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Research Article

On the Weak Characteristic Function Method for a Degenerate Parabolic Equation

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For a nonlinear degenerate parabolic equation, how to impose a suitable boundary value condition to ensure the well-posedness of weak solutions is a very important problem. It is well known that the classical Fichera-Oleinik theory has perfectly solved the problem for the linear case, and the optimal boundary value condition matching up with a linear degenerate parabolic equation can be depicted out by Fechira function. In this paper, a new method, which is called the weak characteristic function method, is introduced. By this new method, the partial boundary condition matching up with a nonlinear degenerate parabolic equation can be depicted out by an inequality from the diffusion function, the convection function, and the geometry of the boundary $\partial\Omega$ itself. Though, by choosing different weak characteristic function, one may obtain the differential partial boundary value conditions, an optimal partial boundary value condition can be prophetic. Moreover, the new method works well in any kind of the degenerate parabolic equations.

1. Introduction

For the earliest movement differential equation of a particle

$$\frac{dx}{dt} = f(t, x), \quad (1)$$

the initial value

$$x(0) = x_0 \quad (2)$$

is the initial position of the particle. For a second order ordinary differential equation

$$\frac{d^2x}{dt^2} = f\left(t, x, \frac{dx}{dt}\right), \quad (3)$$

if we regard it as a its accelerated speed differential equation, we should impose the initial value conditions as

$$x(0) = x_0, \qquad \left.\frac{dx}{dt}\right|_{x=0} = v_0, \quad (4)$$

where v_0 is the initial velocity. If we regard it as describing the motion of a vibrating string, we should impose the boundary value conditions

$$x(0) = 0, \qquad x(1) = 0, \quad (5)$$

which implies that the two ends of the string are fixed at $x = 0$ and $x = 1$. Or even one can impose the following boundary value condition:

$$x(0) = x(1) - x(0.5) = 0, \quad (6)$$

which is called three points boundary value problem. Theoretically, all these conditions are called definite conditions. In other words, in order to solve an explicit differential equation, it is important to find a suitable definite condition. For example, considering the well-known heat conduction equation

$$u_t = \Delta u, \quad (x, t) \in \Omega \times (0, T), \quad (7)$$

besides the initial value

$$u(x, 0) = u_0(x), \quad x \in \Omega, \quad (8)$$

where u_0 is the initial temperature, one of the following boundary value conditions should be imposed.

(i) Dirichlet condition

$$u(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,T). \tag{9}$$

(ii) Neumann condition

$$\frac{\partial u}{\partial n} = 0, \quad (x,t) \in \partial\Omega \times (0,T), \tag{10}$$

where n is the outer normal vector of Ω.

(iii) Robin condition

$$\frac{\partial u}{\partial n} + ku = 0, \quad (x,t) \in \partial\Omega \times (0,T), \tag{11}$$

where k is a positive constant.

But, if one considers the degenerate heat conduction equation

$$u_t = \operatorname{div}(a(x,t)\nabla u) \tag{12}$$

where $a(x,t) \geq 0$, or nonlinear heat conduction equation

$$u_t = \operatorname{div}(k(x,t,u)\nabla u), \tag{13}$$

where $k(x,t,u) \geq 0$, the above three boundary value conditions may be overdetermined. While, for a hyperbolic-parabolic mixed type equation

$$u_t = \operatorname{div}(k(x,t,u)\nabla u) + \operatorname{div}\left(\vec{b}(u)\right), \tag{14}$$

in order to obtain the uniqueness of weak solution, besides one of the above three boundary value conditions is imposed, the entropy condition should be added additionally. In a word, for a degenerate parabolic equation, how to impose a suitable partial boundary value condition to ensure the well-posedness of weak solutions has been an interesting and important problem for a long time. Let us give a basic review of the history.

First studied by Tricomi and Keldyš and later by Fichera and Oleĭnik, the general theory of second order equation with nonnegative characteristic form, which, in particular, contains those degenerating on the boundary had been developed and perfected [1] about in 1960s. By this theory, if one wants to consider the well-posedness problem of a linear degenerate elliptic equation

$$\sum_{r,s=1}^{N+1} a^{rs}(x)\frac{\partial^2 u}{\partial x_r \partial x_s} + \sum_{r=1}^{N+1} b^r(x)\frac{\partial u}{\partial x_r} + c(x)u = f(x), \quad x \in \widetilde{\Omega} \subset \mathbb{R}^{N+1}, \tag{15}$$

only a partial boundary value condition is required. In detail, let $\{n_s\}$ be the unit inner normal vector of $\partial\widetilde{\Omega}$ and denote that

$$\begin{aligned} \Sigma_2 &= \left\{x \in \partial\widetilde{\Omega} : a^{rs}n_r n_s = 0,\ \left(b_r - a^{rs}_{x_s}\right)n_r < 0\right\}, \\ \Sigma_3 &= \left\{x \in \partial\widetilde{\Omega} : a^{rs}n_s n_r > 0\right\}. \end{aligned} \tag{16}$$

Then, the partial boundary value condition is

$$u|_{\Sigma_2 \cup \Sigma_3} = 0. \tag{17}$$

In particular, if the matrix (a^{rs}) is positive definite, (15) is the classical elliptic equation and (17) is just the usual Dirichlet boundary condition.

If the matrix (a^{rs}) is semipositive definite, the most typical is the linear degenerate parabolic equation

$$\begin{aligned} \frac{\partial u}{\partial t} = {} & \sum_{r,s=1}^{N} a^{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{N} b^i(x)\frac{\partial u}{\partial x_i} + c(x)u \\ & - f(x), \quad (x,t) \in Q_T = \Omega \times (0,T). \end{aligned} \tag{18}$$

To study the well-posedness problem of (18), in addition to the initial value condition

$$u(x,0) = u_0(x), \quad x \in \Omega, \tag{19}$$

a partial boundary value condition should be imposed

$$u(x,t) = 0, \quad (x,t) \in \Sigma_p \times (0,T), \tag{20}$$

where

$$\begin{aligned} \Sigma_p = {} & \left\{x \in \partial\Omega : a^{ij}n_i n_j > 0\right\} \\ & \cup \left\{x \in \partial\Omega : a^{ij}n_i n_j = 0,\ \left(b^i - a^{ij}_{x_j}\right)n_i < 0\right\}. \end{aligned} \tag{21}$$

