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Research Article

Multiple Solutions for a Class of Fractional Schrödinger-Poisson System

Lizhen Chen,[1] Anran Li,[2] and Chongqing Wei[2]

[1] *School of Applied Mathematics, Shanxi University of Finance and Economics, Taiyuan 030006, China*
[2] *School of Mathematical Sciences, Shanxi University, Taiyuan 030006, China*

Correspondence should be addressed to Anran Li; lianran@sxu.edu.cn

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We investigate a class of fractional Schrödinger-Poisson system via variational methods. By using symmetric mountain pass theorem, we prove the existence of multiple solutions. Moreover, by using dual fountain theorem, we prove the above system has a sequence of negative energy solutions, and the corresponding energy values tend to 0. These results extend some known results in previous papers.

1. Introduction

We consider the following system via variational methods:

$$\begin{aligned} &(-\Delta)^{\alpha} w + V(x) w + \phi w \\ &\quad = g(x, w) + \lambda h(x) |w|^{q-2} w, \quad x \in R^3, \\ &(-\Delta)^{\beta} \phi = w^2, \quad x \in R^3, \end{aligned} \tag{SP}$$

where $\lambda > 0$, $1 < q < 2$, $\alpha, \beta \in (0, 1]$, $2\beta + 4\alpha > 3$. $(-\Delta)^{\alpha}$ and $(-\Delta)^{\beta}$ represent the Laplace operator of the fractional order. If $\alpha = \beta = 1$, then the system (SP) degenerates into the standard Schrödinger-Poisson system, which describes the interaction between the same charged particles when the magnetic effect can be ignored [1]. In recent years, the existence, multiplicity, and centralization of solutions for the Schrödinger-Poisson system have been deeply studied via variational methods, and a great number of works have been obtained, see, for example, [2–8]. On the other hand, $(-\Delta)^{\alpha}$ is a class of nonlocal pseudo-differential operators. Since nonlocal differential equations can better and more fully describe the physical experimental phenomena than classical local differential operators, the study of nonlinear fractional Laplace equation has become one of the most popular research fields in nonlinear analysis.

In the literature [9], Wei considered the following system:

$$\begin{aligned} (-\Delta)^{\alpha} w + V(x) w + \phi w = g(x, w), \quad x \in \mathbb{R}^3, \\ (-\Delta)^{\alpha} \phi = \gamma_{\alpha} w^2, \quad x \in \mathbb{R}^3. \end{aligned} \tag{1}$$

By using the critical point theory, the author obtained infinitely many solutions when $\alpha \in (0, 1]$. In the literature [10], Teng studied a system of the form

$$\begin{aligned} &(-\Delta)^{\alpha} w + V(x) w + \phi w = \theta |w|^{q-1} w + |w|^{2_{\alpha}^{*}-2} w, \\ &\qquad x \in \mathbb{R}^3, \\ &(-\Delta)^{\beta} \phi = w^2, \quad x \in \mathbb{R}^3, \end{aligned} \tag{2}$$

where $q \in (1, (3 + 2\alpha)/(3 - 2\alpha))$, $\alpha, \beta \in (0, 1)$, $2\beta + 2\alpha > 3$. In [11], Zhang, Marcos, and Squassina used perturbation approach to obtain the existence of solutions for the following system when the nonlinear term is subcritical or critical

$$\begin{aligned} (-\Delta)^{\alpha} w + \lambda \phi w = g(w), \quad x \in \mathbb{R}^3, \\ (-\Delta)^{\beta} \phi = \lambda w^2, \quad x \in \mathbb{R}^3, \end{aligned} \tag{3}$$

where $\lambda > 0$, $\alpha, \beta \in [0,1]$. In [12], Duarte and Souto investigated the following system via variational methods

$$\begin{aligned}(-\Delta)^{\alpha} w + V(x) w + \phi w &= g(w), \quad x \in \mathbb{R}^3, \\ (-\Delta)^{\beta} \phi &= w^2, \quad x \in \mathbb{R}^3,\end{aligned} \tag{4}$$

where $\alpha \in (3/4, 1)$, $\beta \in (0,1)$, $V : \mathbb{R}^3 \longrightarrow \mathbb{R}$ is a periodic potential. A positive solution and a ground state solution were got in [12]. In [13], Li studied a system of the following form:

$$\begin{aligned}(-\Delta)^{\alpha} w + V(x) w + \phi w &= g(x, w), \quad x \in \mathbb{R}^3, \\ (-\Delta)^{\beta} \phi &= w^2, \quad x \in \mathbb{R}^3,\end{aligned} \tag{5}$$

where $\alpha, \beta \in (0,1]$, $2\beta + 4\alpha > 3$. Combining the perturbation method with mountain pass theorem, the existence of nontrivial solutions was obtained in [13]. In [14], Yu, Zhao, and Zhao studied the following fractional Schrödinger–Poisson system with critical growth via variational methods

$$\begin{aligned}\varepsilon^{2\alpha} (-\Delta)^{\alpha} w + V(x) w + \phi w &= |w|^{2_\alpha^* - 2} w + g(w), \\ & \qquad x \in \mathbb{R}^3, \\ \varepsilon^{2\alpha} (-\Delta)^{\alpha} \phi &= w^2, \quad x \in \mathbb{R}^3,\end{aligned} \tag{6}$$

where $\alpha \in (3/4, 1)$, $2_\alpha^* = 6/(3 - 2\alpha)$, the potential V is continuous with positive infimum, g is continuous and subcritical at infinity. Under some Monotone hypothesis on g, the existence of positive ground state solution is got in [14]. For small $\varepsilon > 0$, a multiple result is also got in [14].

