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Research Article

On the Approximation Properties of q – Analogue Bivariate λ-Bernstein Type Operators

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In this article, we establish an extension of the bivariate generalization of the q-Bernstein type operators involving parameter λ and extension of GBS (Generalized Boolean Sum) operators of bivariate q-Bernstein type. For the first operators, we state the Volkov-type theorem and we obtain a Voronovskaja type and investigate the degree of approximation by means of the Lipschitz type space. For the GBS type operators, we establish their degree of approximation in terms of the mixed modulus of smoothness. The comparison of convergence of the bivariate q-Bernstein type operators based on parameters and its GBS type operators is shown by illustrative graphics using MATLAB software.

1. Introduction

Let $h \in C(S)$ with $S = [0, 1], \lambda \in [-1, 1]$, and $m \in \mathbb{N}$. In 2018, Chen et al. [1] proposed a new generalization of Bernstein operators based on a fixed real parameter $\lambda \in [-1, 1]$ as

$$\mathscr{B}_m^{\lambda}(h\,; y) = \sum_{k=0}^{m} \bar{\Omega}_{m,k}^{(\lambda)}(y) h\left(\frac{k}{m}\right), \quad x \in S, \tag{1}$$

where the basis functions $\bar{\Omega}_{m,k}^{(\lambda)}(y)$ are defined as

$$\begin{aligned}
\bar{\Omega}_{m,0}^{(\lambda)}(y) &= \Omega_{m,0}(y) - \frac{\lambda}{m+1}\Omega_{m+1,1}(y),\\
\bar{\Omega}_{m,k}^{(\lambda)}(y) &= \Omega_{m,k}(y) + \lambda\left(\frac{m-2j+1}{m^2-1}\Omega_{m+1,k}(y) - \frac{m-2j-1}{m^2-1}\Omega_{m+1,k+1}(y)\right),\\
&\quad 1 \le k \le m-1,\\
\bar{\Omega}_{m,m}^{(\lambda)}(y) &= \Omega_{m,m}(y) - \frac{\lambda}{m+1}\Omega_{m+1,m}(y),
\end{aligned} \tag{2}$$

The authors studied the established of some Korovkin type approximation properties and the degree of approximation by means of the modulus of continuity, Voronovskaja-type results, and shape-preserving properties for these operators.

This work took the attention of researchers from approximation theory for a short time. Since that time, lots of researchers have put forth many relevant studies on this issue, and numerous articles can be given interrelated with their work [2–5].

In [6] were introduced the bivariate extension of the operators (1) and studied the degree of approximation in terms of the second order Ditzian-Totik modulus of continuity for two variables. A Kantorovich variant of the λ-Bernstein operators (1) was introduced and studied in [7]. Many authors also considered the univariate and bivariate positive linear operators and studied their approximation behavior; we refer the reader to articles (cf. [8–17]) and references therein. Now, we give some basic definitions based on the q-calculus [18], which are used in this paper. Let $0 < q < 1$ and b, x be any real numbers.

The q-number $[b]_q$ is defined as $[b]_q = \{1 - q^b/1 - q, q \neq 1\}$ and for $b = n \in \mathbb{N}$,

$$[n]_q = 1 + q + \cdots + q^{n-1}. \tag{3}$$

The q-number $(1-x)_q^b$ is defined as $(1-x)_q^b = \prod_{j=0}^{\infty} 1-x q^j/1-xq^{j+b}$ and for $b=n\in\mathbb{N}$

$$(1-x)_q^n = (1-x)(1-qx)\cdots\left(1-q^{n-1}x\right). \tag{4}$$

For the integers n, j such that $0\le j\le n$, the q-binomial is defined as

$$\begin{bmatrix} m \\ j \end{bmatrix}_q = \frac{[n]_q!}{[j]_q![n-j]_q!}. \tag{5}$$

For an integer n, the q-factorial is defined as

$$[n]_q! = [n]_q[n-1]_q! \quad \text{if} \quad n\ge 1, \tag{6}$$

and $[n]_q! = 1$ if $n=0$.

Cai et al. [19] considered the generalized Bernstein type operators based on parameters $q-$analogue and for fixed real parameter $\lambda\in[-1,1]$ as

$$\mathscr{B}_m^{\lambda}(h;q,y) = \sum_{j=0}^{m} \bar{\Omega}_{m,j}^{(\lambda,q)}(y) h\left(\frac{[j]_q}{[m]_q}\right), \quad y\in S, \tag{7}$$

where the basis functions $\bar{\Omega}_{m,k}^{(\lambda,q)}(y)$ are defined as

$$\bar{\Omega}_{m,0}^{(\lambda,q)}(y) = \Omega_{m,0}^{q}(y) - \frac{\lambda}{[m]_q+1}\Omega_{m+1,1}^{q}(y), \tag{8}$$

$$\begin{aligned}\bar{\Omega}_{m,j}^{(\lambda,q)}(y) = \Omega_{m,j}^{q}(y) + \lambda\Bigg(&\frac{[m]_q - 2[j]_q + 1}{[m]_q^2-1}\Omega_{m+1,j}(y) \\ &- \frac{[m]_q - 2q[j]_q - 1}{[m]_q^2-1}\Omega_{m+1,j+1}^{q}(y)\Bigg), 1\le j\le m-1,\end{aligned} \tag{9}$$

$$\bar{\Omega}_{m,m}^{(\lambda,q)}(y) = \Omega_{m,m}^{q}(y) - \frac{\lambda}{m+1}\Omega_{m+1,m}^{q}(y), \tag{10}$$

and $\Omega_{m,j}^{q}(y) = \begin{bmatrix} m \\ k \end{bmatrix}_q y^j\prod_{s=0}^{m}(1-q^s y)$. Note that for $q=1$, these operators reduce to λ-Bernstein operators (1) and for $q=\lambda=1$ (7) educes to Bernstein operators defined in [20].

Therefore, linear operators, in particular the limit q-Bernstein operator, are of significant interest for applications.

The purpose of this article is to present an extension of the bivariate λ, q-Bernstein type operators involving parameters and obtain the degree of approximation by means of the Lipschitz type space for two variables. Moreover, we consider the associated Generalized Boolean Sum (GBS) operators and study their degree of approximation in terms of the mixed modulus of smoothness for bivariate functions.

2. Construction of the Bivariate q,λ-Bernstein Type Operators

For $S^2=[0,1]\times[0,1]$, let $C(S^2)$ be the space of all continuous functions on S^2.

For $h\in C(S^2)$ and $\lambda_1,\lambda_2\in[-1,1]$, the bivariate extension of the operator (7) is defined by

$$\begin{aligned}\mathscr{B}_{m_1,m_2,q_{m_1}}^{\lambda_1,\lambda_2,q_{m_2}}(h;y_1,y_2) = \sum_{j_1=0}^{m_1}\sum_{j_2=0}^{m_2}\Omega_{m_1,m_2,j_1,j_2}^{(\lambda_1,\lambda_2)}(y_1,y_2)h \\ \cdot\left(\frac{[j_1]_{q_1}}{[m_1]_{q_1}}, \frac{[j_2]_{q_2}}{[m_2]_{q_2}}\right),\end{aligned} \tag{11}$$

where $\{q_{m_i}\}_{m_i\in N}$ is a sequence in $(0,1)$ satisfying

$$q_{m_i}\longrightarrow 1 \quad \text{and} \quad q_{m_i}^{m_i}\longrightarrow c_i; fori=1,2. \tag{12}$$

Further, $\Omega_{m_1,m_2,j_1,j_2}^{(\lambda_1,\lambda_2)}(y_1,y_2) = \Omega_{m_1,j_1}^{(\lambda_1)}(y_1)\Omega_{m_2,j_2}^{(\lambda_2)}(y_2)$, where $\Omega_{m_1,j_1}^{(\lambda_1)}(y_1)$ and $\Omega_{m_2,j_2}^{(\lambda_2)}(y_2)$ are defined similarly as $\Omega_{m,j}^{(\lambda)}(y)$ in (8).

It is easy to see that $\mathscr{B}_{m_1,m_2,q_{m_1}}^{\lambda_1,\lambda_2,q_{m_2}}(h;y_1,y_2)$ is bounded. We denote

$$\begin{aligned}&\mathscr{B}_{m_1,m_2,q_{m_1}}^{\lambda_1,\lambda_2,q_{m_2}}\left((u-y_1)^s(v-y_2)^t;x,y\right) \\ &= \mathscr{B}_{m_1}^{(\lambda_1)}\left((u-y_1)^s;q_{m_1},y_1\right)\mathscr{B}_{m_2}^{(\lambda_1)}\left((v-y_2)^t;q_{m_2},y_2\right) \\ &= \mu_{m_1,\lambda_1,s}\left(q_{m_1},y_1\right)\mu_{m_2,\lambda_2,t}\left(q_{m_2},y_2\right) s,t\in N\cup\{0\}.\end{aligned} \tag{13}$$

Lemma 1 (see [19]). *For the operators $\mathscr{B}_{m_1}^{\lambda_1}(h,q_1,y_1)$, we have*

(i) $\mathscr{B}_{m_1}^{\lambda_1}(1;q_1,y_1)=1$

(ii) $\mathscr{B}_{m_1}^{(\lambda_1)}(t;q_1,y_1) = y_1 + ([m_1+1]_{q_1}y_1(1-y_1^{m_1})\lambda_1/[m_1]_{q_1}([m_1]_{q_1})) - (2[m+1]_{q_1}y_1\lambda_1/[m]_{q_1}^2-1)[(1-y_1^{m_1}/[m_1]_{q_1}) + q_1y_1(1-y_1^{n_1-1})] + (\lambda_1/q_1[m_1]_{q_1}([m_1]_{q_1}+1))[1-\prod_{i=0}^{m_1}(1-q^iy_1) - y_1^{m_1+1} - [m_1+1]_{q_1}y_1(1-y_1^{m_1-1})] + (\lambda_1/[m_1]_{q_1}^2-1)\{2[m_1+1]_{q_1}y_1^2(1-y_1^{m_1-1}) - (2[m_1+1]_{q_1}y_1/q_1[m_1]_{q_1})(1-y_1^{m_1})\} + (2/q_1[m_1]_{q_1})[1-\prod_{i=0}^{m_1}(1-q_1^iy_1) - y_1^{m_1+1}]$

