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Research Article

Fixed Point Theorems for Ćirić Type $\mathcal{Z}$-Contractions in Generating Spaces of Quasi-Metric Family

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In this paper, the notions of Ćirić type (I) $\mathcal{Z}$-contractions and Ćirić type (II) $\mathcal{Z}$-contractions in generating spaces of quasi-metric family are introduced and new fixed point theorems for such two contractions are established. We give examples to illustrate main results.

1. Introduction and Preliminaries

The Banach contraction principle forms the basis of metric fixed point theory. Because its importance, many authors generalized this contraction principle by generalizing the certain contraction conditions.

Especially, Ćirić [1] proved a result on nonunique fixed points as follows:

If a map T :$T:(X,d)\to(X,d)$, where (X,d) is a metric space, satisfies

$$\min\{d(Tx,Ty),d(x,Tx),d(y,Ty)\} - \min\{d(x,Ty),d(y,Tx)\} \le kd(x,y), \tag{1}$$

for all $x,y\in X$, where $k\in(0,1)$, then T has a fixed point whenever (X,d) is T-orbitally complete.

Also, Ćirić [2] obtained the following result:

If a map $T:(X,d)\to(X,d)$ satisfies the following condition

$$d(Tx,Ty) \le k\max\left\{d(x,y),d(x,Tx),d(y,Ty),\frac{1}{2}[d(x,Ty)+d(y,Tx)]\right\}, \tag{2}$$

for all $x,y\in X$, where $k\in(0,1)$, then T has a unique fixed point provided that (X,d) is T-orbitally complete.

Recently, Khojasteh et al. [3] presented the notion of $\mathcal{Z}$-contraction by using a simulation function and obtained the following result:

If a map $T:(X,d)\to(X,d)$ is $\mathcal{Z}$-contraction with respect to a simulation funtion ζ, that is,

$$\zeta(d(Tx,Ty),d(x,y))\ge 0, \tag{3}$$

for all $x,y\in X$, then T has a unique fixed point when (X,d) is complete.

They unified the some existing metric fixed point results. Afterwards, many authors (for example, [4-8]) obtained generalizations of the result of [3].

Also, a lot of authors generalized the Banach contraction principle by introducing the concepts of generalized metrics, for example, Branciari metric, b-metric, quasi-metric, semi-metric, G-metric, cone metric, fuzzy metric, Menger Probabilistic metric.

In particular, Chang et al. [9, 10] introduced the concept of a generating space of a quasi-metric family and gave some examples and properties of generating space of a quasi-metric family, and they obtained some fixed point and minimization results in such spaces.

In this paper, we introduce the concepts of Ćirić type (I) $\mathcal{Z}$-contraction maps and Ćirić type (II) $\mathcal{Z}$-contraction maps in generating spaces of quasi-metric family, and we establish new fixed point theorems for such two contraction maps.

Let X be a nonempty set and $\{d_a : a\in(0,1]\}$ be a family of mappings d_a; $X\times X\to[0,\infty)$ $\forall a\in(0,1]$.

Then $(X,\{d_a : a\in(0,1]\})$ is called a generating space of quasi-metric family [9, 10] if the following are satisfied:

(QM1) $\forall a \in (0,1], d_a(x,y) = 0$ if and only if $x = y$;
(QM2) $\forall a \in (0,1], d_a(x,y) = d_a(y,x)$;
(QM3) $\forall a \in (0,1], \exists b \in (0,a]$ such that $d_a(x,z) \le d_b(x,y) + d_b(y,z)$;
(QM4) $\forall x, y \in X, d_a(x,y)$ is nonincreasing and left continuous in a.

It follows from (QM3) and (QM4) that $\forall a \in (0,1], \exists b \in (0,a]$ such that

$$d_a(x,z) \le d_c(x,y) + d_c(y,z) \ \forall c \in (0,b]. \tag{4}$$

Example 1. Let (X,d) be a metric space, and let $d_a(x,y) = d(x,y)\ \forall a \in (0,1],\ \forall x, y \in X$.

Then (X,d) is a generating space of quasi-metric family.

Example 2. Let $X = \mathbb{R}$, and let $d_a(x,y) = (1/a)|x-y|\ \forall a \in (0,1],\ \forall x, y \in X$.

Then $(X,\{d_a : a \in (0,1]\})$ is a generating space of quasi-metric family.

Example 3. Let $X = \{a,b,c,d\}$, and let $d : X \times X \to [0,\infty)$ be a map such that

$$\begin{aligned} &d(a,b) = d(b,a) = 3,\\ &d(b,c) = d(c,b) = d(a,c) = d(c,a) = 1,\\ &d(a,d) = d(d,a) = d(b,d) = d(d,b) = d(c,d) = d(d,c) = 4,\\ &d(x,x) = 0\ \forall x \in X. \end{aligned} \tag{5}$$

Let $d_a(x,y) = (1/a)d(x,y)\ \forall a \in (0,1],\ \forall x, y \in X$.

Then $(X,\{d_a : a \in (0,1]\})$ is a generating space of quasi-metric family.

Note that d is not a metric on X. In fact, the following inequality is not satisfied:

$$d(a,b) \le d(a,c) + d(c,b). \tag{6}$$

Let $(X,\{d_a : a \in (0,1]\})$ be a generating space of quasi-metric family, $\{x_n\} \subset X$ be a sequence, and $x \in X$.

