq C\sum_{j'=j}^{\infty}2^{d(1-t_2/p_0)(j'-j)}\left(\sum_{\lambda'\subset\lambda}\left|C_{\lambda'}\right|^{p_0}\right)^{t_2/p_0}2^{-dj'} \\ &= C2^{-d(1-t_2/p_0)j}\sum_{j'=j}^{\infty}\left(2^{-dj'}\sum_{\lambda'\subset\lambda}\left|C_{\lambda'}\right|^{p_0}\right)^{t_2/p_0}. \end{aligned} \tag{91}$$

Since $f \in B_{p_0}^{s_0,q_0}(\Omega)$, then from (81) we get

$$\forall j \geq j_0 \ \forall \lambda \in \Lambda_j$$
$$a_{t_2,\lambda} \leq C2^{-d(1-t_2/p_0)j}\sum_{j'=j}^{\infty}2^{-t_2s_0j'}. \tag{92}$$

Since $s_0 > 0$ then

$$\forall j \geq j_0 \ \forall \lambda \in \Lambda_j$$
$$a_{t_2,\lambda} \leq C2^{-d(1-t_2/p_0)j}2^{-t_2s_0j}. \tag{93}$$

It follows from (60) that

$$\forall j \geq j_0 \ \forall \lambda \in \Lambda_j \qquad \ell_{t_2,\lambda} \leq C 2^{-(s_0 - d/p_0) j}. \tag{94}$$

Then

$$S_{t_2}(p, j) \leq C 2^{-p(s_0 - d/p_0) j}. \tag{95}$$

Therefore

$$f \in O_{p,t_2}^{s_0 - d/p_0}(\Omega). \tag{96}$$

Using (54), both (90) and (96) yield

$$\begin{aligned} f &\in O_{p,t}^{(\theta t_1/t)(s_0 - d/p_0 + d/p) + ((1-\theta) t_2/t)(s_0 - d/p_0)}(\Omega) \\ &= O_{p,t}^{(\theta t_1/t)(d/p) + s_0 - d/p_0}(\Omega) \quad \text{if } p \geq p_0 \\ f &\in O_{p,t}^{(\theta t_1/t) s_0 + ((1-\theta) t_2/t)(s_0 - d/p_0)}(\Omega) \\ &= O_{p,t}^{s_0 - ((1-\theta) t_2/t)(d/p_0)}(\Omega) \quad \text{if } p \leq p_0. \end{aligned} \tag{97}$$

When t_2 tends to 0, fraction $(1-\theta)t_2/t$ tends to 0 too, and $\theta t_1/t$ tends to 1. Hence (76) holds. □

□

4. Generic (p,t)–Oscillation Exponent

For a given Besov or Sobolev space, the residual set that we will construct will be generated from a saturating function F (i.e., for which the lower bounds obtained in Theorem 10 become equality). Thanks to embedding (46), we can choose for saturating function for the Sobolev space the one obtained for the Besov space $B_{p_0}^{s_0,1}(\mathbb{R}^d)$.

4.1. Saturating Function. For $\lambda = k2^{-j} + [0, 2^{-j})^d$, let $J \leq j$ be the unique integer given by the irreducible representation

$$k 2^{-j} = K 2^{-J} \quad \text{with } K \in \mathbb{Z}^d - (2\mathbb{Z})^d. \tag{98}$$

For $L = (\ell_1, \ldots, \ell_d) \in \mathbb{Z}^d$, put

$$\begin{aligned} |L| &= |\ell_1| + \cdots + |\ell_d|, \\ \Omega_L &= L + (0,1)^d. \end{aligned} \tag{99}$$

The following wavelet series will be called a saturating function:

$$F = \sum_{L \in \mathbb{Z}^d} 2^{-|L|} F_L \tag{100}$$

where

$$F_L = \sum_{j=1}^{\infty} \sum_{\lambda \in \Lambda_j(\Omega_L)} j^{-a} 2^{(d/p_0 - s_0) j} 2^{-(d/p_0) J} \psi_\lambda \tag{101}$$

and $a = 2/p_0 + 2/q_0 + 1$.

Remark 12. If $\lambda = k2^{-j} + [0, 2^{-j})^d \in \Lambda_j(\Omega_0)$ and $L \in \mathbb{Z}^d$ then $L + \lambda := L + k2^{-j} + [0, 2^{-j})^d \in \Lambda_j(\Omega_L)$. Both λ and $L + \lambda$ share the same J. The previous function F satisfies

$$C_{L+\lambda} = 2^{-|L|} C_\lambda. \tag{102}$$

This yields

$$\ell_{t, L+\lambda} = 2^{-|L|} \ell_{t,\lambda}. \tag{103}$$

It follows that

$$\zeta_t^{\Omega_L}(p) = \zeta_t^{\Omega_0}(p). \tag{104}$$

It is easy to show that $F_0 \in B_{p_0}^{s_0,q_0}(\Omega_0)$ (see [15], Proposition 2, p. 532). Relation (102) yields that $\|F_L\|_{B_{p_0}^{s_0,q_0}(\Omega_L)} = 2^{-|L|} \|F_0\|_{B_{p_0}^{s_0,q_0}(\Omega_0)}$. Note that the norm of the local Besov space is as the global one but with restriction on

