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*Research Article*

# Strongly Singular Convolution Operators on Herz-Type Hardy Spaces with Variable Exponent

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We investigate the boundedness of the strongly singular convolution operators on Herz-type Hardy spaces with variable exponent.

## 1. Introduction

The theory of function spaces with variable exponents has been extensively studied by researchers since the work of Kováčik and Rákosník [1] appeared in 1991. In [2, 3] the authors defined the Herz-type Hardy spaces with variable exponent and gave some characterizations for them. In [4–7], the authors proved the boundedness of some integral operators on variable function spaces.

Given an open set $E \subset \mathbb{R}^n$ and a measurable function $p(\cdot) : E \longrightarrow [1, \infty)$, $L^{p(\cdot)}(E)$ denotes the set of measurable functions $f$ defined on $E$ such that

$$\int_E \left( \frac{|f(x)|}{\lambda} \right)^{p(x)} dx < \infty \tag{1}$$

holds for some $\lambda > 0$.

The set $L^{p(\cdot)}(E)$ is a Banach function space when it is equipped with the Luxemburg-Nakano norm as follows:

$$\|f\|_{L^{p(\cdot)}(E)} = \inf \left\{ \lambda > 0 : \int_E \left( \frac{|f(x)|}{\lambda} \right)^{p(x)} dx \le 1 \right\}. \tag{2}$$

The space is regarded as the variable $L^p$ space, since it generalized the standard $L^p$ space: if $p(x) = p$ is constant, then $L^{p(\cdot)}(E)$ is isometrically isomorphic to $L^p(E)$.

The space $L^{p(\cdot)}_{\mathrm{loc}}(E)$ is defined by

$$L^{p(\cdot)}_{\mathrm{loc}}(E) := \left\{ f : f \in L^{p(\cdot)}(F) \text{ for all compact subsets } F \subset E \right\}. \tag{3}$$

Define $\mathscr{P}^0(E)$ to be the set of $p(\cdot) : E \longrightarrow (0, \infty)$ such that

$$p^- = \operatorname{ess\,inf} \{ p(x) : x \in E \} > 0, \quad p^+ = \operatorname{ess\,sup} \{ p(x) : x \in E \} < \infty. \tag{4}$$

Define $\mathscr{P}(E)$ to be the set of $p(\cdot) : E \longrightarrow [1, \infty)$ such that

$$p^- = \operatorname{ess\,inf} \{ p(x) : x \in E \} > 1 \tag{5}$$

and

$$p^+ = \operatorname{ess\,sup} \{ p(x) : x \in E \} < \infty. \tag{6}$$

Denote $p'(x) = p(x)/(p(x) - 1)$.

Let $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. The Hardy-Littlewood maximal operator is defined by

$$Mf(x) = \sup_{r>0} \frac{1}{|B_r(x)|} \int_{B_r(x)} |f(y)| \, dy, \tag{7}$$

where $B_r(x) = \{ y \in \mathbb{R}^n : |x - y| < r \}$. Let $\mathscr{B}(\mathbb{R}^n)$ be the set of $p(\cdot) \in \mathscr{P}(\mathbb{R}^n)$ such that the Hardy-Littlewood maximal operator $M$ is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$.

**Lemma 1** (see [8]). *If $p(\cdot) \in \mathscr{P}(\mathbb{R}^n)$ and satisfies*

$$|p(x) - p(y)| \leq \frac{C}{-\log(|x-y|)}, \quad |x-y| \leq \frac{1}{2} \tag{8}$$

*and*

$$|p(x) - p(y)| \leq \frac{C}{\log(|x|+e)}, \quad |y| \geq |x|, \tag{9}$$

*then $(\cdot) \in \mathscr{B}(\mathbb{R}^n)$, that is, the Hardy-Littlewood maximal operator $M$ is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$.*

In addition, we denote the Lebesgue measure and the characteristic function of a measurable set $A \subset \mathbb{R}^n$ by $|A|$ and $\chi_A$, respectively. The notation $f \approx g$ means that there exist two constants $C_1, C_2 > 0$ such that $C_1 g \leq f \leq C_2 g$.

Next we recall the definition of the Herz spaces with variable exponent. Let $B_k = \{x \in \mathbb{R}^n : |x| \leq 2^k\}$ and $A_k = B_k \setminus B_{k-1}$ for $k \in \mathbb{Z}$. Denote $\mathbb{Z}_+$ and $\mathbb{N}$ as the sets of all positive and nonnegative integers, respectively, $\chi_k = \chi_{A_k}$ for $k \in \mathbb{Z}$, $\widetilde{\chi}_k = \chi_k$ if $k \in \mathbb{Z}_+$, and $\widetilde{\chi}_0 = \chi_{B_0}$.

*Definition 2* (see [9]). Let $\alpha \in \mathbb{R}$, $0 < p \leq \infty$, and $q(\cdot) \in \mathscr{P}(\mathbb{R}^n)$. The homogeneous Herz space with variable exponent $\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$ is defined by

$$\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n) = \left\{ f \in L^{q(\cdot)}_{\mathrm{loc}}(\mathbb{R}^n \setminus \{0\}) : \|f\|_{\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} < \infty \right\}, \tag{10}$$

where

$$\|f\|_{\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} = \left\{ \sum_{k=-\infty}^{\infty} 2^{k\alpha p} \|f\chi_k\|^p_{L^{q(\cdot)}(\mathbb{R}^n)} \right\}^{1/p}. \tag{11}$$

The nonhomogeneous Herz space with variable exponent $K^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$ is defined by

$$K^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n) = \left\{ f \in L^{q(\cdot)}_{\mathrm{loc}}(\mathbb{R}^n) : \|f\|_{K^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} < \infty \right\}, \tag{12}$$

where

$$\|f\|_{K^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} = \left\{ \sum_{k=0}^{\infty} 2^{k\alpha p} \|f\widetilde{\chi}_k\|^p_{L^{q(\cdot)}(\mathbb{R}^n)} \right\}^{1/p}. \tag{13}$$

In [2], the authors gave the definition of the Herz-type Hardy space with variable exponent $H\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$ and the atomic decomposition characterizations. $\mathcal{S}(\mathbb{R}^n)$ denotes the space of Schwartz functions, and $\mathcal{S}'(\mathbb{R}^n)$ denotes the dual space of $\mathcal{S}(\mathbb{R}^n)$. Let $G_N(f)$ be the grand maximal function of $f$ defined by

$$G_N(f)(x) = \sup_{\phi \in \mathscr{A}_N} |\phi^*_\nabla(f)(x)|, \tag{14}$$

where

$$\mathscr{A}_N = \left\{ \phi \in \mathcal{S}(\mathbb{R}^n) : \sup_{|\alpha|,|\beta| \leq N} |x^\alpha D^\beta \phi(x)| \leq 1 \right\} \tag{15}$$

and $N > n+1$; $\phi^*_\nabla$ is the nontangential maximal operator defined by

$$\phi^*_\nabla(f)(x) = \sup_{|y-x|<t} |\phi_t * f(y)| \tag{16}$$

with $\phi_t(x) = t^{-n}\phi(x/t)$.

*Definition 3* (see [2]). Let $\alpha \in \mathbb{R}$, $0 < p < \infty$, $q(\cdot) \in \mathscr{P}(\mathbb{R}^n)$, and $N > n+1$.

