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Research Article

Existence and Uniqueness for a System of Caputo-Hadamard Fractional Differential Equations with Multipoint Boundary Conditions

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In this paper, we study existence and uniqueness of solutions for a system of Caputo-Hadamard fractional differential equations supplemented with multi-point boundary conditions. Our results are based on some classical fixed point theorems such as Banach contraction mapping principle, Leray-Schauder fixed point theorems. At last, we have presented two examples for the illustration of main results.

1. Introduction

In recent years, fractional differential equations (FDE) gain enormous attention among scientists due to the applications which were not possible with ordinary or partial differential equations of integer order. FDEs becomes a very successful tool in modeling anomalous diffusion and fractal-like nature. Agrawal discusses diffusion and heat equations of fractional order in [1–3]. Agrawal et al., Baleanu, and others investigated the boundary value problems for fractional differential equations [4]. Fractional dynamic models, fractional control systems, fractional population dynamics models, and fractional fluid dynamics all involve at least one ordinary or partial fractional derivative.

Fractional differential equations have several kinds of fractional derivatives, such as Riemann-Liouville fractional derivative, Caputo fractional derivative, and Grunwald-Letnikov fractional derivative. Another kind of fractional derivative is Hadamard type which was introduced in 1892 [5]. This derivative differs from various derivatives in the sense that the kernel of the integral in the definition of Hadamard derivative contains logarithmic function of arbitrary exponent. A detailed description of Hadamard fractional derivative and integral can be found in [6]. The readers who are interested in the subject of fractional calculus is referred to the books by Kilbas et al. [7], Podlubny [8], Miller and Ross [9], Samko et al. [10], Diethelm [11], and Zhou [12] and the references therein.

Coupled systems of fractional differential equations play a key role in developing differential models such as the synchronization of chaotic systems [13–15], anomalous diffusion [16, 17], disease models [18, 19], ecological models [20], Lorenz system [21], and nonlocal thermoelectricity systems [22, 23]. For recent theoretical results on the topic, we refer the reader to a series of papers [24–37] and the references cited therein. Ahmad and Ntouyas [32, 33] discussed some fractional integral boundary value problems involving Hadamard fractional differential equations/systems and obtained the existence and uniqueness of solutions by applying the Banach fixed point theorem and Leray–Schauder alternative, respectively.

In [35], the authors investigated the existence and uniqueness of solutions for the coupled system of nonlinear fractional differential equations with three-point boundary conditions

$$\begin{cases} \mathscr{D}^{\alpha}u(t) = f(t, v(t), \mathscr{D}^{p}v(t)), t \in (0,1), \\ \mathscr{D}^{\beta}v(t) = g(t, v(t), \mathscr{D}^{q}u(t)), t \in (0,1), \\ u(0) = 0, u(1) = \gamma u(\eta), v(0) = 0, v(1) = \gamma v(\eta), \end{cases} \quad (1)$$

where $1 < \alpha, \beta < 2, p, q, \gamma > 0, 0 < \eta < 1, \alpha - q \geq 1, \beta - p \geq 1, \gamma \eta^{\alpha-1} < 1, \gamma\eta^{\beta-1} < 1$ and $\mathscr{D}^{\alpha}, \mathscr{D}^{\beta}$ are the standard Riemann-Liouville fractional derivative and $f, g : [0,1] \times R \times R \to R$ are given continuous functions.

Recently, Alsulami et al. [36] established the existence and uniqueness results for a nonlinear coupled system of Caputo type fractional differential equations supplemented with nonseparated coupled boundary conditions.

$$\begin{cases} {}^{c}\mathscr{D}^{\alpha}x(t) = f(t, x(t), y(t)), t \in [0,T], 1 < \alpha \leq 2 \\ {}^{c}\mathscr{D}^{\beta}y(t) = g(t, x(t), y(t)), t \in [0,T], 1 < \beta \leq 2, \\ x(0) = \lambda_1 y(T), x'(0) = \lambda_2 y'(T), \\ y(0) = \mu_1 y(T), y'(0) = \mu_2 x'(T), \end{cases} \quad (2)$$

where ${}^{c}\mathscr{D}^{\alpha}, {}^{c}\mathscr{D}^{\beta}$ denote the Caputo fractional derivatives of order α and β, respectively, $f, g : [0,T] \times R \times R \to R$ are appropriately chosen functions, and $\lambda_i, \mu_i, i = 1, 2,$ are real constants with $\lambda_i \mu_i \neq 1, i = 1, 2.$