Then we say that

(1) $\{x_n\}$ *converges* to x (denoted by $\lim_{n\to\infty} x_n = x$) if and only if $\lim_{n\to\infty} d_a(x_n,x) = 0\ \forall a \in (0,1]$;

(2) $\{x_n\}$ is a *Cauchy sequence* if and only if $\forall \epsilon > 0, \exists n_0 \in \mathbb{N}$ such that

$$\forall n,m > n_0, d_a(x_n,x_m) < \epsilon\ \forall a \in (0,1]; \tag{7}$$

(3) $(X,\{d_a : a \in (0,1]\})$ is *complete* if and only if every Cauchy sequence in X is convergent;

(4) $T : X \to X$ is *continuous* at x if and only if $\forall \epsilon > 0, \exists \delta > 0$ such that $\forall a \in (0,1], d_a(Tx,Ty) < \epsilon$, whenever $d_a(x,y) < \delta$;

(5) $T : X \to X$ is *continuous* if and only if it is continuous at each point in X.

Remark 4. Every generating space $(X,\{d_a : a \in (0,1]\})$ of quasi-metric family is a Hausdorff space in the topology $\tau_{\{d_a\}}$ induced by the family of neighborhoods:

$$\{V_x(\epsilon,a) : x \in X, \epsilon > 0, a \in (0,1]\}, \tag{8}$$

where $V_x(\epsilon,a) = \{y \in X : d_a(x,y) < \epsilon\}$ (see [11]).

Thus, every convergent sequence in generating spaces of quasi-metric family has a unique limit.

Note that every convergent sequence is a Cauchy sequence.

Also, note that $T : X \to X$ is continuous at x if and only if

$$\forall a \in (0,1], \lim_{n\to\infty} d_a(Tx_n,Tx) = 0 \text{ whenever } \lim_{n\to\infty} d_a(x_n,x) = 0. \tag{9}$$

Let $(X,\{d_a : a \in (0,1]\})$ be a generating space of quasi-metric family, and let $T : X \to X$ be a mapping.

Then we say that

(1) T is *T-orbitally continuous* if and only if

$$\begin{aligned} &\forall a \in (0,1], \lim_{i\to\infty} d_a\left(T^{n(i)}x, x\right) = 0\\ &\text{implies } \lim_{i\to\infty} d_a\left(TT^{n(i)}x, Tx\right) = 0, \quad \forall x \in X. \end{aligned} \tag{10}$$

(2) $(X,\{d_a : a \in (0,1]\})$ is *T- orbitally complete* if and only if every Cauchy sequence of the form $\{x_{n(i)} = T^{n(i)}x : i = 1,2,3,\cdots\}, x \in X$ is convergent in X.

Remark 5. Let$(X,\{d_a : a \in (0,1]\})$ be a generating space of quasi-metric family, and let $T : X \to X$ be a mapping.

(1) if $T : X \to X$ is continuous, then it is T-orbitally continuous;

(2) if $(X,\{d_a : a \in (0,1]\})$ complete, then it is T-orbitally complete.

Remark 6. A generating space of quasi-metric family induce a fuzzy metric space and Menger probabilistic metric space (see [12] and [13], respectively).

Let $\mathcal{Z}$ be the family of all mappings $\zeta : [0,\infty) \times [0,\infty) \to \mathbb{R}$ such that

(ζ1) $\zeta(0,0) = 0$;
(ζ2) $\zeta(t,s) < s - t \forall s,\ t > 0$;
(ζ3) for any sequence $\{t_n\}, \{s_n\} \subset [0,\infty)$

$$\lim_{n\to\infty} \sup \zeta(t_n,s_n) < 0, \text{whenever } \lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 0. \tag{11}$$

We say that $\zeta \in \mathcal{Z}$ is a *simulation function* [9].

Note that for all $t > 0, \zeta(t,t) < 0$.

Example 7. Let $\zeta_i : [0,\infty) \to [0,\infty), i = 1,2,3,4,5,6,7,8,$ be functions defined as follows:

(1) $\zeta_1(t,s) = \phi_1(s) - \phi_2(t)\ \forall t, s \ge 0,$ where $\phi_1, \phi_2 : [0,\infty) \to [0,\infty)$ are continuous functions such that

$$\phi_1^{-1}(\{0\}) = 0, \phi_2^{-1}(\{0\}) = 0 \text{ and } \phi_2(t) < t \le \phi_1(t) \forall t > 0; \tag{12}$$

(2) $\zeta_2(t,s) = s - (f(t,s)/g(t,s))t\ \forall t,s \geq 0$, where $f,g : [0,\infty)\times[0,\infty) \to [0,\infty)$ are continuous finctions with respect to each variables such that

$$f(t,s) > g(t,s)\ \forall t,s > 0; \tag{13}$$

(3) $\zeta_3(t,s) = s - \phi(s) - t\ \forall t,s \geq 0$, where $\phi : [0,\infty) \to [0,\infty)$ is a lower semi-continuous function with $\phi^{-1}(\{0\}) = 0$;

(4) $\zeta_4(t,s) = s\varphi(s) - t\ \forall t,s \geq 0$, where $\varphi : [0,\infty) \to [0,1]$ is a function such that

$$\lim_{t\to r^+} \sup\varphi(t) < 1\ \forall r > 0; \tag{14}$$

(5) $\zeta_5(t,s) = \psi(s) - t\ \forall t,s \geq 0$, where $\psi : [0,\infty) \to [0,\infty)$ is an upper semicontinuous function such that

$$\psi(t) < t\ \forall t > 0 \text{ and } \psi(0) = 0; \tag{15}$$

(6) $\zeta_6(t,s) = ks - t \forall t,s \geq 0$, where $k \in (0,1]$;

(7) $\zeta_7(t,s) = s - \int_0^t \nu(u)du \forall s,t \geq 0$, where $\nu : [0,\infty) \to [0,\infty)$ is a function such that

$$\int_0^\epsilon \nu(u)du > \epsilon \forall \epsilon > 0; \tag{16}$$

(8) $\zeta_8(t,s) = \frac{s}{1+s} - t\ \forall s,t \geq 0$.