(i) The homogeneous Herz-type Hardy space with variable exponent $H\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$ is defined by

$$H\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n) = \left\{ f \in \mathcal{S}'(\mathbb{R}^n) : G_N(f)(x) \in \dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n) \right\} \tag{17}$$

and

$$\|f\|_{H\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} = \|G_N(f)\|_{\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}. \tag{18}$$

(ii) The nonhomogeneous Herz-type Hardy space with variable exponent $HK^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$ is defined by

$$HK^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n) = \left\{ f \in \mathcal{S}'(\mathbb{R}^n) : G_N(f)(x) \in K^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n) \right\} \tag{19}$$

and

$$\|f\|_{HK^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} = \|G_N(f)\|_{K^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}. \tag{20}$$

For $x \in \mathbb{R}$, we denote by $[x]$ the largest integer less than or equal to $x$. $\delta_2$ is the same as in Lemma 9.

*Definition 4* (see [2]). Let $n\delta_2 \leq \alpha < \infty$, $q(\cdot) \in \mathscr{P}(\mathbb{R}^n)$, and nonnegative integer $s \geq [\alpha - n\delta_2]$.

(i) A function $a$ on $\mathbb{R}^n$ is said to be a central $(\alpha, q(\cdot))$-atom, if it satisfies

(1) supp $a \subset B(0,r) = \{x \in \mathbb{R}^n : |x| < r\}$

(2) $\|a\|_{L^{q(\cdot)}(\mathbb{R}^n)} \leq |B(0,r)|^{-\alpha/n}$

(3) $\int_{\mathbb{R}^n} a(x)x^\beta dx = 0$, $|\beta| \leq s$

(ii) A function $a$ on $\mathbb{R}^n$ is said to be a central $(\alpha, q(\cdot))$-atom of restricted type, if it satisfies conditions (2), (3) and

(1$'$) supp $a \subset B(0,r)$, $r \geq 1$

If $r = 2^k$ for some $k \in \mathbb{Z}$ in Definition 4, then the corresponding central $(\alpha, q(\cdot))$-atom is called a dyadic central $(\alpha, q(\cdot))$-atom.

**Lemma 5** (see [2]). *Let $n\delta_2 \le \alpha < \infty$, $0 < p < \infty$ and $q(\cdot) \in \mathscr{B}(\mathbb{R}^n)$. Then $f \in H\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$ (or $HK^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$) if and only if*

$$f = \sum_{k=-\infty}^{\infty} \lambda_k a_k \quad \left(\text{or } \sum_{k=0}^{\infty} \lambda_k a_k\right), \quad \text{in the sense of } \mathcal{S}'(\mathbb{R}^n), \tag{21}$$

*where each $a_k$ is a central $(\alpha, q(\cdot))$-atom (or central $(\alpha, q(\cdot))$-atom of restricted type) with support contained in $B_k$ and $\sum_{k=-\infty}^{\infty} |\lambda_k|^p < \infty$ (or $\sum_{k=0}^{\infty} |\lambda_k|^p < \infty$). Moreover,*

$$\|f\|_{H\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} \approx \inf \left(\sum_{k=-\infty}^{\infty} |\lambda_k|^p\right)^{1/p} \quad \left(\text{or } \|f\|_{HK^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} \approx \inf \left(\sum_{k=0}^{\infty} |\lambda_k|^p\right)^{1/p}\right), \tag{22}$$

*where the infimum is taken over all above decomposition of $f$.*

*Let $\theta$ be a smooth radial cut-off function such that $\theta(\xi) = 1$ if $|\xi| \ge 1$ and $\theta(\xi) = 0$ if $|\xi| \le 1/2$. Define the multipliers*

$$T_b : \widehat{T_b f}(\xi) = \theta(\xi) \frac{e^{i|\xi|^b}}{|\xi|^{nb/2}} \hat{f}(\xi), \tag{23}$$

*where $0 < b < 1$. The kernel for $T_b$ is very singular. Roughly speaking, it looks like*

$$K_{b'}(x) = \frac{e^{i|x|^{-b'}}}{|x|^n}, \tag{24}$$

*where $b' = b/(1-b)$. Indeed the cancellation is minimal and if one makes a quick computation for $|x| \ge 2|y|$, we have*

$$\left|K_{b'}(x-y) - K_{b'}(x)\right| \le \frac{C|y|}{|x|^{n+b'+1}}. \tag{25}$$

*The study of these operators in the context of $L^q$ spaces was carried out by Hirschman [10] and Wainger [11]. Sharp endpoint estimates were obtained by Fefferman and Stein in [12] via the duality of $H^1$ and BMO. Weighted $L^q$ norm and weak(1,1) estimates were established by Chanillo in [13]. The boundedness of these operators on the weighted Herz-type Hardy spaces was proved by Xiaochun Li and Shanzhen Lu in [14].*

*Motivated by [2, 14], we will study the boundedness of the strongly singular convolution operators $T_b$ on Herz-type Hardy spaces with variable exponent. The main results are as follows.*

**Theorem 6.** *Suppose that $0 < p \le 1$, $q(\cdot) \in \mathscr{P}(\mathbb{R}^n)$ satisfies conditions (8) and (9) in Lemma 1 and $\alpha = n\delta_2$. Then we have*

$$\|T_b(f)\|_{\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} \le C \|f\|_{H\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}, \tag{26}$$

*where $C$ is independent of $f$.*

**Theorem 7.** *Suppose that $0 < p \le 1$, $q(\cdot) \in \mathscr{P}(\mathbb{R}^n)$ satisfies conditions (8) and (9) in Lemma 1 and $n\delta_2 \le \alpha \le n\delta_2 + 1$. Then we have*

$$\|T_b(f)\|_{HK^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} \le C \|f\|_{HK^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}, \tag{27}$$

*where $C$ is independent of $f$.*

## 2. Preliminary Lemmas

Referring to the variable $L^{p(\cdot)}$ space, there are some important lemmas as follows.

**Lemma 8** (see [1]). *Let $p(\cdot) \in \mathscr{P}(\mathbb{R}^n)$. If $f \in L^{p(\cdot)}(\mathbb{R}^n)$ and $g \in L^{p'(\cdot)}(\mathbb{R}^n)$, then $fg$ is integrable on $\mathbb{R}^n$ and*

$$\int_{\mathbb{R}^n} |f(x) g(x)|\, dx \le r_p \|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} \|g\|_{L^{p'(\cdot)}(\mathbb{R}^n)}, \tag{28}$$

*where*

$$r_p = 1 + \frac{1}{p^-} - \frac{1}{p^+}. \tag{29}$$

*The above inequality is named generalized Hölder's inequality with respect to the variable $L^p$ space.*

**Lemma 9** (see [9]). *Let $q(\cdot) \in \mathscr{B}(\mathbb{R}^n)$. Then there exists a positive constant $C$ such that, for all balls $B$ in $\mathbb{R}^n$ and all measurable subsets $S \subset B$,*

$$\frac{\|\chi_B\|_{L^{q(\cdot)}(\mathbb{R}^n)}}{\|\chi_S\|_{L^{q(\cdot)}(\mathbb{R}^n)}} \le C \frac{|B|}{|S|}, \qquad \frac{\|\chi_S\|_{L^{q(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{q(\cdot)}(\mathbb{R}^n)}} \le C \left(\frac{|S|}{|B|}\right)^{\delta_1}, \tag{30}$$

*and*

$$\frac{\|\chi_S\|_{L^{q'(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{q'(\cdot)}(\mathbb{R}^n)}} \le C \left(\frac{|S|}{|B|}\right)^{\delta_2} \tag{31}$$

*hold, where $\delta_1$ and $\delta_2$ are constants with $0 < \delta_1, \delta_2 < 1$.*

Throughout this paper $\delta_2$ is the same as in Lemma 9.