Now, if one considers the well-posedness problem of a nonlinear degenerate parabolic equation, it is naturally to conjecture that only a partial boundary value condition should be imposed. For example, considering the nonlinear parabolic equation

$$\frac{\partial u}{\partial t} = \operatorname{div}\left(a(x,t)|\nabla u|^{p(x)-2}\nabla u\right), \quad (x,t) \in Q_T, \tag{22}$$

with

$$a(x,t) = 0, \quad (x,t) \in \partial\Omega \times [0,T), \tag{23}$$

if $u_1(x,t) \in BV(Q_T), u_2(x,t) \in BV(Q_T)$ are two weak solutions of (22) with the initial values $u_1(x), u_2(x)$, respectively, then it is easily to show that

$$\begin{aligned} & \int_\Omega \left|u_1(x,t) - v_2(x,t)\right|^2 dx \\ & \quad \leq c\int_\Omega \left|u_1(x) - u_2(x)\right|^2 dx, \quad t \in [0,T), \end{aligned} \tag{24}$$

even without any boundary value condition. In other words, for a general nonlinear degenerate parabolic equation,

$$\frac{\partial u}{\partial t} = \operatorname{div}(a(x,t,u,\nabla u)\nabla u) + f(x,t,u,\nabla u), \tag{25}$$

though we can expect that only a partial boundary value condition like (20) is enough to ensure the stability of weak solutions (or uniqueness of weak solution), since Fichera-Oleinik theory is invalid, if we insist on the partial boundary value condition (20) is still imposed in the sense of the trace, then it is difficult to assign the geometry of the partial boundary Σ_p appearing in (20). In this paper, we will try to find a new method to solve this problem. For the sake of convenience, we can call the new method as the weak characteristic function method. We first introduce the related definitions.

Definition 1. If $g(x)$ is a nonnegative continuous function in $\mathbb{R}^N$, when x is near to the boundary $\partial\Omega$, $g(x)$ is a C^2 function and satisfies

$$\partial\Omega = \left\{x \in \mathbb{R}^N : g(x) = 0\right\}, \qquad \Omega = \left\{x \in \mathbb{R}^N : g(x) > 0\right\}, \tag{26}$$

then we say $g(x)$ is a weak characteristic function of Ω.

Only if Ω is with a C^2 smooth boundary, the distance function $d(x) = \operatorname{dist}(x, \partial\Omega)$ is a weak characteristic function of Ω, and its square d^2 is another weak characteristic function of Ω. Certainly, if $u_0(x)$ is a continuous function with $u_0(x)|_{x\in\partial\Omega} = 0$, then the function $a(u_0(x)) + d(x)$ also is a weak characteristic function of Ω.

Definition 2. By the weak characteristic function method it means that one can find the explicit geometric expression of Σ_p in the partial boundary value condition (20) by choosing a suitable test function related to a weak characteristic function of Ω.

We will choose two special nonlinear parabolic equations of (25) to verify the new method. The first one is

$$\frac{\partial u}{\partial t} = \Delta A(u) + \operatorname{div}(b(u)), \quad (x,t) \in Q_T, \tag{27}$$

where $\Omega \subset \mathbb{R}^N$ is an open bounded domain, $b(u) = \{b^i(u)\}$, and

$$A(u) = \int_0^u a(s)\,ds, \quad a(s) \ge 0. \tag{28}$$

The second type is the evolutionary $p(x)$−Laplacian equation similar to (22) (see below please). We will introduce the backgrounds of these two kinds of equations, respectively.

Equation (27) arises from heat flow in materials with temperature dependent on conductivity, flow in a porous medium, the conservation law, the one-dimensional Euler equation, and the boundary layer theory. It is with hyperbolic-parabolic mixed type and might have discontinuous solution. For the Cauchy problem, the well-posedness theory has been established perfectly, one can refer to [2–10] and the references therein. For the initial-boundary value problem, also there are many important papers devoting to its well-posedness problem; one can see [11–16] and the references therein. However, unlike the Cauchy problem, how to impose a suitable boundary value condition to match up with (27) has been an interesting and difficult problem for a long time. Actually, for the completely degenerate case, i.e., $A \equiv 0$, (25) becomes a first order hyperbolic equation, and it is well known that a smooth solution is constant along the maximal segment of the characteristic line in Q_T. When this segment intersects both $\{0\} \times \Omega$ and $\partial\Omega$, then the usual boundary value condition

$$u(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,T), \tag{29}$$

is overdetermined if (27) is fulfilled in the traditional trace sense. Thus one needs to work within a suitable framework of entropy solutions and entropy boundary conditions. In the BV setting, the authors of [11] gave an interpretation of the boundary condition (29) as an entropy inequality on $\partial\Omega$, which is the so-called BLN condition. However, since the trace of solutions is involved in the formulation of the BLN condition, it makes no sense if the solution is merely in L^∞. The author of [12] extended the Dirichlet problem for hyperbolic equations to the L^∞ setting and proved the uniqueness of the entropy solution by introducing an integral formulation of the boundary condition. This idea had been generalized to deal with the strongly degenerate parabolic equations [13–16], in which the boundary condition is not directly shown as (27) in sense of the trace but is implicitly contained in a family of entropy inequalities.

If we still comprehend the boundary value condition is true in the sense of the trace, when the domain $\Omega = \mathbb{R}^N_+$ is the half space of $\mathbb{R}^N$, in our previous work [17], we probed the initial-boundary value problem of (27) in the half space $\mathbb{R}^N_+ \times (0,T)$. We have proved that if $b_N'(0) < 0$, we can give the general Dirichlet boundary condition

$$u(x,t) = 0, \quad (x,t) \in \partial\mathbb{R}^N_+ \times (0,T). \tag{30}$$

But if $b_N'(0) \ge 0$, then no boundary condition is necessary, and the solution of the equation is free from any limitation of the boundary condition.

When Ω is a bounded smooth domain, in [18], by the parabolically regularized method, we had proved the existence of the entropy solution [18], but we could not obtain the stability based on the partial boundary value condition (20). At that time, we could not find a valid way to depict out the geometric expression of Σ_p in (20).

The first discovery of this paper is that, by the weak characteristic new method, we find that the partial boundary value condition (20) admits the form as

$$\Sigma_p = \left\{x \in \partial\Omega : \Delta g + \gamma \sum_{i=1}^N \left|g_{x_i}\right| \ge 0\right\}, \tag{31}$$

where the constant γ satisfies

$$\left|b^i(u) - b^i(v)\right| \le \gamma |u - v|, \tag{32}$$

and when x is near to the boundary $\partial\Omega$, $g(x)$ is a weak characteristic function of Ω.