Then $\zeta_i \in \mathcal{Z},\ \forall i = 1,2,3,4,5,6,7,8$.

For more examples of simulation functions, we can find in [8, 9, 11, 12].

Lemma 8. *Let $(X,\{d_a : a \in (0,1]\})$ be a generating space of quasi-metric family. Suppose that $\{x_n\} \subset X$ is not a Cauchy sequence.*

Then there exists an $\epsilon > 0$ for which we can find two subsequences $\{x_{m(k)}\}$ and $\{x_{n(k)}\}$ of $\{x_n\}$ such that $m(k)$ is the smallest index with $m(k) > n(k) > k$

$$d_a(x_{m(k)}, x_{n(k)}) \geq \epsilon, \text{ where } a \in (0,1] \text{ and } d_s(x_{m(k)-1}, x_{n(k)}) < \epsilon\ \forall s \in (0,1]. \tag{17}$$

Further if

$$\lim_{n\to\infty} d_s(x_{n-1}, x_n) = 0\ \forall s \in (0,1], \tag{18}$$

then we have

(1) $\lim_{n\to\infty} d_c(x_{m(k)}, x_{n(k)}) = \epsilon\ \forall c \in (0,a]$;

(2) $\lim_{n\to\infty} d_b(x_{m(k)+1}, x_{n(k)}) = \epsilon$, *where* $b \in (0,a]$;

(3) $\lim_{n\to\infty} d_b(x_{m(k)}, x_{n(k)+1}) = \epsilon$, *where* $b \in (0,a]$;

(4) $\lim_{n\to\infty} d_b(x_{m(k)+1}, x_{n(k)+1}) = \epsilon$, *where* $b \in (0,a]$;

Proof. If $\{x_n\} \subset X$ is not a Cauchy sequence, then from definition (17) holds.

Suppose that (18) is satisfied.

Then from (17) we have that for each $c \in (0,a]$

$$\begin{aligned}\epsilon \leq d_a(x_{m(k)}, x_{n(k)}) &\leq d_c(x_{m(k)}, x_{n(k)}) \\ &\leq d_b(x_{m(k)}, x_{m(k)-1}) + d_b(x_{m(k)-1}, x_{n(k)}) \\ &< d_b(x_{m(k)}, x_{m(k)-1}) + \epsilon, \text{ for some } b \in (0,c].\end{aligned} \tag{19}$$

By taking $k\to\infty$ in above inequality

$$\lim_{k\to\infty} d_c(x_{m(k)}, x_{n(k)}) = \epsilon\ \forall c \in (0,a]. \tag{20}$$

Hence (1) is proved.

We show that (2) holds.

It follows from (QM3) and (QM4) that for $a \in (0,1], \exists b \in (0,a]$ such that $\forall c \in (0,b]$,

$$d_a(x_{m(k)}, x_{n(k)}) \leq d_c(x_{m(k)}, x_{m(k)+1}) + d_c(x_{m(k)+1}, x_{n(k)}), \tag{21}$$

$$d_c(x_{m(k)+1}, x_{n(k)}) \leq d_e(x_{m(k)+1}, x_{m(k)}) + d_e(x_{m(k)}, x_{n(k)}) \quad \text{for some } e \in (0,c], \tag{22}$$

Letting $k\to\infty$ in Equations (21 and 22), and using Equations (18 and 19) we obtain

$$\lim_{k\to\infty} d_c(x_{m(k)+1}, x_{n(k)}) = \epsilon\ \forall c \in (0,b]. \tag{23}$$

In particular, we have

$$\lim_{k\to\infty} d_b(x_{m(k)+1}, x_{n(k)}) = \epsilon, \text{ where } b \in (0,a]. \tag{24}$$

Thus proof of Equation (2) is complete.

In similar with the proof of Equation (2), we have

$$\lim_{k\to\infty} d_c(x_{m(k)}, x_{n(k)+1}) = \epsilon\ \forall c \in (0,b], \tag{25}$$

and so

$$\lim_{k\to\infty} d_b(x_{m(k)}, x_{n(k)+1}) = \epsilon, \text{ where } b \in (0,a]. \tag{26}$$

We now show that Equation (4) holds.

For $a, \exists b \in (0,a]$ such that $\forall c \in (0,b]$,

$$d_a(x_{m(k)+1}, x_{n(k)}) \leq d_c(x_{m(k)+1}, x_{n(k)+1}) + d_c(x_{n(k)+1}, x_{n(k)}), \tag{27}$$

and

$$\begin{aligned}d_c(x_{m(k)+1}, x_{n(k)+1}) &\leq d_e(x_{m(k)+1}, x_{m(k)}) \\ &\quad + d_e(x_{m(k)}, x_{n(k)+1})\ \forall e \in (0,c].\end{aligned} \tag{28}$$

By taking $k\to\infty$ in Equations (17) and (29), and using Equations (18), (23) and (25) we obtain

$$\lim_{k\to\infty} d_c(x_{m(k)+1}, x_{n(k)+1}) = \epsilon\ \forall c \in (0,b], \tag{29}$$

and so

$$\lim_{k\to\infty} d_b(x_{m(k)+1}, x_{n(k)+1}) = \epsilon, \text{ where } b \in (0,a]. \tag{30}$$

Hence (4) is proved.