$$\Lambda(\Omega) = \bigcup_{j \in \mathbb{N}_0} \Lambda_j(\Omega). \tag{105}$$

Therefore $F \in B_{p_0}^{s_0,q_0}(\mathbb{R}^d)$ and $\|F\|_{B_{p_0}^{s_0,q_0}(\mathbb{R}^d)} = \|F_0\|_{B_{p_0}^{s_0,q_0}(\Omega_0)}$.

Theorem 13. *Let $s_0, q_0, p_0 > 0$ and $t > 0$ be such that $s_0 - d/p_0 > -d/t$. Then the saturating function F satisfies*

$$\forall L \in \mathbb{Z}^d \qquad \zeta_t(p) = \zeta_t^{\Omega_L}(p) = \begin{cases} p\left(s_0 - \dfrac{d}{p_0}\right) + d & \forall p \geq p_0 \\ p s_0 & \forall p \leq p_0. \end{cases} \tag{106}$$

Proof. Thanks to both (104) and Theorem 10, it suffices to show that

$$\zeta_t^{\Omega_0}(p) \leq \begin{cases} p\left(s_0 - \dfrac{d}{p_0}\right) + d & \forall p \geq p_0 \\ p s_0 & \forall p \leq p_0. \end{cases} \tag{107}$$

We know that, for $j \in \mathbb{N}$,

$$\Lambda_j(\Omega_0) = \left\{\lambda = k 2^{-j} + \left[0, 2^{-j}\right)^d \text{ with } k \in \left\{0, \ldots, 2^j - 1\right\}^d\right\}. \tag{108}$$

It is easy to show that

$$\forall p > 0 \qquad \eta^{\Omega_0}(p) = \begin{cases} d + p\left(s_0 - \dfrac{d}{p_0}\right) & \text{if } p \geq p_0 \\ s_0 p & \text{if } p \leq p_0 \end{cases} \tag{109}$$

(the proof is the same as in [15], Proposition 4, p. 540-541, and we do not need the assumption $s_0 > d/p_0$).

The following result estimates the t–wavelet leaders of F_0 and allows the computation of its local (p,t)– oscillation exponent on Ω_0.

Proposition 14. *Let $s_0, p_0, q_0 > 0$ and $t > 0$ such that $s_0 - d/p_0 > -d/t$. Let F_0 be as in (101). Then there exist two positive constants $C_1(t)$ and $C_2(t)$ such that*

$$\begin{gathered}\forall \varepsilon > 0\ \exists j_\varepsilon\ \forall j \geq j_\varepsilon\ \forall \lambda \in \Lambda_j(\Omega_0) \\ C_1(t)\, 2^{-\varepsilon j} j^a C_\lambda \leq \ell_{t,\lambda} \leq C_2(t)\, j^a C_\lambda.\end{gathered} \tag{110}$$

Proof. For $\lambda' \subset \lambda$, either $J' = J$ or $J' > j$. There exists $C > 0$ such that, at each scale $J' > j$, there are $C2^{d(J'-j)}$ subdyadic cubes λ' of λ that have the irreducible representation $K'2^{-J'}$. Then

$$\begin{aligned} a_{t,\lambda} &= \sum_{j'=j}^{\infty} \sum_{\lambda' \subset \lambda} j'^{-at} 2^{(d/p_0 - s_0)tj'} 2^{-(dt/p_0)J'} 2^{-dj'} \\ &= \left(\sum_{j'=j}^{\infty} j'^{-at} 2^{((d/p_0 - s_0)t - d)j'} \right) 2^{-(dt/p_0)J} \\ &\quad + \sum_{J'=j+1}^{\infty} 2^{d(J'-j)} \sum_{j'=J'}^{\infty} \left(j'^{-at} 2^{((d/p_0 - s_0)t - d)j'} \right) 2^{-(dt/p_0)J'}. \end{aligned} \tag{111}$$

The left-hand series corresponds to $\lambda' = k2^{-j} + [0, 2^{-j'})^d$ if $\lambda = k2^{-j} + [0, 2^{-j})^d$.