For example, $N = 2$, if the domain Ω is the disc $D_1 = \{(x, y) : x^2 + y^2 < 1\}$, a weak characteristic function can be chosen as $g(x) = 1 - (x^2 + y^2)$,

$$g_x = -2x, \quad g_y = -2y, \quad \Delta g = -4 \tag{33}$$

then

$$\Delta g + \gamma \left(|g_x| + |g_y|\right) = -4 + 2\gamma (|x| + |y|), \tag{34}$$

and

$$\Sigma_p = \{x \in \partial D_1 : \gamma (|x| + |y|) \geq 2\}, \tag{35}$$

which implies that if $\gamma \leq 1$, then $\Sigma_p = \emptyset$; if $1 < \gamma < 2$, Σ_p is a proper subset of $\partial\Omega$; if $\gamma \geq 2$, then $\Sigma_p = \partial\Omega$.

It is well known and very important in applications that the boundary conditions usually stand for some physical meanings. At least from my own perspective, if we regard (27) as a nonlinear heat conduction (or heat diffusion) process, then $\Sigma_p = \emptyset$ means that $u(x, t) = 0$ occurs before x attains the boundary value $\partial\Omega$.

From mathematical theory, the partial boundary value condition (20) with the form as (27) is just as a definite condition. Since condition (31) includes $g(x)$ and γ, we can say condition (31) is determined by the degeneracy of a, the weak characteristic function of Ω, and the first order derivative term in a special sense; this fact seems more or less likely to that (21). We will prove the stability of the entropy solutions to (27) under the partial boundary value condition (20) with expression (31).

The second degenerate parabolic equation considered in this paper is

$$u_t = \operatorname{div}\left(a(x) |\nabla u|^{p(x)-2} \nabla u\right) + \operatorname{div}(b(u)), \tag{36}$$

which comes from a new kind of fluids: the so-called electrorheological fluids (see [19, 20]). If $a(x) \equiv 1$, this kind of equations has been researched widely recently. One can refer to [21–29], etc. If $a(x) \equiv 1$ and $p(x) = p$ are constant, (36) is the well-known non-Newtonian fluid equation [10]. If $a(x)$ is a $C^1(\overline{\Omega})$ function, $p(x) = p$; the author of [30] considered the nonlinear equation

$$\frac{\partial u}{\partial t} - \operatorname{div}\left(a(x) |\nabla u|^{p-2} \nabla u\right) - f_i(x) D_i u + c(x, t) u = 0, \tag{37}$$

and made important progress on its study. They classified the boundary into three parts: the nondegenerate boundary, the weakly degenerate boundary, and the strongly degenerate boundary, by means of a reasonable integral description. The boundary value condition should be supplemented definitely on the nondegenerate boundary and the weakly degenerate boundary. On the strongly degenerate boundary, they formulated a new approach to prescribe the boundary value condition rather than defining the Fichera function as treating the linear case. Moreover, they formulated the boundary value condition on this strongly degenerate boundary in a much weak sense since the regularity of the solutions much weaker near this boundary.

In this paper, we assume that $a(x) \in C^1(\overline{\Omega})$ satisfies condition

$$a(x) > 0, \quad x \in \Omega; \qquad a(x) = 0, \quad x \in \partial\Omega, \tag{38}$$

and $b^i(s)$ is a C^1 function on $\mathbb{R}$. The second discovery of this paper is that, by choosing $a(x)$ as the weak characteristic function of Ω, we deduce that Σ_p can be depicted out by

$$\Sigma_p = \left\{x \in \partial\Omega : \sum_{i=1}^{N} a_{x_i}(x) \neq 0\right\}. \tag{39}$$

By (39), we can prove the stability of the entropy solutions of (36) under the partial boundary value condition (20) with the expression (38).

Let us give a simple summary. For a nonlinear degenerate parabolic equation, to the best knowledge of the author, there are three ways to deal with the boundary value condition. The traditional way is to comprehend (29) (also (20)) in the sense of the trace as in [2, 4, 10, 17, 18, 31]. The second way, the boundary value condition (29) is understood in weaker sense than the trace and is elegantly implicitly contained in family entropy inequalities [11–16]. In this way, if the equation is completely degenerate, then the boundary value condition is replaced by BLN condition. Moreover, in [12–16], the entropy solutions are in L^∞ space, the existence of the traditional trace on the boundary is not guaranteed, and it is impossible to depict out Σ_p in a geometric way. The third way, the boundary value condition (29) is decomposed into two parts; on one part (the nondegenerate part and the weak degenerate part in [30]) the boundary value condition is true in the sense of trace, while on the other part (the strongly degenerate part in [30]), the boundary value condition is true in a much weaker sense than the trace. In this paper, we still use the traditional way to deal with the boundary value condition. The most innovation lies in the fact that if one chooses the different weak characteristic function $\phi(x)$ of Ω, then one obtains the different partial boundary value condition

$$u(x, t) = 0, \quad (x, t) \in \Sigma_\phi \times (0, T), \tag{40}$$

where $\Sigma_\phi \subseteq \partial\Omega$ depends on $\phi(x)$. Thus, we can predict that the optimal partial boundary value condition matching up with a nonlinear degenerate parabolic equation should have the form

$$u(x, t) = 0, \quad (x, t) \in \Sigma_\phi \times (0, T), \tag{41}$$

with that

$$\Sigma = \bigcap_{\phi} \Sigma_\phi. \tag{42}$$

But we can not prove this conjecture for the time being.

2. Main Results

For small $\eta > 0$, let

$$S_\eta(s) = \int_0^s h_\eta(\tau)\,d\tau, \quad h_\eta(s) = \frac{2}{\eta}\left(1 - \frac{|s|}{\eta}\right)_+. \tag{43}$$

Obviously $h_\eta(s) \in C(\mathbb{R})$, and

$$h_\eta(s) \ge 0, \quad \left|sh_\eta(s)\right| \le 1, \quad \left|S_\eta(s)\right| \le 1; \quad \lim_{\eta\to 0} S_\eta(s) = \operatorname{sgn} s, \quad \lim_{\eta\to 0} sS'_\eta(s) = 0. \tag{44}$$

Definition 3. A function u is said to be the entropy solution of (27) with the initial value condition (19), if

(1) u satisfies

$$u \in BV(Q_T) \cap L^\infty(Q_T), \quad \frac{\partial}{\partial x_i}\int_0^u \sqrt{a(s)}ds \in L^2(Q_T). \tag{45}$$

(2) For any $\varphi \in C_0^2(Q_T)$, $\varphi \ge 0$, for any $k \in \mathbb{R}$, for any small $\eta > 0$, u satisfies

$$\iint_{Q_T}\left[I_\eta(u-k)\varphi_t - B_\eta^i(u,k)\varphi_{x_i} + A_\eta(u,k)\Delta\varphi - S'_\eta(u-k)\left|\nabla\int_0^u \sqrt{a(s)}ds\right|^2\varphi\right]dxdt \ge 0. \tag{46}$$

(3) The initial value is true in the sense of

$$\lim_{t\to 0}\int_\Omega \left|u(x,t) - u_0(x)\right|dx = 0. \tag{47}$$

(4) If the partial boundary value condition (20) is true in the sense of the trace, then we say u is the solution of (27) with the initial-boundary value conditions (19) and (20).