Inspired by [9–16], in this paper, we prove the existence of multiple solutions for system (SP) by symmetric mountain pass theorem. Moreover, we prove the system (SP) has a sequence of negative energy solutions by dual fountain theorem. The assumptions on V and nonlinearity g in this paper are given below:

(V) $V \in C(\mathbb{R}^3, \mathbb{R})$, $V_0 := \inf_{x \in \mathbb{R}^3} V(x) > 0$ and $\lim_{|x| \longrightarrow +\infty} V(x) = +\infty$;

(H1) $g \in C(\mathbb{R}^3 \times \mathbb{R}, \mathbb{R})$, and there exists $C_1 > 0$ such that $|g(x, w)| \le C_1(|w| + |w|^{p-1})$, where $p \in (4, 2_\alpha^*)$, $2_\alpha^* = 6/(3 - 2\alpha)$;

(H2) there exist $\kappa > 4$ and $r > 0$ such that $0 < \kappa G(x, w) := \kappa \int_0^w g(x, t) dt \le g(x, w) w$, for $|w| \ge r$. Moreover, $\inf_{x \in \mathbb{R}^3, |w| = r} G(x, w) > 0$;

(H3) $g(x, -w) = -g(x, w)$, $x \in \mathbb{R}^3$, $w \in \mathbb{R}$;

(H4) $h : \mathbb{R}^3 \longrightarrow \mathbb{R}^+$, and $h \in L^{2/(2-q)}(\mathbb{R}^3)$.

2. Preliminaries

For $1 \le \nu < \infty$, $L^{\nu}(\mathbb{R}^3)$ denotes the usual Lebesgue space with norm $\|w\|_{\nu} = (\int_{\mathbb{R}^3} |w|^{\nu} dx)^{1/\nu}$. Fix $\alpha \in (0,1)$, fractional Sobolev space $H^{\alpha}(\mathbb{R}^3)$ denoted as

$$\begin{aligned}& H^{\alpha}(\mathbb{R}^3) \\ & := \left\{ w \in L^2(\mathbb{R}^3) : \int_{\mathbb{R}^3} \left(1 + |l|^{2\alpha}\right) |\mathscr{F} w(l)|^2 dl < \infty \right\},\end{aligned} \tag{7}$$

equipped with the norm

$$\|w\|_{H^{\alpha}} = \left(\int_{\mathbb{R}^3} \left(|\mathscr{F} w(l)|^2 + |l|^{2\alpha} |\mathscr{F} w(l)|^2 \right) dl \right)^{1/2}, \tag{8}$$

where $\mathscr{F} w$ denotes the Fourier transform of function w. Let $g \in C_0^{\infty}(\mathbb{R}^3)$; the fractional Laplacian operator $(-\Delta)^{\alpha} : C_0^{\infty}(\mathbb{R}^3) \longrightarrow (C_0^{\infty}(\mathbb{R}^3))'$ is defined by

$$(-\Delta)^{\alpha} g = \mathscr{F}^{-1} \left(|l|^{2\alpha} (\mathscr{F} g) \right), \quad l \in \mathbb{R}^3. \tag{9}$$

According to Plancherel theorem [17], one has $\|\mathscr{F} w\|_2 = \|w\|_2$, $\||l|^{\alpha} \mathscr{F} w\|_2 = \|(-\Delta)^{\alpha/2} w\|_2$. By (8), we define the equivalent norm

$$\|w\|_{H^{\alpha}} = \left(\int_{\mathbb{R}^3} \left(\left| (-\Delta)^{\alpha/2} w(x) \right|^2 + |w(x)|^2 \right) dx \right)^{1/2}. \tag{10}$$

$D^{\alpha,2}(\mathbb{R}^3)$ is denoted as

$$D^{\alpha,2}(\mathbb{R}^3) = \left\{ w \in L^{2_\alpha^*}(\mathbb{R}^3) : |l|^{\alpha} \mathscr{F} w(l) \in L^2(\mathbb{R}^3) \right\}. \tag{11}$$

In particular, $D^{\alpha,2}(\mathbb{R}^3)$ is the completion of $C_0^{\infty}(\mathbb{R}^3)$, with respect to the norm

$$\begin{aligned}\|w\|_{D^{\alpha,2}} &= \left(\int_{\mathbb{R}^3} \left| (-\Delta)^{\alpha/2} w \right|^2 dx \right)^{1/2} \\ &= \left(\int_{\mathbb{R}^3} |l|^{2\alpha} |\mathscr{F} w(l)|^2 dl \right)^{1/2}.\end{aligned} \tag{12}$$

$L^q(\mathbb{R}^3, h)$, $q \in (1,2)$ represents a weighted Lebesgue space, that is,

$$\begin{aligned}L^q(\mathbb{R}^3, h) &= \Big\{ w : \mathbb{R}^3 \\ &\longrightarrow \mathbb{R} \text{ is measurable and } \int_{\mathbb{R}^3} h(x) |w|^q dx \\ &< +\infty \Big\},\end{aligned} \tag{13}$$

equipped with the norm

$$\|w\|_{L^q(\mathbb{R}^3, h)} = \left(\int_{\mathbb{R}^3} h(x) |w|^q dx \right)^{1/q}. \tag{14}$$

