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Research Article

A Note on Integral Inequalities on Time Scales Associated with Ostrowski's Type

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Inequalities become a hot topic for researcher due to its wide applications in means and sum, numerical integration, quantum calculus. Different generalizations and refinements are made by researchers. Here, in this article, we give another generalization of integral inequalities and harmonizing them on time scale $\mathbb{T}$ from $\mathbb{R}$.

1. Introduction

Inequalities have a great contribution in mathematical analysis. In nonlinear analysis, these inequalities are very useful. Ostrowski's inequalities have various coatings in numerical integration and in the theory of probability. In 1938, a mathematician A. Ostrowski gave an inequality named as Ostrowski inequality, since then a large number of results related to this inequality have been investigated by many researchers. In literature, many research papers appeared which contains refinements, elongations, generalizations and many similiar results of this inequality.

Theorem 1 (see [1]). *Let $f : [b_1, b_2] \to \mathbb{R}$ be differentiable on (b_1, b_2), then we have*

$$\left| f(r) - \frac{1}{b_1 - b_2} \int_{b_1}^{b_2} f(s)ds \right| \leq M(b_2 - b_1)\left[\frac{(r - (b_1 + b_2)/2)^2}{(b_2 - b_1)^2} + \frac{1}{4} \right], \tag{1}$$

where $M = \sup_{b_1 < r < b_2} |f'(r)| < \infty$ holds for all $r \in [b_1, b_2]$.

This is the Ostrowski inequality here the constant $1/4$ is best possible. Ostrowski's inequality plays a vital role in theory of special means. This inequality has multiple ises in a variety of settings. Lately there have been elongations and many new results of this inequality. This inequality has significant and remarkable background in mathematical analysis. All the work related to this inequality is not possible to list here.

If you want to study discrete and continuous analysis together you will need the theory of time scale. S. Hilger competed the great task of harmonizing continuous and discrete calculus in one result, in his PhD research. Now we are able to give one definition for discrete and continuous analysis and if we change the range of function in the result we will come to different cases of time scale.

Time Scales is defined as a closed subset of $\mathbb{R}$ by Stefan Hilger, which is symbolize as $\mathbb{T}$. A point of $\mathbb{T}$ is defined as $r : r \in \mathbb{T}$. If we consider $\mathbb{T} = \mathbb{R}$ then, $T^{\Delta}(r) = T^{\nabla}(r) = T'(r)$. However, if $\mathbb{T} = \mathbb{Z}$ then, $T^{\Delta}(r) = \Delta T(r)$, where $T^{\Delta}(r) = T(r+1) - T(r)$ and $T^{\nabla}(r) = T(r) - T(r-1)$ are forward and backward difference operators used in difference equability. The mappings $\sigma, \rho : \mathbb{T} \to \mathbb{T}$ defined as $\sigma(r) = \inf\{s \in \mathbb{T} : s > r\}$ and $\rho(r) = \sup\{s \in \mathbb{T} : s < r\}$ are the jump operators. S. Hilger gave a new definition of derivative which was denoted by T^{Δ}; T^{Δ} exists if and only if for every $\varepsilon > 0$ $\exists$ a neighborhood U of r s.t

$$\left| T^{\sigma}(r) - T(s) - T^{\Delta}(r)(\sigma(r) - s) \right| \leq \varepsilon \left| \sigma(r) - s \right| \forall\, s \in U. \tag{2}$$

Also a differentiable mapping $T : \mathbb{T} \to \mathbb{R}$ is known as anti-derivative of T on $\mathbb{T}$ provided that $T^{\Delta}(r) = \mathcal{T}(r)$, then

$$\int_{b_1}^{b_2} T(r)\Delta r = \mathcal{T}(b_2) - \mathcal{T}(b_1), \forall r \in \mathbb{T}. \tag{3}$$

Let $T : \mathbb{T} \to \mathbb{R}$ and $r \in \mathbb{T}$,

(1) If T is differentiable at r then T is continuous at r.