**Lemma 10** (see [9]). *Suppose $q(\cdot) \in \mathscr{B}(\mathbb{R}^n)$. Then there exists a constant $C > 0$ such that, for all balls $B$ in $\mathbb{R}^n$,*

$$\frac{1}{|B|} \|\chi_B\|_{L^{q(\cdot)}(\mathbb{R}^n)} \|\chi_B\|_{L^{q'(\cdot)}(\mathbb{R}^n)} \le C. \tag{32}$$

**Lemma 11** (see [15]). *Define a variable exponent $\tilde{q}(\cdot)$ by $1/p(x) = 1/\tilde{q}(x) + 1/q$ for $x \in \mathbb{R}^n$. Then we have*

$$\|fg\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le C \|f\|_{L^{\tilde{q}(\cdot)}(\mathbb{R}^n)} \|g\|_{L^q(\mathbb{R}^n)}, \tag{33}$$

*for all measurable functions $f$ and $g$.*

**Lemma 12** (see [16]). *Let $p(\cdot) \in \mathscr{P}(\mathbb{R}^n)$ satisfy conditions (8) and (9) in Lemma 1. Then*

$$\|\chi_Q\|_{L^{p(\cdot)}(\mathbb{R}^n)} \approx \begin{cases} |Q|^{1/p(x)} & \text{if } |Q| \le 2^n \text{ and } x \in Q, \\ |Q|^{1/p(\infty)} & \text{if } |Q| \ge 1 \end{cases} \tag{34}$$

*for every cube (or ball) $Q \subset \mathbb{R}^n$, where $p(\infty) = \lim_{x \to \infty} p(x)$.*

*A nonnegative locally integrable function $\omega(x)$ on $\mathbb{R}^n$ is said to belong to $A_p (1 < p < \infty)$, if there is a constant $C > 0$ such that*

$$\sup_Q \left(\frac{1}{|Q|}\int_Q \omega(x)\,dx\right)\left(\frac{1}{|Q|}\int_Q \omega(x)^{1-p'}\,dx\right)^{p-1} \le C < \infty, \tag{35}$$

*where $p' = p/(p-1)$; $Q$ denotes a cube in $\mathbb{R}^n$ with its sides parallel to the coordinate axes.*

*The weighted $(L^p, L^p)$ boundedness of $T_b$ has been proved by Chanillo [13].*

**Lemma 13** (see [13]). *Let $\omega \in A_p$, $1 < p < \infty$. Then*

$$\int_{\mathbb{R}^n} |T_b(f)(x)|^p \omega(x)\,dx \le C \int_{\mathbb{R}^n} |f(x)|^p \omega(x)\,dx. \tag{36}$$

**Lemma 14** (see [5]). *Given a family $\mathscr{F}$ and an open set $E \subset \mathbb{R}^n$, assume that for some $p_0$, $0 < p_0 < \infty$ and for every $\omega \in A_\infty$,*

$$\int_E f(x)^{p_0} \omega(x)\,dx \le C_0 \int_E g(x)^{p_0} \omega(x)\,dx, \quad (f,g) \in \mathscr{F}. \tag{37}$$

*Given $p(\cdot) \in \mathscr{P}^0(E)$ such that $p(\cdot)$ satisfies (8) and (9) in Lemma 1, then for all $(f,g) \in \mathscr{F}$ such that $f \in L^{p(\cdot)}(E)$*

$$\|f\|_{L^{p(\cdot)}(E)} \le C \|g\|_{L^{p(\cdot)}(E)}. \tag{38}$$

*Since $A_p \subset A_\infty$, by Lemmas 13 and 14 it is easy to get the $(L^{p(\cdot)}(\mathbb{R}^n), L^{p(\cdot)}(\mathbb{R}^n))$-boundedness of the strongly singular convolution operators $T_b$.*

To prove our main results, we also need the following lemmas.

**Lemma 15** (see [11]). *The kernel for the multiplier operator $T_b(f)(x)$ is given by*

$$C\frac{e^{i\alpha_b|x|^{-b'}}}{|x|^n}\chi(|x| \le 1) + h(x), \quad b' = \frac{b}{(1-b)}, \tag{39}$$

*with $|h(x)| \le C(1+|x|)^{-(n+1)} + C|x|^{-n+\varepsilon}\chi(|x| \le 1)$, $\varepsilon > 0$. Here $\alpha_b = b^{b/(1-b)} - b^{1/(1-b)}$ and $\varepsilon$ depend only on $b$.*

**Lemma 16** (see [13]). *Let $\widetilde{K}_{b',s}(x) = e^{i\alpha_b|x|^{-b'}}/|x|^{n(b'+2)/s}$ and $(b'+2)/s < 1$. Then*

$$\left\|\widetilde{K}_{b',s} * f\right\|_s \le C\|f\|_{s'}, \quad \frac{1}{s} + \frac{1}{s'} = 1. \tag{40}$$

## 3. The Proof of Main Results

Firstly we give the proof of Theorem 6.

*Proof of Theorem 6.* Let $f \in H\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$. By Lemma 5, we have

$$f(x) = \sum_{j=-\infty}^{\infty} \lambda_j a_j, \tag{41}$$

where

$$\|f\|_{H\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} \approx \inf\left(\sum_{j=-\infty}^{\infty} \left|\lambda_j\right|^p\right)^{1/p}, \tag{42}$$

the infimum is taken over the above decomposition of $f$, and $a_j$ is a dyadic central $(\alpha, q(\cdot))$-atom with the support $B_j$. Then we have

$$\begin{aligned} \|T_b(f)\|^p_{\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} &= \sum_{k=-\infty}^{\infty} 2^{k\alpha p} \|T_b(f)\chi_k\|^p_{L^{q(\cdot)}(\mathbb{R}^n)} \\ &\le C\sum_{k=-\infty}^{\infty} 2^{k\alpha p}\left(\sum_{j=-\infty}^{k-2}\left|\lambda_j\right|\left\|T_b(a_j)\chi_k\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}\right)^p \\ &\quad + C\sum_{k=-\infty}^{\infty} 2^{k\alpha p}\left(\sum_{j=k-1}^{\infty}\left|\lambda_j\right|\left\|T_b(a_j)\chi_k\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}\right)^p \\ &=: CI_1 + CI_2. \end{aligned} \tag{43}$$

We first estimate $I_2$; by $0 < p \le 1$ and the $(L^{q(\cdot)}(\mathbb{R}^n), L^{q(\cdot)}(\mathbb{R}^n))$-boundedness of $T_b$ we have

$$\begin{aligned} I_2 &\le C\sum_{k=-\infty}^{\infty} 2^{k\alpha p}\left(\sum_{j=k-1}^{\infty}\left|\lambda_j\right|\left\|a_j\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}\right)^p \\ &\le C\sum_{k=-\infty}^{\infty}\left(\sum_{j=k-1}^{\infty}\left|\lambda_j\right|2^{(k-j)\alpha}\right)^p \\ &\le C\sum_{j=-\infty}^{\infty}\left|\lambda_j\right|^p\left(\sum_{k=-\infty}^{j+1}2^{(k-j)\alpha p}\right) \le C\sum_{j=-\infty}^{\infty}\left|\lambda_j\right|^p \\ &\le C\|f\|_{H\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}. \end{aligned} \tag{44}$$

Now we estimate $I_1$. Let

$$K_{b'}(x) = C\frac{i\alpha_b|x|^{-b'}}{|x|^n}\chi(|x| \le 1). \tag{45}$$

By Lemma 15 and the Minkowski inequality, we have

$$\begin{aligned} I_1 &\le \sum_{k=-\infty}^{\infty} 2^{k\alpha p}\left(\sum_{j=-\infty}^{k-2}\left|\lambda_j\right|\left\|\left(K_{b'} * a_j\right)\chi_k\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}\right)^p \\ &\quad + \sum_{k=-\infty}^{\infty} 2^{k\alpha p}\left(\sum_{j=-\infty}^{k-2}\left|\lambda_j\right|\left\|\left(h * a_j\right)\chi_k\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}\right)^p \\ &=: I_{11} + I_{12}. \end{aligned} \tag{46}$$

To estimate the term $I_{12}$, we need the pointwise estimate for $h * a_j(x)$.