For convenience, we use C to represent any positive constants which may change from line to line. According to [18], the embedding $H^{\alpha}(\mathbb{R}^3) \hookrightarrow L^{\nu}(\mathbb{R}^3)$ is continuous for all $\nu \in [2, 2_\alpha^*]$, i.e., there exists $M_{\nu} > 0$ satisfying

$$\|w\|_{\nu} \le M_{\nu} \|w\|_{H^{\alpha}}, \quad w \in H^{\alpha}(\mathbb{R}^3). \tag{15}$$

So, the embedding $H^{\alpha}(\mathbb{R}^3) \hookrightarrow L^{\nu}(\mathbb{R}^3)$ is continuous when $2\beta + 4\alpha > 3$. Fix $w \in H^{\alpha}(\mathbb{R}^3)$; we define the nonlinear operator $L_w : D^{\beta,2}(\mathbb{R}^3) \longrightarrow \mathbb{R}$ by

$$L_w(\nu) = \int_{\mathbb{R}^3} w^2 \nu dx. \tag{16}$$

Hence

$$\begin{aligned}\left|L_w(v)\right| &\le \left(\int_{\mathbb{R}^3} |w(x)|^{12/(3+2\beta)}\,dx\right)^{(3+2\beta)/6} \\ &\cdot \left(\int_{\mathbb{R}^3} |v(x)|^{2_\beta^*}\,dx\right)^{1/2_\beta^*} \\ &\le C\left(\int_{\mathbb{R}^3} |w(x)|^{12/(3+2\beta)}\,dx\right)^{(3+2\beta)/6} \|v\|_{D^{\beta,2}} \\ &\le C\|w\|^2 \|v\|_{D^{\beta,2}}.\end{aligned} \tag{17}$$

By Lax-Milgram theorem, we can find a unique $\phi_w^\beta \in D^{\beta,2}(\mathbb{R}^3)$ such that

$$\int_{\mathbb{R}^3} (-\Delta)^{\beta/2}\phi_w^\beta (-\Delta)^{\beta/2} v\,dx = \int_{\mathbb{R}^3} w^2 v\,dx, \quad \forall v \in D^{\beta,2}\left(\mathbb{R}^3\right), \tag{18}$$

and ϕ_w^β is expressed as

$$\phi_w^\beta(x) = c_\beta \int_{\mathbb{R}^3} \frac{w^2(y)}{|x-y|^{3-2\beta}}\,dy, \quad \text{where } c_\beta = \frac{\Gamma(3/2-2\beta)}{\pi^{3/2}2^{2\beta}\Gamma(\beta)}. \tag{19}$$

According to (19), $\phi_w^\beta \ge 0$ for all $x \in \mathbb{R}^3$. Since $\beta \in (0,1]$, $2\beta + 4\alpha > 3$, we can also get $12/(3+2\beta) \in (2, 2_\alpha^*)$. Together with (17) and (18),

$$\begin{aligned}\left\|\phi_w^\beta\right\|_{D^{\beta,2}} &= \int_{\mathbb{R}^3} \left|(-\Delta)^{\beta/2}\phi_w^\beta\right|^2 dx \\ &\le C\left(\int_{\mathbb{R}^3} |w(x)|^{12/(3+2\beta)}\,dx\right)^{(3+2\beta)/6} \\ &\le C\|w\|^2.\end{aligned} \tag{20}$$

From Hölder's inequality and (19),

$$\begin{aligned}&\int_{\mathbb{R}^3} \phi_w^\beta w^2\,dx \\ &\le C\left(\int_{\mathbb{R}^3} |w(x)|^{(12/(3+2\alpha))dx}\right)^{(3+2\alpha)/6} \left\|\phi_w^\beta\right\|_{D^{\beta,2}} \\ &\le C\|w\|^2 \left\|\phi_w^\beta\right\|_{D^{\beta,2}} \le C\|w\|^4.\end{aligned} \tag{21}$$

Evidently,

$$\int_{\mathbb{R}^3} \phi_w^\beta w^2\,dx \le C\|w\|^4. \tag{22}$$

Substituting (18) into system (SP), system (SP) is equivalent to

$$\begin{aligned}&(-\Delta)^\alpha w + V(x)w + \phi_w^\beta w \\ &= g(x,w) + \lambda h(x)|w|^{q-2}w.\end{aligned} \tag{S}$$

For the equation (S), we define the work space E as

$$E := \left\{w \in H^\alpha\left(\mathbb{R}^3\right) : \int_{\mathbb{R}^3} V(x)w^2\,dx < \infty\right\}, \tag{23}$$

with

$$\|w\| = \left(\int_{\mathbb{R}^3} \left(\left|(-\Delta)^{\alpha/2}w\right|^2 + V(x)w^2\right)dx\right)^{1/2}. \tag{24}$$

Lemma 1 (see [19]). *Assume condition (V) holds, $\alpha \in (0,1)$, $\nu \in [2, 2_\alpha^*)$; then the embedding $E \hookrightarrow L^\nu(\mathbb{R}^3)$ is compact.*