Motivated by the research going on in this direction, in this paper, we study existence and uniqueness of solutions for a coupled system of Caputo-Hadamard fractional differential equations.

$$\begin{cases} {}^{c}\mathscr{D}^{\alpha}_{a^+}u(t) = f(t, u(t), v(t)), t \in [a,b], \\ {}^{c}\mathscr{D}^{\beta}_{a^+}v(t) = g(t, u(t), v(t)), t \in [a,b], \end{cases} \quad (3)$$

with multipoint boundary conditions

$$\begin{cases} u(a) = \lambda_1 v(b), \lambda_2 {}^{\mathscr{C}}\mathscr{D}^{\gamma_1}_{a^+}u(b) = \mu_2 \sum_{i=1}^{N} {}^{\mathscr{C}}\mathscr{D}^{\delta_1}_{a^+}v(\eta_i), \\ v(a) = \mu_1 u(b), \lambda_3 {}^{\mathscr{C}}\mathscr{D}^{\gamma_2}_{a^+}v(b) = \mu_3 \sum_{i=1}^{M} {}^{\mathscr{C}}\mathscr{D}^{\delta_2}_{a^+}v(\xi_i), \end{cases} \quad (4)$$

where $\alpha, \beta \in (1,2], \gamma_i, \delta_i \in (0,1], i = 1, 2, \eta_i \in R,$ for $i = 1, 2, \cdots N (N \in N), a < \eta_1 < \eta_2 < \cdots < b, \xi_i \in R,$ for $i = 1, 2, \cdots M (M \in N), a < \xi_1 < \xi_2 < \cdots < b, \lambda_i, \mu_i, i = 1, 2, 3$ are real positive constants $\mathscr{D}^{\kappa}_{a^+}$ denotes the Caputo-Hadamard fractional derivatives of order κ for $(\kappa = \alpha, \beta, \gamma_i, \delta_i, \text{for } i = 1, 2)$, $f, g \in [a,b] \times R \times R \to R$ are appropriately chosen functions.

The paper is organized as follows. In Sect. 2, we present some preliminary concepts of fractional calculus. Sect. 3 contains main results concerning the existence and uniqueness of solutions for the given problem (3), (4). The Leray-Schauder alternative theorem is applied to prove existence, while the uniqueness result was obtained via the Banach contraction mapping principle. Finally, we also discuss some examples for illustration of the existence-uniqueness results.

2. Preliminaries

For the convenience of the reader, we present some concepts of Hadamard type fractional calculus to facilitate the analysis of system (3), (4).

Definition 1 [7]. The Hadamard fractional integral of order $q > 0$ of a function $x(t)$ for all $t > a > 0$ is defined by

$${}^{H}\mathscr{I}^{q}_{a^+}x(t) = \frac{1}{\Gamma(q)} \int_a^t \left(\ln \frac{t}{s}\right)^{q-1} x(s) \frac{ds}{s}, \quad (5)$$

where $\Gamma(q) = \int_0^{\infty} t^{q-1} e^{-t} dt$ is the gamma function,s provided the right side is pointwise defined on R^+.

Definition 2 [7]. The Hadamard fractional derivative of order $q > 0$ of a function $x(t)$ for all $t > a > 0$ is defined by

$${}^{H}\mathscr{D}^{q}_{a^+}x(t) = \frac{1}{\Gamma(n-q)} \left(t \frac{d}{dt}\right)^{n} \int_a^t \left(\ln \frac{t}{s}\right)^{n-q-1} x(s) \frac{ds}{s}, \quad (6)$$

where $n = [q] + 1$ with $[q]$ denotes the integral part of the real number q and $\ln(\cdot) = \ln_e(\cdot)$.