Consequently, there exists h_j such that

$$\begin{aligned} a_{t,\lambda} = h_j \Bigg[&\left(\sum_{j'=j}^{\infty} 2^{((d/p_0 - s_0)t - d)j'} \right) 2^{-(dt/p_0)J} \\ &+ \sum_{J'=j+1}^{\infty} 2^{d(J'-j)} \sum_{j'=J'}^{\infty} \left(2^{((d/p_0 - s_0)t - d)j'} \right) 2^{-(dt/p_0)J'} \Bigg] \end{aligned} \tag{112}$$

with

$$\forall j\ h_j \leq 1 \tag{113}$$

and

$$\begin{gathered}\forall \varepsilon > 0\ \exists j_\varepsilon\ \forall j \geq j_\varepsilon \\ h_j \geq 2^{-t\varepsilon j}.\end{gathered} \tag{114}$$

Since $s_0 - d/p_0 > -d/t$, we get

$$\begin{aligned} a_{t,\lambda} &\approx h_j \left(2^{((d/p_0 - s_0)t - d)j} 2^{-(dt/p_0)J} + 2^{-(s_0 t + d)j} \right) \\ &\approx h_j 2^{-(s_0 t + d)j} 2^{(dt/p_0)(j-J)} \end{aligned} \tag{115}$$

where $\approx$ means that the left quantity is bounded from below and above by positive constants times the right quantity.

It follows that

$$\ell_{t,\lambda} = 2^{(d/t)j} a_\lambda^{1/t} \approx h_j^{1/t} 2^{(d/p_0 - s_0)j} 2^{-(d/p_0)J}. \tag{116}$$

Hence (110) holds. □

Relation (110) together with (23) yields (107). □

4.2. The Residual Set. We will need the following lemma.

Lemma 15. *Let $s_0, p_0, q_0 > 0$ and $t > 0$ such that $s_0 - d/p_0 > -d/t$. Then, for all $\Omega \in \mathcal{O}$, for all $f \in B_{p_0}^{s_0,q_0}(\Omega)$,*

$$\begin{gathered}\forall j\ \forall \lambda \in \Lambda_j(\Omega) \\ \ell_{t,\lambda} \leq C(t)\, 2^{(d/p_0 - s_0)j} \|f\|_{B_{p_0}^{s_0,q_0}(\Omega)}\end{gathered} \tag{117}$$

where $C(t) = \max(1/(1 - 2^{s_0 t}), 1/(1 - 2^{-d(1-t/p_0) - s_0 t}))^{1/t}$.

Proof. Let $\Omega \in \mathcal{O}$ and $f \in B_{p_0}^{s_0,q_0}(\Omega)$. Write $\|f\|$ instead of $\|f\|_{B_{p_0}^{s_0,q_0}(\Omega)}$. Clearly

$$\begin{gathered}\forall j \\ \sum_{\lambda \in \Lambda_j(\Omega)} |C_\lambda|^{p_0} \leq 2^{(d - s_0 p_0)j} \|f\|^{p_0}.\end{gathered} \tag{118}$$

Let $\lambda \in \Lambda_j(\Omega)$ and $t > 0$.

(i) Suppose that $t \leq p_0$. Then

$$\begin{aligned} &\sum_{\lambda' \in \Lambda_{j'}, \lambda' \subset \lambda} |C_{\lambda'}|^t 2^{-dj'} \\ &\leq 2^{-dj'} 2^{d(1-t/p_0)(j'-j)} \left(\sum_{\lambda' \in \Lambda_{j'}, \lambda' \subset \lambda} |C_{\lambda'}|^{p_0} \right)^{t/p_0}. \end{aligned} \tag{119}$$

Using (118)

$$\begin{aligned} &\sum_{\lambda' \in \Lambda_{j'}, \lambda' \subset \lambda} |C_{\lambda'}|^t 2^{-dj'} \\ &\leq 2^{-d(1-t/p_0)j} 2^{-d(t/p_0)j'} \left(2^{(d - s_0 p_0)j'} \|f\|^{p_0} \right)^{t/p_0} \\ &\leq 2^{-d(1-t/p_0)j} 2^{-s_0 t j'} \|f\|^t. \end{aligned} \tag{120}$$

Then

$$\begin{aligned} a_{t,\lambda} &\leq 2^{-d(1-t/p_0)j} \left(\sum_{j'=j}^{\infty} 2^{-s_0 t j'} \right) \|f\|^t \\ &\leq C_1(t)\, 2^{(-d(1-t/p_0) - s_0 t)j} \|f\|^t \end{aligned} \tag{121}$$

where $C_1(t) = 1/(1 - 2^{s_0 t})$.