Lemma 9. *Let $(X, \{d_a : a \in (0,1]\})$ be a generating space of quasi-metric family. Suppose that*

$$\lim_{n\to\infty} d_a(x_n, x) = 0 \text{ and } \lim_{n\to\infty} d_a(y_n, y) = 0 \ \forall a \in (0,1]. \quad (31)$$

Then we have

$$\lim_{n\to\infty} d_a(x_n, y_n) = d_a(x, y) \ \forall a \in (0,1]. \quad (32)$$

Proof. Since

$$d_a(x_n, y_n) \le d_b(x_n, x) + d_b(x, y_n), b \in (0,a], \quad (33)$$

$$d_a(x, y) \le d_c(x, y_n) + d_c(y_n, y), c \in (0,a], \quad (34)$$

we have

$$d_a(x_n, y_n) \le d_b(x_n, x) + d_b(x, y_n) \le d_b(x_n, x) + d_{bc}(x, y_n), \quad (35)$$

$$d_a(x, y) \le d_c(x, y_n) + d_c(y_n, y) \le d_{bc}(x, y_n) + d_c(y_n, y), \quad (36)$$

where $d_{bc}(y_n, y) = \max\{d_b(y_n, y), d_c(y_n, y)\}$ and $d_{bc}(x, y_n) = \max\{d_b(x, y_n), d_c(x, y_n)\}$.

Hence we have

$$\begin{aligned} 0 &\le |d_a(x_n, y_n) - d_a(x, y)| = |d_a(x_n, y_n) - d_{bc}(x, y_n) \\ &\quad + d_{bc}(x, y_n) - d_a(x, y)| \le |d_a(x_n, y_n) - d_{bc}(x, y_n)| \\ &\quad + |d_{bc}(x, y_n) - d_a(x, y)| \le d_b(x_n, x) + d_c(y_n, y) \\ &\le d_e(x_n, x) + d_e(y_n, y) \text{ where } 0 < e < \min\{b, c\}. \end{aligned} \quad (37)$$

By taking $n \to \infty$, we obtain

$$\lim_{n\to\infty} d_a(x_n, y_n) = d_a(x, y) \ \forall a \in (0,1]. \quad (38)$$

□

2. Fixed Point Theorems

2.1. Nonunique Fixed Point Results. Let $(X, \{d_a : a \in (0,1]\})$ be a generating space of quasi-metric family.

A mapping $T : X \to X$ is called Ćirić type (I) $\mathcal{Z}$-contraction with respect to $\zeta \in \mathcal{Z}$ if

$$\zeta(m(a, x, y), d_a(x, y)) \ge 0, \forall a \in (0,1], \quad (39)$$

for all $x, y \in X$, where $m(a, x, y) = d_a(Tx, Ty)$

$$-\min\{d_a(x, Ty), d_a(y, Tx), d_a(x, Tx), d_a(y, Ty)\}. \quad (40)$$

Now, we prove our first fixed point result.

Theorem 10. *Let $(X, \{d_a : a \in (0,1]\})$be a generating space of quasi-metric family, and let $T : X \to X$ be a Ćirić type (I) $\mathcal{Z}$-contraction mapping with respect to ζ. IfX is T-orbitally complete, then T has a fixed point.*

Proof. Let $x_0 \in X$ be any fixed point. Define a sequence $\{x_n\} \subset X$ by $x_n = Tx_{n-1} = T^n x_0 \ \forall n = 1, 2, 3 \cdots$.

If $x_{n_0} = x_{n_0+1}$ for some $n_0 \in \mathbb{N}$, then x_{n_0} is a fixed point of T, and the proof is finished.

Assume that $x_{n-1} \ne x_n \ \forall n = 1, 2, 3 \cdots$.

We infer that $\forall n = 1, 2, 3, \cdots$

$$\begin{aligned} m(a, x_{n-1}, x_n) &= d_a(Tx_{n-1}, Tx_n) - \min\{d_a(x_{n-1}, Tx_n), \\ &\quad d_a(x_n, Tx_{n-1}), d_a(x_{n-1}, Tx_{n-1}), d_a(x_n, Tx_n)\} \\ &= d_a(x_n, x_{n+1}) - \min\{d_a(x_{n-1}, x_{n+1}), d_a(x_n, x_n), \\ &\quad d_a(x_{n-1}, x_n), d_a(x_n, x_{n+1})\} = d_a(x_n, x_{n+1}). \end{aligned} \quad (41)$$

It follows from (41) and (42) that $\forall a \in (0,1]$, $\forall n = 1, 2, 3, \cdots$

$$\begin{aligned} 0 &\le \zeta(m(a, x_{n-1}, x_n), d_a(x_{n-1}, x_n)) \\ &= \zeta(d_a(x_n, x_{n+1}), d_a(x_{n-1}, x_n)) \\ &< d_a(x_{n-1}, x_n) - d_a(x_n, x_{n+1}). \end{aligned} \quad (42)$$

Consequently, we obtain that $\forall a \in (0,1]$, $\forall n = 1, 2, 3, \cdots$

$$d_a(x_n, x_{n+1}) < d_a(x_{n-1}, x_n). \quad (43)$$

Since $\{d_a(x_{n-1}, x_n)\}$ is a decreasing sequence bounded from below by 0, there exists $r \ge 0$ such that

$$\lim_{n\to\infty} d_a(x_{n-1}, x_n) = r, \ \forall a \in (0,1]. \quad (44)$$

We now show that $\lim_{n\to\infty} d_a(x_{n-1}, x_n) = 0$, $\forall a \in (0,1]$.