(2) If T is differentiable at r, then

$$f^{\sigma}(r) = f(r) + \mu(r) f^{\Delta}(r). \tag{4}$$

In the resent years, calculus of time scales has enchanted scientists due to its tremendous practical applications in many branches, e.g., quantum calculus, dynamical system, information theory, etc., see [2–4]. During the last decennia, the progression of integral and differential equation have been revealed. The convenient discoveries concern a consequential part in many areas of research of mathematics (can be seen in [5, 6]). S. Hilger has proposed the time scale theory in the terms “a theory that combines differential and difference calculus in the most worldly wise manner”. Concludingly, a number of researchers have discussed the new assorted fact of the dynamic inequations on time scales comprehensively [5, 7–11].

Lemma 2 (see [1]). *Let $b_1, b_2, s, r \in \mathbb{T}$ and $b_1 < b_2$. If $f : [b_1, b_2] \to \mathbb{R}$ be differentiable, then $\forall s \in [b_1, b_2]$ and $\lambda \in [0, 1]$, then.*

$$\begin{aligned}\left(1 - \frac{\lambda}{2}\right) f(r) = {} & \frac{1}{b_2 - b_1} \int_{b_1}^{b_2} f(\sigma(s))\Delta s \\ & - \lambda \frac{(r - b_1) f(b_1) + (b_2 - r) f(b_2)}{2(b_2 - b_1)} \\ & + \frac{1}{b_2 - b_1} \int_{b_1}^{b_2} f^{\Delta}(s) K(r, s)\Delta s,\end{aligned} \tag{5}$$

where

$$K(r, s) := \begin{cases} s - \left(b_1 + \lambda \dfrac{r - b_1}{2}\right), b_1 \le s < r, \\ s - \left(b_2 - \lambda \dfrac{b_2 - r}{2}\right), r \le s \le b_2. \end{cases} \tag{6}$$

Theorem 3 (see[1]). *Let $b_1, b_2, s, r \in \mathbb{T}$ and If $f : [b_1, b_2] \to \mathbb{R}$ be differentiable, then*

$$\left| f(r) - \frac{1}{b_2 - b_1} \int_{b_1}^{b_2} f(\sigma(s))\Delta s \right| \le \frac{M}{b_2 - b_1} \left(h_2(r, b_1) + h_2(r, b_2)\right), \tag{7}$$

where $M = \sup_{b_1 < r < b_2} \left| f^{\Delta}(r) \right| < \infty$.

This is sharp because the R.H.S of this inequality can’t be changed by any smaller number. In this paper, we also get a generalization of this inequality. In this article first of all we will prove a generalize form of montgomery identity and then discuss the case for $w = b_2 - b_1$ In our next result get a generalized version of (7), we have also discussed its continuous, discrete and quantum calculus cases by choosing time scale as $\mathbb{R}, \mathbb{T}$ *and* $q_0^{\mathbb{Z}}$.

2. Main Results

For points $a, b \in \mathbb{T}$ *such that* $a < b$. The interval $[a, b]$ is distinguished as a real interval and $[a, b]_{\mathbb{T}}$ is distinguished as $[a, b] \cap \mathbb{T}$. In this sense $[a, b]_{\mathbb{T}}$ is a nonempty, closed and bounded set having points from $\mathbb{T}$. In this paper, by the interval $[a, b]$ we mean $[a, b] \cap \mathbb{T}$. Now we first prove an identity which is the generalize form of Montgomery identity and then use this identity in our next theorems to get new generalizations of Owstrowski’s inequality.

Lemma 4. *Let $b_1, b_2, w \in \mathbb{T}, b_1 < b_2$. If $f : [b_1, b_2] \to \mathbb{R}$ be differentiable, then $\forall s \in [b_1, b_2]$, we have*

$$f(r) - \frac{1}{w} \int_{b_1}^{b_2} f(\sigma(s))\Delta s = \frac{1}{w} \int_{b_1}^{b_2} f^{\Delta}(s) k_w(r, s)\Delta s, \tag{8}$$

where

$$K_w(r, s) = \begin{cases} s - \dfrac{(b_2 - w) f(b_2) - b_1 f(b_1)}{f(b_2) - f(b_1)} = s - \theta_1, & r \in [b_1, s], \\ s - \dfrac{b_2 f(b_2) - (b_1 + w) f(b_1)}{f(b_2) - f(b_1)} = s - \theta_2, & r \in (s, b_2], \end{cases} \tag{9}$$

with $\theta_2 - \theta_1 = w$.