(iii) $\mathscr{B}_{m_1}^{\lambda_1}(t^2;q_1,y_1) = y_1^2 + (y_1(1-y_1)/[m_1]_{q_1}) + (y_1\lambda_1[m_1+1]_{q_1}/[m_1]_{q_1}([m_1]_{q_1}-1))[q_1y_1(1-y_1^{m_1-1}) + (1-y_1^{m_1}/[m_1]_{q_1})] - (2[m_1+1]_{q_1}\lambda_1/[m_1]_{q_1}([m_1]_{q_1}^2-1))[(y_1(1-y_1^{m_1})/[m_1]_{q_1}) + q_1(2+q_1)y_1^2(1-y_1^{m_1-1}) + q_1^3[m_1-1]_{q_1}y_1^3(1-y_1^{m_1-2})] - (\lambda_1/q_1[m_1]_{q_1}([m_1]_{q_1}+1))\{[m_1+1]_{q_1}y_1^2(1-y_1^{m_1-1}) - ([m_1+1]_{q_1}y_1(1-y_1^{m_1})/q_1[m_1]_{q_1}) + (1-\prod_{i=0}^{m_1}(1-q_1^iy_1) - y_1^{m_1+1}/q_1[m_1]_{q_1})\} +$

$(2\lambda_1/[m_1]_{q_1}([m_1]_{q_1}^2-1))\{q_1[m_1-1]_{q_1}[m_1+1]_{q_1}y_1^3(1-y_1^{m_1-2})-((1-q_1)[m_1+1]_{q_1}y_1^2(1-y_1^{m_1-1})/q_1)+([m_1+1]_{q_1}y_1(1-y_1^{m_1})/q_1^2[m_1]_{q_1})-1-\prod_{i=0}^{m_1}(1-q_1^iy_1)-y_1^{m_1+1}/q_1^2[m_1]_{q_1}\}$

Let $e_{rs}(y_1,y_2)=t_1^rt_2^s,(r,s)\in\mathbb{N}_0\times\mathbb{N}_0,$ with $r+s\leq 2$. In order to obtain the main results, we need the following lemmas:

Lemma 2. *For the operators* $\mathscr{B}_{m_1,m_2,q_{m_1}}^{\lambda_1,\lambda_2,q_{m_2}}$, *we have*

(i) $\mathscr{B}_{m_1,m_2,q_{m_1}}^{\lambda_1,\lambda_2,q_{m_2}}(1;y_1,y_2)=1$

(ii) $\mathscr{B}_{m_1,m_2,q_{m_1}}^{\lambda_1,\lambda_2,q_{m_2}}(e_{10};y_1,y_2)=y_1+([m_1+1]_{q_1}y_1(1-y_1^{m_1})\lambda_1/[m_1]_{q_1}([m_1]_{q_1}-1))-(2[m_1+1]_{q_1}y_1\lambda_1/[m]_{q_1}^2-1)[(1-y_1^{m_1}/[m_1]_{q_1})+q_1y_1(1-y_1^{m_1-1})]+(\lambda_1/q_1[m_1]_{q_1}([m_1]_{q_1}+1))[1-\prod_{i=0}^{m_1}(1-q_1^iy_1)-y_1^{m_1+1}-[m_1+1]_{q_1}y_1(1-y_1^{m_1})]+(\lambda_1/[m_1]_{q_1}^2-1)\{2[m_1+1]_{q_1}y_1^2(1-y_1^{m_1-1})-(2[m_1+1]_{q_1}y_1/q_1[m_1]_{q_1})(1-y_1^{m_1})+(2/q_1[m_1]_{q_1})[1-\prod_{i=0}^{m_1}(1-q_1^iy_1)-y_1^{m_1+1}]\}$

(iii) $\mathscr{B}_{m_1,m_2,q_{m_1}}^{\lambda_1,\lambda_2,q_{m_2}}(e_{20};y_1,y_2)=y_1^2+(y_1(1-y_1)/[m_1]_{q_1})+(y_1\lambda_1[m_1+1]_{q_1}/[m_1]_{q_1})[q_1y_1(1-y_1^{m_1-1})+(1-y_1^{m_1}/[m_1]_{q_1})]-(2[m_1+1]_{q_1}\lambda_1/[m_1]_{q_1}([m_1]_{q_1}^2-1))[(y_1(1-y_1^{m_1})/[m_1]_{q_1})+q_1(2+q_1)y_1^2(1-y_1^{m_1-1})+q_1^3[m_1-1]_{q_1}y_1^3(1-y_1^{m_1-1})]-(\lambda_1/q_1[m_1]_{q_1}([m_1]_{q_1}+1))\{[m_1+1]_{q_1}y_1^2(1-y_1^{m_1-1})-([m_1+1]_{q_1}y_1(1-y_1^{m_1})/q_1[m_1]_{q_1})+(1-\prod_{i=0}^{m_1}(1-q_1^iy_1)-y_1^{m_1+1}/q_1[m_1]_{q_1})\}+(2\lambda_1/[m_1]_{q_1}([m_1]_{q_1}^2-1))\{q_1[m_1-1]_{q_1}[m_1+1]_{q_1}y_1^3(1-y_1^{m_1-2})-((1-q_1)[m_1+1]_{q_1}y_1^2(1-y_1^{m_1-1})/q_1)+([m_1+1]_{q_1}y_1(1-y_1^{m_1})/q_1^2[m_1]_{q_1})-1-\prod_{i=0}^{m_1}(1-q_1^iy_1)-y_1^{m_1+1}/q_1^2[m_1]_{q_1}\}+(2/q_1[m_1]_{q_1})[1-\prod_{i=0}^{m_1}(1-q_1^iy_1)-y_1^{m_1+1}]$

(iv) $\mathscr{B}_{m_1,m_2,q_{m_1}}^{\lambda_1,\lambda_2,q_{m_2}}(e_{01};x,y)=y_2+([m_2+1]_{q_2}y_2(1-y_2^{m_2})\lambda_2/[m_2]_{q_2}([m_2]_{q_2}-1))-(2[m_2+1]_{q_2}y_2\lambda_2/[m]_{q_2}^2-1)[(1-y_2^{m_2}/[m_2]_{q_2})+q_2y_2(1-y_2^{m_2-1})]+(\lambda_2/q_2[m_2]_{q_2}([m_2]_{q_2}+1))[1-\prod_{i=0}^{m_2}(1-q_2^iy_2)-y_2^{m_2+1}-[m_2+1]_{q_2}y_2(1-y_2^{m_2})]+(\lambda_2/[m_2]_{q_2}^2-1)\{2[m_2+1]_{q_2}y_2^2(1-y_2^{m_2-1})-(2[m_2+1]_{q_2}y_2/q_2[m_2]_{q_2})(1-y_2^{m_2})+(2/q_2[m_2]_{q_2})[1-\prod_{i=0}^{m_2}(1-q_2^iy_2)-y_2^{m_2+1}]\}$

(v) $\mathscr{B}_{m_1,m_2,q_{m_1}}^{\lambda_1,\lambda_2,q_{m_2}}(e_{02};y_1,y_2)=y_2^2+(y_2(1-y_2)/[m_2]_{q_2})+(y_2\lambda_2[m_2+1]_{q_2}/[m_2]_{q_2}([m_2]_{q_2}-1))[q_2y_2(1-y_2^{m_2-1})+(1-y_2^{m_2}/[m_2]_{q_2})]-(2[m_2+1]_{q_2}\lambda_2/[m_2]_{q_2}([m_2]_{q_2}^2-1))[(y_2(1-y_2^{m_2})/[m_2]_{q_2})+q_2(2+q_2)y_2^2(1-y_2^{m_2-1})+q_2^3[m_2-1]_{q_2}y_2^3(1-y_2^{m_2-1})]-(\lambda_2/q_2[m_2]_{q_2}([m_2]_{q_2}+1))\{[m_2+1]_{q_2}y_2^2(1-y_2^{m_2-1})-([m_2+1]_{q_2}y_2(1-y_2^{m_2})/q_2[m_2]_{q_2})+(1-\prod_{i=0}^{m_2}(1-q_2^iy_2)-y_2^{m_2+1}/q_2[m_2]_{q_2})\}+(2\lambda_2/[m_2]_{q_2}([m_2]_{q_2}^2-1))\{q_2[m_2-1]_{q_2}[m_2+1]_{q_2}y_2^3(1-y_2^{m_2-2})-((1-q_2)[m_2+1]_{q_2}y_2^2(1-y_2^{m_2-1})/q_2)+([m_2+1]_{q_2}y_2(1-y_2^{m_2})/q_2^2[m_2]_{q_2})-1-\prod_{i=0}^{m_2}(1-q_2^iy_2)-y_2^{m_2+1}/q_2^2[m_2]_{q_2}\}$