From (V) and (H1)-(H4), $I : E \longrightarrow \mathbb{R}$

$$\begin{aligned}I(w) &= \frac{1}{2}\int_{\mathbb{R}^3} \left(\left|(-\Delta)^{\alpha/2}w\right|^2 + V(x)w^2\right)dx \\ &+ \frac{1}{4}\int_{\mathbb{R}^3} \phi_w^\beta w^2\,dx - \int_{\mathbb{R}^3} G(x,w)\,dx \\ &- \frac{\lambda}{q}\int_{\mathbb{R}^3} h(x)|w|^q\,dx.\end{aligned} \tag{25}$$

is well defined. Moreover, by Lemma 1, $I \in C^1(E,\mathbb{R})$ with

$$\begin{aligned}&\left\langle I'(w), v\right\rangle \\ &= \int_{\mathbb{R}^3} \left((-\Delta)^{\alpha/2}w(-\Delta)^{\alpha/2}v + V(x)wv\right)dx \\ &+ \int_{\mathbb{R}^3} \phi_w^\beta wv\,dx - \int_{\mathbb{R}^3} g(x,w)v\,dx \\ &- \lambda\int_{\mathbb{R}^3} h(x)|w|^{q-2}wv\,dx, \quad v \in E\end{aligned} \tag{26}$$

Proposition 2. *(i) If equation* (S) *has a solution $w \in E$, then system* (SP) *has a solution $(w,\phi) \in E \times D^{\beta,2}(\mathbb{R}^3)$.*

(ii) If for every $v \in E$, the following equation

$$\begin{aligned}&\int_{\mathbb{R}^3} \left((-\Delta)^{\alpha/2}w(-\Delta)^{\alpha/2}v + V(x)wv\right)dx \\ &+ \int_{\mathbb{R}^3} \phi_w^\beta wv\,dx - \int_{\mathbb{R}^3} g(x,w)v\,dx \\ &- \lambda\int_{\mathbb{R}^3} h(x)|w|^{q-2}wv\,dx = 0\end{aligned} \tag{27}$$

holds, then $w \in E$ is a solution of (S).

Set $\{e_i\}_{i=1}^\infty$ as a set of normalized orthogonal basis of E, $X_i = \mathbb{R}e_i$. $Y_k = \oplus_{i=1}^k X_i$, $Z_k = \overline{\oplus_{i=k+1}^\infty X_i}$, $k \in \mathbb{N}$. Obviously, $E = Y_k \oplus Z_k$.

Definition 3 (see [20, 21]). Set $I \in C^1(E,\mathbb{R})$, $c \in \mathbb{R}$. If any sequence $\{w_n\} \subset E$ satisfying

$$\begin{aligned}I(w_n) &\longrightarrow c, \\ I'(w_n) &\longrightarrow 0, \\ n &\longrightarrow \infty,\end{aligned} \tag{28}$$

has a convergent subsequence, then I satisfies the $(PS)_c$ condition. Any sequence satisfying (28) is called the $(PS)_c$ sequence.

Definition 4 (see [21, 22]). Set $I \in C^1(E, \mathbb{R})$, $c \in \mathbb{R}$. If every sequence $\{w_{n_j}\} \subset E$, satisfying

$$\begin{gathered} w_{n_j} \in Y_{n_j}, \\ I\left(w_{n_j}\right) \longrightarrow c, \\ I|'_{Y_{n_j}} \longrightarrow 0, \\ n \longrightarrow \infty, \end{gathered} \tag{29}$$

has a convergent subsequence, then I satisfies the $(PS)_c^*$ condition.

Proposition 5 (see [23]). *Let $E = Y \oplus Z$ be a Banach space with $\dim Y < \infty$. Assume $I \in C^1(E, \mathbb{R})$ is an even functional and satisfies the $(PS)_c$ condition and*

(A1) there exist ω, $\mu > 0$ satisfying $I_{\partial B_\omega \cap Z} = \inf_{w \in Z, \|w\| = \omega} I(w) \geq \mu$;

(A2) for every linear subspace $U \subset E$ with $\dim U < \infty$, there exists a constant $L = L(U)$ such that $\max_{w \in U, \|w\| \geq L} I(w) < 0$,

then I has a list of unbounded critical points.

Proposition 6 (see [22]). *Assume that $I \in C^1(E, \mathbb{R})$ is an even functional, $k_0 \in \mathbb{N}$. If for any $k > k_0$, there exist $r_k > \gamma_k > 0$ such that*

(C1) $b_k = \inf\{I(w) : w \in Z_k, \|w\| = r_k\} \geq 0$;

(C2) $a_k = \max\{I(w) : w \in Y_k, \|w\| = \gamma_k\} < 0$;

(C3) $c_k = \inf\{I(w) : w \in Z_k, \|w\| \leq r_k\} \longrightarrow 0$, $k \longrightarrow \infty$;

(C4) *for every $c \in [c_{k_0}, 0]$, I satisfies the $(PS)_c^*$ condition,*

then I has a sequence of negative critical points that converge to 0.

3. Main Results

Lemma 7. *If hypotheses (V) and (H1)-(H4) hold, then for any $c \in \mathbb{R}$, I satisfies the $(PS)_c$ condition.*

Proof. First, we prove the $(PS)_c$ sequence $\{w_n\}$ of I is bounded. According to (H4), it is easy to get that

$$\begin{aligned} &\int_{\mathbb{R}^3} h(x) |w_n|^q dx \\ &\quad \leq \left(\int_{\mathbb{R}^3} |h(x)|^{2/(2-q)} dx\right)^{(2-q)/2} \left(\int_{\mathbb{R}^3} |w_n|^2 dx\right)^{q/2} \\ &\quad \leq \|h\|_{2/(2-q)} \|w\|_2^q \leq M_2^q \|h\|_{2/(2-q)} \|w_n\|^q. \end{aligned} \tag{30}$$