Proof. We initiated with

$$\begin{aligned}\int_{b_1}^{b_2} & f^{\Delta}(s) K_w(r, s)\Delta s \\ & = \int_{b_1}^{r} f^{\Delta}(s) \left\{ s - \frac{(b_2 - w) f(b_2) - b_1 f(b_1)}{f(b_2) - f(b_1)} \right\} \Delta s \\ & + \int_{r}^{b_2} f^{\Delta}(s) \left\{ s - \frac{b_2 f(b_2) - (b_1 + w) f(b_1)}{f(b_2) - f(b_1)} \right\} \Delta s.\end{aligned} \tag{10}$$

We can rewrite after calculations

$$\begin{aligned}\int_{b_1}^{b_2} & f^{\Delta}(s) K_w(r, s)\Delta s \\ & = f(r) \left\{ r - \frac{(b_2 - w) f(b_2) - b_1 f(b_1)}{f(b_2) - f(b_1)} \right\} \\ & - f(b_1) \left\{ b_1 - \frac{(b_2 - w) f(b_2) - b_1 f(b_1)}{f(b_2) - f(b_1)} \right\} \\ & + f(b_2) \left\{ b_2 - \frac{b_2 f(b_2) - (b_1 + w) f(b_1)}{f(b_2) - f(b_1)} \right\} \\ & - f(r) \left\{ r - \frac{b_2 f(b_2) - (b_1 + w) f(b_1)}{f(b_2) - f(b_1)} \right\} \\ & - \int_{b_1}^{b_2} f^{<3;}(r)\Delta r = w f(r) - \int_{b_1}^{b_2} f^{<3;}(r)\Delta(r).\end{aligned} \tag{11}$$

Eventually, we come to the required result, i.e.,

$$f(r) - \frac{1}{w} \int_{b_1}^{b_2} f(\sigma(s))\Delta s = \frac{1}{w} \int_{b_1}^{b_2} f^{\Delta}(s) k_w(r, s)\Delta s. \tag{12}$$

Remark 5. Let $w = b_2 - b_1$ then the above equation (8) becomes

$$f(r)-\frac{1}{b_2-b_1}\int_{b_1}^{b_2}f(\sigma(s))\Delta s=\frac{1}{b_2-b_1}\int_{b_1}^{b_2}f^{\Delta}(s)K_{b_2-b_1}(r,s)\Delta s, \tag{13}$$

which is the Montgomery identity on $\mathbb{T}$ talked in [9], also discussed in [1] with continuous, discrete and quantum cases.

Theorem 6. *With supposition: for a time scale $\mathbb{T}$, $b_1, b_2, w \in \mathbb{T}$ such that $b_1 < b_2$. If $f : [b_1, b_2] \to \mathbb{R}$ be differentiable, then $\forall\, s \in [b_1, b_2]$.*

$$\left|f(r)-\frac{1}{w}\int_{b_1}^{b_2}f(\sigma(s))\Delta s\right|\le\frac{M}{w}\left[h_2(r,b_1)-h_2(r,b_2)+\frac{b_1-b_2+w}{f(b_2)-f(b_1)}\left[(r-b_1)f(b_2)+(b_2-r)f(b_1)\right]\right]. \tag{14}$$

holds where $M=\sup_{b_1<r<b_2}\left|f^{\Delta}(s)\right|$.

Proof. We can rescript Lemma 4 as

$$\begin{aligned}&\left|f(r)-\frac{1}{w}\int_{b_1}^{b_2}f(\sigma(s))\Delta s\right|\\&\le\frac{1}{w}\left|\int_{b_1}^{b_2}f^{\Delta}(s)K_w(r,s)\Delta s\right|\\&\le\frac{1}{+w}\left[\int_{b_1}^{r}\left|f^{\Delta}(s)\left(s-\frac{(b_2-w)f(b_2)-b_1f(b_1)}{f(b_2)-f(b_1)}\right)\right|\Delta s+\int_{r}^{b_2}\left|f^{\Delta}(s)\left(s-\frac{b_2f(b_2)-(b_1+w)f(b_1)}{f(b_2)-f(b_1)}\right)\right|\Delta s\right]\\&\le\frac{1}{w}\left[\int_{b_1}^{r}\left|f^{\Delta}(s)\right|\left(s-\frac{(b_2-w)f(b_2)-b_1f(b_1)}{f(b_2)-f(b_1)}\right)\Delta s+\int_{r}^{b_2}\left|f^{\Delta}(s)\right|\left(\frac{b_2f(b_2)-(b_1+w)f(b_1)}{f(b_2)-f(b_1)}-s\right)\Delta s\right]\\&\le\frac{M}{w}\left[h_2(r,b_1)-h_2(r,b_2)+\int_{b_1}^{r}\left(b_1-\frac{(b_2-w)f(b_2)-b_1f(b_1)}{f(b_2)-f(b_1)}\right)\Delta s+\int_{r}^{b_2}\left(\frac{b_2f(b_2)-(b_1+w)f(b_1)}{f(b_2)-f(b_1)}-b_2\right)\Delta s\right].\end{aligned} \tag{15}$$