Corollary 3. *Applying Lemma 2, we have*

(i) $\mathscr{B}_{m_1,m_2,q_{m_1}}^{\lambda_1,\lambda_2,q_{m_2}}(u-y_1;y_1,y_2)=[m_1+1]_{q_1}y_1(1-y_1^{m_1})\lambda_1/[m_1]_{q_1}([m_1]_{q_1}-1)-(2[m+1]_{q_1}y_1\lambda_1/[m]_{q_1}^2-1)[(1-y_1^{m_1}/[m_1]_{q_1})+q_1y_1(1-y_1^{m_1-1})]+(\lambda_1/q_1[m_1]_{q_1}([m_1]_{q_1}+1))[1-\prod_{i=0}^{m_1}(1-q_1^iy_1)-y_1^{m_1+1}-[m_1+1]_{q_1}y_1(1-y_1^{m_1})]+(\lambda_1/[m_1]_{q_1}^2-1)\{2[m_1+1]_{q_1}y_1^2(1-y_1^{m_1-1})-(2[m_1+1]_{q_1}y_1/q_1[m_1]_{q_1})(1-y_1^{m_1})+(2/q_2[m_2]_{q_2})[1-\prod_{i=0}^{m_2}(1-q_2^iy_2)-y_2^{m_2+1}]\}$

(ii) $\mathscr{B}_{m_1,m_2,q_{m_1}}^{\lambda_1,\lambda_2,q_{m_2}}(v-y_2;y_1,y_2)=[m_2+1]_{q_2}y_2(1-y_2^{m_2})\lambda_2/[m_2]_{q_2}([m_2]_{q_2}-1)-(2[m_2+1]_{q_2}y_2\lambda_2/[m]_{q_2}^2-1)[(1-y_2^{m_2}/[m_2]_{q_2})+q_2y_2(1-y_2^{m_2-1})]+(\lambda_2/q_2[m_2]_{q_2}([m_2]_{q_2}+1))[1-\prod_{i=0}^{m_2}(1-q_2^iy_2)-y_2^{m_2+1}-[m_2+1]_{q_2}y_2(1-y_2^{m_2})]+(\lambda_2/[m_2]_{q_2}^2-1)\{2[m_2+1]_{q_2}y_2^2(1-y_2^{m_2-1})-(2[m_2+1]_{q_2}y_2/q_2[m_2]_{q_2})(1-y_2^{m_2})+(2/q_2[m_2]_{q_2})[1-\prod_{i=0}^{m_2}(1-q_2^iy_2)-y_2^{m_2+1}]\}$

(iii) $\mathscr{B}_{m_1,m_2,q_{m_1}}^{\lambda_1,\lambda_2,q_{m_2}}((u-y_1)^2;y_1,y_2)=(y_1(1-y_1)/[m_1]_{q_1})+([m_1+1]_{q_1}y_1\lambda_1/[m_1]_{q_1}([m_1]_{q_1}-1))[q_1y_1(1-y_1^{m_1-1})+(1-y_1^{m_1}/[m_1]_{q_1})-2y_1(1-y_1^{m_1})]-(2[m_1+1]_{q_1}\lambda_1/[m_1]_{q_1}([m_1]_{q_1}^2-1))[(y_1(1-y_1^{m_1})/[m_1]_{q_1})+q_1(2+q_1)y_1^2(1-y_1^{m_1-1})+q_1^3[m_1-1]_{q_1}y_1^3(1-y_1^{m_1-1})+(4y_1[m_1+1]_{q_1}y_1\lambda_1/[m_1]_{q_1}^2-1)[(1-y_1^{m_1}/[m_1]_{q_1})+q_1y_1(1-y_1^{m_1-1})]]-(\lambda_1/q_1[m_1]_{q_1}([m_1]_{q_1}+1))\{[m_1+1]_{q_1}y_1^2(1-y_1^{m_1-1})-([m_1+1]_{q_1}y_1(1-y_1^{m_1})/q_1[m_1]_{q_1})+(1-\prod_{i=0}^{m_1}(1-q_1^iy_1)-y_1^{m_1+1}/q_1[m_1]_{q_1})-2y_1[1-\prod_{i=0}^{m_1}(1-q_1^iy_1)-y_1^{m_1+1}-[m_1+1]_{q_1}y_1(1-y_1^{m_1})]\}+(2\lambda_1/[m_1]_{q_1}([m_1]_{q_1}^2-1))\{q_1[m_1-1]_{q_1}[m_1+1]_{q_1}y_1^3(1-y_1^{m_1-2})-((1-q_1)[m_1+1]_{q_1}y_1^2(1-y_1^{m_1-1})/q_1)+([m_1+1]_{q_1}y_1(1-y^{m_1})/q_1^2[m_1]_{q_1})-1-\prod_{i=0}^{m_1}(1-q_1^iy_1)-y_1^{m_1+1}/q_1^2[m_1]_{q_1}\}-2y_1(\lambda_1/[m_1]_{q_1}^2-1)\{2[m_1+1]_{q_1}y_1^2(1-y_1^{m_1-1})-(2[m_1+1]_{q_1}y_1/q_1[m_1]_{q_1})(1-y_1^{m_1})+(2/q_1[m_1]_{q_1})[1-\prod_{i=0}^{m_1}(1-q_1^iy_1)-y_1^{m_1+1}]\}$

(iv) $\mathscr{B}^{\lambda_1,\lambda_2,q_{m_2}}_{m_1,m_2,q_{m_1}}((u-y_2)^2;y_1,y_2)=(y_2(1-y_2)/[m_2]_{q_2})+([m_2+1]_{q_2}y_2\lambda_2/[m_2]_{q_2}([m_2]_{q_2}-1))[q_2y_2(1-y_2^{m_2-1})+(1-y_2^{m_2}/[m_2]_{q_2})-2y_2(1-y_2^{m_2})]-(2[m_2+1]_{q_2}\lambda_2/[m_2]_{q_2}([m_2]^2_{q_2}-1))[(y_2(1-y_2^{m_2})/[m_2]_{q_2})+q_2(2+q_2)y_2^2(1-y_2^{m_2-1})+q_2^3[m_2-1]_{q_2}y_2^3(1-y_2^{m_2-1})+(4y_2[m_2+1]_{q_2}y_2\lambda_2/[m_2]^2_{q_2}-1)[(1-y_2^{m_2}/[m_2]_{q_2})+q_2y_2(1-y_2^{m_2-1})]]-(\lambda_2/q_2[m_2]_{q_2}([m_2]_{q_2}+1))\{[m_2+1]_{q_2}y_2^2(1-y_2^{m_2-1})-([m_2+1]_{q_2}y_2(1-y_2^{m_2})/q_2[m_2]_{q_2})+(1-\prod_{i=0}^{m_2}(1-q_2^iy_2)-y_2^{m_2+1}/q_2[m_2]_{q_2})-2y_2[1-\prod_{i=0}^{m_2}(1-q_2^iy_2)-y_2^{m_2+1}-[m_2+1]_{q_2}y_2(1-y_2^{m_2})]\}+(2\lambda_2/[m_2]_{q_2}([m_2]^2_{q_2}-1))\{q_2[m_2-1]_{q_2}[m_2+1]_{q_2}y_2^3(1-y_2^{m_2-2})-((1-q_2)[m_2+1]_{q_2}y_2^2(1-y_2^{m_2-1})/q_2)+([m_2+1]_{q_2}y_2(1-y_2^{m_2})/q_2^2[m_2]_{q_2})-1-\prod_{i=0}^{m_2}(1-q_2^iy_2)-y_2^{m_2+1}/q_2^2[m_2]_{q_2}\}-2y_2(\lambda_2/[m_2]^2_{q_2}-1)\{2[m_2+1]_{q_2}y_2^2(1-y_2^{m_2-1})-(2[m_2+1]_{q_2}y_2/q_2[m_2]_{q_2})(1-y_2^{m_2})+(2/q_2[m_2]_{q_2})[1-\prod_{i=0}^{m_2}(1-q_2^iy_2)-y_2^{m_2+1}]\}$

(v) $\lim_{m_1\to\infty}[m_1]_{q_{m_1}}\mathscr{B}^{\lambda_1,\lambda_2,q_{m_2}}_{m_1,m_2,q_{m_1}}(t_1-y_1;y_1,y_2)=0$

(vi) $\lim_{m_2\to\infty}[m_2]_{q_{m_2}}\mathscr{B}^{\lambda_1,\lambda_2,q_{m_2}}_{m_1,m_2,q_{m_1}}(t_2-y_2;y_1,y_2)=0$

(vii) $\lim_{m_1\to\infty}[m_1]_{q_{m_1}}\mathscr{B}^{\lambda_1,\lambda_2,q_{m_2}}_{m_1,m_2,q_{m_1}}((t_1-y_1)^2;y_1,y_2)=y_1(1-y_1), y_1\in(0,1)$

(viii) $\lim_{m_2\to\infty}[m_2]_{q_{m_2}}\mathscr{B}^{\lambda_1,\lambda_2,q_{m_2}}_{m_1,m_2,q_{m_1}}((t_2-y_2)^2;y_1,y_2)=y_2(1-y_2), y_2\in(0,1)$

In what follows, the Volkov-type approximation theorem is proved for $\mathscr{B}^{\lambda_1,\lambda_2,q_{m_2}}_{m_1,m_2,q_{m_1}}$.

Theorem 4. *Let* $h\in C(S^2)$. *Then, we have*

$$\lim_{m_1,m_2\to\infty}\left\|\mathscr{B}^{\lambda_1,\lambda_2,q_{m_2}}_{m_1,m_2,q_{m_1}}(h)-h\right\|=0. \tag{14}$$

Proof. Using Lemma 2, it is obvious

$$\lim_{m_1,m_2\to\infty}\mathscr{B}^{\lambda_1,\lambda_2,q_{m_2}}_{m_1,m_2,q_{m_1}}(e_{rs};y_1,y_2)=e_{rs},(r,s)\in\{(0,0),(0,1),(0,1)\}, \tag{15}$$

and

$$\lim_{m_1,m_2\to\infty}\mathscr{B}^{\lambda_1,\lambda_2,q_{m_2}}_{m_1,m_2,q_{m_1}}(e_{20}+e_{02};y_1,y_2)=e_{20}+e_{02}, \tag{16}$$

uniformly on S^2. The result follows using [[21], Thm 2.1].