By condition (H2), there exists $r > 0$ such that

$$g(x, w) w \geq \kappa G(x, w), \quad |w| \geq r. \tag{31}$$

Moreover, for any given $C_0 \in (0, (1/16)V_0)$, we can choose a constant $\delta > 0$ such that

$$\left|\frac{1}{\kappa} g(x, w) w - G(x, w)\right| \leq C_0 w^2, \quad \text{for } |w| \leq \delta. \tag{32}$$

From condition (H1), when $\delta \leq |u| \leq r$, one has

$$\left|\frac{1}{\kappa} g(x, w) w - G(x, w)\right| \leq C\left(\frac{1}{\delta} + r^{p-2}\right) w^2. \tag{33}$$

So, for any $|w| \leq r$,

$$\left|\frac{1}{\kappa} g(x, w) w - G(x, w)\right| \leq C_0 w^2 + C\left(\frac{1}{\delta} + r^{p-2}\right) w^2. \tag{34}$$

For $\lim_{|x| \longrightarrow +\infty} V(x) = +\infty$, there exists $L > r > 0$ such that

$$\frac{1}{16} V(x) > C\left(\frac{1}{\delta} + r^{p-2}\right), \quad |x| \geq L. \tag{35}$$

Now (34) implies

$$\begin{aligned} &\frac{1}{4} \int_{R^3} V(x) w_n^2 dx \\ &\quad + \int_{|w_n(x)| \leq r} \left[\frac{1}{\kappa} g(x, w_n) w_n - G(x, w_n)\right] dx \\ &\quad \geq \frac{1}{4} \int_{\mathbb{R}^3} V(x) w_n^2 dx \\ &\quad - \int_{|w_n(x)| \leq r} \left[C_0 |w_n|^2 + C\left(\frac{1}{\delta} + r^{p-2}\right) |w_n|^2\right] dx \\ &\quad = \frac{1}{8} \int_{\mathbb{R}^3} V(x) w_n^2 dx - \int_{|w_n(x)| \leq r} C_0 |w_n|^2 dx \\ &\quad + \frac{1}{8} \int_{\mathbb{R}^3} V(x) w_n^2 dx \\ &\quad - \int_{|w_n(x)| \leq r} C\left(\frac{1}{\delta} + r^{p-2}\right) |w_n|^2 dx \\ &\quad \geq \int_{|w_n(x)| \leq r} \left[\frac{1}{16} V_0 - C_0\right] w_n^2 dx \\ &\quad + \frac{1}{8} \int_{\mathbb{R}^3} V(x) w_n^2 dx - C\left(\frac{1}{\delta} + r^{p-2}\right) r^2 \\ &\quad \cdot meas\left\{x \in \mathbb{R}^3 \mid |x| \leq L\right\} \geq \frac{1}{8} \int_{\mathbb{R}^3} V(x) w_n^2 dx \\ &\quad - C\left(\frac{1}{\delta} + r^{p-2}\right) r^2 \cdot meas\left\{x \in \mathbb{R}^3 \mid |x| \leq L\right\}. \end{aligned} \tag{36}$$

Since $\{w_n\}$ is the $(PS)_c$ sequence, when n is large enough,

$$\begin{aligned}
c+\|w_n\| &\geq I(w_n)-\frac{1}{\kappa}\left\langle I'(w_n),w_n\right\rangle \\
&=\left(\frac{1}{2}-\frac{1}{\kappa}\right)\int_{\mathbb{R}^3}\left(|\nabla w_n|^2+V(x)w_n^2\right)dx \\
&\quad+\left(\frac{1}{4}-\frac{1}{\kappa}\right)\int_{\mathbb{R}^3}\phi_{w_n}^2 w_n^2dx \\
&\quad+\int_{\mathbb{R}^3}\left[\frac{1}{\kappa}g(x,w_n)w_n-G(x,w_n)\right]dx \\
&\quad+\left(\frac{1}{\kappa}-\frac{1}{q}\right)\lambda\int_{\mathbb{R}^3}h(x)|w_n|^q dx \\
&\geq\frac{1}{4}\int_{\mathbb{R}^3}\left(|\nabla w_n|^2+V(x)w_n^2\right)dx \\
&\quad+\int_{\mathbb{R}^3}\left[\frac{1}{\kappa}g(x,w_n)w_n-G(x,w_n)\right]dx \\
&\quad-\left(\frac{1}{q}-\frac{1}{\kappa}\right)\lambda M_2^q\|h\|_{2/(2-q)}\|w_n\|^q \\
&\geq\frac{1}{4}\int_{\mathbb{R}^3}|\nabla w_n|^2dx+\frac{1}{4}\int_{\mathbb{R}^3}V(x)w_n^2dx \\
&\quad+\int_{|w_n(x)|\leq r}\left[\frac{1}{\kappa}g(x,w_n)w_n-G(x,w_n)\right]dx \\
&\quad-\left(\frac{1}{q}-\frac{1}{\kappa}\right)\lambda M_2^q\|h\|_{2/(2-q)}\|w_n\|^q.
\end{aligned} \tag{37}$$