And further

$$\left|f(r)-\frac{1}{w}\int_{b_1}^{b_2}f(\sigma(s))\Delta s\right|\le\frac{M}{w}\left[h_2(r,b_1)-h_2(r,b_2)+\frac{b_1-b_2+w}{f(b_2)-f(b_1)}\left[(r-b_1)f(b_2)+(b_2-r)f(b_1)\right]\right]. \tag{16}$$

Remark 7. When $w = b_2 - b_1$, then inequality (14) reduces to

$$\left|f(r)-\frac{1}{b_2-b_1}\int_{b_1}^{b_2}f(\sigma(s))\Delta s\right|\le\frac{M}{b_2-b_1}\left[h_2(r,b_1)-h_2(r,b_2)\right]. \tag{17}$$

which is the Ostrowski inequality on time scales as stated in (7).

Remark 8. Further choosing $r=(b_1+b_2)/2$ and $r=b_2$, respectively in inequality (14) with assumption of Theorem 6, we come to

$$\begin{aligned}&\left|f\left(\frac{b_1+b_2}{2}\right)-\frac{1}{w}\int_{b_1}^{b_2}f(\sigma(s))\Delta s\right|\\&\le\frac{M}{w}\left[h_2\left(\frac{b_1+b_2}{2},b_1\right)-h_2\left(\frac{b_1+b_2}{2},b_2\right)+\frac{(b_1-b_2+w)(b_2-b_1)}{2(f(b_2)-f(b_1))}\left[f(b_2)+f(b_1)\right]\right],\end{aligned} \tag{18}$$

$$\left|f(b_2)-\frac{1}{w}\int_{b_1}^{b_2}f(\sigma(s))\Delta s\right|\le\frac{M}{w}\left[h_2(b_2,b_1)+\frac{b_1-b_2+w}{f(b_2)-f(b_1)}\left[(b_2-b_1)f(b_2)\right]\right]. \tag{19}$$

Corollary 9 (Continuous Case). *Let $\mathbb{T}=\mathbb{R}$, then $h_2(r,s)=(r-s)^2/2$, for all $r,s\in\mathbb{R}$, $\sigma(s)=s$ and in the case Δ-integral becomes usual Riemann integral, as Cauchy's integral is a particular case of Riemann integral thus the inequality in (17) becomes*

$$\left|f(r)-\frac{1}{w}\int_{b_1}^{b_2}f(s)ds\right|\le\frac{M}{(b_2-b_1)}\left(r-\frac{b_1+b_2}{2}\right). \tag{20}$$

Corollary 10 (Discrete Case). *Let $\mathbb{T}=\mathbb{Z}$ then $b_1=0, b_2=n$, $r=i$ and $f(k)=x_k$. Also $h_2(r,0)=(r,2)=r(r-1)/2, h_2(r,n)=(r-n,2)=(r-n)(r-n-1)/2$, thus the inequality in (17) becomes*

$$\left|x_i-\frac{1}{n}\sum_{j=1}^{n}x_j\right|\le\frac{M}{n}\left(i-\frac{n}{2}\right). \tag{21}$$