Here the pairs of equal indices imply a summation from 1 up to N, and

$$B_\eta^i(u,k) = \int_k^u \left(b^i\right)'(s)S_\eta(s-k)\,ds, \quad A_\eta(u,k) = \int_k^u a(s)S_\eta(s-k)\,ds, \quad I_\eta(u-k) = \int_0^{u-k} S_\eta(s)\,ds. \tag{48}$$

On one hand, if (27) has a classical solution u, multiplying (27) by $\varphi S_\eta(u-k)$ and integrating over Q_T, we are able to show that u satisfies Definition 3.

On the other hand, let $\eta \longrightarrow 0$ in (46). We have

$$\iint_{Q_T}\left[|u-k|\varphi_t - \operatorname{sgn}(u-k)\left(b^i(u) - b^i(k)\right)\varphi_{x_i}\right]dxdt + \iint_{Q_T}\operatorname{sgn}(u-k)(A(u)-A(k))\Delta\varphi dxdt \ge 0. \tag{49}$$

Thus if u is the entropy solution in Definition 3, then u is a entropy solution defined in [2, 10], etc.

The existence of the entropy solution in the sense of Definition 3 can be proved similar to Theorem 2.3 in [18]; we omit the details here.

Theorem 4. *Suppose that $A(s)$ is a $C^2(\mathbb{R})$ function and $b^i(s)$ is a $C^1(\mathbb{R})$ function; u and v are two solutions of (27) with the different initial values $u_0(x)$, $v_0(x) \in L^\infty(\Omega)$, respectively. If u and v are with the same homogeneous partial boundary value condition (20), then*

$$\int_\Omega |u(x,t) - v(x,t)|\,dx \le \int_\Omega \left|u_0(x) - v_0(x)\right|dx. \tag{50}$$

Definition 5. A function $u(x,t)$ is said to be a weak solution of (36) with the initial value (18), provided that

$$u \in L^\infty(Q_T), \quad u_t \in L^2(Q_T), \quad a(x)|\nabla u|^{p(x)} \in L^1(Q_T), \tag{51}$$

and for any function $\varphi_1 \in C_0^1(Q_T)$ and $\varphi_2 \in L^\infty(0,T; W_{loc}^{1,p(x)}(\Omega))$ there holds

$$\iint_{Q_T}\left[\frac{\partial u}{\partial t}(\varphi_1\varphi_2) + a(x)|\nabla u|^{p(x)-2}\nabla u\nabla(\varphi_1\varphi_2) + b^i(u)(\varphi_1\varphi_2)_{x_i}\right]dxdt = 0. \tag{52}$$

The initial value (18) is satisfied in the sense of (47). If the partial boundary value condition (20) is true in the sense of the trace, then we say u is the solution of (36) with the initial-boundary value conditions (19) and (20).

Here, $W^{1,p(x)}(\Omega)$ is the variable exponent Sobolev space [23]. Suppose that $p_- = \min_{x\in\overline{\Omega}} > 1$, $a(x)$ satisfies (38), and $b^i(s)$ is a C^1 function on $\mathbb{R}$. If

$$u_0(x) \in L^\infty(\Omega), \quad a(x)u_0(x) \in W^{1,p(x)}(\Omega), \tag{53}$$

and there are some other restrictions in $a(x)$ and $b^i(x)$, in a similar way as that of Theorem 2.5 of [32], we can prove the existence of a weak solution of (36) with the initial value (19) in the sense of Definition 5. We omit the details here. We mainly pay attentions to the stability of the weak solutions.

According to Lemma 3.2 of [32], if

$$\int_{\Omega} a(x)^{-1/(p(x)-1)}\,dx < \infty, \quad i = 1, 2, \ldots, N, \tag{54}$$

then

$$\int_{\Omega} |\nabla u|\,dx < \infty, \tag{55}$$

and u can be defined the trace on the boundary $\partial\Omega$. If the homogeneous boundary value condition (29) is imposed, the stability can be established in a way analogous to the one of the evolutionary p–Laplacian equation [10]. In this paper, we will use the weak characteristic function method to prove the following stability theorems based on the partial boundary value condition (20).

Theorem 6. *Let $u(x,t)$ and $v(x,t)$ be two weak solutions of (36) with the initial values $u_0(x)$ and $v_0(x)$ respectively, with the same partial boundary value condition*

$$u(x,t) = v(x,t) = 0, \quad (x,t) \in \Sigma_p \times [0,T). \tag{56}$$

If $a(x)$ satisfies (38) and (54), $b^i(s)$ is a Lipschitz function, and

$$\left(\frac{1}{\eta}\right)^{(p^+-1)/p_-} \left(\int_{\Omega\setminus D_\eta} |\nabla a|^{p(x)}\,dx\right)^{1/p^-} \le c, \tag{57}$$

then the stability (50) is true, where Σ_p has the form as (39),

$$D_\eta = \{x \in \Omega : a(x) > \eta\} \tag{58}$$

for the sufficiently small η.

The last but not least, we would like to suggest that the weak characteristic function method introduced in this paper can be widely used to study the boundary value problem of any kind of the degenerate parabolic or hyperbolic equations.

3. The Proof of Theorem 4

Let Γ_u be the set of all jump points of $u \in BV(Q_T)$, ν be the normal of Γ_u at $X = (x,t)$, and $u^+(X)$ and $u^-(X)$ be the approximate limits of u at $X \in \Gamma_u$ with respect to $(\nu, Y-X) > 0$ and $(\nu, Y-X) < 0$, respectively. For the continuous function $p(u,x,t)$ and $u \in BV(Q_T)$, we define

$$\widehat{p}(u,x,t) = \int_0^1 p(\tau u^+ + (1-\tau)u^-, x, t)\,d\tau, \tag{59}$$

which is called the composite mean value of p. For a given t, we denote Γ_u^t, H^t, $(\nu_1^t, \ldots, \nu_N^t)$ and $u_\pm^t$ as all jump points of $u(\cdot,t)$, Housdorff measure of Γ_u^t, the unit normal vector of Γ_u^t, and the asymptotic limit of $u(\cdot,t)$, respectively. Moreover, if $f(s) \in C^1(\mathbb{R})$, $u \in BV(Q_T)$, then $f(u) \in BV(Q_T)$ and

$$\frac{\partial f(u)}{\partial x_i} = \widehat{f'}(u)\frac{\partial u}{\partial x_i}, \quad i = 1, 2, \ldots, N, N+1, \tag{60}$$

where $x_{N+1} = t$.