Let $|x| \geq 2^j$. By generalized Hölder's inequality we have

$$\begin{aligned}
&\left|h * a_j(x)\right| \leq \int_{|t|\leq r} |h(x-t)| \left|a_j(t)\right| dt \\
&\leq C \int_{|t|\leq r} \left|a_j(t)\right| \\
&\cdot \left[\frac{1}{(1+|x-t|)^{n+1}} + \frac{\chi(|x-t|\leq 1)}{|x-t|^{n-\varepsilon}}\right] dt \\
&\leq C \left(\int_{|t|\leq r} \left|a_j(t)\right| dt\right) \\
&\cdot \left[\frac{1}{(1+|x|)^{n+1}} + \frac{\chi(|x|\leq 2)}{|x|^{n-\varepsilon}}\right] \leq C \left\|a_j\right\|_{L^{q(\cdot)}(\mathbb{R}^n)} \\
&\cdot \left\|\chi_j\right\|_{L^{q'(\cdot)}(\mathbb{R}^n)} \left[\frac{1}{(1+|x|)^{n+1}} + \frac{\chi(|x|\leq 2)}{|x|^{n-\varepsilon}}\right].
\end{aligned} \tag{47}$$

Therefore, by $n\delta_2 = \alpha$, $0 < p \leq 1$, Lemmas 9 and 10, the Minkowski inequality, and generalized Hölder's inequality we have

$$\begin{aligned}
I_{12} &\leq C \sum_{k=-\infty}^{\infty} 2^{k\alpha p} \left(\sum_{j=-\infty}^{k-2} \left|\lambda_j\right| \left\|a_j\right\|_{L^{q(\cdot)}(\mathbb{R}^n)} \left\|\chi_j\right\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\right. \\
&\cdot \left.\left\|\frac{1}{(1+|\cdot|)^{n+1}} \chi_k(\cdot)\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}\right)^p + C \sum_{k=-\infty}^{\infty} 2^{k\alpha p} \\
&\cdot \left(\sum_{j=-\infty}^{k-2} \left|\lambda_j\right| \left\|a_j\right\|_{L^{q(\cdot)}(\mathbb{R}^n)} \left\|\chi_j\right\|_{L^{q'(\cdot)}(\mathbb{R}^n)} \left\|\frac{\chi(|\cdot|\leq 2)}{|\cdot|^{n-\varepsilon}}\right.\right. \\
&\cdot \left.\left.\chi_k(\cdot)\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}\right)^p \leq C \sum_{k=-\infty}^{\infty} \left(\sum_{j=-\infty}^{k-2} \left|\lambda_j\right| 2^{k\alpha}\right. \\
&\cdot \frac{1}{(1+2^k)^{n+1}} \left\|a_j\right\|_{L^{q(\cdot)}(\mathbb{R}^n)} \left\|\chi_{B_j}\right\|_{L^{q'(\cdot)}(\mathbb{R}^n)} \\
&\cdot \left.\left\|\chi_{B_k}\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}\right)^p + C \sum_{k=-\infty}^{1} \left(\sum_{j=-\infty}^{k-2} \left|\lambda_j\right| 2^{k\alpha} \frac{1}{2^{k(n-\varepsilon)}}\right. \\
&\cdot \left.\left\|a_j\right\|_{L^{q(\cdot)}(\mathbb{R}^n)} \left\|\chi_{B_j}\right\|_{L^{q'(\cdot)}(\mathbb{R}^n)} \left\|\chi_{B_k}\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}\right)^p \\
&\leq C \sum_{k=-\infty}^{\infty} \left(\sum_{j=-\infty}^{k-2} \left|\lambda_j\right| \frac{2^{k\alpha+kn-j\alpha}}{(1+2^k)^{n+1}} \frac{\left\|\chi_{B_j}\right\|_{L^{q'(\cdot)}(\mathbb{R}^n)}}{\left\|\chi_{B_k}\right\|_{L^{q'(\cdot)}(\mathbb{R}^n)}}\right)^p \\
&+ C \sum_{k=-\infty}^{1} \left(\sum_{j=-\infty}^{k-2} \left|\lambda_j\right| \frac{2^{k\alpha+kn-j\alpha}}{2^{k(n-\varepsilon)}} \frac{\left\|\chi_{B_j}\right\|_{L^{q'(\cdot)}(\mathbb{R}^n)}}{\left\|\chi_{B_k}\right\|_{L^{q'(\cdot)}(\mathbb{R}^n)}}\right)^p \\
&\leq C \sum_{k=-\infty}^{\infty} \left(\sum_{j=-\infty}^{k-2} \left|\lambda_j\right| \frac{2^{kn+(j-k)(n\delta_2-\alpha)}}{(1+2^k)^{n+1}}\right)^p \\
&+ C \sum_{k=-\infty}^{1} \left(\sum_{j=-\infty}^{k-2} \left|\lambda_j\right| \frac{2^{kn+(j-k)(n\delta_2-\alpha)}}{2^{k(n-\varepsilon)}}\right)^p \\
&\leq C \sum_{k=-\infty}^{\infty} \left(\sum_{j=-\infty}^{k-2} \left|\lambda_j\right| \frac{2^{kn}}{(1+2^k)^{n+1}}\right)^p \\
&+ C \sum_{k=-\infty}^{1} \left(\sum_{j=-\infty}^{k-2} \left|\lambda_j\right| \frac{2^{kn}}{2^{k(n-\varepsilon)}}\right)^p \leq C \sum_{j=-\infty}^{\infty} \left|\lambda_j\right|^p \\
&\cdot \left(\sum_{k=j+2}^{\infty} \frac{2^{knp}}{(1+2^k)^{(n+1)p}}\right) + C \sum_{j=-\infty}^{-1} \left|\lambda_j\right|^p \\
&\cdot \left(\sum_{k=j+2}^{1} \frac{2^{knp}}{2^{k(n-\varepsilon)p}}\right) \leq C \sum_{j=-\infty}^{\infty} \left|\lambda_j\right|^p \left(\sum_{k=j+2}^{1} 2^{knp}\right. \\
&+ \left.\sum_{k=1}^{\infty} \frac{1}{2^{kp}}\right) + C \sum_{j=-\infty}^{-1} \left|\lambda_j\right|^p \left(\sum_{k=j+2}^{1} 2^{k\varepsilon p}\right) \\
&\leq C \sum_{j=-\infty}^{\infty} \left|\lambda_j\right|^p.
\end{aligned} \tag{48}$$