Corollary 11 (Quantum Calculus Case). *Let $\mathbb{T}=q_0^{\mathbb{Z}}$ with $q>1, b_1=q^m, b_2=q^n$ with $m<n$. In this situation we have $h_k(r,s)=\Pi_{\upsilon=0}^{k-1}r-q^{\upsilon}/\sum_{\mu=0}^{\upsilon}q^{\mu}$, for all $r,s\in\mathbb{T}$, therefore $h_2(r,q^m)=(r-q^m)(r-q^{m+1})/(1+q)$ and $h_2(r,q^n)=(r-q^n)(r-q^{n+1})/(1+q)$ thus the inequality in (17) becomes*

$$\left|f(r)-\frac{1}{q^{n-q^m}}\int_{q^m}^{q^m}f(\sigma(s))\Delta s\right|\le\frac{M}{q^{n-q^m}}\left(r-\frac{q^{2n+1}-q^{2m+1}}{q+1}\right). \tag{22}$$

Theorem 12. *Let $b_1, b_2, w \in \mathbb{T}\, b_1 < b_2$. If $f : [b_1, b_2] \to \mathbb{R}$ be differentiable, and if f^{Δ} is rd-continuous and $\gamma \le f^{\Delta}(r) \le \Gamma\, \forall\, r \in [b_1, b_2]$, then $\forall\, s \in [b_1, b_2]$,*

$$\begin{aligned}&\left|f(r)-\frac{1}{w}\int_{b_1}^{b_2}f(<3;(s))\Delta s-\frac{f(b_2)-f(b_1)}{(b_2-b_1)^2}\left\{h_2(r,b_1)-h_2(r,b_2)+\frac{b_1-b_2+w}{f(b_2)-f(b_1)}\left\{(r-b_1)f(b_2)+(b_2-r)f(b_1)\right\}\right\}\right|\\&\le\frac{\Gamma-\gamma}{2w}\int_{b_1}^{b_2}\left|K_w(r,s)-\frac{1}{b_2-b_1}\left\{h_2(r,b_1)-h_2(r,b_2)+\frac{b_1-b_2+w}{f(b_2)-f(b_1)}\left\{(r-b_1)f(b_2)+(b_2-r)f(b_1)\right\}\right\}\right|\Delta s,\end{aligned} \tag{23}$$

holds where

$$K_w(r,s) = \begin{cases} s - \dfrac{(b_2 - w)f(b_2) - b_1 f(b_1)}{f(b_2) - f(b_1)} = s - \theta_1, & r \in [b_1, s], \\ s - \dfrac{b_2 f(b_2) - (b_1 + w) f(b_1)}{f(b_2) - f(b_1)} = s - \theta_2, & r \in (s, b_2], \end{cases} \tag{24}$$

with $\theta_2 - \theta_1 = w$.

Proof. Choosing $f(r) = K_w(r,s)$ *and* $g(r) = f^{\Delta}(r)$ in theorem 3.1 of [12], we have.

$$\left| \int_{b_1}^{b_2} K_w(r,s) f^{\Delta}(s)\Delta s - \frac{1}{w}\int_{b_1}^{b_2} K_w(r,s) \int_{b_1}^{b_2} f^{\Delta}(s)\Delta s \right| \leq \frac{\Gamma - \gamma}{2} \int_{b_1}^{b_2} \left| K_w(r,\tau)\Delta\tau \right| \Delta r. \tag{25}$$

By solving $K_w(r,s)$ and $f^{\Delta}(r)$ on $[b_1, b_2]$, we get

$$\begin{aligned} \int_{b_1}^{b_2} K_w(r,s)\Delta s &= \left[\int_{b_1}^{r} \left(s - \frac{(b_2 - w)f(b_2) - b_1 f(b_1)}{f(b_2) - f(b_1)} \right)\Delta s + \int_{r}^{b_2} \left(s - \frac{b_2 f(b_2) - (b_1 + w) f(b_1)}{f(b_2) - f(b_1)} \right)\Delta s \right] \\ &= \left[h_2(r,b_1) - h_2(r,b_2) + \left\{ b_1 - \frac{(b_2 - w)f(b_2) - b_1 f(b_1)}{f(b_2) - f(b_1)} \right\}(r - b_1) + \left\{ b_2 - \frac{b_2 f(b_2) - (b_1 + w) f(b_1)}{f(b_2) - f(b_1)} \right\}(b_2 - r) \right], \end{aligned} \tag{26}$$

that is

$$\int_{b_1}^{b_2} K_w(r,s)\Delta s = \left[h_2(r,b_1) - h_2(r,b_2) + (b_1 - b_2 + w)\left\{ r + \frac{b_2 f(b_1) - b_1 f(b_2)}{f(b_2) - f(b_1)} \right\} \right]. \tag{27}$$