In order to discuss the next results, let us recall the definitions of modulus of continuity and partial modulus of continuity

Definition 5 (see [22]). For $h\in S^2$ and $\delta>0$, the full modulus of continuity in the bi-variate case is defined as

$$\omega_c(h;\delta)=\max_{\sqrt{\sum_{i=1}^2(t_i-y_i)^2}\le\delta}\left\{|h(t_1,t_2)-h(y_1,y_2)|:(t_1,t_2),(y_1,y_2)\in S^2\right\}. \tag{17}$$

For each fixed $i=1,2$, the partial modulus of continuity of h with respect to y_i is defined

$$\omega^{(1)}(h;\delta)=\sup\{|h(x_1,y)-h(x_2,y)|:y\in[0,1]|x_1-x_2|\le\delta\}, \tag{18}$$

and

$$\omega^{(2)}(h;\delta)=\sup\{|h(x,y_1)-h(x_2,y)|:x\in[0,1]\,|y_1-y_2|\le\delta\}, \tag{19}$$

respectively.

Theorem 6. *For* $h\in C(S^2)$, *we have*

$$\left\|\mathscr{B}^{\lambda_1,\lambda_2,q_{m_2}}_{m_1,m_2,q_{m_1}}(h)-h\right\|\le2\omega_c(h;\delta), \tag{20}$$

where $\delta^2=\mu_{m_1,\lambda_1,2}(q_{m_1},y_1)+\mu_{m_2,\lambda_2,2}(q_{m_2},y_2)$.

Proof. Using the facts that

$$\begin{aligned}&\left|\mathscr{B}^{\lambda_1,\lambda_2,q_{m_2}}_{m_1,m_2,q_{m_1}}(h(t_1,t_2);y_1,y_2)-h(y_1,y_2)\right|\\&\quad\le\sum_{j_1=0}^{m_1}\sum_{j_2=0}^{m_2}\Omega^{(\lambda_1,\lambda_2)}_{m_1,m_2,j_1,j_2}(y_1,y_2)\left|h\left(\frac{[j_1]_{q_1}}{[m_1]_{q_1}},\frac{[j_2]_{q_2}}{[m_2]_{q_2}}\right)-h(y_1,y_2)\right|.\end{aligned} \tag{21}$$

Then, we use the following property of the complete modulus of continuity:

$$|h(t_1,t_2)-h(y_1,y_2)|\le\omega^{(c)}(f;\delta)\left(1+\frac{\sqrt{(t_1-y_1)^2+(t_2-y_2)^2}}{\delta}\right), \tag{22}$$

we get

$$\begin{aligned}&\left|\mathscr{B}^{\lambda_1,\lambda_2,q_{m_2}}_{m_1,m_2,q_{m_1}}(h(t_1,t_2);y_1,y_2)-h(y_1,y_2)\right|\\&\quad\le\left[1+\frac{\mathscr{B}^{\lambda_1,\lambda_2,q_{m_2}}_{m_1,m_2,q_{m_1}}\left(\sqrt{(t_1-y_1)^2+(t_2-y_2)^2};y_1,y_2\right)}{\delta}\right]\omega^{(c)}(h;\delta).\end{aligned} \tag{23}$$

Applying Cauchy Schwarz inequality, we have

$$\begin{aligned}&\left|\mathscr{B}_{m_1,m_2,q_{m_1}}^{\lambda_1,\lambda_2,q_{m_2}}(h(t_1,t_2);y_1,y_2)-h(y_1,y_2)\right|\\&\leq\left[1+\frac{1}{\delta}\left\{\mathscr{B}_{m_1,m_2,q_{m_1}}^{\lambda_1,\lambda_2,q_{m_2}}\left((t-x)^2;y_1,y_2\right)\right.\right.\\&\quad\left.\left.+\mathscr{B}_{m_1,m_2,q_{m_1}}^{\lambda_1,\lambda_2,q_{m_2}}\left((s-y)^2;y_1,y_2\right)\right\}^{1/2}\right]\omega_c(h;\delta)\\&\leq\left[1+\frac{1}{\delta}\left\{\mu_{m_1,\lambda_1,2}\left(q_{m_1},y_1\right)+\mu_{m_2,\lambda_2,2}\left(q_{m_2},y_2\right)\right\}^{1/2}\right]\omega_c(h;\delta);\forall(y_1,y_2)\in S^2.\end{aligned}\tag{24}$$

By choosing

$$\delta=\left\{\mu_{m_1,\lambda_1,2}\left(q_{m_1},y_1\right)+\mu_{m_2,\lambda_2,2}\left(q_{m_2},y_2\right)\right\}^{1/2},\tag{25}$$

we obtain the desired result.

Theorem 7. *Let $h\in C(S^2)$, then the following inequality holds:*

$$\begin{aligned}&\left|\mathscr{B}_{m_1,m_2,q_{m_1}}^{\lambda_1,\lambda_2,q_{m_2}}(h(t_1,t_2);y_1,y_2)-h(y_1,y_2)\right|\\&\leq2\left\{\omega^{(1)}\left(h;\sqrt{\mu_{m_1,\lambda_1,2}\left(q_{m_1},y_1\right)}\right)+\omega^{(2)}\left(h;\sqrt{\mu_{m_2,\lambda_2,2}\left(q_{m_2},y_2\right)}\right)\right\}.\end{aligned}\tag{26}$$

Proof. Similarly to previous theorems, using relations (19) we obtain

$$\begin{aligned}&\left|\mathscr{B}_{m_1,m_2,q_{m_1}}^{\lambda_1,\lambda_2,q_{m_2}}(h(t_1,t_2);y_1,y_2)-h(y_1,y_2)\right|\\&\leq\left\{\mathscr{B}_{m_1,m_2,q_{m_1}}^{\lambda_1,\lambda_2,q_{m_2}}\left(\omega^{(2)}(h;|t_2-y_2|);y_1,y_2\right)\right.\\&\quad\left.+\mathscr{B}_{m_1,m_2,q_{m_1}}^{\lambda_1,\lambda_2,q_{m_2}}\left(\omega^{(1)}(h;|t_1-y_1|);y_1,y_2\right)\right\}.\end{aligned}\tag{27}$$

Applying Cauchy Schwarz inequality with $\delta_1=(\mu_{m_1,\lambda_1,2}(q_{m_1},y_1))^{1/2}$ and $\delta_2=(\mu_{m_2,\lambda_2,2}(q_{m_2},y_2))^{1/2}$, we obtain the desired result.

Now, we want to give the quantitative result in terms of the Lipschitz class functionals. For any functions $h:S^2\to\mathbb{R}$ and $0<\mathrm{u}\leq1$, the function h is said to be in Lipschitz class $Lip_M\mathrm{u}(S^2)$ if $\exists$ a $M>0$ such that

$$Lip_M\mathrm{u}:=\left\{h:|h(s_1,s_2)-h(y_1,y_2)|\leq M\|s-t\|^{\mathrm{u}}\right\},\tag{28}$$

$\forall s=(s_1,s_2),t=(y_1,y_2)\in S^2$, where $\|s-t\|=\{(s_1-y_1)^2+(s_2-y_2)^2\}^{1/2}$ is the Euclidean norm and M is a positive real constant.

The following theorem yields us an estimate of error for functions in $Lip_M\mathrm{u}$, by the operators $\mathscr{B}_{m_1,m_2,q_{m_1}}^{\lambda_1,\lambda_2,q_{m_2}}$.

Theorem 8. *Let $h\in Lip_M u$. Then for sufficiently large m and n and for all $y_1,y_2\in S^2$, there holds the inequality*

$$\left\|\mathscr{B}_{m_1,m_2,q_{m_1}}^{\lambda_1,\lambda_2,q_{m_2}}(h)-h\right\|_{C(I^2)}\leq M\left\{[m]_{q_m}^{-1}+[n]_{q_n}^{-1}\right\}^{\frac{u'}{2}},\tag{29}$$

where $M>0$ is a constant.

Proof. From hypothesis, we have

$$\begin{aligned}&\left|\mathscr{B}_{m_1,m_2,q_{m_1}}^{\lambda_1,\lambda_2,q_{m_2}}(h;y_1,y_2)-h(y_1,y_2)\right|\\&\leq\mathscr{B}_{m_1,m_2,q_{m_1}}^{\lambda_1,\lambda_2,q_{m_2}}\left(|h(s_1,s_2)-h(y_1,y_2)|;y_1,y_2\right)\\&\leq M\mathscr{B}_{m_1,m_2,q_{m_1}}^{\lambda_1,\lambda_2,q_{m_2}}\left(\|s-t\|^{\mathrm{u}};y_1,y_2\right),\end{aligned}\tag{30}$$

where $s=(s_1,s_2)$, $t=(y_1,y_2)\in S^2$. Applying Hölder's inequality and Corollary 3, we obtain

$$\begin{aligned}&\left|\mathscr{B}_{m_1,m_2,q_{m_1}}^{\lambda_1,\lambda_2,q_{m_2}}(h;y_1,y_2)-h(y_1,y_2)\right|\\&\leq\mathrm{M}\left\{\mathscr{B}_{m_1,m_2,q_{m_1}}^{\lambda_1,\lambda_2,q_{m_2}}\left(\|s-t\|^2;y_1,y_2\right)\right\}^{u'/2}\\&\leq M\left\{\mu_{m_1,\lambda_1,2}\left(q_{m_2},y_1\right)+\mu_{m_2,\lambda_2,2}\left(q_{m_2},y_2\right)\right\}^{\mathrm{u}/2},\end{aligned}\tag{31}$$

which leads to the required result on applying Corollary 3.