What remains is estimating $I_{11}$. Let $2^{j_0-1} < 2^{j(1-b)} \leq 2^{j_0}$ for some $j_0 \in \mathbb{Z}$, where $b$ is the same as the above. Then it follows that

$$\begin{aligned}
I_{11} &= \sum_{k=-\infty}^{j_0} 2^{k\alpha p} \left(\sum_{j=-\infty}^{k-2} \left|\lambda_j\right| \left\|\left(K_{b'} * a_j\right) \chi_k\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}\right)^p \\
&+ \sum_{k=j_0+1}^{\infty} 2^{k\alpha p} \left(\sum_{j=-\infty}^{k-2} \left|\lambda_j\right| \left\|\left(K_{b'} * a_j\right) \chi_k\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}\right)^p \\
&=: I_{13} + I_{14}.
\end{aligned} \tag{49}$$

To estimate the term $I_{14}$, we need the pointwise estimate for $K_{b'} * a_j(x)$. Let $|x| \geq 2^j$. Then, by the vanishing moment condition on $a_j(x)$, we have

$$\left|K_{b'} * a_j(x)\right| \leq \int_{B_j} \left|K_{b'}(x-y) - K_{b'}(x)\right| \left|a_j(y)\right| dy. \tag{50}$$

From the condition of $K_{b'}(x)$, $|K_{b'}(x-y)-K_{b'}(x)| \le C(|y|/|x|^{n+b'+1})$, if $|x|\ge 2|y|$, it follows that

$$\begin{aligned}\left|K_{b'} * a_j(x)\right| &\le \frac{C2^j}{|x|^{n+b'+1}}\int_{B_j}\left|a_j(y)\right|dy \\ &\le \frac{C2^j}{|x|^{n+b'+1}}\left\|a_j\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}\left\|\chi_{B_j}\right\|_{L^{q'(\cdot)}(\mathbb{R}^n)}.\end{aligned} \tag{51}$$

Note that $b'=b/(1-b)$; that is, $(1-b)(b'+1)=1$. Since $n\delta_2=\alpha$, $0<p\le 1$, and $2^{j_0-1}<2^{j(1-b)}\le 2^{j_0}$, then by Lemmas 9 and 10 we have

$$\begin{aligned}I_{14} &\le C\sum_{k=j_0+1}^{\infty}2^{k\alpha p}\left(\sum_{j=-\infty}^{k-2}\left|\lambda_j\right|\frac{2^j}{2^{k(n+b'+1)}}\left\|a_j\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}\right. \\ &\quad\left.\cdot\left\|\chi_{B_j}\right\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\left\|\chi_k\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}\right)^p \\ &\le C\sum_{k=j_0+1}^{\infty}\left(\sum_{j=-\infty}^{k-2}\left|\lambda_j\right|\frac{2^{k\alpha-j\alpha}2^j2^{kn}}{2^{k(n+b'+1)}}\right. \\ &\quad\left.\cdot\frac{\left\|\chi_{B_j}\right\|_{L^{q'(\cdot)}(\mathbb{R}^n)}}{\left\|\chi_{B_k}\right\|_{L^{q'(\cdot)}(\mathbb{R}^n)}}\right)^p \le C\sum_{j=-\infty}^{\infty}\left|\lambda_j\right|^p\sum_{k=j_0+1}^{\infty}\frac{2^{jp}}{2^{k(b'+1)p}} \\ &\le C\sum_{j=-\infty}^{\infty}\left|\lambda_j\right|^p\frac{2^{jp}}{2^{j_0(b'+1)p}} \le C\sum_{j=-\infty}^{\infty}\left|\lambda_j\right|^p \\ &\quad\cdot\frac{2^{jp}}{2^{j(1-b)(b'+1)p}} \le C\sum_{j=-\infty}^{\infty}\left|\lambda_j\right|^p.\end{aligned} \tag{52}$$

Now to estimate $I_{13}$, we split $K_{b'} * a_j(x)$ as follows:

$$\begin{aligned}K_{b'} * a_j(x) &= C\int_{B_j}\frac{e^{i\alpha_b|x-y|^{-b'}}}{|x-y|^{n(b'+2)/s}} \\ &\quad\cdot\left[\frac{1}{|x-y|^{n(1-(b'+2)/s)}}-\frac{1}{|x|^{n(1-(b'+2)/s)}}\right]a_j(y)\,dy \\ &\quad + C\left(\widetilde{K}_{b',s} * a_j(x)\right)\frac{1}{|x|^{n(1-(b'+2)/s)}} =: E(x)+F(x),\end{aligned} \tag{53}$$

where $\widetilde{K}_{b',s}$ is the same as in Lemma 16 and let $s>\max\{q^+,2\}$ satisfy $(b'+2)/s<1$.

Applying the mean value theorem to the term brackets in the integrand of $E(x)$, then for $|x|\ge 2^j$ we have the pointwise estimate for $E(x)$ as follows:

$$\begin{aligned}|E(x)| &\le C\int_{B_j}\frac{|y|}{|x|^{n+1}}\left|a_j(y)\right|dy \\ &\le C\frac{2^j}{|x|^{n+1}}\left\|a_j\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}\left\|\chi_{B_j}\right\|_{L^{q'(\cdot)}(\mathbb{R}^n)}.\end{aligned} \tag{54}$$

On the other hand, since $0<p\le 1$, by the Minkowski inequality we get

$$\begin{aligned}I_{13} &\le \sum_{k=-\infty}^{j_0}2^{k\alpha p}\sum_{j=-\infty}^{k-2}\left|\lambda_j\right|^p\left\|E\chi_k\right\|^p_{L^{q(\cdot)}(\mathbb{R}^n)} \\ &\quad+\sum_{k=-\infty}^{j_0}2^{k\alpha p}\sum_{j=-\infty}^{k-2}\left|\lambda_j\right|^p\left\|F\chi_k\right\|^p_{L^{q(\cdot)}(\mathbb{R}^n)} =: I_{15}+I_{16}.\end{aligned} \tag{55}$$

For $I_{15}$, using $n\delta_2=\alpha$, the pointwise estimate for $E(x)$, and Lemmas 9 and 10 we have

$$\begin{aligned}I_{15} &\le C\sum_{k=-\infty}^{j_0}2^{k\alpha p}\sum_{j=-\infty}^{k-2}\left|\lambda_j\right|^p\frac{2^{jp}}{2^{k(n+1)p}}\left\|a_j\right\|^p_{L^{q(\cdot)}(\mathbb{R}^n)} \\ &\quad\cdot\left\|\chi_{B_j}\right\|^p_{L^{q'(\cdot)}(\mathbb{R}^n)}\left\|\chi_k\right\|^p_{L^{q(\cdot)}(\mathbb{R}^n)} \le C\sum_{j=-\infty}^{\infty}\left|\lambda_j\right|^p \\ &\quad\cdot\sum_{k=j+2}^{j_0}\frac{2^{jp}}{2^{k(n+1)p}}2^{(k-j)\alpha p}\left(2^{kn}\frac{\left\|\chi_{B_j}\right\|_{L^{q'(\cdot)}(\mathbb{R}^n)}}{\left\|\chi_{B_k}\right\|_{L^{q'(\cdot)}(\mathbb{R}^n)}}\right)^p \\ &\le C\sum_{j=-\infty}^{\infty}\left|\lambda_j\right|^p.\end{aligned} \tag{56}$$