We therefore get

$$\ell_{t,\lambda} \leq (C_1(t))^{1/t} 2^{(d/p_0 - s_0)j} \|f\|. \tag{122}$$

(ii) If $t > p_0$ and $s_0 - d/p_0 > -d/t$, then

$$\begin{aligned}\sum_{\lambda'\in\Lambda_{j'},\lambda'\subset\lambda} |C_{\lambda'}|^t 2^{-dj'} &\leq 2^{-dj'} \left(\sum_{\lambda'\in\Lambda_{j'},\lambda'\subset\lambda} |C_{\lambda'}|^{p_0} \right)^{t/p_0} \\ &\leq 2^{-dj'} \left(2^{(d-s_0p_0)j'} \|f\|^{p_0} \right)^{t/p_0} \\ &= 2^{(-d(1-t/p_0)-s_0t)j'} \|f\|^t. \end{aligned} \tag{123}$$

Since $s_0 - d/p_0 > -d/t$, we get

$$\begin{aligned} a_{t,\lambda} &\leq \left(\sum_{j'=j}^{\infty} 2^{(-d(1-t/p_0)-s_0t)j'} \right) \|f\|^t \\ &\leq C_2(t)\, 2^{(-d(1-t/p_0)-s_0t)j} \|f\|^t \end{aligned} \tag{124}$$

where $C_2(t) = 1/(1 - 2^{-d(1-t/p_0)-s_0t})$.

Then

$$\ell_{t,\lambda} \leq (C_2(t))^{1/t} 2^{(d/p_0-s_0)j} \|f\|. \tag{125}$$

Therefore (117) holds with $C(t) = \max\{C_1(t), C_2(t)\}^{1/t}$.

Hence Lemma 15 holds. □

Theorem 16. *Let $s_0, p_0, q_0 > 0$ (resp., $s_0 > 0$ and $p_0 \geq 1$) and $t > 0$ such that $s_0 - d/p_0 > -d/t$.*

(1) *For all $L \in \mathbb{Z}^d$, there exists a residual set $\mathscr{A}(\Omega_L)$ of $B^{s_0,q_0}_{p_0}(\Omega_L)$ (resp., $L^{p_0,s_0}(\Omega_L)$) such that for all $f \in \mathscr{A}(\Omega_L)$*

$\forall p > 0$

$$\zeta_t^{\Omega_L}(p) = \begin{cases} d + p\left(s_0 - \dfrac{d}{p_0}\right) & \text{if } p \geq p_0 \\ s_0 p & \text{if } p \leq p_0 \end{cases} \tag{126}$$

and

$$\inf_{0<p<\infty} \left(d + hp - \zeta_t^{\Omega_L}(p)\right) = \begin{cases} -\infty & \text{if } h < s_0 - \dfrac{d}{p_0} \\ p_0(h - s_0) + d & \text{if } s_0 - \dfrac{d}{p_0} \leq h \leq s_0 \\ d & \text{if } h > s_0. \end{cases} \tag{127}$$

(2) *There exists a residual set $\mathscr{A}$ of $B^{s_0,q_0}_{p_0}(\mathbb{R}^d)$ (resp., $L^{p_0,s_0}(\mathbb{R}^d)$) such that for all $f \in \mathscr{A}$*

$\forall p > 0$

$$\zeta_t(p) = \begin{cases} d + p\left(s_0 - \dfrac{d}{p_0}\right) & \text{if } p \geq p_0 \\ s_0 p & \text{if } p \leq p_0 \end{cases} \tag{128}$$

and

$$\inf_{0<p<\infty} \left(d + hp - \zeta_t(p)\right) = \begin{cases} -\infty & \text{if } h < s_0 - \dfrac{d}{p_0} \\ p_0(h - s_0) + d & \text{if } s_0 - \dfrac{d}{p_0} \leq h \leq s_0 \\ d & \text{if } h > s_0. \end{cases} \tag{129}$$

Relations (126) and (127) also hold on $\mathscr{A}$.

Proof.

(1) (i) First consider $L = 0$. By Theorem 10, it suffices to prove the upper bound in (126). Thanks to embedding (46), it suffices to write the proof in Besov spaces.

We will follow the idea of [64] in the construction of the residual set since it does not depend on the separability of the space $B^{s_0,q_0}_{p_0}(\Omega_0)$. Let F_0 be the saturating function defined in (101).

From now on, when it is necessary, we will make the dependency on the function in the previous notations (for example, we write $C_\lambda(f)$ instead of C_λ in (10), $\ell_{t,\lambda}(f)$ instead of $\ell_{t,\lambda}$, and $\zeta^{\Omega_0}_{t,f}(p)$ instead of $\zeta^{\Omega_0}_t(p)$).

If $n \in \mathbb{N}$, set

$$E_n(\Omega_0) = \left\{ g_n \in B^{s_0,q_0}_{p_0}(\Omega_0) : \forall\lambda \in \Lambda(\Omega_0)\ \exists M \in \mathbb{Z}\setminus\{0\} : C_\lambda(g_n) = \frac{M}{n} C_\lambda(F_0) \right\}, \tag{130}$$

where $\Lambda(\Omega)$ is as in (105).