Let $C'(S^2)$ denote the space of continuous functions $h(y_1,y_2)$ on S^2 whose first-order partial derivatives g'_{y_1} and g'_{y_2} are also continuous on S^2.

Our next result yields us the rate of approximation for continuously differentiable functions on S^2 by the operators $\mathscr{B}_{m_1,m_2,q_{m_1}}^{\lambda_1,\lambda_2,q_{m_2}}$.

Theorem 9. *Let $h\in C'(S^2)$. Then for sufficiently large m and n, we have*

$$\left\|\mathscr{B}_{m_1,m_2,q_{m_1}}^{\lambda_1,\lambda_2,q_{m_2}}(h)-h\right\|_{C(S^2)}\leq A\left(\|h'_{y_1}\|_{C(S^2)}[m_1]_{q_{m_1}}^{-1/2}+\|h'_{y_2}\|_{C(S^2)}[m_2]_{q_{m_2}}^{-1/2}\right),\tag{32}$$

where A is some positive constant.

Proof. For $y_1,y_2\in S^2$ be arbitrary, we may write

$$\begin{aligned}h(t_1,t_2)-h(y_1,y_2)&=\int_{y_1}^{t_1}h'_\eta(\eta,t_2)d\eta\\&\quad+\int_{y_2}^{t_2}h'_\phi(y_1,\phi)d\phi,\ \text{for}(t_1,t_2)\in S^2.\end{aligned}\tag{33}$$

Hence, applying the operator $\mathscr{B}_{m_1,m_2,q_{m_1}}^{\lambda_1,\lambda_2,q_{m_2}}$ on both sides of the above equation, we obtain

$$\begin{aligned}\mathscr{B}_{m_1,m_2,q_{m_1}}^{\lambda_1,\lambda_2,q_{m_2}}(h;y_1,y_2)-h(y_1,y_2)&=\mathscr{B}_{m_1,m_2,q_{m_1}}^{\lambda_1,\lambda_2,q_{m_2}}\left(\int_{y_1}^{t_1}h'_{\eta}(\eta,t_2)d\eta;y_1,y_2\right)\\&+\mathscr{B}_{m_1,m_2,q_{m_1}}^{\lambda_1,\lambda_2,q_{m_2}}\left(\int_{y_2}^{t_2}h'_{\phi}(y_1,\phi)d\phi;y_1,y_2\right).\end{aligned} \tag{34}$$

By using sup-norm on S^2

$$\begin{aligned}\left|\int_{y_1}^{t_1}h'_{\eta}(\eta,t_2)d\eta\right|\le\left\|h'_{y_1}\right\|_{C(S^2)}|t_1-y_1|,\left|\int_{y_2}^{t_2}h'_{\phi}(y_1,\phi)d\phi\right|\\\le\left\|h'_{y_2}\right\|_{C(S^2)}|t_2-y_2|.\end{aligned} \tag{35}$$

we get

$$\begin{aligned}&\left|\mathscr{B}_{m_1,m_2,q_{m_1}}^{\lambda_1,\lambda_2,q_{m_2}}(h;y_1,y_2)-h(y_1,y_2)\right|\\&\le\left\|h'_{y_1}\right\|_{C(S^2)}\mathscr{B}_{m_1,m_2,q_{m_1}}^{\lambda_1,\lambda_2,q_{m_2}}(|t_1-y_1|;y_1,y_2)\\&+\left\|h'_{y_2}\right\|_{C(S^2)}\mathscr{B}_{m_1,m_2,q_{m_1}}^{\lambda_1,\lambda_2,q_{m_2}}(|t_2-y_2|;y_1,y_2).\end{aligned} \tag{36}$$

Hence, applying the Cauchy-Schwarz inequality and Corollary 3, we obtain

$$\begin{aligned}&\left|\mathscr{B}_{m_1,m_2,q_{m_1}}^{\lambda_1,\lambda_2,q_{m_2}}(h;y_1,y_2)-h(y_1,y_2)\right|\\&\le\left\|h'_{y_1}\right\|_{C(S^2)}\left\{\mathscr{B}_{m_1,m_2,q_{m_1}}^{\lambda_1,\lambda_2,q_{m_2}}\left((t-y)^2;y_1,y_2\right)\right\}^{1/2}\\&+\|h'_{y_2}\|_{C(S^2)}\left\{\mathscr{B}_{m_1,m_2,q_{m_1}}^{\lambda_1,\lambda_2,q_{m_2}}\left((t_2-y_2)^2;y_1,y_2\right)\right\}^{1/2},\end{aligned} \tag{37}$$

from which the desired result is immediate.

The following result yields the degree of approximation of h by $\mathscr{B}_{m_1,m_2,q_{m_1}}^{\lambda_1,\lambda_2,q_{m_2}}$ in terms of the partial modul of continuity of the partial derivatives of h.

Theorem 10. *Let $h\in C'(S^2)$. Then for sufficiently large m and n, we have*

$$\left\|\mathscr{B}_{m_1,m_2,q_{m_1}}^{\lambda_1,\lambda_2,q_{m_2}}(h)-h\right\|_{C(S^2)}\le A\sum_{i=1}^{2}[m_i]_{q_{m_i}}^{-\frac{1}{2}}\left\{1+2\omega^{(i)}\left(h'_{y_i};[m_i]_{q_{m_i}}^{-\frac{1}{2}}\right)\right\}, \tag{38}$$

where $\omega^{(i)}(h'_{y_i};.)$ are the partial modul of continuity of h'_{y_i} for $i=1,2$ and A is some positive constant.

Proof. If we use the mean value theorem in the following form, we have

$$\begin{aligned}h(t_1,t_2)-h(y_1,y_2)&=(t_1-y_1)h'_{y_1}(\eta,y_2)+(t_2-y_2)h'_{y_2}(y_1,\mathrm{u})\\&=(t_1-y_1)h'_{y_1}(y_1,y_2)+(t_1-y_1)\\&\cdot\left(h'_{y_1}(\eta,y_2)-h'_{y_1}(y_1,y_2)\right)\\&+(t_2-y_2)h'_{y_2}(y_1,y_2)+(t_2-y_2)\\&\cdot\left(h'_{y_2}(y_1,\mathrm{u})-h'_{y_2}(y_1,y_2)\right),\end{aligned} \tag{39}$$

where $y_1<\eta<t_1$ and $y_2<\mathrm{u}<t_2$. Applying the operator $\mathscr{B}_{m_1,m_2,q_{m_1}}^{\lambda_1,\lambda_2,q_{m_2}}$ to both sides, we deduce that

$$\begin{aligned}&\mathscr{B}_{m_1,m_2,q_{m_1}}^{\lambda_1,\lambda_2,q_{m_2}}(h;y_1,y_2)-h(y_1,y_2)\\&=h'_{y_1}(y_1,y_2)\mathscr{B}_{m_1,m_2,q_{m_1}}^{\lambda_1,\lambda_2,q_{m_2}}(t_1-y_1;y_1,y_2)\\&+\mathscr{B}_{m_1,m_2,q_{m_1}}^{\lambda_1,\lambda_2,q_{m_2}}\left((t_1-y_1)\left(h'_{y_1}(\eta,y_2)-h'_{y_1}(y_1,y_2)\right);y_1,y_2\right)\\&+h'_{y_2}(y_1,y_2)\mathscr{B}_{m_1,m_2,q_{m_1}}^{\lambda_1,\lambda_2,q_{m_2}}(t_2-y_2;y_1,y_2)\\&+\mathscr{B}_{m_1,m_2,q_{m_1}}^{\lambda_1,\lambda_2,q_{m_2}}\left((t_2-y_2)\left(h'_{y_2}(y_1,\mathrm{u})-h'_{y_2}(y_1,y_2)\right);y_1,y_2\right).\end{aligned} \tag{40}$$

Since h'_{y_1} and h'_{y_2} are continuous in S^2, there exist positive constants A_1 and A_2 such that $|h'_{y_1}|\le A_1$ and $|h'_{y_1}|\le A_2$, for all $(y_1,y_2)\in S^2$. Hence, applying the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}&\left|\mathscr{B}_{m_1,m_2,q_{m_1}}^{\lambda_1,\lambda_2,q_{m_2}}(h;y_1,y_2)-h(y_1,y_2)\right|\\&\le\sum_{i=1}^{2}\left|h'_{y_i}\right|\mathscr{B}_{m_1,m_2,q_{m_1}}^{\lambda_1,\lambda_2,q_{m_2}}(|t_i-y_i|;y_1,y_2)\\&+\sum_{i=1}^{2}\mathscr{B}_{m_1,m_2,q_{m_1}}^{\lambda_1,\lambda_2,q_{m_2}}\left(|t_i-y_i|\left(1+\frac{|t_i-y_i|}{\delta_i}\right);y_1,y_2\right)\omega^{(i)}\left(h'_{y_i};\delta_i\right)\\&\le\sum_{i=1}^{2}\left\{M_i\left\{\mathscr{B}_{m_1,m_2,q_{m_1}}^{\lambda_1,\lambda_2,q_{m_2}}\left((t_i-y_i)^2;y_1,y_2\right)\right\}^{\frac{1}{2}}\right.\\&+\omega^{(i)}\left(h'_{y_i};\delta_i\right)\left[\left\{\mathscr{B}_{m_1,m_2,q_{m_1}}^{\lambda_1,\lambda_2,q_{m_2}}\left((t_i-y_i)^2;y_1,y_2\right)\right\}^{\frac{1}{2}}\right.\\&\left.\left.+\frac{1}{\delta_i}\mathscr{B}_{m_1,m_2,q_{m_1}}^{\lambda_1,\lambda_2,q_{m_2}}\left((t_i-y_i)^2;y_1,y_2\right)\right]\right\}.\end{aligned} \tag{41}$$

Choosing $\delta_i=([m_i]_{q_{m_i}})^{-1/2}$, $i=1,2$ we get the required result.