Finally, we estimate $I_{16}$. Noting that $x\in A_k$, we get

$$\begin{aligned}I_{16} &= \sum_{k=-\infty}^{j_0}2^{k\alpha p}\sum_{j=-\infty}^{k-2}\left|\lambda_j\right|^p\left\|F\chi_k\right\|^p_{L^{q(\cdot)}(\mathbb{R}^n)} \le C\sum_{k=-\infty}^{j_0}2^{k\alpha p} \\ &\quad\cdot\sum_{j=-\infty}^{k-2}\left|\lambda_j\right|^p \\ &\quad\cdot\left\|\left(\widetilde{K}_{b',s} * a_j(\cdot)\right)\frac{1}{|\cdot|^{n(1-(b'+2)/s)}}\chi_k(\cdot)\right\|^p_{L^{q(\cdot)}(\mathbb{R}^n)} \\ &\le C\sum_{k=-\infty}^{j_0}2^{k\alpha p}\sum_{j=-\infty}^{k-2}\left|\lambda_j\right|^p \\ &\quad\cdot\frac{1}{2^{kn(1-(b'+2)/s)p}}\left\|\left(\widetilde{K}_{b',s} * a_j(\cdot)\right)\chi_k(\cdot)\right\|^p_{L^{q(\cdot)}(\mathbb{R}^n)}.\end{aligned} \tag{57}$$

Noting $s>\max\{q^+,2\}$, $s'<\min\{q^-,2\}$, $1/s+1/s'=1$, we denote $\widetilde{q}(\cdot)>s/(s-2)$ and $1/q(x)=1/\widetilde{q}(x)+1/s$.

When $|B_k|\le 2^n$ and $x_k\in B_k$, by Lemma 12 we have

$$\left\|\chi_{B_k}\right\|_{L^{\widetilde{q}(\cdot)}(\mathbb{R}^n)} \approx \left|B_k\right|^{1/\widetilde{q}(x_k)} \approx \left\|\chi_{B_k}\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}\left|B_k\right|^{-1/s}. \tag{58}$$

When $|B_k|\ge 1$ we have

$$\left\|\chi_{B_k}\right\|_{L^{\widetilde{q}(\cdot)}(\mathbb{R}^n)} \approx \left|B_k\right|^{1/\widetilde{q}(\infty)} \approx \left\|\chi_{B_k}\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}\left|B_k\right|^{-1/s}. \tag{59}$$

So we obtain

$$\left\|\chi_{B_k}\right\|_{L^{\widetilde{q}(\cdot)}(\mathbb{R}^n)} \approx \left\|\chi_{B_k}\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}\left|B_k\right|^{-1/s}. \tag{60}$$

In similar method we can obtain

$$\left\|\chi_{B_j}\right\|_{L^{q(\cdot)/(q(\cdot)-s')}(\mathbb{R}^n)} \approx \left|B_j\right|^{(q(\cdot)-s')/q(\cdot)} \quad (61)$$

and

$$\left\|\chi_{B_j}\right\|_{L^{q'(\cdot)}(\mathbb{R}^n)} \approx \left|B_j\right|^{1/q'(\cdot)}. \quad (62)$$

Thus by Lemmas 11, 12, and 16 we have

$$\begin{aligned}
&\left\|\left(\widetilde{K}_{b',s} * a_j(\cdot)\right)\chi_k(\cdot)\right\|_{L^{q(\cdot)}(\mathbb{R}^n)} \\
&\quad \le C\left\|\left(\widetilde{K}_{b',s} * a_j(\cdot)\right)\chi_k(\cdot)\right\|_{L^{s}(\mathbb{R}^n)}\left\|\chi_k\right\|_{L^{\tilde{q}(\cdot)}(\mathbb{R}^n)} \\
&\quad \le C\left\|a_j\right\|_{L^{s'}(\mathbb{R}^n)}\left\|\chi_{B_k}\right\|_{L^{\tilde{q}(\cdot)}(\mathbb{R}^n)} \le C\left\|a_j\right\|_{L^{s'}(\mathbb{R}^n)} \\
&\quad \cdot \left\|\chi_{B_k}\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}\left|B_k\right|^{-1/s} \le C\left\|\left|a_j\right|^{s'}\right\|^{1/s'}_{L^{q(\cdot)/s'}(\mathbb{R}^n)} \\
&\quad \cdot \left\|\chi_{B_j}\right\|^{1/s'}_{L^{q(\cdot)/(q(\cdot)-s')}(\mathbb{R}^n)}\left\|\chi_{B_k}\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}\left|B_k\right|^{-1/s} \\
&\quad \le C\left\|a_j\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}\left|B_j\right|^{((q(\cdot)-s')/q(\cdot))(1/s')}\left\|\chi_{B_k}\right\|_{L^{q(\cdot)}(\mathbb{R}^n)} \\
&\quad \cdot \left|B_k\right|^{-1/s} = C\left\|a_j\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}\left|B_j\right|^{1-1/q(\cdot)}\left|B_j\right|^{-1/s} \\
&\quad \cdot \left\|\chi_{B_k}\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}\left|B_k\right|^{-1/s} \le C\left\|a_j\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}\left\|\chi_{B_j}\right\|_{L^{q'(\cdot)}(\mathbb{R}^n)} \\
&\quad \cdot \left|B_j\right|^{-1/s}\left\|\chi_{B_k}\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}\left|B_k\right|^{-1/s}.
\end{aligned} \quad (63)$$

So by using $0 < p \le 1$, $n\delta_2 = \alpha$, $2^{j_0-1} < 2^{j(1-b)} \le 2^{j_0}$, $(1-b)(b'+1) = 1$, and Lemmas 9 and 10 we have

$$\begin{aligned}
I_{16} &\le C\sum_{k=-\infty}^{j_0} 2^{k\alpha p}\sum_{j=-\infty}^{k-2}\left|\lambda_j\right|^p \frac{1}{2^{kn(1-(b'+2)/s)p}} \\
&\quad \times \left(\left\|a_j\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}\left\|\chi_{B_j}\right\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\left|B_j\right|^{-1/s}\left\|\chi_{B_k}\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}\right. \\
&\quad \left.\cdot \left|B_k\right|^{-1/s}\right)^p \le C\sum_{k=-\infty}^{j_0}\sum_{j=-\infty}^{k-2}\left|\lambda_j\right|^p 2^{(k-j)\alpha p}2^{knp(b'+1)/s} \\
&\quad \cdot \left(\frac{\left\|\chi_{B_j}\right\|_{L^{q'(\cdot)}(\mathbb{R}^n)}}{\left\|\chi_{B_k}\right\|_{L^{q'(\cdot)}(\mathbb{R}^n)}}\right)^p\left|B_j\right|^{-p/s} \le C\sum_{j=-\infty}^{\infty}\left|\lambda_j\right|^p \\
&\quad \cdot \sum_{k=j+2}^{j_0} 2^{knp(b'+1)/s}\left|B_j\right|^{-p/s} \le C\sum_{j=-\infty}^{\infty}\left|\lambda_j\right|^p \\
&\quad \cdot \frac{2^{j_0np(b'+1)/s}}{2^{jnp/s}} \le C\sum_{j=-\infty}^{\infty}\left|\lambda_j\right|^p\frac{2^{jnp(1-b)(b'+1)/s}}{2^{jnp/s}} \\
&\quad \le C\sum_{j=-\infty}^{\infty}\left|\lambda_j\right|^p.
\end{aligned} \quad (64)$$

Therefore, by (43), (44), (46), (48), (49), (52), (55), (56), and (64) we complete the proof of Theorem 6. □

Similar to the method of Theorem 6, next we give the proof of Theorem 7.