Thus, according to (36), when n is large enough,

$$\begin{aligned}
&c+\|w_n\| \\
&\quad\geq\frac{1}{4}\int_{\mathbb{R}^3}|\nabla w_n|^2dx+\frac{1}{4}\int_{\mathbb{R}^3}V(x)w_n^2dx \\
&\qquad+\int_{|w_n(x)|\leq r}\left[\frac{1}{\kappa}g(x,w_n)w_n-G(x,w_n)\right]dx \\
&\qquad-\left(\frac{1}{q}-\frac{1}{\kappa}\right)\lambda M_2^q\|h\|_{2/(2-q)}\|w_n\|^q \\
&\quad\geq\frac{1}{8}\|w_n\|^2-C\left(\frac{1}{\delta}+r^{p-2}\right)r^2 \\
&\qquad\cdot meas\left\{x\in\mathbb{R}^3\mid|x|\leq L\right\} \\
&\qquad-\left(\frac{1}{q}-\frac{1}{\kappa}\right)\lambda M_2^q\|h\|_{2/(2-q)}\|w_n\|^q.
\end{aligned} \tag{38}$$

Therefore, $\{w_n\}$ is a bounded sequence in E. By Lemma 1, there exists $\widehat{w}\in E$ such that

$$\begin{aligned}
&w_n\rightharpoonup\widehat{w}\quad\text{in } E,\\
&w_n\longrightarrow\widehat{w}\quad\text{in } L^\nu\left(\mathbb{R}^3\right),\\
&w_n(x)\longrightarrow\widehat{w}(x)\quad\text{a.e. on }\mathbb{R}^3.
\end{aligned} \tag{39}$$

Next, we define the linear operator $B_\varphi : E\longrightarrow\mathbb{R}$ as

$$B_\varphi(\nu)=(-\Delta)^{\alpha/2}\varphi(-\Delta)^{\alpha/2}\nu. \tag{40}$$

From Hölder's inequality, we obtain

$$\left|B_\varphi(\nu)\right|\leq\|\varphi\|\,\|\nu\|,\quad \nu\in E. \tag{41}$$

Now by Lemma 1 and (22),

$$\begin{aligned}
&\left|\int_{\mathbb{R}^3}\phi_{w_n}^\beta w_n(w_n-\widehat{w})dx\right| \\
&\quad\leq\left\|\phi_{w_n}^\beta\right\|_{2_\beta^*}\|w_n\|_{12/(3+2\beta)}\|w_n-\widehat{w}\|_{12/(3+2\beta)} \\
&\quad\leq C\left\|\phi_{w_n}^\beta\right\|_{\mathscr{D}^{\beta,2}}\|w_n\|_{12/(3+2\beta)}\|w_n-\widehat{w}\|_{12/(3+2\beta)} \\
&\quad\leq C\|w_n\|_{12/(3+2\beta)}^3\|w_n-\widehat{w}\|_{12/(3+2\beta)} \\
&\quad\leq C\|w_n\|^3\|w_n-\widehat{w}\|_{12/(3+2\beta)}.
\end{aligned} \tag{42}$$

Similarly, we can also prove

$$\left|\int_{\mathbb{R}^3}\phi_{\widehat{w}}^\beta\widehat{w}(w_n-\widehat{w})dx\right|\leq C\|\widehat{w}\|^3\|w_n-\widehat{w}\|_{12/(3+2\beta)}. \tag{43}$$

Since $w_n\longrightarrow\widehat{w}$ in $L^\nu(\mathbb{R}^3)(\nu\in[2,2_\alpha^*))$, $\lim_{n\to\infty}\int_{\mathbb{R}^3}(\phi_{w_n}^\beta w_n-\phi_{\widehat{w}}^\beta\widehat{w})(w_n-\widehat{w})dx=0$.

At last, combining Hölder's inequality with (H1) and (H4), we can easily get

$$\begin{aligned}
&\lim_{n\longrightarrow\infty}\int_{\mathbb{R}^3}\left(g(x,w_n)-g(x,\widehat{w})\right)(w_n-\widehat{w})dx=0,\\
&\lim_{n\longrightarrow\infty}\int_{\mathbb{R}^3}\left(h(x)|w_n|^{q-2}w_n-h(x)|\widehat{w}|^{q-2}\widehat{w}\right)\\
&\qquad\cdot(w_n-\widehat{w})dx=0.
\end{aligned} \tag{44}$$

Thus

$$\begin{aligned}
o(1)&=\left\langle I'(w_n)-I'(\widehat{w}),w_n-\widehat{w}\right\rangle \\
&=B_{w_n}(w_n-\widehat{w})-B_w(w_n-\widehat{w}) \\
&\quad+\int_{\mathbb{R}^3}V(x)w_n(w_n-\widehat{w})-V(x)\widehat{w}(w_n-\widehat{w})dx \\
&\quad+\int_{\mathbb{R}^3}\left(g(x,w_n)-g(x,\widehat{w})\right)(w_n-\widehat{w})dx \\
&\quad+\lambda\int_{\mathbb{R}^3}h(x)\left(|w_n|^{q-2}w_n-|\widehat{w}|^{q-2}w\right)(w_n-\widehat{w})dx \\
&\quad+\int_{\mathbb{R}^3}\left(\phi_{w_n}^\beta w_n-\phi_{\widehat{w}}^\beta\widehat{w}\right)(w_n-\widehat{w})dx \\
&=B_{w_n}(w_n-\widehat{w})-B_w(w_n-\widehat{w}) \\
&\quad+\int_{\mathbb{R}^3}V(x)w_n(w_n-\widehat{w})-V(x)\widehat{w}(w_n-\widehat{w})dx \\
&\quad+o(1),
\end{aligned} \tag{45}$$

that is,

$$\|w_n - \widehat{w}\|^2 = B_{w_n}(w_n - \widehat{w}) - B_w(w_n - \widehat{w}) + \int_{\mathbb{R}^3} V(x)(w_n - \widehat{w})^2 dx \longrightarrow 0. \quad (46)$$