3. Construction of GBS Operator of Generalized Bernstein Type

In the last two decades, the study of generalized Boolean sum (GBS) operators of certain linear positive operators has attracted very much attention in the approximation theory. In early 1937 with Bögel [23], a great number of studies are performed related to these operators. To make an analysis in multidimensional spaces, Bögel [23] introduced the concepts of continuity and differentiability in a Bögel space. There are still many authors working on this subject. Agrawal et al. [24] studied the degree of approximation for bivariate Lupas-Durrmeyer type operators based on Pólya distribution with associated GBS operators. Further, Gupta et al. [25] introduced some GBS operators of discrete and integral types and examined the properties of approximations in a Bögel space. Recently, Kajla and Miclacus [16] studied the rate of approximation of Bögel continuous and Bögel differentiable functions by the GBS operators of Bernstein-Durrmeyer type operators. In the last year, Kumar and Shivam [26] constructed the bivariate Kantorovich-type sampling operator, involved with GBS operators, as well as estimation of the rate of convergence of the sequences of these operators.

A function $h : S^2 \to \mathbb{R}$ is called B-continuous at point $(x_0, y_0) \in S^2$ if

$$\lim_{(x,y)\to\left(y_0^1, y_0^2\right)} \Delta_{(y_1,y_2)} h\left[y_0^1, y_0^2 ; y_1, y_2\right] = 0, \tag{42}$$

for any $(y_1, y_2) \in S^2$, (see [23]). The function $h : S^2 \to \mathbb{R}$ is B-bounded on S^2 if there exists $M > 0$ such that $|\Delta_{(x,y)} h[t_1, t_2 ; y_1, y_2]| \le M$ for every $(y_1, y_2), (t_1, t_2) \in S^2$.

Throughout this article, $B_b(S^2)$ denotes all B-bounded functions on S^2. The space of all B-continuous functions is denoted by $C_b(S^2)$.

Motivated by the above authors, we construct the GBS operator of $\mathscr{B}_{m_1,m_2}^{(\lambda_1,\lambda_2)}$, who is defined as follows:

$$\begin{aligned}\mathscr{G}\mathscr{B}\mathscr{S}_{m_1,m_2}^{(\lambda_1,\lambda_2)}&(h(t_1, t_2) ; x, y)\\ &\coloneqq \mathscr{B}_{m_1,m_2}^{(\lambda_1,\lambda_2)}(h(t_1, y_2) + h(y_1, t_2) - h(t_1, t_2) ; y_1, y_2),\end{aligned} \tag{43}$$

for all $(y_1, y_2) \in S^2$. More precisely, the q-analogue λ-Bernstein type GBS operator is defined as follows:

$$\begin{aligned}\mathscr{G}\mathscr{B}\mathscr{S}_{m_1,m_2}^{(\lambda_1,\lambda_2)}(h(t_1, t_2) ; y_1, y_2) &= \sum_{j_1=0}^{m_1} \sum_{j_2=0}^{m_2} \Omega_{m_1,m_2,j_1,j_2}^{(\lambda_1,\lambda_2)}(y_1, y_2)\\ &\times \left[h\left(\frac{[j_1]_{q_1}}{[m_1]_{q_1}}, y_2\right) + h\left(y_1, \frac{[j_2]_{q_2}}{[m_2]_{q_2}}\right) - h\left(\frac{[j_1]_{q_1}}{[m_1]_{q_1}}, \frac{[j_2]_{q_2}}{[m_2]_{q_2}}\right)\right],\end{aligned} \tag{44}$$

where the operator $\mathscr{G}\mathscr{B}\mathscr{S}_{m_1,m_2}^{(\lambda_1,\lambda_2)}$ is well-defined on the space $C_b(S^2)$ into $C(S^2)$ and $h \in C_b(S^2)$.

4. Degree of Approximation by $\mathscr{G}\mathscr{B}\mathscr{S}_{m_1,m_2,q_{m_1},q_{m_2}}^{(\lambda_1,\lambda_2)}$

For $(y_1, y_2), (t_1, t_2) \in S^2$, the mixed modulus of smoothness of $h \in C_b(S^2)$ is defined by

$$\begin{aligned}\omega_B(h ; \delta_1, \delta_2) \coloneqq \sup \Big\{ \left|\Delta_{(y_1,y_2)} h[t_1, t_2 ; y_1, y_2]\right| &: |y_1 - t_1|\\ &< \delta_1, |y_2 - t_2| < \delta_2 \Big\},\end{aligned} \tag{45}$$

and for any $(\delta_{1,}\delta_2) \in (0,\infty) \times (0,\infty)$. Using (45), we have

$$\omega_B(h ; P_1\delta_1, P_2\delta_2) \le (1 + P_1)(1 + P_2)\omega_B(f ; \delta_1, \delta_2) ; P_1, P_2 > 0, \tag{46}$$

The basic results of ω_B were studied by Badea et al. [27, 28] and are similar to the properties of the usual modulus of continuity for bivariate functions. We shall obtain the rate of approximation of the operators (44) to $h \in C_b(S^2)$ in terms of the mixed modulus of continuity for two variables. For this, we apply the Shisha-Mond theorem for B-continuous functions defined by Gonska [29] and Badea and Cottin [28].

Theorem 11. *For every $h \in C_b(S^2)$, at each point $(y_1, y_2) \in S^2$ and sufficiently large m and n, the operator (44) satisfy the following results*

$$\left\|\mathscr{G}\mathscr{B}\mathscr{S}_{m_1,m_2,q_{m_1}}^{(\lambda_1,\lambda_2),q_{m_2}}(h) - h\right\|_B \le C_{\lambda_1}^{\lambda_2}\omega_B\left(h ; [m_1]_{q_{m_1}}^{-1/2}, [m_2]_{q_{m_2}}^{-1/2}\right), \tag{47}$$

where $C_{\lambda_1}^{\lambda_2}$ is a positive constant depending on parameters λ_1 and λ_2.

Proof. Using (45) and applying the inequality (46), we have

$$\begin{aligned}\left|\Delta_{(y_1,y_2)} h[t_1, t_2 ; y_1, y_2]\right| &\le \omega_B(h ; |t_1 - y_1|, |t_2 - y_2|)\\ &\le \prod_{i=1}^{2}\left(1 + \frac{|t_i - y_i|}{\delta_i}\right)\omega_B(g ; \delta_1, \delta_2)\\ &\le \big(1+|t_1 - y_1|\,\delta_1^{-1}+|t_2 - y_2|\,\delta_2^{-1}\\ &\quad + (\delta_1\delta_2)^{-1}(|t_1 - y_1||t_2 - y_2|)\big)\omega_B\\ &\quad \cdot (g ; \delta_1, \delta_2),\end{aligned} \tag{48}$$

for every $(y_1, y_2), (t_1, t_2) \in S^2$ and for any $\delta_1, \delta_2 > 0$. Taking the definition of $\Delta_{(y_1,y_2)} h[t_1, t_2 ; y_1, y_2]$, we may write

$$h(y_1, t_2) + h(t_1, y_2) - h(t_1, t_2) = h(y_1, y_2) - \Delta_{(y_1,y_2)} h[t_1, t_2 ; y_1, y_2]. \tag{49}$$

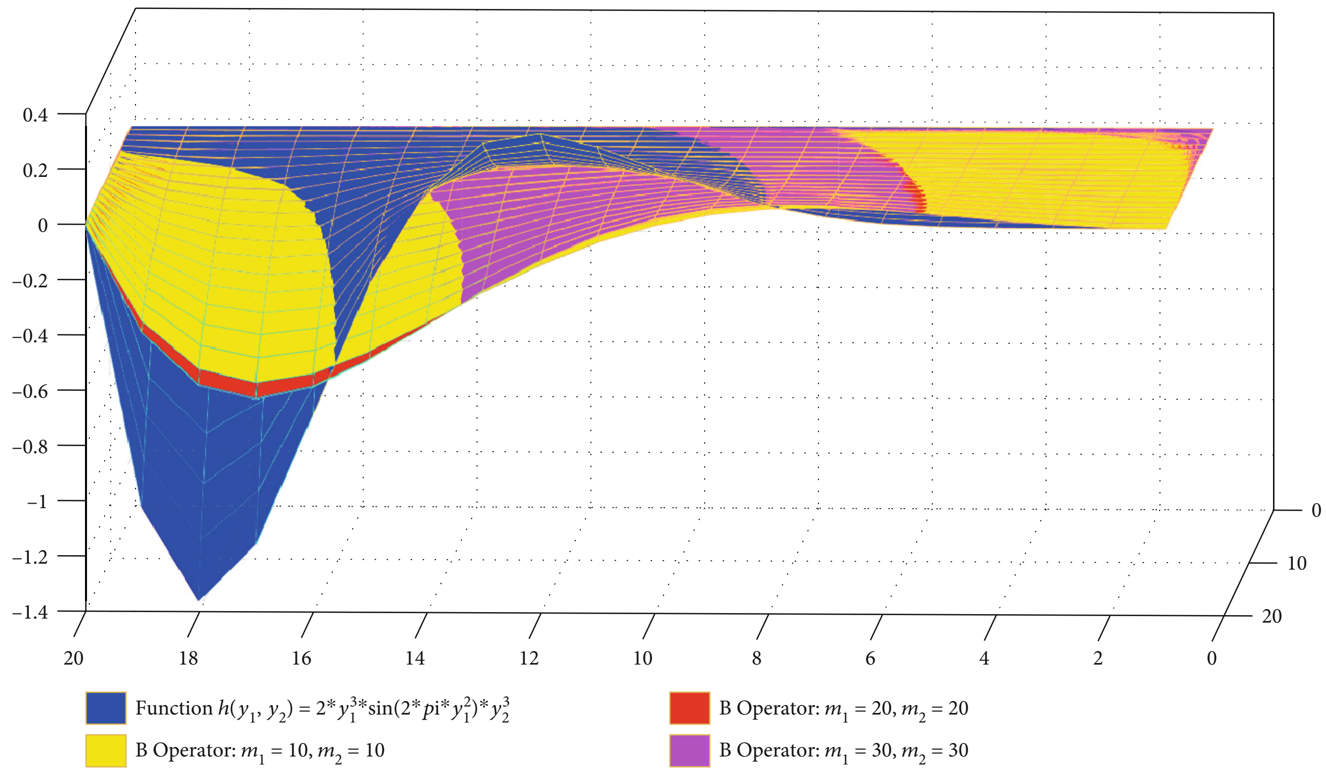

FIGURE 1: The convergence of $\mathscr{B}_{m_1,m_2,q_{m_1}}^{\lambda_1,\lambda_2,q_{m_2}}(h; y_1, y_2)$ to $h(y_1, y_2)$.