*Proof of Theorem 7.* Let $f \in HK^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$. By Lemma 5, we have

$$f(x) = \sum_{j=0}^{\infty}\lambda_j a_j, \quad (65)$$

where

$$\|f\|_{HK^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} \approx \inf\left(\sum_{j=0}^{\infty}\left|\lambda_j\right|^p\right)^{1/p}, \quad (66)$$

the infimum is taken over the above decomposition of $f$, and $a_j$ is a dyadic central $(\alpha, q(\cdot))$-atom of restricted type with the support $B_j$. Then we have

$$\begin{aligned}
\left\|T_b(f)\right\|^p_{HK^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} &= \left\|G_N(T_b f)\right\|^p_{K^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} \\
&= \sum_{k=0}^{\infty} 2^{k\alpha p}\left\|G_N(T_b f)\chi_k\right\|^p_{L^{q(\cdot)}(\mathbb{R}^n)} \\
&\le C\sum_{k=0}^{\infty} 2^{k\alpha p}\left(\sum_{j=0}^{k-1}\left|\lambda_j\right|\left\|G_N(T_b a_j)\chi_k\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}\right)^p \\
&\quad + C\sum_{k=0}^{\infty} 2^{k\alpha p}\left(\sum_{j=k}^{\infty}\left|\lambda_j\right|\left\|G_N(T_b a_j)\chi_k\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}\right)^p \\
&=: CJ_1 + CJ_2.
\end{aligned} \quad (67)$$

We first estimate $J_2$; by $0 < p \le 1$ and the $(L^{q(\cdot)}(\mathbb{R}^n), L^{q(\cdot)}(\mathbb{R}^n))$-boundedness of $M$ and $T_b$ we have

$$\begin{aligned}
J_2 &= \sum_{k=0}^{\infty} 2^{k\alpha p}\left(\sum_{j=k}^{\infty}\left|\lambda_j\right|\left\|G_N(T_b a_j)\chi_k\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}\right)^p \\
&\le C\sum_{k=0}^{\infty} 2^{k\alpha p}\left(\sum_{j=k}^{\infty}\left|\lambda_j\right|\left\|M(T_b a_j)\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}\right)^p \\
&\le C\sum_{k=0}^{\infty} 2^{k\alpha p}\left(\sum_{j=k}^{\infty}\left|\lambda_j\right|\left\|T_b a_j\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}\right)^p \\
&\le C\sum_{k=0}^{\infty} 2^{k\alpha p}\left(\sum_{j=k}^{\infty}\left|\lambda_j\right|\left\|a_j\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}\right)^p \\
&\le C\sum_{j=0}^{\infty}\left|\lambda_j\right|^p\left(\sum_{k=0}^{j} 2^{(k-j)\alpha p}\right) \le C\|f\|_{HK^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}.
\end{aligned} \quad (68)$$

To estimate $J_1$, we need the pointwise estimate for $G_N(T_b a_j)(x)$.

Suppose that $y, t$ satisfy $|x-y|<t$. Let $|x|>2^{j+2}$ and $\phi \in \mathscr{K}_m$, where $m \in \mathbb{N}$ and $\mathscr{K}_m = \{\phi \in \mathcal{S}(\mathbb{R}^n) : \sup_{u\in\mathbb{R}^n, |\alpha|\le m}(1+|u|)^{m+n}|D^\alpha \phi(u)| \le 1\}$. By the vanishing moment condition on $a_j(x)$, it is easy to prove that $\int_{\mathbb{R}^n} T_b a_j(x)dx = 0$. So we have

$$\begin{aligned} &\left|\left(T_b a_j * \phi_t\right)(y)\right| \\ &= \left|\int_{\mathbb{R}^n} t^{-n} T_b a_j(z)\left(\phi\left(\frac{y-z}{t}\right)-\phi\left(\frac{y}{t}\right)\right)dz\right| \\ &\le \left|\int_{|z|\le 2^{j+1}} t^{-n} T_b a_j(z)\left(\phi\left(\frac{y-z}{t}\right)-\phi\left(\frac{y}{t}\right)\right)dz\right| \\ &\quad + \left|\int_{|z|>2^{j+1}} t^{-n} T_b a_j(z)\left(\phi\left(\frac{y-z}{t}\right)-\phi\left(\frac{y}{t}\right)\right)dz\right| \\ &=: J_{11} + J_{12}. \end{aligned} \tag{69}$$

For $J_{11}$, by Lemma 9, the generalized Hölder inequality, and the mean value theorem, we obtain

$$\begin{aligned} J_{11} &\le C\left\|T_b a_j\right\|_{L^{q(\cdot)}(\mathbb{R}^n)} \\ &\quad \cdot t^{-n}\left\|\left(\phi\left(\frac{y-\cdot}{t}\right)-\phi\left(\frac{y}{t}\right)\right)\chi_{B_{j+1}}\right\|_{L^{q'(\cdot)}(\mathbb{R}^n)} \\ &\le C\left\|a_j\right\|_{L^{q(\cdot)}(\mathbb{R}^n)} \\ &\quad \cdot t^{-n}\left\|\sum_{|\beta|=1}\left|D^\beta \phi\left(\frac{y-\theta\cdot}{t}\right)\right|\frac{|\cdot|}{t}\chi_{B_j}\right\|_{L^{q'(\cdot)}(\mathbb{R}^n)} \\ &\le C\left\|a_j\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}\left\|\frac{|\cdot|\,\chi_{B_j}}{\left(|x-y|+|y-\theta\cdot|\right)^{n+1}}\right\|_{L^{q'(\cdot)}(\mathbb{R}^n)} \\ &\le C2^{-j\alpha}\frac{1}{|x|^{n+1}}\left\||\cdot|\,\chi_{B_j}\right\|_{L^{q'(\cdot)}(\mathbb{R}^n)} \le C2^{-j\alpha+j} \\ &\quad \cdot \frac{1}{|x|^{n+1}}\left\|\chi_{B_j}\right\|_{L^{q'(\cdot)}(\mathbb{R}^n)}, \end{aligned} \tag{70}$$

where $0 \le \theta \le 1$.

For $J_{12}$, by Lemma 15 we have

$$\begin{aligned} &J_{12} \\ &\le t^{-n}\int_{|z|>2^{j+1}}\left|K_{b'} * a_j(z)\right|\left|\phi\left(\frac{y-z}{t}\right)-\phi\left(\frac{y}{t}\right)\right|dz \\ &\quad + t^{-n}\int_{|z|>2^{j+1}}\left|h * a_j(z)\right|\left|\phi\left(\frac{y-z}{t}\right)-\phi\left(\frac{y}{t}\right)\right|dz \\ &=: J_{13} + J_{14}. \end{aligned} \tag{71}$$

Noting that $2^j \ge 1$, then $|z| > 2^{j+1} \ge 2$. Since $|z-w| \ge |z|-|w| > 2^j \ge 1$ for $|w| \le 2^j$, we obtain

$$\begin{aligned} &\left|K_{b'} * a_j(z)\right| \\ &= \left|\int_{B_j}\frac{e^{i\alpha_b|z-w|^{-b'}}}{|z-w|^n}\chi\left(|z-w|\le 1\right)a_j(w)\,dw\right| = 0. \end{aligned} \tag{72}$$

So we have $J_{13} = 0$.