Definition 3 [38]. Let $q \geq 0$ and $n = [q] + 1$. If $y(x) \in AC^n_{\delta}[a,b]$, where $0 < a < b < \infty$ and

$$AC^n_{\delta}[a,b] = \left\{ g : [a,b] \to C : \delta^{n-1} g(x) \in AC[a,b], \delta = x \frac{d}{dx} \right\}. \quad (7)$$

The Caputo type modification of the Hadamard fractional derivative of order q is defined by

$${}^{\mathscr{C}}\mathscr{D}^{q}_{a^+}y(x) = \mathscr{D}^{q}_{a^+} \left[y(t) - \sum_{k=0}^{n-1} \frac{\delta^k y(a)}{k!} \left(\ln \frac{t}{a}\right)^{k} \right](x). \quad (8)$$

Theorem 4 [38]. *Let $q \geq 0$, and $n = [q] + 1$. If $y(t) \in AC^n_{\delta}[a,b]$, where $0 < a < b < \infty$. Then ${}^{\mathscr{C}}\mathscr{D}^{q}_{a^+}f(t)$ exist everywhere on $[a,b]$ and*

(i) if $q \notin N_0$, ${}^{\mathscr{C}}\mathscr{D}^{q}_{a^+}f(t)$ can be represented by

$${}^{\mathscr{C}}\mathscr{D}^{q}_{a^+}y(t) = \mathscr{I}^{n-q}_{a^+}\delta^n y(t) = \frac{1}{\Gamma(n-q)} \int_a^t \left(\ln \frac{t}{s}\right)^{n-q-1} \delta^n y(s) \frac{ds}{s}, \quad (9)$$

(ii) if $q \in N_0$, then ${}^{\mathscr{C}}\mathscr{D}^{q}_{a^+}y(t) = \delta^n y(t)$

Remark 5. If $a, \alpha, \beta > 0$ then

$$\left({}^{H}\mathscr{D}^{\alpha}_{a}\left(\ln \frac{t}{a}\right)^{\beta-1}\right)(x) = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)} \left(\ln \frac{x}{a}\right)^{\beta-\alpha-1}. \quad (10)$$

Lemma 6 [38]. *Let $q \geq 0$ and $n = [q] + 1$. If $x(t) \in AC_\delta^n[a, b]$, then the Caputo-Hadamard fractional differential equation ${}^{\mathscr{C}}\mathscr{D}_{a^+}^q x(t) = 0$ has a solution:*

$$x(t) = \sum_{k=0}^{n-1} c_k \left(\log \frac{t}{a}\right)^k, \tag{11}$$

and the following formula holds:

$$\mathscr{I}_{a^+}^q {}^{\mathscr{C}}\mathscr{D}_{a^+}^q x(t) = x(t) + \sum_{k=0}^{n-1} c_k \left(\ln \frac{t}{a}\right)^k, \tag{12}$$

where $c_k \in R$, $k = 1, 2, \cdots, n-1$.

Now, we present an auxiliary lemma for boundary value problem of linear fractional differential equation with Caputo-Hadamard derivative.

Lemma 7. *Let $\Delta = (\lambda_2\lambda_3/\Gamma(2-\gamma_1)\Gamma(2-\gamma_2))(\ln(b/a))^{2-\gamma_1-\gamma_2} - \mu_2\mu_3/\Gamma(2-\delta_1)\Gamma(2-\delta_2)\sum_{i=1}^N(\ln(\eta_i/a))^{1-\delta_1}\sum_{i=1}^M(\ln(\xi_i/a))^{1-\delta_2} \neq 0$ and $\mu_1\lambda_1 \neq 1$. Let $x, y \in AC_n^\delta[a, b]$. Then, the solution of the linear Caputo-Hadamard fractional differential system*

$$\begin{cases} {}^c\mathscr{D}_{a^+}^\alpha u(t) = x(t),\ t \in [a, b],\ 1 < \alpha \leq 2, \\ {}^c\mathscr{D}_{a^+}^\beta v(t) = y(t),\ t \in [a, b],\ 1 < \beta \leq 2, \\ u(a) = \lambda_1 v(b),\ \lambda_2 {}^c\mathscr{D}_{a^+}^{\gamma_1} u(b) = \mu_2 \sum_{i=1}^N {}^c\mathscr{D}_{a^+}^{\delta_1} v(\eta_i), \\ v(a) = \mu_1 u(b),\ \lambda_3 {}^c\mathscr{D}_{a^+}^{\gamma_2} v(b) = \mu_3 \sum_{i=1}^M {}^c\mathscr{D}_{a^+}^{\delta_2} v(\xi_i), \end{cases} \tag{13}$$