On the contrary, assume that

$$\lim_{n\to\infty} d_a(x_{n-1}, x_n) = r > 0, \text{ where } a \in (0,1]. \quad (45)$$

Let $t_n := m(a, x_{n-1}, x_n)$ and $s_n := d_a(x_{n-1}, x_n), a \in (0,1]$, $\forall n = 1, 2, 3, \cdots$.

Then

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} t_n = r > 0. \quad (46)$$

From condition (ζ_3) we have

$$0 \le \lim \sup_{n\to\infty} \zeta(t_n, s_n) < 0, \quad (47)$$

which is a contradiction. Hence we have $r = 0$, and hence

$$\lim_{n\to\infty} d_a(x_n, x_{n-1}) = 0, \ \forall a \in (0,1]. \quad (48)$$

We now show that $\{x_n\}$ is a Cauchy sequence.

Suppose that $\{x_n\}$ is not a Cauchy sequence.

By Lemma 8, there exist $\epsilon > 0$ and two subsequences $\{x_{n(k)}\}, \{x_{n(k)}\}$ of $\{x_n\}$ satisfying (17).

Since (49) holds, from Lemma 8 we have that

$$\begin{aligned} \lim_{n\to\infty} d_b(x_{m(k)}, x_{n(k)}) &= \varepsilon, \text{and} \lim_{n\to\infty} d_b(x_{m(k)+1}, x_{n(k)}) \\ &= \lim_{n\to\infty} d_b(x_{m(k)}, x_{n(k)+1}) \\ &= \lim_{n\to\infty} d_b(x_{m(k)+1}, x_{n(k)+1}) \\ &= \varepsilon \text{where} b \in (0,a] \end{aligned} \quad (49)$$

Let

$$\begin{aligned} t_k &= m(b, x_{m(k)}, x_{n(k)}) = d_b(x_{m(k)+1}, x_{n(k)+1}) \\ &\quad - \min\{d_b(x_{m(k)}, x_{n(k)+1}), d_b(x_{m(k)+1}, x_{n(k)}), \\ &\quad d_b(x_{m(k)}, x_{m(k)+1}), d_b(x_{n(k)}, x_{n(k)+1})\}, s_k = d_b(x_{m(k)}, x_{n(k)}). \end{aligned} \quad (50)$$

Then

$$\lim_{k\to\infty} t_k = \lim_{k\to\infty} s_k = \varepsilon > 0. \tag{51}$$

It follows from (ξ3) that

$$0 \le \limsup_{k\to\infty} \zeta(t_k, s_k) < 0, \tag{52}$$

which is a contradiction.

Thus $\{x_n = T^n x_0\}$ is a Cauchy sequence. It follows from T-orbitally completness of X that there exists $x_* \in X$ such that

$$\lim_{n\to\infty} d_a(x_n, x_*) = 0,\ \forall a \in (0,1). \tag{53}$$

From Lemma 9

$$\lim_{n\to\infty} d_a(x_n, Tx_*) = d_a(x_*, Tx_*),\ \forall a \in (0,1]. \tag{54}$$

Thus it follows from (40) that

$$0 \le \zeta(m(a, x_n, x_*), d_a(x_n, x_*)) < d_a(x_n, x_*) - m(a, x_n, x_*), \tag{55}$$

where

$$\begin{aligned} m(a, x_n, x_*) &= d_a(Tx_n, Tx_*) - \min\{d_a(x_n, Tx_*), \\ &\quad d_a(x_*, Tx_n), d_a(x_n, Tx_n), d_a(x_*, Tx_*)\} \\ &= d_a(x_{n+1}, Tx_*) - \min\{d_a(x_n, Tx_*), \\ &\quad d_a(x_*, x_{n+1}), d_a(x_n, x_{n+1}), d_a(x_*, Tx_*)\}. \end{aligned} \tag{56}$$

Hence

$$m(a, x_n, x_*) < d_a(x_n, x_*). \tag{57}$$

Letting $n \to \infty$ in above inequality, and using Equations (49), (54) and (55), we have

$$d_a(x_*, Tx_*) = 0\ \forall a \in (0,1]. \tag{58}$$

Thus $x_* = Tx_*$.

Example 11. Let $X = [0, \infty)$ with $d_a(x, y) = (1/a)|x - y|\ \forall a \in (0,1],\ \forall x, y \in X$.

Then $(X, \{d_a : a \in (0,1]\})$ is a generating space of quasi-metric family.

Define a mapping $T : X \to X$ as follows:

$$Tx = \frac{1}{8}x. \tag{59}$$

Let $\zeta(t, s) = \phi(s) - \psi(t)$, where

$$\psi(t) = 2t \text{ and } \phi(t) = \frac{1}{4}t\ \forall t \ge 0. \tag{60}$$

Then $\zeta \in \mathcal{Z}$.

We have that

$$\begin{aligned} &\zeta(m(a, x, y), d_a(x, y)) = \phi(d_a(x, y)) - \psi(m(a, x, y)) \\ &= \phi\left(\frac{1}{a}|x - y|\right) - \psi\left(\frac{1}{8}\frac{1}{a}|x - y| - \min\{d_a(x, Tx), \right. \\ &\quad \left. d_a(y, Ty), d_a(x, Ty), d_a(y, Tx)\}\right) = 2\min\{d_a(x, Tx), \\ &\quad d_a(y, Ty), d_a(x, Ty), d_a(y, Tx)\} \ge 0. \end{aligned} \tag{61}$$

Thus T is a Ćirić type (I) $\mathcal{Z}$-contraction with respect to ζ.