Thus, the R.H.S of inequality (25) becomes

$$\begin{aligned} &\frac{\Gamma - \gamma}{2} \int_{b_1}^{b_2} \left| K_w(r,s) - \frac{1}{w}\int_{b_1}^{b_2} K_w(r,\tau)\Delta\tau \right| \Delta s \\ &= \frac{\Gamma - \gamma}{2} \int_{b_1}^{b_2} \left| K_w(r,s) - \frac{1}{w}\left[h_2(r,b_1) - h_2(r,b_2) + (b_1 - b_2 + w)\left\{ r + \frac{b_2 f(b_1) - b_1 f(b_2)}{f(b_2) - f(b_1)} \right\} \right] \right| \Delta s. \end{aligned} \tag{28}$$

From (25) and (27), we get

$$\begin{aligned} &\left| \int_{b_1}^{b_2} K_w(r,s) f^{\Delta}(s) - \frac{1}{w}\int_{b_1}^{b_2} K_w(r,s)\Delta s \int_{b_1}^{b_2} f^{\Delta}(s)\Delta s \right| \\ &\leq \frac{\Gamma - \gamma}{2} \int_{b_1}^{b_2} \left| K_w(r,s) - \frac{1}{w}\left[h_2(r,b_1) - h_2(r,b_2) + (b_1 - b_2 + w)\left\{ r + \frac{b_2 f(b_1) - b_1 f(b_2)}{f(b_2) - f(b_1)} \right\} \right] \right| \Delta s. \end{aligned} \tag{29}$$

Now, from Lemma 4,

$$\begin{aligned} &f(r) = \frac{1}{w}\int_{b_1}^{b_2} f(\sigma(s))\Delta s + \frac{1}{w}\int_{b_1}^{b_2} f(\sigma(s))\Delta s = \int_{b_1}^{b_2} f^{\Delta}(s) K_w(r,s)\Delta s, \\ &\left| w f(r) - \int_{b_1}^{b_2} f(\sigma(s))\Delta s - \frac{1}{w}\int_{b_1}^{b_2} K_w(r,s)\Delta s \int_{b_1}^{b_2} f^{\Delta}(s)\Delta s \right| \\ &= \left| \int_{b_1}^{b_2} f^{\Delta}(s) K_w(r,s)\Delta s - \frac{1}{w}\int_{b_1}^{b_2} f^{\Delta}(s)\Delta s \right|. \end{aligned} \tag{30}$$

By using these inequalities, we come to the following result.

$$\begin{aligned} &\left| w f(r) - \int_{b_1}^{b_2} f(\sigma(s))\Delta s - \frac{1}{w}\left[h_2(r,b_1) - h_2(r,b_2) + (b_1 - b_2 + w)\left\{ r + \frac{b_2 f(b_1) - b_1 f(b_2)}{f(b_2) - f(b_1)}(f(b_2) - f(b_1)) \right\} \right] \right| \\ &\leq \frac{\Gamma - \gamma}{2} \int_{b_1}^{b_2} \left| K_w(r,s) - \frac{1}{w}\left[h_2(r,b_1) - h_2(r,b_2) + (b_1 - b_2 + w)\left\{ r + \frac{b_2 f(b_1) - b_1 f(b_2)}{f(b_2) - f(b_1)} \right\} \right] \right| \Delta s. \end{aligned} \tag{31}$$

$$\begin{aligned} &\left| f(r) - \frac{1}{w}\int_{b_1}^{b_2} f(\sigma(s))\Delta s - \frac{1}{w}[h_2(r,b_1) - h_2(r,b_2) + (b_1 - b_2 + w)\{(r - b_1) f(b_2) + (b_2 - r) f(b_1)\}] \right| \\ &\leq \frac{\Gamma - \gamma}{2w} \int_{b_1}^{b_2} \left| K_w(r,s) - \frac{1}{w}\left[h_2(r,b_1) - h_2(r,b_2) + (b_1 - b_2 + w)\left\{ r + \frac{b_2 f(b_1) - b_1 f(b_2)}{f(b_2) - f(b_1)} \right\} \right] \right| \Delta s. \end{aligned} \tag{32}$$