Clearly

$$\forall n\ \forall g_n \in E_n(\Omega_0)\ \forall\lambda \in \Lambda(\Omega_0) \quad \ell_{t,\lambda}(g_n) \geq \frac{1}{n} \ell_{t,\lambda}(F_0). \tag{131}$$

Write $\|f\|$ instead of $\|f\|_{B^{s_0,q_0}_{p_0}(\Omega_0)}$. In [64] (Proof of Lemma 2.9, p. 1520-1521), it is proved that there exists $C > 0$ such that

$$\forall f \in B^{s_0,q_0}_{p_0}(\Omega_0)\ \forall n \in \mathbb{N}\ \exists g_n \in E_n(\Omega_0) \quad \|f - g_n\| \leq \frac{C}{n}, \tag{132}$$

and so, if $N \in \mathbb{N}$, then the set $D_N(\Omega_0) := \bigcup_{n\geq N} E_n(\Omega_0)$ is dense in $B^{s_0,q_0}_{p_0}(\Omega_0)$.

Let $(r_n)_n$ be a sequence of positive numbers which converges to 0. Put

$$A_n(\Omega_0) = \left\{ f \in B^{s_0,q_0}_{p_0}(\Omega_0) : \exists g_n \in E_n(\Omega_0)\ \|f - g_n\| < r_n \right\}. \tag{133}$$

Then

$$\mathscr{A}(\Omega_0) := \bigcap_{N\in\mathbb{N}} \bigcup_{n\geq N} A_n(\Omega_0) \tag{134}$$

is a residual set of $B^{s_0,q_0}_{p_0}(\Omega_0)$.

If $\widetilde{C}(t) = \max(2^{1/t-1}, 1)$, then the following trivial relation holds:

$$\begin{gathered} \forall (f,g) \ \forall\lambda \\ \ell_{t,\lambda}(f+g) \leq \widetilde{C}(t)\left(\ell_{t,\lambda}(f) + \ell_{t,\lambda}(g)\right). \end{gathered} \tag{135}$$

□

Choose

$$r_n = \frac{C_1(t)}{2n^{2a}\widetilde{C}(t)\,C(t)} 2^{-dn/p_0} \tag{136}$$

where $C(t)$ is as in Lemma 15 and $C_1(t)$ is any constant satisfying (110). Let $\mathscr{A}(\Omega_0)$ be the residual set given by (141) associated with the sequence (r_n).

Let $f \in \mathscr{A}(\Omega_0)$. Then for infinitely many integers n, there exists $g_n \in E_n(\Omega_0)$ such that $\|f - g_n\| < r_n$.

By (135)

$$\begin{gathered} \forall\lambda \in \Lambda \\ \ell_{t,\lambda}(f) \geq \frac{1}{\widetilde{C}(t)}\ell_{t,\lambda}(g_n) - \ell_{t,\lambda}(g_n - f). \end{gathered} \tag{137}$$

By Lemma 15 and Proposition 14,

$$\begin{aligned} &\forall\lambda \in \Lambda_n(\Omega_0) \\ &\ell_{t,\lambda}(g_n - f) \leq C(t)\, 2^{(d/p_0 - s_0)n} \|f - g_n\| \\ &\qquad \leq C(t)\, 2^{(d/p_0 - s_0)n} r_n \\ &\qquad \leq \frac{1}{2n^a\widetilde{C}(t)}\ell_{t,\lambda}(F_0). \end{aligned} \tag{138}$$

It follows from (131) that

$$\begin{gathered} \forall\lambda \in \Lambda_n(\Omega_0) \\ \ell_{t,\lambda}(f) \geq \frac{1}{2n\widetilde{C}(t)}\ell_{t,\lambda}(F_0). \end{gathered} \tag{139}$$

Since (139) holds for infinitely many scales n and for all $\lambda \in \Lambda_n$, and thanks to the fact that the liminf in $\zeta_{t,F_0}(p)$ is actually a limit, we deduce that

$$\begin{gathered} \forall p > 0 \\ \zeta^{\Omega_0}_{t,f}(p) \leq \zeta^{\Omega}_{t,F_0}(p). \end{gathered} \tag{140}$$

Result (106) allows achieving the proof of Theorem 16.

(ii) Now if $L \neq 0$, it suffices to replace F_0 by F_L defined in (101) and r_n by $2^{-|L|}r_n$. This means that the residual set of $B^{s_0,q_0}_{p_0}(\Omega_0)$ on which (126) will hold is

$$\mathscr{A}(\Omega_L) := \bigcap_{N\in\mathbb{N}} \bigcup_{n\geq N} A_n(\Omega_L), \tag{141}$$

where

$$\begin{aligned} A_n(\Omega_L) = \Big\{ &f \in B^{s_0,q_0}_{p_0}(\Omega_L) : \exists g_n \\ &\in E_n(\Omega_L)\ \|f - g_n\|_{B^{s_0,q_0}_{p_0}(\Omega_L)} < 2^{-|L|} r_n \Big\} \end{aligned} \tag{142}$$

and

$$\begin{aligned} E_n(\Omega_L) = \Big\{ &g_n \in B^{s_0,q_0}_{p_0}(\Omega_L) : \forall\lambda \in \Lambda(\Omega_L)\ \exists M \\ &\in \mathbb{Z}\setminus\{0\} : C_\lambda(g_n) = \frac{M}{n} C_\lambda(F_L) \Big\}. \end{aligned} \tag{143}$$

(2) Clearly

$$\mathscr{A} = \bigcap_{L\in\mathbb{Z}^d} \mathscr{A}(\Omega_L) \tag{144}$$

is a residual set of $B^{s_0,q_0}_{p_0}(\mathbb{R}^d)$ on which (128) and (126) hold (using (24) and the first result in this theorem).