Lemma 7. *Let u be a solution of (27). Then in the sense of Hausdorff measure $H_N(\Gamma_u)$, we have*

$$a(s) = 0, \quad s \in I(u^+(x,t), u^-(x,t)) \ \text{a.e. on } \Gamma_u, \tag{61}$$

where $I(\alpha,\beta)$ denotes the closed interval with endpoints α and β.

This lemma can be proved in a similar way as described in [9]; we omit the details here.

Proof of Theorem 4. Let u, v be two entropy solutions of (27) with initial values

$$\begin{aligned} u(x,0) &= u_0(x), \\ v(x,0) &= v_0(x). \end{aligned} \tag{62}$$

By Definition 3, for $\varphi \in C_0^2(Q_T)$, we have

$$\begin{aligned} &\iint_{Q_T} \left[I_\eta(u-k)\varphi_t - B_\eta^i(u,k)\varphi_{x_i} \right.\\ &\quad \left. + A_\eta(u,k)\Delta\varphi\right] dxdt - \iint_{Q_T} S_\eta'(u-k) \\ &\quad \cdot \left|\nabla \int_0^u \sqrt{a(s)}ds\right|^2 \varphi dxdt \ge 0, \end{aligned} \tag{63}$$

$$\begin{aligned} &\iint_{Q_T} \left[I_\eta(v-l)\varphi_\tau - B_\eta^i(v,l)\varphi_{y_i} + A_\eta(v,l)\Delta\varphi\right] dyd\tau \\ &\quad - \iint_{Q_T} S_\eta'(v-l)\left|\nabla \int_0^v \sqrt{a(s)}ds\right|^2 \varphi dyd\tau \ge 0. \end{aligned} \tag{64}$$

Let $\varphi = \psi(x,t,y,\tau) = \phi(x,t) j_h(x-y, t-\tau)$, where $\phi(x,t) \ge 0$, $\phi(x,t) \in C_0^\infty(Q_T)$, and

$$j_h(x-y, t-\tau) = \omega_h(t-\tau)\Pi_{i=1}^N \omega_h(x_i - y_i), \tag{65}$$

$$\begin{aligned} &\omega_h(s) = \frac{1}{h}\omega\left(\frac{s}{h}\right), \\ &\omega(s) \in C_0^\infty(R), \\ &\omega(s) \ge 0, \\ &\omega(s) = 0, \quad \text{if } |s| > 1, \quad \int_{-\infty}^{\infty} \omega(s)\,ds = 1. \end{aligned} \tag{66}$$

We choose $k = v(y,\tau)$, $l = u(x,t)$, and $\varphi = \psi(x,t,y,\tau)$ in (63) and (64) and integrate it over Q_T. It yields

$$\begin{aligned} &\iint_{Q_T}\iint_{Q_T} \left[I_\eta(u-v)(\psi_t + \psi_\tau) - \left(B_\eta^i(u,v)\psi_{x_i} \right.\right.\\ &\quad \left.\left. + B_\eta^i(v,u)\psi_{y_i}\right)\right] dxdtdyd\tau \\ &\quad + \iint_{Q_T}\iint_{Q_T} \left[A_\eta(u,v)\Delta_x\psi \right.\\ &\quad \left. + A_\eta(v,u)\Delta_y\psi\right] dxdtdyd\tau \end{aligned}$$

$$-\iint_{Q_T}\iint_{Q_T} S'_\eta(u-v)\left(\left|\nabla_x\int_0^u\sqrt{a(s)}ds\right|^2 + \left|\nabla_y\int_0^v\sqrt{a(s)}ds\right|^2\right)\psi dxdtdyd\tau \geq 0. \quad (67)$$

Here Δ_x is the usual Laplacian operator corresponding to the variable x, and ∇_x is the gradient operator corresponding to the variable x.

By the basic relations

$$\frac{\partial j_h}{\partial t}+\frac{\partial j_h}{\partial \tau}=0,$$
$$\frac{\partial j_h}{\partial x_i}+\frac{\partial j_h}{\partial y_i}=0, \quad i=1,\cdots,N; \quad (68)$$
$$\frac{\partial \psi}{\partial t}+\frac{\partial \psi}{\partial \tau}=\frac{\partial \phi}{\partial t}j_h,$$
$$\frac{\partial \psi}{\partial x_i}+\frac{\partial \psi}{\partial y_i}=\frac{\partial \phi}{\partial x_i}j_h,$$

using Lemma 7, just by the same calculations as in the proof of Theorem 2.4 in [18], letting $\eta \longrightarrow 0, h \longrightarrow 0$ in (67), we can deduce that

$$\iint_{Q_T}\left[|u(x,t)-v(x,t)|\phi_t - \operatorname{sgn}(u-v)\left(b^i(u)-b^i(v)\right)\phi_{x_i}\right]dxdt + \iint_{Q_T}|A(u)-A(v)|\Delta\phi dxdt \geq 0. \quad (69)$$

If we let

$$\phi(x,t)=\eta(t)\xi(x), \quad (70)$$

where $\eta(t)\in C_0^\infty(0,T)$ and $\xi(x)\in C_0^\infty(\Omega)$, then

$$\iint_{Q_T}\left[|u(x,t)-v(x,t)|\eta_t\xi(x)-\operatorname{sgn}(u-v)\cdot\left(b^i(u)-b^i(v)\right)\eta(t)\xi_{x_i}\right]dxdt+\iint_{Q_T}|A(u)-A(v)|\eta(t)\Delta\xi dxdt \geq 0. \quad (71)$$

For $0<\tau<s<T$, we choose

$$\eta(t)=\int_{\tau-t}^{s-t}\alpha_\epsilon(\sigma)d\sigma, \quad \epsilon<\min\{\tau,T-s\}, \quad (72)$$

where $\alpha_\epsilon(t)$ is the kernel of mollifier with $\alpha_\epsilon(t)=0$ for $t\notin(-\epsilon,\epsilon)$.