For $J_{14}$, by the pointwise estimate for $h * a_j(z)$ in the proof of Theorem 6, we obtain

$$\begin{aligned} J_{14} &\le C\left\|a_j\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}\left\|\chi_j\right\|_{L^{q'(\cdot)}(\mathbb{R}^n)} t^{-n} \\ &\quad \times \int_{|z|>2^{j+1}}\left|\phi\left(\frac{y-z}{t}\right)-\phi\left(\frac{y}{t}\right)\right| \\ &\quad \cdot \left[\frac{1}{(1+|z|)^{n+1}} + \frac{\chi(|z|\le 2)}{|z|^{n-\varepsilon}}\right]dz \\ &\le C\left\|a_j\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}\left\|\chi_j\right\|_{L^{q'(\cdot)}(\mathbb{R}^n)} \\ &\quad \cdot t^{-n}\int_{|x|/2>|z|>2^{j+1}}\left|\phi\left(\frac{y-z}{t}\right)-\phi\left(\frac{y}{t}\right)\right| \\ &\quad \cdot \frac{1}{(1+|z|)^{n+1}}dz + C\left\|a_j\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}\left\|\chi_j\right\|_{L^{q'(\cdot)}(\mathbb{R}^n)} \\ &\quad \cdot t^{-n}\int_{|z|\ge|x|/2}\left|\phi\left(\frac{y-z}{t}\right)-\phi\left(\frac{y}{t}\right)\right|\frac{1}{(1+|z|)^{n+1}}dz \\ &=: J_{15} + J_{16}. \end{aligned} \tag{73}$$

Using the mean value theorem, we get

$$\begin{aligned} &J_{15} \\ &\le C\left\|a_j\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}\left\|\chi_j\right\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\int_{|x|/2>|z|>2^{j+1}}\sum_{|\beta|=1}\left|D^\beta\phi\left(\frac{y-\theta z}{t}\right)\right| \\ &\quad \cdot \left|\frac{z}{t}\right|\frac{t^{-n}}{(1+|z|)^{n+1}}dz \\ &\le C\left\|a_j\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}\left\|\chi_j\right\|_{L^{q'(\cdot)}(\mathbb{R}^n)} \\ &\quad \cdot \int_{|x|/2>|z|>2^{j+1}}\frac{|z|}{\left(|x-y|+|y-\theta z|\right)^{n+1}}\frac{1}{(1+|z|)^{n+1}}dz \\ &\le C\left\|a_j\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}\left\|\chi_j\right\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\frac{1}{|x|^{n+1}} \\ &\quad \cdot \int_{|x|/2>|z|>2^{j+1}}\frac{|z|}{(1+|z|)^{n+1}}dz \\ &\le C\left\|a_j\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}\left\|\chi_j\right\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\frac{\ln|x|}{|x|^{n+1}}, \end{aligned} \tag{74}$$

where $0 \le \theta \le 1$.

For $J_{16}$, noting that $\phi \in \mathscr{K}_m$, we get

$$\begin{aligned} J_{16} &\le C\left\|a_j\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}\left\|\chi_j\right\|_{L^{q'(\cdot)}(\mathbb{R}^n)} \\ &\quad \cdot t^{-n}\int_{|z|\ge|x|/2}\left(\left|\phi\left(\frac{y-z}{t}\right)\right|+\left|\phi\left(\frac{y}{t}\right)\right|\right) \\ &\quad \cdot \frac{1}{(1+|z|)^{n+1}}dz \le C\left\|a_j\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}\left\|\chi_j\right\|_{L^{q'(\cdot)}(\mathbb{R}^n)} \end{aligned}$$

$$\cdot\left(\frac{1}{|x|^{n+1}}+\frac{t^{-n}}{(1+|y|/|t|)^{n}}\int_{|z|\geq|x|/2}\frac{1}{(1+|z|)^{n+1}}dz\right)$$
$$\leq C\left\|a_j\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}\left\|\chi_j\right\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\left(\frac{1}{|x|^{n+1}}\right.$$
$$\left.+\frac{1}{|x|\left(|x-y|+|y|\right)^{n}}\right)\leq C\left\|a_j\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}\left\|\chi_j\right\|_{L^{q'(\cdot)}(\mathbb{R}^n)}$$
$$\cdot\frac{1}{|x|^{n+1}}. \tag{75}$$

Thus, for $|x|>2^{j+2}$, we get

$$\left|G_N\left(T_b a_j\right)(x)\right| \leq C\left\|a_j\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}\left\|\chi_{B_j}\right\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\left(2^j+\ln|x|\right)\frac{1}{|x|^{n+1}}. \tag{76}$$

So by using $0<p\leq 1$, $\alpha\leq 1+n\delta_2$, and Lemmas 9 and 10 we have

$$J_1=\sum_{k=0}^{\infty}2^{k\alpha p}\left(\sum_{j=0}^{k-1}\left|\lambda_j\right|\left\|G_N\left(T_b a_j\right)\chi_k\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}\right)^p$$
$$\leq C\sum_{k=0}^{\infty}\left(\sum_{j=0}^{k-1}\left|\lambda_j\right|2^{k\alpha}\left\|a_j\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}\left\|\chi_{B_j}\right\|_{L^{q'(\cdot)}(\mathbb{R}^n)}2^j\right.$$
$$\left.\cdot\frac{1}{2^{k(n+1)}}\left\|\chi_{B_k}\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}\right)^p$$
$$+C\sum_{k=0}^{\infty}\left(\sum_{j=0}^{k-1}\left|\lambda_j\right|2^{k\alpha}\left\|a_j\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}\left\|\chi_{B_j}\right\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\ln 2^k\right.$$
$$\left.\cdot\frac{1}{2^{k(n+1)}}\left\|\chi_{B_k}\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}\right)^p$$
$$\leq C\sum_{k=0}^{\infty}\left(\sum_{j=0}^{k-1}\left|\lambda_j\right|2^{(j-k)(1+n\delta_2-\alpha)}\right)^p \tag{77}$$
$$+C\sum_{k=0}^{\infty}\left(\sum_{j=0}^{k-1}\left|\lambda_j\right|2^{j(n\delta_2-\alpha)}\frac{k}{2^{k(1+n\delta_2-\alpha)}}\right)^p$$
$$\leq C\sum_{j=0}^{\infty}\left|\lambda_j\right|^p\sum_{k=j+1}^{\infty}2^{(j-k)(1+n\delta_2-\alpha)p}+C\sum_{j=0}^{\infty}\left|\lambda_j\right|^p$$
$$\cdot 2^{j(n\delta_2-\alpha)p}\sum_{k=j+1}^{\infty}\frac{k^p}{2^{k(1+n\delta_2-\alpha)p}}\leq C\sum_{j=0}^{\infty}\left|\lambda_j\right|^p$$
$$+C\sum_{j=0}^{\infty}\left|\lambda_j\right|^p 2^{j(n\delta_2-\alpha)p}\frac{j^p}{2^{j(1+n\delta_2-\alpha)p}}\leq C\sum_{j=0}^{\infty}\left|\lambda_j\right|^p$$
$$\leq C\left\|f\right\|_{HK_{q(\cdot)}^{\alpha,p}(\mathbb{R}^n)}.$$

Therefore, by (67), (68), and (77) we complete the proof of Theorem 7. □

## Data Availability

No data were used to support this study.

## Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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