5. Generic Validity of the t−Multifractal Formalisms

We first show that condition $\eta(t) > 0$ in both result (15) and upper bound (31) holds for all functions in Besov spaces $B^{s_0,q_0}_{p_0}(\mathbb{R}^d)$ for $s_0, q_0, p_0 > 0$ and Sobolev spaces $L^{p_0,s_0}(\mathbb{R}^d)$ for $s_0 > 0$ and $p_0 \geq 1$, for all $t > 0$ such that $s_0 - d/p_0 > -d/t$.

Lemma 17. *Let f be any function in either the Besov space $B^{s_0,q_0}_{p_0}(\mathbb{R}^d)$ for $s_0, q_0, p_0 > 0$ or the Sobolev space $L^{p_0,s_0}(\mathbb{R}^d)$ for $s_0 > 0$ and $p_0 \geq 1$. Then, for all $t > 0$ such that $s_0 - d/p_0 > -d/t$,*

$$\eta(t) > 0. \tag{145}$$

Proof. Embedding (56) implies that $\eta(p) \geq \zeta_t(p)$. It follows that $\eta(t) \geq \zeta_t(t)$. Since $s_0 - d/p_0 > -d/t$, then Theorem 10 yields $\eta(t) > 0$. □

Lemma 17 together with both Theorem 10 and upper bound (31) yields the following corollary.

Corollary 18. *Let f be any function in either the Besov space $B^{s_0,q_0}_{p_0}(\mathbb{R}^d)$ for $s_0, q_0, p_0 > 0$ or the Sobolev space $L^{p_0,s_0}(\mathbb{R}^d)$ for $s_0 > 0$ and $p_0 \geq 1$. Then, for all $t > 0$ such that $s_0 - d/p_0 > -d/t$*

$$\begin{gathered} \forall h \\ d_t(h) \leq D_t(h) \leq \inf_{0<p<\infty}\left(d + hq - \zeta_t(p)\right) \\ \begin{cases} = -\infty & \text{if } h < s_0 - \dfrac{d}{p_0} \\ \leq p_0(h - s_0) + d & \text{if } s_0 - \dfrac{d}{p_0} \leq h \leq s_0 \\ \leq d & \text{if } h > s_0. \end{cases} \end{gathered} \tag{146}$$

The following result shows that the Baire generic t-spectrum is the same as the prevalent generic one obtained in [49].

Theorem 19. *Let $s_0, p_0, q_0 > 0$ (resp., $s_0 > 0$ and $p_0 \geq 1$) and $t > 0$ such that $s_0 - d/p_0 > -d/t$.*

(1) *For all $L \in \mathbb{Z}^d$, for all functions f in the residual set $\mathscr{A}(\Omega_L)$ of $B^{s_0,q_0}_{p_0}(\Omega_L)$ (resp., $L^{p_0,s_0}(\Omega_L)$) constructed in Theorem 16,*

$$d_t^{\Omega_L}(h) = \begin{cases} -\infty & \text{if } h < s_0 - \dfrac{d}{p_0} \text{ or } h > s_0 \\ p_0(h - s_0) + d & \text{if } s_0 - \dfrac{d}{p_0} \leq h \leq s_0 \end{cases} \tag{147}$$

and

$$D_t^{\Omega_L}(h) = \begin{cases} -\infty & \text{if } h < s_0 - \dfrac{d}{p_0} \\ p_0(h - s_0) + d & \text{if } s_0 - \dfrac{d}{p_0} \leq h \leq s_0 \\ d & \text{if } h > s_0. \end{cases} \tag{148}$$

(2) *For all functions f in the residual set $\mathscr{A}$ of $B^{s_0,q_0}_{p_0}(\mathbb{R}^d)$ (resp., $L^{p_0,s_0}(\mathbb{R}^d)$) constructed in Theorem 16,*

$$d_t(h) = \begin{cases} -\infty & \text{if } h < s_0 - \dfrac{d}{p_0} \text{ or } h > s_0 \\ p_0(h - s_0) + d & \text{if } s_0 - \dfrac{d}{p_0} \leq h \leq s_0 \end{cases} \tag{149}$$

and

$$D_t(h) = \begin{cases} -\infty & \text{if } h < s_0 - \dfrac{d}{p_0} \\ p_0(h - s_0) + d & \text{if } s_0 - \dfrac{d}{p_0} \leq h \leq s_0 \\ d & \text{if } h > s_0. \end{cases} \tag{150}$$

Both (147) and (148) also hold on $\mathscr{A}$.