By (71), since $|b^i(u)-b^i(v)|\leq\gamma|u-v|$, we have

$$\int_\Omega|u(x,s)-v(x,s)|\xi(x)dx \leq \int_\Omega|u(x,\tau)-v(x,\tau)|\xi(x)dx + \int_s^\tau\int_\Omega|u-v|\left(\Delta\xi+\gamma\sum_{i=1}^N\left|\xi_{x_i}\right|\right)dxdt. \quad (73)$$

For any small $\lambda>0$, we choose

$$\xi(x)=\begin{cases}1, & g(x)>\lambda,\\ 1-\dfrac{(g(x)-\lambda)^2}{\lambda^2}, & 0\leq g(x)\leqslant\lambda,\end{cases} \quad (74)$$

where $g(x)$ is a weak characteristic function of Ω. Then

$$\xi_{x_i}=-\frac{2(g(x)-\lambda)}{\lambda^2}g_{x_i},$$
$$\Delta\xi=-\frac{2}{\lambda^2}|\nabla g|^2-\frac{2(g-\lambda)}{\lambda^2}\Delta g. \quad (75)$$

By (73), we have

$$\int_\Omega|u(x,s)-v(x,s)|\xi(x)dx \leq \int_\Omega|u(x,\tau)-v(x,\tau)|\xi(x)dx + \int_s^\tau\int_{\Omega_\lambda}|u-v|\left(\Delta\xi+\sum_{i=1}^N\gamma\left|\xi_{x_i}\right|\right)dxdt, \quad (76)$$

where $\Omega_\lambda=\{x\in\Omega: g(x)<\lambda\}$.

Then, since $-2(g-\lambda)/\lambda^2>0$ in Ω_λ, we have

$$\begin{aligned}\int_\Omega|u(x,s)-v(x,s)|\xi(x)dx &\leq \int_\Omega|u(x,\tau)-v(x,\tau)|\xi(x)dx+\int_s^\tau\int_{\Omega_\lambda}|u-v|\cdot\left(-\frac{2(g-\lambda)}{\lambda^2}\Delta g+\gamma\sum_{i=1}^N\left|\frac{2(g(x)-\lambda)}{\lambda^2}g_{x_i}\right|\right)dxdt\\ &=\int_\Omega|u(x,\tau)-v(x,\tau)|\xi(x)dx+\int_s^\tau\int_{\Omega_\lambda}|u-v|\left[-\frac{2(g-\lambda)}{\lambda^2}\left(\Delta g+\gamma\sum_{i=1}^N\left|g_{x_i}\right|\right)\right]dxdt\\ &\leq\int_\Omega|u(x,\tau)-v(x,\tau)|\xi(x)dx+\frac{c}{\lambda}\int_s^\tau\int_{\Omega_\lambda}|u-v|\left(\Delta g+\gamma\sum_{i=1}^N\left|g_{x_i}\right|\right)dxdt.\end{aligned} \quad (77)$$

Since by (20) and (31),

$$\begin{aligned}\lim_{\lambda\longrightarrow 0}\frac{1}{\lambda}\int_{\Omega_\lambda}|u-v|\left(\Delta g+\gamma\sum_{i=1}^{N}\left|g_{x_i}\right|\right)dx &= \int_{\partial\Omega}|u-v|\left(\Delta g+\gamma\sum_{i=1}^{N}\left|g_{x_i}\right|\right)d\Sigma \\ &= \int_{\Sigma_p\cup\Sigma_p'}|u-v|\left(\Delta g+\gamma\sum_{i=1}^{N}\left|g_{x_i}\right|\right)d\Sigma \\ &\le \int_{\Sigma_p}|u-v|\left(\Delta g+\gamma\sum_{i=1}^{N}\left|g_{x_i}\right|\right)d\Sigma = 0.\end{aligned} \tag{78}$$

Accordingly, letting $\lambda \longrightarrow 0$, we have

$$\int_\Omega |u(x,s)-v(x,s)|\,dx \le \int_\Omega |u(x,\tau)-v(x,\tau)|\,dx. \tag{79}$$

Let $\tau \longrightarrow 0$. Then

$$\int_\Omega |u(x,s)-v(x,s)|\,dx \le \int_\Omega |u(x,0)-v(x,0)|\,dx. \tag{80}$$

Theorem 4 is proved. □

4. The Proof of Theorem 6

Let $W^{1,p(x)}(\Omega)$ be the variable exponent Sobolev space. One can refer to [22–24] for the following lemma.

Lemma 8. *(i) The space $L^{p(x)}(\Omega)$, $W^{1,p(x)}(\Omega)$, and $W_0^{1,p(x)}(\Omega)$ are reflexive Banach spaces.*

(ii) $p(x)$-Hölder's inequality. Let $q_1(x)$ and $q_2(x)$ be real functions with $1/q_1(x)+1/q_2(x)=1$ and $q_1(x)>1$. Then, the conjugate space of $L^{q_1(x)}(\Omega)$ is $L^{q_2(x)}(\Omega)$. For any $u\in L^{q_1(x)}(\Omega)$ and $v\in L^{q_2(x)}(\Omega)$,

$$\left|\int_\Omega uv\,dx\right| \le 2\|u\|_{L^{q_1(x)}(\Omega)}\|v\|_{L^{q_2(x)}(\Omega)}. \tag{81}$$

(iii)

$$\begin{aligned}&\text{If } \|u\|_{L^{p(x)}(\Omega)}=1, \\ &\text{then } \int_\Omega |u|^{p(x)}dx = 1. \\ &\text{If } \|u\|_{L^{p(x)}(\Omega)}>1, \\ &\text{then } |u|^{p^-}_{L^{p(x)}(\Omega)} \le \int_\Omega |u|^{p(x)}dx \le |u|^{p^+}_{L^{p(x)}(\Omega)}. \\ &\text{If } \|u\|_{L^{p(x)}(\Omega)}<1, \\ &\text{then } |u|^{p^+}_{L^{p(x)}(\Omega)} \le \int_\Omega |u|^{p(x)}dx \le |u|^{p^-}_{L^{p(x)}(\Omega)}.\end{aligned} \tag{82}$$

(iv) If $p_1(x)\le p_2(x)$, then

$$L^{p_1(x)}(\Omega)\supset L^{p_2(x)}(\Omega). \tag{83}$$

(v) If $p_1(x)\le p_2(x)$, then

$$W^{1,p_2(x)}(\Omega)\hookrightarrow W^{1,p_1(x)}(\Omega). \tag{84}$$

(vi) $p(x)$-Poincarés inequality. If $p(x)\in C(\Omega)$, then there is a constant $C>0$, such that

$$\|u\|_{L^{p(x)}(\Omega)} \le C\|\nabla u\|_{L^{p(x)}(\Omega)}, \quad \forall u\in W_0^{1,p(x)}(\Omega). \tag{85}$$

This implies that $\|\nabla u\|_{L^{p(x)}(\Omega)}$ and $\|u\|_{W^{1,p(x)}(\Omega)}$ are equivalent norms of $W_0^{1,p(x)}(\Omega)$.