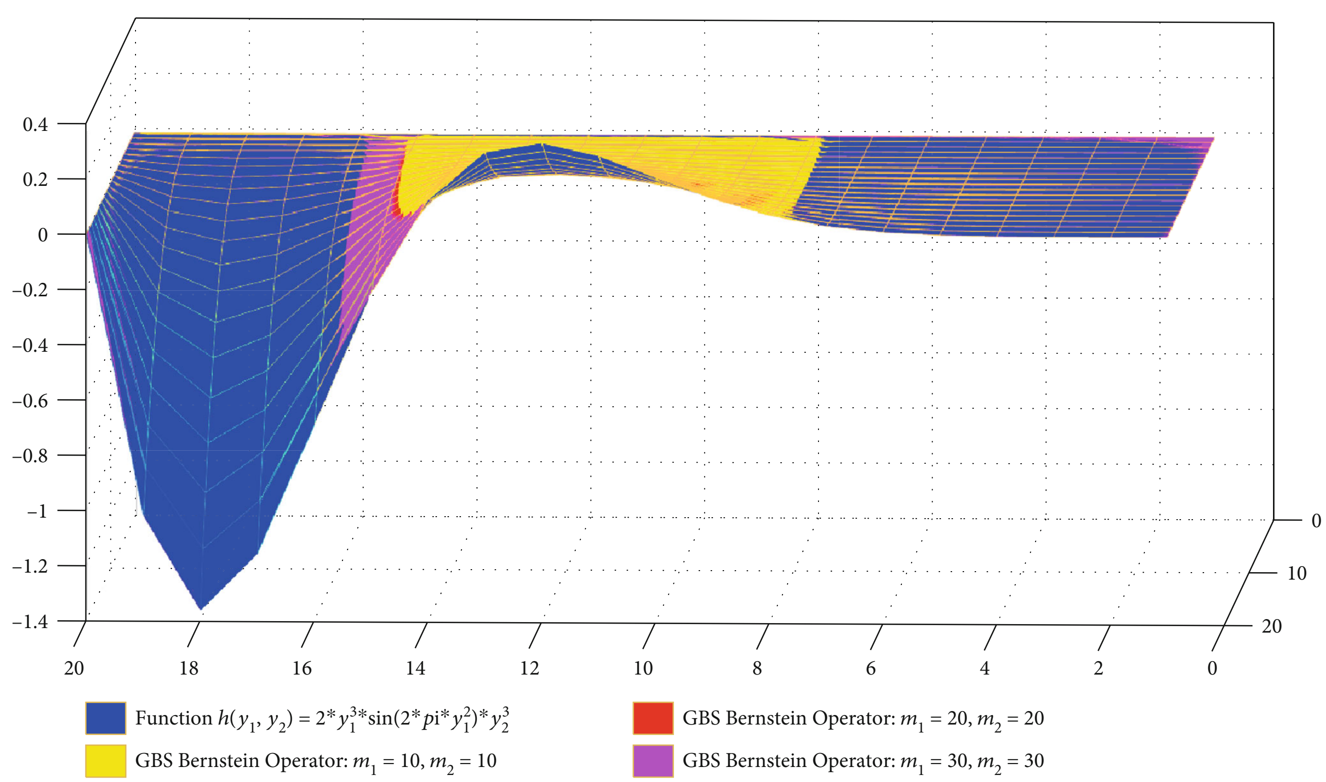

FIGURE 2: The convergence of $\mathscr{B}_{m_1,m_2,q_{m_1}}^{\lambda_1,\lambda_2,q_{m_2}}(h; y_1, y_2)$ to $h(y_1, y_2)$.

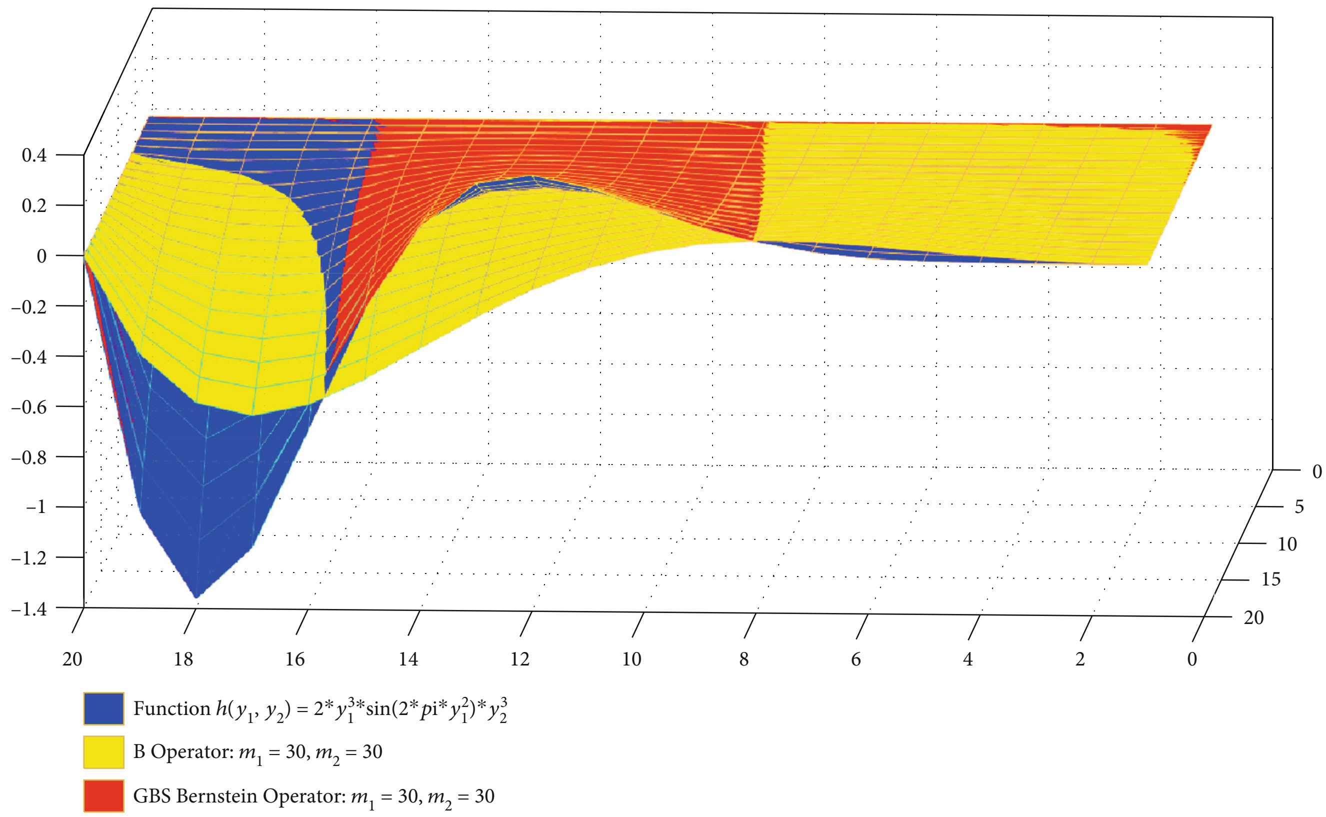

FIGURE 3: The convergence of $\mathscr{GBS}_{30,30,0.8}^{(0.6,0.6),0.8}(h;y_1,y_2)$ and $\mathscr{B}_{30,30,0.8}^{0.6,0.6,0.8}(h;y_1,y_2)$ to $h(y_1,y_2)$.