All conditions of Theorem 10 are satisfied. Hence, from Theorem 10 T has a fixed point, $x_* = 0$.

Corollary 12. *Let (X, d) be a metric space, and let $T : X \to X$ be a mapping such that*

$$\zeta(m(x, y), d(x, y)) \ge 0. \tag{62}$$

for all $x, y \in X$, where $m(x, y) = d(Tx, Ty)$

$$-\min\{d(x, Ty), d(y, Tx), d(x, Tx), d(y, Ty)\}. \tag{63}$$

If X is T-orbitally complete, then T has a fixed point.

2.2. Unique Fixed Point Results. Let $(X, \{d_a : a \in (0,1]\})$ be a generating space of quasi-metric family.

A mapping $T : X \to X$ is called Ćirić type (II) $\mathcal{Z}$-contraction with respect to $\zeta \in \mathcal{Z}$ if

$$\forall a \in (0,1],\ \forall x, y \in X, \zeta(d_a(Tx, Ty), n(a, x, y)) \ge 0, \tag{64}$$

where $n(a, x, y) = \max\{d_a(x, y), d_a(x, Tx), d_a(y, Ty), (1/2)[d_a(x, Ty) + d_a(y, Tx)]\}$ and $\zeta \in \mathcal{Z}$ is nondecreasing with respect to the second variable.

Theorem 13. *Let $(X, \{d_a : a \in (0,1]\})$ be a generating space of quasi-metric family, and let $T : X \to X$ be a Ćirić type (II) $\mathcal{Z}$-contraction with respect to $\zeta \in \mathcal{Z}$. If X is T-orbitally complete, then T has a unique fixed point.*

Proof. Firstly, we show the uniqueness of fixed point whenever it exists.

Suppose that T has two fixed points, say $u, v \in X$ such that $u \ne v$.

Then from (54) we have

$$\begin{aligned} 0 &\le \zeta(d_a(Tu, Tv), n(a, u, v)) = \zeta(d_a(u, v), d_a(u, v)) \\ &< d_a(u, v) - d_a(u, v) = 0, \end{aligned} \tag{65}$$

which is a contradiction. Thus T has a unique fixed point if it exists.

Secondly, we show the existence of fixed point.

As in proof of Theorem 10, let us define a sequence $\{x_n\} \subset X$ by $x_n = Tx_{n-1} = T^n x_0\ \forall n = 1, 2, 3\cdots$, where $x_0 \in X$ is any fixed point, such that

$$x_{n-1} \ne x_n\ \forall n = 1, 2, 3\cdots. \tag{66}$$

We infer that

$$\begin{aligned} (a, x_{n-1}, x_n) &= \max\left\{d_a(x_{n-1}, x_n), d_a(x_{n-1}, Tx_{n-1}), d_a(x_n, Tx_n), \right. \\ &\quad \left. \frac{1}{2}[d_a(x_{n-1}, Tx_n) + d_a(x_n, Tx_{n-1})]\right\} \\ &= \max\left\{d_a(x_{n-1}, x_n), d_a(x_{n-1}, x_n), d_a(x_n, x_{n+1}), \right. \\ &\quad \left. \frac{1}{2}[d_a(x_{n-1}, x_{n+1}) + d_a(x_n, x_n)]\right\} \\ &\le \max\left\{d_a(x_{n-1}, x_n), d_a(x_{n-1}, x_n), d_a(x_n, x_{n+1}), \right. \\ &\quad \left. \frac{1}{2}[d_b(x_{n-1}, x_n) + d_b(x_n, x_{n+1})]\right\} \\ &\le \max\left\{d_b(x_{n-1}, x_n), d_b(x_{n-1}, x_n), d_b(x_n, x_{n+1}), \right. \\ &\quad \left. \frac{1}{2}[d_b(x_{n-1}, x_n) + d_b(x_n, x_{n+1})]\right\} \\ &= \max\left\{d_b(x_{n-1}, x_n), d_b(x_n, x_{n+1})\right\} \text{where} b \in (0, a]. \end{aligned} \tag{67}$$

Hence

$$n(b, x_{n-1}, x_n) \leq \max\{d_b(x_{n-1}, x_n), d_b(x_n, x_{n+1})\}. \quad (68)$$

It follows from 65 that

$$\begin{aligned} 0 &\leq \zeta(d_b(Tx_{n-1}, Tx_n), n(b, x_n, x_{n+1})) \\ &= \zeta(d_b(x_n, x_{n+1}), n(b, x_n, x_{n+1})) \\ &< n(b, x_n, x_{n+1}) - d_b(x_n, x_{n+1}) \\ &\leq \max\{d_b(x_{n-1}, x_n), d_b(x_n, x_{n+1})\} \\ &\quad - d_b(x_n, x_{n+1}), \end{aligned} \quad (69)$$

which implies

$$d_b(x_n, x_{n+1}) < d_b(x_{n-1}, x_n) \forall n = 1, 2, 3, \cdots. \quad (70)$$

Thus there exists $r \geq 0$ such that

$$\lim_{n\to\infty} d_b(x_{n-1}, x_n) = r. \quad (71)$$

We show that $r = 0$.

Assume that $r \neq 0$.

Let

$$\begin{aligned} s_n &= \max\{d_b(x_{n-1}, x_n), d_b(x_n, x_{n+1})\} \text{ and} \\ t_n &= d_b(x_n, x_{n+1}) \ \forall n = 1, 2, 3, \cdots. \end{aligned} \quad (72)$$

Then $\lim_{n\to\infty} s_n = \lim_{n\to\infty} t_n = r > 0$.