In order to prove Theorem 6, we let $g(x)$ be a weak characteristic function of Ω and define $S_\eta(s)$, $h_\eta(s)$, and $I_\eta(s)$ as in Section 2.

Theorem 9. *Let $u(x,t)$ and $v(x,t)$ be two weak solutions of (36) with the initial values $u_0(x)$ and $v_0(x)$, respectively, and with the same partial boundary value condition*

$$\Sigma_p = \left\{x\in\partial\Omega : \sum_{i=1}^{N} g_{x_i}(x)\ne 0\right\}. \tag{86}$$

If $b^i(s)$ is a Lipschitz function, $a(x)$ satisfies (38),

$$\eta^{-p^+/p^-}\left(\int_{\Omega\setminus D_\eta} a(x)\left|\frac{\partial g}{\partial x_i}\right|^{p(x)}dx\right)^{1/p^-} \le c, \quad i=1,2,\ldots,N, \tag{87}$$

then there holds

$$\int_\Omega |u(x,t)-v(x,t)|\,dx \le c\int_\Omega |u_0(x)-v_0(x)|\,dx, \quad t\in[0,T), \tag{88}$$

where $D_\eta = \{x\in\Omega : g(x)>\eta\}$ and $g(x)$ is a weak characteristic function of Ω.

Proof. For any given weak characteristic function $g(x)$, we define

$$\varphi_\eta(x) = \begin{cases}\dfrac{1}{\eta}g(x), & g(x)<\eta,\\ 1, & g(x)\ge\eta,\end{cases} \tag{89}$$

where η is a positive constant small enough.

In view of the definition of weak solution, by a process of limit, letting $\varphi_1=\chi_{s,t}\varphi_n(x)$ and $\varphi_2=g_n(u-v)$, we can choose $\chi_{s,t}\varphi_\eta S_\eta(u-v)$ as the test function, where $[s,t]\subseteq(0,T)$, and $\chi_{s,t}$ is its characteristic function. Then we have

$$\int_s^t\int_\Omega \varphi_\eta(x)S_\eta(u-v)\frac{\partial(u-v)}{\partial t}dx\,dt + \int_s^t\int_\Omega a(x) \cdot\left(|\nabla u|^{p(x)-2}\nabla u - |\nabla v|^{p(x)-2}\nabla v\right)\nabla(u-v)\,h_\eta(u$$

$$
\begin{aligned}
&- v)\varphi_\eta(x)\,dxdt + \int_s^t\int_\Omega a(x)\left(|\nabla u|^{p(x)-2}\nabla v\right.\\
&\left.- |\nabla v|^{p(x)-2}\nabla v\right)S_\eta(u-v)\nabla\varphi_\eta dxdt\\
&+ \sum_{i=1}^N\int_s^t\int_\Omega\left[b^i(u)-b^i(v)\right]\cdot(u-v)_{x_i}h_\eta(u-v)\\
&\cdot\varphi_\eta(x)\,dxdt + \sum_{i=1}^N\int_s^t\int_\Omega\left[b^i(u)-b^i(v)\right]\\
&\cdot S_\eta(u-v)\varphi_{\eta x_i}(x)\,dxdt = 0.
\end{aligned}
\tag{90}
$$

As $n \longrightarrow \infty$, we have

$$
\begin{aligned}
&\lim_{\eta\longrightarrow 0}\int_\tau^s\int_\Omega\varphi_\eta(x)S_\eta(u-v)\frac{\partial(u-v)}{\partial t}dxdt\\
&= \lim_{\eta\longrightarrow 0}\int_\tau^s\int_\Omega\frac{\partial\left(\varphi_\eta(x)I_\eta(u-v)\right)}{\partial t}dxdt\\
&= \lim_{\eta\longrightarrow 0}\int_\Omega\varphi_\eta(x)\\
&\cdot\left[I_\eta(u-v)(x,s)-I_\eta(u-v)(x,\tau)\right]dx\\
&= \int_\Omega|u-v|(x,s)\,dx - \int_\Omega|u-v|(x,\tau)\,dx.
\end{aligned}
\tag{91}
$$

Denote

$$
\begin{gathered}
D_\eta = \{x\in\Omega : g(x)>\eta\}\\
\text{and } q(x) = \frac{p(x)}{p(x)-1}.
\end{gathered}
\tag{92}
$$

Note that $|\varphi_{\eta x_i}| = (1/\eta)|g_{x_i}|$ and $x\in\Omega\setminus D_\eta$. We may assume that

$$
\left\|\frac{1}{\eta}[a(x)]^{1/p(x)}S_\eta(u-v)g_{x_i}\right\|_{L^{p(x)}(\Omega\setminus D_\eta)} > 1,
\tag{93}
$$

without loss the generality. Using (ii) of Lemma 8, we have

$$
\begin{aligned}
&\left\|\frac{1}{\eta}[a(x)]^{1/p(x)}S_\eta(u-v)g_{x_i}\right\|_{L^{p(x)}(\Omega\setminus D_\eta)}\\
&\le\left\|\frac{1}{\eta}[a(x)]^{1/p(x)}g_{x_i}\right\|_{L^{p(x)}(\Omega\setminus D_\eta)}\\
&\le\left(\int_{\Omega\setminus D_\eta}a(x)\left(\frac{1}{\eta}\right)^{p(x)}\left|\frac{\partial g}{\partial x_i}\right|^{p(x)}dx\right)^{1/p^-}\\
&\le\left(\frac{1}{\eta}\right)^{p^+/p^-}\left(\int_{\Omega\setminus D_\eta}a(x)\left|\frac{\partial g}{\partial x_i}\right|^{p(x)}dx\right)^{1/p^-}\le c.
\end{aligned}
\tag{94}
$$