Proof. As in the proof of Theorem 16, it suffices to give the proof for $L = 0$; let $f \in \mathscr{A}(\Omega_0)$. Relation (139) is satisfied for infinitely many scales n. Let $\alpha \geq 1$. For each such scale n, write $\lambda \in \Gamma_n(\alpha)$ if $\lambda \in \Lambda_n$ satisfies (98) with $J = [n/\alpha]$. Let $K(\alpha)$ be the set of points $x \in \Omega_0$ that belong to $\lambda \in \Gamma_n(\alpha)$, for an infinite number of values of the above n's. In [15], p. 535-536, we can find the following result (see also [65]).

Proposition 20. *For $\alpha \geq 1$, the Hausdorff dimension of $K(\alpha)$ is d/α. Moreover, there exists a σ-finite measure m_α carried by $K(\alpha)$ such that $m_\alpha(\Omega) = 0$ for any $\Omega \subset K(\alpha)$ with $\dim\Omega < d/\alpha$.*

The following proposition bounds the pointwise L^t regularity of $f \in \mathscr{A}(\Omega_0)$ on $K(\alpha)$.

Proposition 21. *Let $s_0, p_0, q_0 > 0$ (resp., $s_0 > 0$ and $p_0 \geq 1$) and $t > 0$ such that $s_0 - d/p_0 > -d/t$. For all $f \in \mathscr{A}(\Omega_0)$, for all $x \in K_\alpha$,*

$$u_t(x) \leq s_0 - \frac{d}{p_0}\left(1 - \frac{1}{\alpha}\right). \tag{151}$$

Proof. Let $x \in K(\alpha)$. Result (15), relation (139), and result (145) imply that

$$\begin{aligned} u_{t,f}(x) &\leq \liminf_{n\to\infty} \frac{\log\left(\ell_{t,\lambda_n(x)}(f)\right)}{\log(2^{-n})} \\ &\leq \liminf_{n\to\infty} \frac{\log\left(\ell_{t,\lambda_n(x)}(F_0)\right)}{\log(2^{-n})} \\ &= s_0 - \frac{d}{p_0}\left(1 - \frac{1}{\alpha}\right). \end{aligned} \tag{152}$$

□

Put $H(\alpha) = s_0 - (d/p_0)(1 - 1/\alpha)$. Take $\alpha = d/(p_0(h - s_0) + d)$ if $h \in [s_0 - d/p_0, s_0]$ (resp., $\alpha = 1$ if $h > s_0$). Then $H(\alpha) = h$ if $h \in [s_0 - d/p_0, s_0]$ (resp., $H(\alpha) = s_0 < h$ if $h > s_0$). Using Proposition 21, for all $x \in K(\alpha)$, $u_t(x) \leq h$. It follows that $K(\alpha) \subset B_t(h)$. Therefore, for all $h \geq s_0 - d/p_0$,

$$\min\{p_0(h - s_0) + d, d\} \leq D_t^{\Omega_0}(h). \tag{153}$$

Thus, by (20) and (146), for all $h \geq s_0 - d/p_0$,

$$D_t^{\Omega_0}(h) = \min\{p_0(h - s_0) + d, d\}. \tag{154}$$

For $h \in [s_0 - d/p_0, s_0]$, consider the set $J(h) = \{x \in \Omega_0; u_{t,f}(x) < H(\alpha) = h\}$, where $\alpha = d/(p_0(h - s_0) + d)$. Clearly $J(\alpha) = \bigcup_{n\geq 1} B_t^{\Omega_0}(h - 1/n)$. By (150), for all $n \geq 1$, $\dim B_t^{\Omega_0}(h - 1/n) < \min\{p_0(h - s_0) + d, d\}$. Then, by Proposition 20, $m_\alpha(B_t^{\Omega_0}(h - 1/n)) = 0$. Using the σ−additivity of the measure m_α, we get $m_\alpha(J(h)) = 0$.

Since $K(\alpha) \subset B_t^{\Omega_0}(h) = E_t^{\Omega_0}(h) \cup J(h)$, it follows that $m_\alpha(E_t^{\Omega_0}(h)) > 0$. Thus,

$$p_0(h - s_0) + d \leq d_t(h). \tag{155}$$

By (150), for $h \in [s_0 - d/p_0, s_0]$, we get

$$d_t^{\Omega_0}(h) = p_0(h - s_0) + d. \tag{156}$$

For $\alpha = 1$, $K(1) = \Omega_0$. Then for all $x \in \Omega_0$, $u_t(x) \leq s_0$. Thus, for $h > s_0$, $E_t^{\Omega_0}(h) = \emptyset$.