It follows from (54 and 55) that

$$\begin{aligned} 0 &\leq \limsup \zeta(d_b(Tx_{n-1}, Tx_n), n(b, x_{n-1}, x_n)) \\ &\leq \limsup \zeta(t_n, s_n) < 0, \end{aligned} \quad (73)$$

which is a contradiction. Hence $r = 0$, and hence

$$\lim_{n\to\infty} d_b(x_{n-1}, x_n) = 0, \ \forall b \in (0, a]. \quad (74)$$

We now show that $\{x_n\}$ is a Cauchy sequence.

Suppose that $\{x_n\}$ is not a Cauchy sequence.

By Lemma 1.1, there exist $\epsilon > 0$ and two subsequences $\{x_{n(k)}\}, \{x_{n(k)}\}$ of $\{x_n\}$ satisfying (17).

Since (73) holds, it follows from Lemma 1.1 that

$$\begin{aligned} \lim_{n\to\infty} d_c(x_{m(k)}, x_{n(k)}) &= \varepsilon, \text{and} \lim_{n\to\infty} d_c(x_{m(k)+1}, x_{n(k)}) \\ &= \lim_{n\to\infty} d_c(x_{m(k)}, x_{n(k)+1}) \\ &= \lim_{n\to\infty} d_c(x_{m(k)+1}, x_{n(k)+1}) \\ &= \varepsilon \text{ for some } c \in (0, b). \end{aligned} \quad (75)$$

Let $s_k = n(c, x_{m(k)}, x_{n(k)})$ and $t_n = d_c(x_{m(k)+1}, x_{n(k)+1})$.

Then

$$\lim_{k\to\infty} n(c, x_{m(k)}, x_{n(k)}) = \lim_{k\to\infty} d_c(x_{m(k)}, x_{n(k)}) = \epsilon. \quad (76)$$

Thus from ($\zeta 3$) we have

$$0 \leq \limsup \zeta(t_k, s_k) < 0, \quad (77)$$

which is a contradiction. Thus $\{x_n\}$ is a Cauchy sequence.

Hence there exists $x_* \in X$ such that

$$\lim_{n\to\infty} d_a(x_n, x_*) = 0 \ \forall a \in (0, 1]. \quad (78)$$

Let $s_n = n(a, x_n, x_*)$ and $t_n = d_a(x_n, Tx_*), a \in (0, 1]$, $\forall n = 1, 2, 3, \cdots$.

If $d_a(Tx_*, x_*) > 0$, where $a \in (0, 1]$, then from Lemma 9 we have

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} t_n = d_a(x_*, Tx_*) > 0. \quad (79)$$

Hence

$$0 \leq \limsup \zeta(t_n, s_n) < 0, \quad (80)$$

which is a contradiction.

Thus

$$d_a(Tx_*, x_*) = 0 \ \forall a \in (0, 1]. \quad (81)$$

Hence $x_* = Tx_*$.

Corollary 14. *Let $(X, \{d_a : a \in (0, 1]\})$ be a generating space of quasi-metric family, and let $T : X \to X$ be a mapping such that*

$$\forall a \in (0, 1), \ \forall x, y \in X, \zeta(d_a(Tx, Ty), l(a, x, y)) \geq 0, \quad (82)$$

where $l(a, x, y) = \max\{d_a(x, y), (1/2)[d_a(x, Tx) + d_a(y, Ty)], (1/2)[d_a(x, Ty) + d_a(y, Tx)]\}$ and $\zeta \in \mathcal{Z}$ is nondecreasing with respect to the second variable.

If X is T-orbitally complete, then T has a unique fixed point.

Corollary 15. *Let $(X, \{d_a : a \in (0, 1]\})$ be a generating space of quasi-metric family, and let $T : X \to X$ be a mapping such that*

$$\forall a \in (0, 1], \ \forall x, y \in X, \zeta(d_a(Tx, Ty), d_a(x, y)) \geq 0, \quad (83)$$

where $\zeta \in \mathcal{Z}$ is nondecreasing with respect to the second variable.

If X is T-orbitally complete, then T has a unique fixed point.

By taking $\zeta = \zeta_6$ in Corollary 2.5, we have the following result.

Corollary 16. (Banach) *Let $(X, \{d_a : a \in (0, 1]\})$ be a generating space of quasi-metric family, and let $T : X \to X$ be a mapping such that*

$$\forall a \in (0, 1], \ \forall x, y \in X, d_a(Tx, Ty) \leq k d_a(x, y), \text{ where } k \in (0, 1]. \quad (84)$$

If X is T-orbitally complete, then T has a unique fixed point.

Corollary 17. *Let (X, d) be a metric space, and let $T : X \to X$ be a mapping such that*

$$\forall x, y \in X, \zeta(d(Tx, Ty), n(x, y)) \geq 0, \quad (85)$$

where $n(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), (1/2)[d(x, Ty) + d(y, Tx)]\}$ and $\zeta \in \mathcal{Z}$ is nondecreasing with respect to the second variable.

If X is T-orbitally complete, then T has a unique fixed point.

Example 18. Let $X = \{2/n : n = 2, 3, 4, \cdots\} \cup \{0\}$, and let

$$d_a(x, y) = \frac{1}{a}|x - y| \ \forall a \in (0, 1]. \quad (86)$$

Let $T : X \to X$ be a map defined as

$$Tx = \begin{cases} \dfrac{2}{n+1}, & \left(x = \dfrac{2}{n}, n = 2, 3, 4, \cdots\right), \\ x = 0, & 0. \end{cases} \tag{87}$$

Obviously, $(X, \{d_a : a \in (0, 1]\})$ is complete, and hence it is T-orbitally complete.