Applying the operator $\mathscr{B}_{m_1,m_2,q_{m_1}}^{\lambda_1,\lambda_2,q_{m_2}}$ on both sides of the above inequality

$$\mathscr{GBS}_{m_1,m_2,q_{m_1}}^{(\lambda_1,\lambda_2),q_{m_2}}(h;y_1,y_2) = h(y_1,y_2)\mathscr{B}_{m_1,m_2,q_{m_1}}^{\lambda_1,\lambda_2,q_{m_2}}(1;y_1,y_2) - \mathscr{B}_{m_1,m_2,q_{m_1}}^{\lambda_1,\lambda_2,q_{m_2}} \cdot \left(\Delta_{(y_1,y_2)}h[t_1,t_2;y_1,y_2];y_1,y_2\right). \tag{50}$$

Now, from Lemma 2, with the help of Cauchy-Schwarz inequality and Remark 1 (in that order), we obtain

$$\begin{aligned}
&\left|\mathscr{GBS}_{m_1,m_2,q_{m_1},q_{m_2}}^{(\lambda_1,\lambda_2)}(h;y_1,y_2) - h(y_1,y_2)\right| \\
&\le \mathscr{B}_{m_1,m_2,q_{m_1}}^{\lambda_1,\lambda_2,q_{m_2}}\left(\left|\Delta_{(y_1,y_2)}h[t_1,t_2;y_1,y_2]\right|;y_1,y_2\right) \\
&\le \omega_B(g;\delta_1,\delta_2)\Bigg(\mathscr{B}_{m_1,m_2,q_{m_1}}^{\lambda_1,\lambda_2,q_{m_2}}(1;y_1,y_2) \\
&\quad + \frac{1}{\delta_1}\sqrt{\mathscr{B}_{m_1,m_2,q_{m_1}}^{\lambda_1,\lambda_2,q_{m_2}}\left((t_1-y_1)^2;y_1,y_2\right)} \\
&\quad + \frac{1}{\delta_2}\sqrt{\mathscr{B}_{m_1,m_2,q_{m_1}}^{\lambda_1,\lambda_2,q_{m_2}}\left((t_2-y_2)^2;y_1,y_2\right)} \\
&\quad + \frac{1}{\delta_1\delta_2}\sqrt{\mathscr{B}_{m_1,m_2,q_{m_1}}^{\lambda_1,\lambda_2,q_{m_2}}\left((t_1-y_1)^2;y_1,y_2\right)} \\
&\quad \sqrt{\mathscr{B}_{m_1,m_2,q_{m_1}}^{\lambda_1,\lambda_2,q_{m_2}}\left((t_2-y_2)^2;y_1,y_2\right)}\Bigg).
\end{aligned} \tag{51}$$

TABLE 1: Error of approximation $\mathscr{E}_{m_1,m_2}^{\lambda_1,\lambda_2}$ for $m_1 = m_2 = 10, 20$ and 30; $q_{m_1} = q_{m_2} = 0.8$ and $\lambda_1 = \lambda_2 = 0.6$.

(y_1,y_2)	$\mathscr{E}_{10,10}^{0.6,0.6}$	$\mathscr{E}_{20,20}^{0.6,0.6}$	$\mathscr{E}_{30,30}^{0.6,0.6}$
(0.1,0.1)	0.000160	0.000120	0.000117
(0.1,0.4)	0.001862	0.001518	0.001486
(0.2,0.6)	0.012728	0.012002	0.011916
(0.3,0.2)	0.002130	0.002100	0.002094
(0.6,0.5)	0.005007	0.008771	0.009194
(0.75,0.7)	0.094096	0.104575	0.105493
(0.9,0.7)	0.246766	0.226662	0.225008
(0.9,0.9)	0.594899	0.553166	0.549529

TABLE 2: Error of approximation $\mathscr{E}_{m_1,m_2}^{*\lambda_1,\lambda_2}$ for $m_1 = m_2 = 10, 20$ and 30 and $q_1 = q_2 = 0.8$ and $\lambda_1 = \lambda_2 = 0.6$.

(y_1,y_2)	$\mathscr{E}_{10,10}^{*0.6,0.6}$	$\mathscr{E}_{20,20}^{*0.6,0.6}$	$\mathscr{E}_{30,30}^{*0.6,0.6}$
(0.1,0.1)	0.000144	0.000107	0.000103
(0.1,0.4)	0.000912	0.000701	0.000682
(0.2,0.6)	0.001831	0.001428	0.001385
(0.3,0.2)	0.000588	0.000600	0.000598
(0.6,0.5)	0.040427	0.035837	0.035374
(0.75,0.7)	0.013232	0.014782	0.014782
(0.9,0.7)	0.071823	0.062288	0.061427
(0.9,0.9)	0.035840	0.032438	0.032209

Now, setting $\delta_i = ([m_i]_{q_{m_i}})^{-1/2}, i = 1, 2$, the required result is obtained.

For $0 < u \le 1$, the Lipschitz class of Bögel continuous functions is defined as

$$Lip_M u = \left\{ h \in C_b(S^2) : \left| \Delta_{(y_1,y_2)} h[t_1, t_2 ; y_1, y_2] \right| \le L\|t - s\|^{u} \right\}, \quad (52)$$

where $t = (t_1, t_2), s = (y_1, y_2) \in S^2$ and $\|t - s\| = \{(t_1 - y_1)^2 + (t_2 - y_2)^2\}^{1/2}$ are the Euclidean norm.

In the next result, we obtain the degree of approximation of the operators $\mathscr{GBS}^{(\lambda_1,\lambda_2),q_{m_2}}_{m_1,m_2,q_{m_1}}$ for functions in the Lipschitz-class of Bögel continuous functions.

Theorem 12. *If $h \in Lip_L u'$, then for sufficiently large m_1 and m_2, we have*

$$\left\| \mathscr{GBS}^{(\lambda_1,\lambda_2),q_{m_2}}_{m_1,m_2,q_{m_1}}(h ; y_1, y_2) - (h) \right\|_B \le L \left\{ [m_1]^{-1}_{q_{m_1}} + [m_2]^{-1}_{q_{m_2}} \right\}^{u/2}, \quad (53)$$

where L is a positive constant.

Proof. Using the equation (50), Hölder's inequality, Lemma 2, and Lemma 1, we get

$$\begin{aligned} &\left| \mathscr{GBS}^{(\lambda_1,\lambda_2),q_{m_2}}_{m_1,m_2,q_{m_1}}(h ; y_1, y_2) - h(y_1, y_2) \right| \\ &\le \mathscr{B}^{\lambda_1,\lambda_2,q_{m_2}}_{m_1,m_2,q_{m_1}} \left(\left| \Delta_{(t,r)} g(y, z) \right| ; y, z \right) \\ &\le L \mathscr{B}^{\lambda_1,\lambda_2,q_{m_2}}_{m_1,m_2,q_{m_1}} \left(\|t - s\|^{u'} ; y_1, y_2 \right) \\ &\le L \left\{ \mathscr{B}^{(\lambda_1,\lambda_2)}_{m_1,m_2,q_m,q_n} \left(\|t - s\|^2 ; y_1, y_2 \right) \right\}^{u/2} \\ &\le L \left\{ \mu_{m_1,\lambda_1,2}\left(q_{m_1}, y_1\right) + \mu_{m_2,\lambda_2,2}\left(q_{m_2}, y_2\right) \right\}^{u/2} \\ &\le L \left\{ [m_1]^{-1}_{q_{m_1}} + [m_2]^{-1}_{q_{m_2}} \right\}^{u/2}. \end{aligned} \quad (54)$$

Thus, we get the desired result.

5. Numerical Results and Discussions

Example 13. Let us choose $m_1 = m_2 = 10, 20$ and 30, $q_{m_1} = q_{m_2} = 0.8$, and $\lambda_1 = \lambda_2 = 0.6$. Denote $\mathscr{E}^{\lambda_1,\lambda_2}_{m_1,m_2} = |\mathscr{B}^{\lambda_1,\lambda_2,q_{m_2}}_{m_1,m_2,q_{m_1}}(h(s_1, s_2) ; y_1, y_2) - h(y_1, y_2)|$, the error function of approximation by operators. The convergence of the bivariate Bernstein operators: $\mathscr{B}^{\lambda_1,\lambda_2,q_{m_2}}_{10,10,q_{m_1}}(h ; y_1, y_2)$ (yellow), then $\mathscr{B}^{\lambda_1,\lambda_2,q_{m_2}}_{20,20,q_{m_1}}(h ; y_1, y_2)$ (red) and $\mathscr{B}^{\lambda_1,\lambda_2,q_{m_2}}_{30,30,q_{m_1}}(h ; y_1, y_2)$ (magenta) to $h(y_1, y_2) = 2 y_1^3 \sin(2\pi y_1^2) y_2^3$ (blue) will be illustrated in Figure 1.

Example 14. For $m_1 = m_2 = 10, 20$ and 30, $\lambda_1 = \lambda_2 = 0.6$ and $q_{m_1} = q_{m_2} = 0.8$, the convergence of $\mathscr{GBS}^{(\lambda_1,\lambda_2),q_{m_2}}_{m_1,m_2,q_{m_1}}(h ; y_1, y_2)$ *to* $h(y_1, y_2) = 2y_1^3 \sin(2\pi y_1^2) y_2^3$ is illustrated in Figure 2. Denote $\mathscr{E}^{*\lambda_1,\lambda_2}_{m_1,m_2} = |\mathscr{GBS}^{(\lambda_1,\lambda_2),q_{m_2}}_{m_1,m_2,q_{m_1}}(h(s_1, s_2) ; y_1, y_2) - h(y_1, y_2)|$, the error function of approximation by operators. This example explains the convergence of the operators $\mathscr{GBS}^{(\lambda_1,\lambda_2),q_{m_2}}_{m_1,m_2,q_{m_1}}(h(y_1, y_2))$ that are going to the function $h(y_1, y_2)$ *if* the values of m_1, m_2 are increasing.
Comparative results are given in Figure 3, Tables 1 and 2, for the errors of the approximation of $\mathscr{GBS}^{(\lambda_1,\lambda_2),q_{m_2}}_{m_1,m_2,q_{m_1}}(h ; y_1, y_2)$ and $\mathscr{B}^{\lambda_1,\lambda_2,q_{m_2}}_{m_1,m_2,q_{m_1}}(h ; y_1, y_2)$ to the functions $h(y_1, y_2) = 2y_1^3 \sin(2\pi y_1^2) y_2^3$ for $m_1 = m_2 = 30$, $\lambda_1 = \lambda_2 = 0.6$, and $q_1 = q_2 = 0.8$. Note that (see Tables 1 and 2 and Figure 3) the GBS-Bernstein operator approximation outperforms others.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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