Both (147) and (148) also hold on $\mathscr{A}$. This achieves the proof of Theorem 19. □

6. Conclusion

The following theorem summarizes the range of Baire validity of both t−multifractal formalism and t−multifractal formalism for upper t−sets, locally on the Ω_L and also globally on $\mathbb{R}^d$. It follows directly from both Theorems 16 and 19.

Theorem 22. *Let $s_0, p_0, q_0 > 0$ (resp., $s_0 > 0$ and $p_0 \geq 1$) and $p > 0$ such that $s_0 - d/p_0 > -d/t$. Baire generically in $B^{s_0,q_0}_{p_0}(\mathbb{R}^d)$ (resp., $L^{p_0,s_0}(\mathbb{R}^d)$), $E_t(h) = \emptyset$ for all $h > s_0$, the t−multifractal formalism is valid for $h \leq s_0$, and the t−multifractal formalism for upper t−sets is valid for all h.*

The same results hold for the local t−multifractal formalism on Ω_L.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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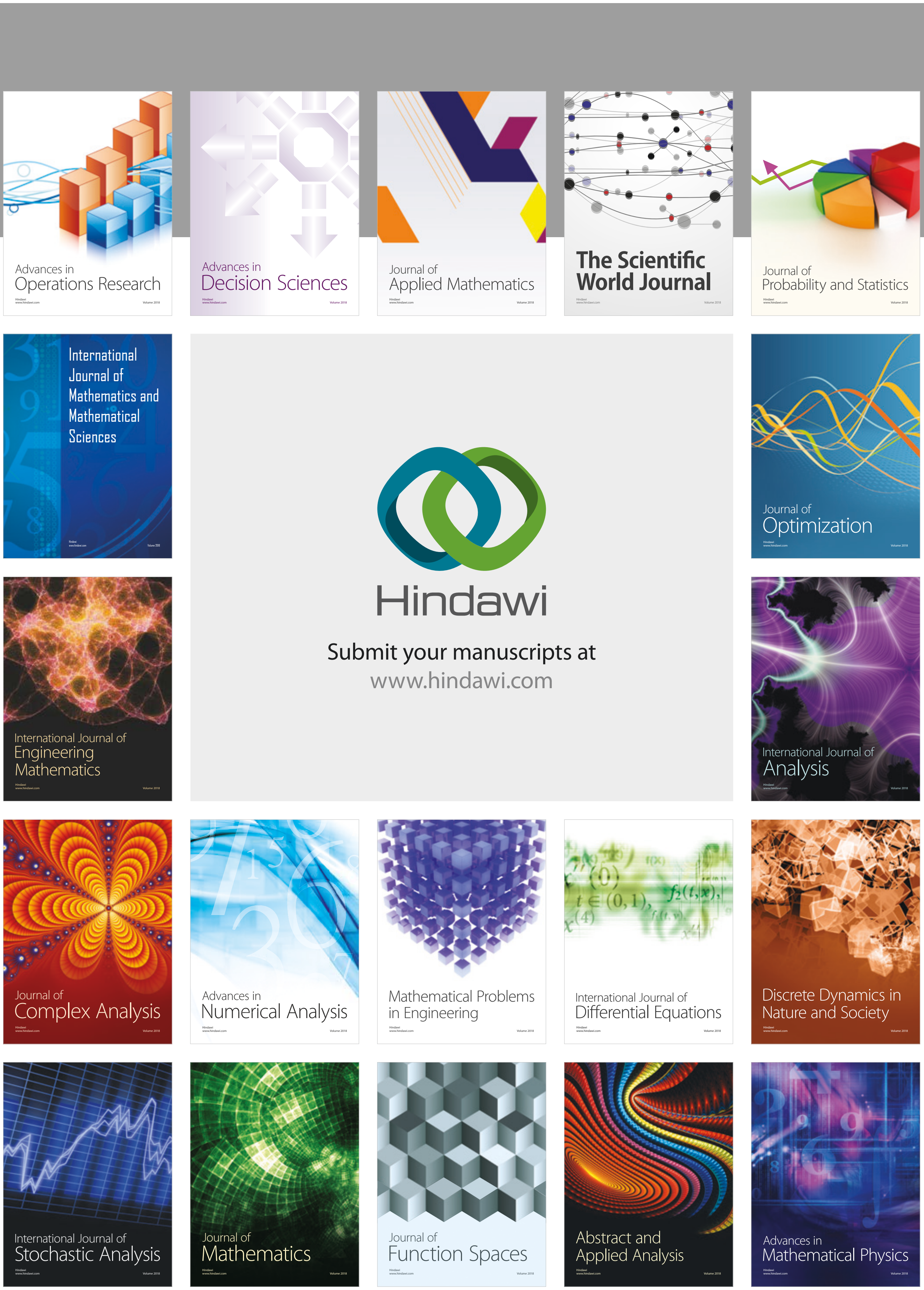

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