We now show that T is a Ćirić type (II) $\mathcal{Z}$-contraction with respect to $\zeta_7(t, s) = s - \int_0^t \nu(u)du$ with $\nu(u) = 1 + 2u$, $\forall u \geq 0$.

Case 1: Let $x = 2/n$, $y = 0$.

Then we have

$$\begin{aligned}
&\zeta_7\left(d_a\left(T\frac{2}{n}, T0\right), n\left(a, \frac{2}{n}, 0\right)\right) \\
&= \zeta_7\left(\frac{1}{a}\cdot\frac{2}{n+1}, \frac{1}{a}\cdot\frac{2}{n}\right) \\
&= \frac{1}{a}\cdot\frac{2}{n} - \int_0^{1/a\cdot 2/n+1}(1+2u)du \\
0 &= \frac{1}{a}\cdot\frac{2}{n} - \left(\frac{1}{a}\cdot\frac{2}{n+1} + \left(\frac{1}{a}\cdot\frac{2}{n+1}\right)^2\right) \\
&= \frac{1}{a}\cdot\frac{4}{n(n+1)^3} > 0.
\end{aligned} \tag{88}$$

Case 2: Let $x = 2/n$, $y = 2/n(m < n)$.

$$\begin{aligned}
\zeta_7\left(d_a\left(T\frac{2}{n}, T\frac{2}{m}\right), n\left(a, \frac{2}{n}, \frac{2}{m}\right)\right) &= \zeta_7\left(\frac{1}{a}\cdot\left(\frac{2}{n+1} - \frac{2}{m+1}\right), \frac{1}{a}\cdot\left(\frac{2}{n} - \frac{2}{m}\right)\right) \\
&= \frac{1}{a}\cdot\left(\frac{2}{n} - \frac{2}{m}\right) - \int_0^{1/a\cdot(2/n+1-2/m+1)}(1+2u)du \\
&= \frac{1}{a}\cdot\left(\frac{2}{n} - \frac{2}{m}\right) - \left(\frac{1}{a}\cdot\left(\frac{2}{n+1} - \frac{2}{m+1}\right) + \left(\frac{1}{a}\cdot\left(\frac{2}{n+1} - \frac{2}{m+1}\right)\right)^2\right) \\
&= \frac{1}{a}\cdot\frac{2(m-n)(n+1)(m+1)\left((n+1)(m+1) - mn + 4(m-n)^2\right)}{mn(n+1)^2(m+1)^2} \\
&= \frac{1}{a}\cdot\frac{2(m-n)\left((n+1)(m+1) - mn + 4(m-n)^2\right)}{mn(n+1)(m+1)} > 0.
\end{aligned} \tag{89}$$

Thus T is a Ćrić type (II) $\mathcal{Z}$-contraction with respect to ζ_6. All conditions of Theorem 13 are satisfied, and T has a unique fixed point theorem $x_* = 0$.

However, Banach contraction principle in the setting of generating spaces of quasi-metric family, i.e. Corollary 16 is not satisfied.

In fact, if for $x = 2/n$, $y = 2/(n+1)$ $\forall n = 2, 3, 4, \cdots$

$$\forall a \in (0,1], d_a(Tx, Ty) \leq kd_a(x, y), \text{where } k \in (0, 1], \tag{90}$$

then we have that

$$\frac{1}{a}\left|\frac{2}{n+1} - \frac{2}{n+2}\right| \leq k\frac{1}{a}\left|\frac{2}{n} - \frac{2}{n+1}\right|, \tag{91}$$

which implies

$$k \geq \frac{n(n+1)}{(n+1)(n+2)}\ \forall n = 2, 3, 4, \cdots. \tag{92}$$

By taking $n \to \infty$, we have $k \geq 1$ which is a contradiction.

Thus Theorem 13 is a generalization of the Banach contraction principle in generating space of quasi-metric family.

3. Consequence

By applying simulation functions of Example 3 to Theorem 10 and Theorem 13, we have some fixed point results.

Especially, taking ζ_8 for ζ in Theorem 10 and Theorem 13, we have the following results.

Corollary 19. *Let $(X, \{d_a : a \in (0, 1]\})$ be a generating space of quasi-metric family, and let $T : X \to X$ be a mapping such that*

$$\forall a \in (0,1],\ \forall x, y \in X, m(a, x, y) \leq \frac{d_a(x, y)}{1 + d_a(x, y)}. \tag{93}$$

If X is T-orbitally complete, then T has a fixed point.

Corollary 20. *Let (X, d)be a metric space, and let $T : X \to X$ be a mapping such that*

$$\forall x, y \in X, m(x, y) \leq \frac{d(x, y)}{1 + d(x, y)}. \tag{94}$$

If X is T-orbitally complete, then T has a fixed point.

Corollary 21. *Let $(X, \{d_a : a \in (0, 1]\})$ be a gessnerating space of quasi-metric family, and let $T : X \to X$ be a mapping such that*

$$\forall a \in (0,1],\ \forall x, y \in X, d_a(Tx, Ty) \leq \frac{n(a, x, y)}{1 + n(a, x, y)}. \tag{95}$$

If X is T-orbitally complete, then T has a unique fixed point.

Corollary 22. *Let (X, d) be a metric space, and let $T : X \to X$ be a mapping such that*

$$\forall x, y \in X, d(Tx, Ty) \leq \frac{n(x, y)}{1 + n(x, y)}. \tag{96}$$

If X is T-orbitally complete, then T has a unique fixed point.

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares that there are no conficts of interest regarding the publication of this paper.

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