Remark 13. Let $w = b_2 - b_1$, then from (23), we have

$$\begin{aligned} &\left| f(r) - \frac{1}{b_2 - b_1}\int_{b_1}^{b_2} f(\sigma(s))\Delta s - \frac{f(b_2) - f(b_1)}{(b_2 - b_1)^2}\{h_2(r,b_1) - h_2(r,b_2)\} \right| \\ &\leq \frac{\Gamma - \gamma}{2} \int_{b_1}^{b_2} \left| K_{b_2 - b_1}(r,s) - \frac{1}{b_2 - b_1}\{h_2(r,b_1) - h_2(r,b_2)\} \right| \Delta s. \end{aligned} \tag{33}$$

Theorem 14. *Let* $b_1, b_2, w \in \mathbb{T}$ *with* $b_1 < b_2$. *If* $f : [b_1, b_2] \to \mathbb{R}$ *be differentiable, then for all* $s \in [b_1, b_2]$,

$$\begin{aligned} &\left| w f(r) - \int_{b_1}^{b_2} f^{\sigma}(s)\Delta s - \frac{\Gamma + \gamma}{2}\left[h_2(r,b_1) - h_2(r,b_2) + \frac{(b_1 - b_2 + w)}{f(b_2) - f(b_1)}\{(r - b_1) f(b_2) + (b_2 - r) f(b_1)\} \right] \right| \\ &\leq \frac{\Gamma - \gamma}{2}\left\{ h_2\left(b_1, \frac{(b_2 - w)f(b_2) - b_1 f(b_1)}{f(b_2) - f(b_1)} \right) + h_2\left(r, \frac{(b_2 - w)f(b_2) - b_1 f(b_1)}{f(b_2) - f(b_1)} \right) \right. \\ &\quad \left. + h_2\left(b_2, \frac{b_2 f(b_2) - (b_1 + w) f(b_1)}{f(b_2) - f(b_1)} \right) + h_2\left(r, \frac{b_2 f(b_2) - (b_1 + w) f(b_1)}{f(b_2) - f(b_1)} \right) \right\}. \end{aligned} \tag{34}$$

Proof. By using Lemma 4, we know that

$$f(r)=\frac{1}{w}\int_{b_1}^{b_2}f^{\sigma}(s)\Delta s+\frac{1}{w}\int_{b_1}^{b_2}f^{\Delta}(s)K_w(r,s)\Delta s,$$
$$\int_{b_1}^{b_2}f^{\Delta}(s)K_w(r,s)\Delta s=wf(r)-\int_{b_1}^{b_2}f^{\sigma}(s)\Delta s, \quad (35)$$

where

$$K_w(r,s)=\begin{cases} s-\dfrac{(b_2-w)f(b_2)-b_1f(b_1)}{f(b_2)-f(b_1)}=s-\theta_1, & r\in[b_1,s],\\ s-\dfrac{b_2f(b_2)-(b_1+w)f(b_1)}{f(b_2)-f(b_1)}=s-\theta_2, & r\in(s,b_2]. \end{cases} \quad (36)$$

Also

$$\int_{b_1}^{b_2}K_w(r,s)\Delta s=h_2(r,b_1)-h_2(r,b_2)+\frac{(b_1-b_2+w)}{f(b_2)-f(b_1)}\{(r-b_1)f(b_2)+(b_2-r)f(b_1)\}. \quad (37)$$

Let $C=(\Gamma+\gamma)/2$

$$\int_{b_1}^{b_2}K_w(r,s)\left[f^{\Delta}(s)-C\right]\Delta s=wf(r)-\int_{b_1}^{b_2}f^{\sigma}(s)\Delta s-\frac{\Gamma+\gamma}{2}\left[h_2(r,b_1)-h_2(r,b_2)+\frac{(b_1-b_2+w)}{f(b_2)-f(b_1)}\{(r-b_1)f(b_2)+(b_2-r)f(b_1)\}\right]. \quad (38)$$