□

Lemma 8. *If hypotheses (V) and (H1)-(H4) hold, then I satisfies $(PS)_c^*$ condition for all $c \in \mathbb{R}$.*

Proof. By Definition 4, we just prove the following fact: if for any $c \in \mathbb{R}$, $\{w_{n_j}\} \subset E$, and $w_{n_j} \in Y_{n_j}$, $I(w_{n_j}) \longrightarrow c$, $I|'_{Y_{n_j}} \longrightarrow 0$, as $n_j \longrightarrow \infty$, then $\{w_{n_j}\}$ has a convergence subsequence. The proof method is similar to Lemma 7. □

Lemma 9 (see [24]). *For $2 \le \nu < 2_\alpha^*$, $k \in \mathbb{N}$, set*

$$\beta_\nu(k) := \sup\{\|w\|_\nu : w \in Z_k, \|w\| = 1\}, \quad (47)$$

and then $\beta_\nu(k) \longrightarrow 0$, $k \longrightarrow \infty$.

Theorem 10. *If hypotheses (V) and (H1)-(H4) hold, then we can find $\lambda_0 > 0$, such that system (SP) has multiple solutions for every $\lambda < \lambda_0$. Moreover, the corresponding energy values tend to infinity.*

Proof. According to Lemma 7, I satisfies $(PS)_c$ condition. We only need to prove that I satisfies (A1) and (A2). By virtue of (H1),

$$|G(x,w)| \le \frac{C_1}{2}|w|^2 + \frac{C_1}{p}|w|^p, \quad (x,w) \in \mathbb{R}^3 \times \mathbb{R}. \quad (48)$$

From Lemma 9, we can get

$$\begin{aligned} I(w) &= \frac{1}{2}\int_{\mathbb{R}^3}\left(\left|(-\Delta)^{\alpha/2} w\right|^2 + V(x)w^2\right)dx + \frac{1}{4} \cdot \int_{\mathbb{R}^3}\phi_w^\beta w^2 dx - \int_{\mathbb{R}^3} G(x,w)\,dx - \frac{\lambda}{q} \\ &\cdot \int_{\mathbb{R}^3} h(x)|w|^q dx \ge \frac{1}{2}\|w\|^2 - \frac{C_1}{2}\int_{\mathbb{R}^3}|w|^2 dx \\ &- \frac{C_1}{p}\int_{\mathbb{R}^3}|w|^p dx - M_2^q\lambda\|h\|_{2/(2-q)}\|w\|^q \ge \frac{1}{2}\|w\|^2 \\ &- \frac{C_1}{2}\beta_2^2(k)\|w\|^2 - \frac{C_1}{p}M_p^p\|w\|^p - M_2^q\lambda\|h\|_{2/(2-q)} \\ &\cdot \|w\|^q \ge \|w\|^2\left[\frac{1}{2} - \frac{C_1}{2}\beta_2^2(k) - C_1M_p^p\|w\|^{p-2} \right. \\ &\left. - M_2^q\lambda\|h\|_{2/(2-q)}\|w\|^{q-2}\right]. \end{aligned} \quad (49)$$

Take a sufficiently large k such that $\beta_2^2(k) < 1/2C_1$. Combining the above inequality, we obtain

$$I(w) \ge \|w\|^2 \cdot \left[\frac{1}{4} - C_1M_p^p\|w\|^{p-2} - M_2^q\lambda\|h\|_{2/(2-q)}\|w\|^{q-2}\right]. \quad (50)$$

Set

$$\eta(t) = \frac{1}{4} - C_1M_p^pt^{p-2} - M_2^q\lambda\|h\|_{2/(2-q)}t^{q-2}, \quad t > 0. \quad (51)$$

Since $1 < q < 2 < p$, there exists

$$\omega_\lambda := \left(\frac{\lambda(2-q)M_2^q\|h\|_{2/(2-q)}}{C_1M_p^p(p-2)}\right)^{1/(p-q)} > 0, \quad (52)$$

such that $\max_{t\in\mathbb{R}^+}\eta(t) = \eta(\omega_\lambda)$. Therefore, for every $\lambda < \lambda_0 := ((2-q)/4C_1M_p^p(p-q))^{(p-q)/(p-2)} \cdot (C_1(p-2)M_p^p/M_2^q(2-q)\|h\|_{2/(2-q)})$,

$$I(w) \ge \omega_\lambda^2\eta(\omega_\lambda) := \mu > 0, \quad \text{with } \|w\| = \omega_\lambda. \quad (53)$$

On the other hand, by conditions (H1) and (H2), there exist positive constants C_2, C_3 such that

$$G(x,w) \ge C_2|w|^\kappa - C_3|w|^2, \quad (x,w) \in \mathbb{R}^3 \times \mathbb{R}. \quad (54)$$

Since all the norms are equivalent in every finite linear subspace $U \subset E$, then for $w \in U$

$$\begin{aligned} I(w) &= \frac{1}{2}\int_{\mathbb{R}^3}\left(\left|(-\Delta)^{\alpha/2} w\right|^2 + V(x)w^2\right)dx \\ &+ \frac{1}{4}\int_{\mathbb{R}^3}\phi_w^\beta w^2 dx - \int_{\mathbb{R}^3} G(x,w)\,dx \\ &- \frac{\lambda}{q}\int_{\mathbb{R}^3} h(x)|w|^q dx \\ &\le \frac{1}{2}\|w\|^2 + C\|w\|^4 - C_2\|w\|_\kappa^\kappa - C_3\|w\|_2^2 \\ &- \frac{\lambda}{q}\|w\|_{L^q(\mathbb{R}^3,h)}^q. \end{aligned} \quad (55)$$