We further have

$$
\begin{aligned}
&\left|\int_\Omega a(x)\right.\\
&\left.\cdot\left(|\nabla u|^{p(x)-2}\nabla u-|\nabla v|^{p(x)-2}\nabla v\right)\nabla\varphi_\eta S_\eta(u-v)\,dx\right|\\
&=\left|\int_{\Omega\setminus D_\eta}a(x)\right.\\
&\left.\cdot\left(|\nabla u|^{p(x)-2}\nabla u-|\nabla v|^{p(x)-2}\nabla v\right)\nabla\varphi_\eta S_\eta(u-v)\,dx\right|\\
&\le\left\|[a(x)]^{(p(x)-1)/p(x)}\right.\\
&\left.\cdot\left(|\nabla u|^{p(x)-1}+|\nabla v|^{p(x)-1}\right)\right\|_{L^{q(x)}(\Omega\setminus D_\eta)}\\
&\cdot\left\|\frac{1}{\eta}[a(x)]^{1/p(x)}S_\eta(u-v)\nabla g\right\|_{L^{p(x)}(\Omega\setminus D_\eta)}\\
&\le c\left[\left(\int_{\Omega\setminus D_\eta}a(x)|\nabla u|^{p(x)}dx\right)^{1/q_1}\right.\\
&\left.+\left(\int_{\Omega\setminus D_\eta}a(x)|\nabla u|^{p(x)}dx\right)^{1/q_1}\right],
\end{aligned}
\tag{95}
$$

which goes to zero as $n\longrightarrow 0$, where q_1 is taken to be q^- (or q^+) if

$$
\left\||\nabla u|^{p(x)-1}\right\|_{L^{q(x)}(\Omega\setminus D_\eta)} > 1 \quad (\text{or} \le 1).
\tag{96}
$$

Consider the convection term

$$
\begin{aligned}
&\int_{\{x\in\Omega:|u-v|<\eta\}}[a(x)]^{1/(1-p(x))}\\
&\cdot\left(\left|\frac{b^i(u)-b^i(v)}{u-v}\right|\right)^{p(x)/(p(x)-1)}dx\\
&\leqslant c\int_\Omega[a(x)]^{1/(1-p(x))}dx\leqslant c,
\end{aligned}
\tag{97}
$$

since b^i is a Lipschitz function.

If $\{x\in\Omega : |u-v|=0\}$ is a set with measure zero, it has

$$
\begin{aligned}
&\lim_{\eta\longrightarrow 0}\int_{\{x\in\Omega:|u-v|<\eta\}}[a(x)]^{1/(1-p(x))}dx\\
&=\int_{\{x\in\Omega:|u-v|=0\}}[a(x)]^{1/(1-p(x))}dx = 0.
\end{aligned}
\tag{98}
$$

If the set $\{x\in\Omega : |u-v|=0\}$ has a positive measure, due to the fact that $a(x)|\nabla|^{p(x)}$, $a(x)|\nabla v|^{p(x)}\in L^1(Q_T)$, we have

$$
\begin{aligned}
&\lim_{\eta\longrightarrow 0}\int_{\{x\in\Omega:|u-v|<\eta\}}a(x)|\nabla(u-v)|^{p(x)}dx\\
&=\int_{\{x\in\Omega:|u-v|=0\}}a(x)|\nabla(u-v)|^{p(x)}dx = 0.
\end{aligned}
\tag{99}
$$

According to (81), when $|s| \leqslant \eta$, it has

$$\left|sS_\eta{}'(s)\right| \leqslant c. \tag{100}$$

By Lemma 8, we have

$$\begin{aligned}
&\left|\int_{\{x\in\Omega:|u-v|<\eta\}} \varphi_\eta \left[b^i(u) - b^i(v)\right] h_\eta(u-v) \cdot (u-v)_{x_i}\, dx\right| \\
&\leqslant c\int_{\{x\in\Omega:|u-v|<\eta\}} \left|\frac{b^i(u)-b^i(v)}{u-v}(u-v)_{x_i}\right| dx \\
&\leqslant c\left\|[a(x)]^{-1/p(x)}\frac{b^i(u)-b^i(v)}{u-v}\right\|_{L^{q(x)}(\Omega_\eta)} \cdot \left\|[a(x)]^{1/p(x)}(u-v)_{x_i}\right\|_{L^{p(x)}(\Omega_\eta)} \\
&\leqslant c\left\{\int_{\{x\in\Omega:|u-v|<\eta\}} [a(x)]^{1/(1-p(x))} \cdot \left|\frac{b_i(u)-b_i(v)}{u-v}\right|^{p(x)/(p(x)-1)} dx\right\}^{1/q_1} \\
&\cdot \left\{\int_{\{x\in\Omega:|u-v|<\eta\}} a(x)\left|(u-v)_{x_i}\right|^{p(x)} dx\right\}^{1/p_1},
\end{aligned} \tag{101}$$

where $\Omega_\eta = \{x \in \Omega : |u(x,t) - v(x,t)| < \eta\}$, q_1 is taken to be q^- (or q^+) according to (iii) of Lemma 8, and p_1 is p^- (or p^+) similarly.

By (97)-(99) and (101), we have

$$\lim_{\eta\longrightarrow 0}\int_\Omega \left[b^i(u) - b^i(v)\right]\varphi_\eta(x) h_\eta(u-v)(u-v)_{x_i}\, dx = 0. \tag{102}$$

Meanwhile,

$$\begin{aligned}
&\lim_{\eta\longrightarrow 0}\left|\int_\Omega \left[b^i(u)-b^i(v)\right] S_\eta(u-v)\varphi_{\eta x_i}(x)\, dx\right| \\
&\leq c\lim_{\eta\longrightarrow 0}\frac{1}{\eta}\int_{\Omega\setminus D_\eta}\left|(u-v)\sum_{i=1}^N \frac{\partial g}{\partial x_i}\right| dx \\
&\leq c\int_{\Sigma_p} |u-v|\, dx = 0.
\end{aligned} \tag{103}$$

Let $\eta \longrightarrow 0$ in (90). Using (91) and (95) in combination with (102)-(103), we obtain

$$\frac{d}{dt}\|u-v\|_{L^1(\Omega)} \leqslant 0 \tag{104}$$

and thus arrive at

$$\int_\Omega |u(x,t) - v(x,t)|\, dx \leqslant \int_\Omega \left|u_0(x) - v_0(x)\right| dx, \quad \forall t \in [0,T). \tag{105}$$

□

Proof of Theorem 6. If we choose the weak characteristic function $g(x) = a(x)$, then the part of the boundary Σ_p of (39) is just the same as (86) and condition (57) is just the same as (87). Theorem 6 follows from Theorem 9 directly. □

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares that he have no conflicts of interest.

Authors' Contributions

The author reads and approves the final manuscript.

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