On the other hand

$$\left|\int_{b_1}^{b_2}K_w(r,s)\left[f^{\Delta}(s)-C\right]\Delta s\right|\le\max_{s\in[b_1,b_2]}\left|f^{\Delta}(s)-C\right|\int_{b_1}^{b_2}|K_w|(r,s)|\Delta s|, \quad (39)$$

we have also

$$\max_{s\in[b_1,b_2]}\left|f^{\Delta}(s)-C\right|\le\frac{\Gamma-\gamma}{2},$$
$$\int_{b_1}^{b_2}|K_w(r,s)|=\left[h_2\left(b_1,\frac{(b_2-w)f(b_2)-b_1f(b_1)}{f(b_2)-f(b_1)}\right)+h_2\left(r,\frac{(b_2-w)f(b_2)-b_1f(b_1)}{f(b_2)-f(b_1)}\right)+h_2\left(r,\frac{b_2f(b_2)-(b_1+w)f(b_1)}{f(b_2)-f(b_1)}\right)+h_2\left(b_2,\frac{b_2f(b_2)-(b_1+w)f(b_1)}{f(b_2)-f(b_1)}\right)\right]. \quad (40)$$

therefore

$$\left|\int_{b_1}^{b_2}|K_w(r,s)|\left[f^{\Delta}(s)-C\right]\Delta s\right|\le\frac{\Gamma-\gamma}{2}\left\{h_2\left(b_1,\frac{(b_2-w)f(b_2)-b_1f(b_1)}{f(b_2)-f(b_1)}\right)+h_2\left(r,\frac{(b_2-w)f(b_2)-b_1f(b_1)}{f(b_2)-f(b_1)}\right)+h_2\left(b_2,\frac{b_2f(b_2)-(b_1+w)f(b_1)}{f(b_2)-f(b_1)}\right)+h_2\left(r,\frac{b_2f(b_2)-(b_1+w)f(b_1)}{f(b_2)-f(b_1)}\right)\right\}. \quad (41)$$

Remark 15. Let $r=(b_1+b_2)/2\in\mathbb{T}$ and $w=b_2-b_1$, then.

$$f\left(\frac{a_1+b_2}{2}\right)-\frac{1}{b_2-b_1}\int_{b_1}^{b_2}f(\sigma(s))\Delta s-\frac{\Gamma+\gamma}{2}\frac{1}{b_2-b_1}\left\{h_2\left(\frac{b_1+b_2}{2},b_1\right)-h_2\left(\frac{b_1+b_2}{2},b_2\right)\right\}\le\frac{\Gamma-\gamma}{2(b_2-b_1)}\left\{h_2\left(\frac{b_1+b_2}{2},b_1\right)-h_2\left(\frac{b_1+b_2}{2},b_2\right)\right\}. \quad (42)$$

Corollary 16 (Continuous Case). *Let* $\mathbb{T}=\mathbb{R}$, *then* $h_2(r,s)=(r-s)^2/2$, *for all* $r,s\in\mathbb{R}$, $\sigma(s)=s$ *and in the case* Δ*-integral becomes usual Riemann integral, thus the inequality*

$$\left|f\left(\frac{b_1+b_2}{2}\right)-\frac{1}{b_2-b_1}\int_{b_1}^{b_2}f(s)ds-\frac{\Gamma-\gamma}{2}\left[\frac{(b_1-(b_1+b_2)/2)^2}{2}-\frac{((b_1+b_2)/2-b_2)^2}{2}\right]\right|\le\frac{\Gamma-\gamma}{2(b_2-b_1)}\left[\frac{(b_1-(b_1+b_2)/2)^2}{2}-\frac{((b_1+b_2)/2-b_2)^2}{2}\right],$$
$$\left|f\left(\frac{b_1+b_2}{2}\right)-\frac{1}{b_2-b_1}\int_{b_1}^{b_2}f(s)ds\right|\le\frac{(\Gamma-\gamma)(b_2-b_1)}{8}. \quad (43)$$

3. Concluding Remarks

The study of inequalities on $\mathbb{T}$ is the genom of mathematics which is most recently gaining a substantial attention. The given article is the description of some general statements regarding Ostrowski's type inequalities on $\mathbb{T}$. The results demonstrated here are some stimulus generalization of Ostrowski's type inequalities via Δ-integrals and generalizing the results of articles [8, 10–13]. These results will be very useful in the study of quantum calculus and dynamical system related differential equations which bring difference and differential equations together [14–18].

Data Availability

The authors confirm that the data supporting the findings of this article are available within the article and are available on request from the corresponding author.

Conflicts of Interest

The author declares that they have no conflicts of interest.

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