For $q < 2 < 4 < \kappa$, $I(w) \longrightarrow -\infty$ as $\|w\| \longrightarrow \infty$. Then there exists $L = L(U) > 0$ such that $\max_{w\in U,\|w\|\ge L} I(w) < 0$. Thus, according to Proposition 5, the system (SP) has a list of solutions $\{(w_n, \phi_n)\} \subset E \times D^{\beta,2}(\mathbb{R}^3)$, and the corresponding energy values tend to infinity. □

Theorem 11. *If hypotheses (V) and (H1)-(H4) hold, then the system (SP) has a sequence of negative energy solutions for all $\lambda > 0$, and the energy values tend to 0.*

Proof. By Lemma 8, for all $c \in \mathbb{R}$, I satisfies the $(PS)_c^*$ condition. It now remains to show that (C1)-(C3) are satisfied. According to Lemma 9, for every $\nu \in [2, 2_\alpha^*)$, $\beta_\nu(k) \longrightarrow 0$, as $k \longrightarrow \infty$. Thus there exists $k_1 > 0$ such that $\beta_2(k) \le \sqrt{1/2C_1}$ for $k > k_1$. For $4 < p < 2_\alpha^*$, there exists $L \in (0,1)$ such that

$$\frac{1}{8}\|w\|^2 \ge \frac{C_1}{p}M_p^p\|w\|^p, \quad \text{with } \|w\| \le L. \quad (56)$$

Hence, for $w \in Z_k$ with $\|w\| \le L$, it follows that

$$\begin{aligned} I(w) &= \frac{1}{2}\int_{\mathbb{R}^3}\left(\left|(-\Delta)^{\alpha/2} w\right|^2 + V(x) w^2\right) dx \\ &\quad + \frac{1}{4}\int_{\mathbb{R}^3} \phi_w^\beta w^2 dx - \int_{\mathbb{R}^3} G(x,w)\, dx \\ &\quad - \frac{\lambda}{q}\int_{\mathbb{R}^3} h(x)|w|^q dx \\ &\ge \frac{1}{2}\|w\|^2 - \frac{C_1}{2}\int_{\mathbb{R}^3}|w|^2 dx - \frac{C_1}{p}\int_{\mathbb{R}^3}|w|^p dx \\ &\quad - \frac{\lambda}{q}\int_{\mathbb{R}^3} h(x)|w|^q dx \\ &\ge \frac{1}{2}\|w\|^2 - \frac{C_1}{2}\beta_2^2(k)\|u\|^2 - \frac{C_1}{p}M_p^p\|w\|^p \\ &\quad - \frac{\lambda}{q}\int_{\mathbb{R}^3} h(x)|w|^q dx \\ &\ge \frac{1}{4}\|w\|^2 - \frac{C_1}{p}M_p^p\|w\|^p \\ &\quad - \frac{\lambda}{q}\|h\|_{2/(2-q)}\beta_2^q(k)\|w\|^q \\ &\ge \frac{1}{8}\|w\|^2 - \frac{\lambda}{q}\|h\|_{2/(2-q)}\beta_2^q(k)\|w\|^q. \end{aligned} \tag{57}$$

For every $k > k_1$, let $r_k = ((8/q)\lambda\|h\|_{2/(2-q)}\beta_2^q(k))^{1/(2-q)}$. By Lemma 9, $r_k \longrightarrow 0$, as $k \longrightarrow \infty$. Thus, there exists $k_0 > k_1$, such that for every $k \ge k_0$, $I(w) \ge 0$, for $w \in Z_k$ with $\|w\| = r_k$.

Secondly, since for every fixed $k \in \mathbb{N}$, the norms are equivalent in Y_k, when k is sufficiently large, there exists a small enough γ_k such that $0 < \gamma_k < r_k$ and $I(w) < 0$ for $w \in Y_k$ with $\|w\| = \gamma_k$.

Finally, according to (C3), when $k \ge k_0$, for $u \in Z_k$, with $\|w\| \le r_k$, one has

$$\begin{aligned} I(w) &\ge -\frac{\lambda}{q}\|h\|_{2/(2-q)}\beta_2^q(k)\|w\|^q \\ &\ge -\frac{\lambda}{q}\|h\|_{2/(2-q)}\beta_2^q(k)\, r_k^q. \end{aligned} \tag{58}$$

Since $\beta_2(k) \longrightarrow 0, r_k \longrightarrow 0$, as $k \longrightarrow \infty$, therefore (C3) holds. By Proposition 6, I has a list of solutions $\{(w_n, \phi_n)\} \subset E \times D^{\beta,2}(\mathbb{R}^3)$ such that

$$\begin{aligned} &\frac{1}{2}\int_{\mathbb{R}^3}\left(\left|(-\Delta)^{\alpha/2} w_n\right|^2 + V(x) w_n^2\right) dx \\ &\quad + \frac{1}{4}\int_{\mathbb{R}^3}\phi_n^\beta w_n^2 dx - \int_{\mathbb{R}^3} G(x,w_n)\, dx \\ &\quad - \frac{\lambda}{q}\int_{\mathbb{R}^3} h(x)|w_n|^q dx \longrightarrow 0. \end{aligned} \tag{59}$$

□

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

All authors read and approved the final manuscript.

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