is equivalent to the system of integral equations

$$\begin{aligned} u(t) = {} & \frac{\mu_3}{\Delta}\left[\frac{\mu_1\mu_2\lambda_1(\ln(b/a))\sum_{i=1}^N(\ln(\eta_i/a))^{1-\delta_1}}{\Gamma(2-\delta_1)(1-\mu_1\lambda_1)} + \frac{\lambda_1\lambda_2(\ln(b/a))^{2-\gamma_1}}{\Gamma(2-\gamma_1)(1-\mu_1\lambda_1)} + \frac{\mu_2(\ln(t/a))\sum_{i=1}^N(\ln(\eta_i/a))^{1-\delta_1}}{\Gamma(2-\delta_1)}\right]B_3 \\ & - \frac{\lambda_3}{\Delta}\left[\frac{\mu_1\mu_2\lambda_1(\ln(b/a))\sum_{i=1}^N(\ln(\eta_i/a))^{1-\delta_1}}{\Gamma(2-\delta_1)(1-\mu_1\lambda_1)} + \frac{\lambda_1\lambda_2(\ln(b/a))^{2-\gamma_1}}{\Gamma(2-\gamma_1)(1-\mu_1\lambda_1)} + \frac{\mu_2(\ln(t/a))\sum_{i=1}^N(\ln(\eta_i/a))^{1-\delta_1}}{\Gamma(2-\delta_1)}\right]A_3 \\ & + \frac{\mu_2}{\Delta}\left[\frac{\mu_1\lambda_1\lambda_3(\ln(b/a))^{2-\gamma_2}}{\Gamma(2-\gamma_2)(1-\mu_1\lambda_1)} + \frac{\lambda_1\mu_3(\ln(b/a))\sum_{i=1}^M(\ln(\xi_i/a))^{1-\delta_2}}{\Gamma(2-\delta_2)(1-\mu_1\lambda_1)} + \frac{\lambda_3(\ln(t/a))(\ln(b/a))^{1-\gamma_2}}{\Gamma(2-\gamma_2)}\right]A_2 \\ & - \frac{\lambda_2}{\Delta}\left[\frac{\mu_1\lambda_1\lambda_3(\ln(b/a))^{2-\gamma_2}}{\Gamma(2-\gamma_2)(1-\mu_1\lambda_1)} + \frac{\lambda_1\mu_3(\ln(b/a))\sum_{i=1}^M(\ln(\xi_i/a))^{1-\delta_2}}{\Gamma(2-\delta_2)(1-\mu_1\lambda_1)} + \frac{\lambda_3(\ln(t/a))(\ln(b/a))^{1-\gamma_2}}{\Gamma(2-\gamma_2)}\right]B_2 + \frac{\lambda_1}{1-\mu_1\lambda_1}(\mu_1B_1 + A_1) \\ & + \int_a^t \frac{(\ln(t/s))^{\alpha-1}}{\Gamma(\alpha)}x(s)\frac{ds}{s}, \end{aligned} \tag{14}$$

$$\begin{aligned} v(t) = {} & \frac{\mu_3}{\Delta}\left[\frac{\mu_1\lambda_1\lambda_2(\ln(b/a))^{2-\gamma_1}}{\Gamma(2-\gamma_1)(1-\mu_1\lambda_1)} + \frac{\mu_1\mu_2(\ln(b/a))\sum_{i=1}^N(\ln(\eta_i/a))^{1-\delta_1}}{\Gamma(2-\delta_1)(1-\mu_1\lambda_1)} + \frac{\lambda_2(\ln(t/a))(\ln(b/a))^{1-\gamma_1}}{\Gamma(2-\gamma_1)}\right]B_3 - \frac{\lambda_3}{\Delta}\left[\frac{\mu_1\lambda_1\lambda_2(\ln(b/a))^{2-\gamma_1}}{\Gamma(2-\gamma_1)(1-\mu_1\lambda_1)}\right. \\ & \left. + \frac{\mu_1\mu_2(\ln(b/a))\sum_{i=1}^N(\ln(\eta_i/a))^{1-\delta_1}}{\Gamma(2-\delta_1)(1-\mu_1\lambda_1)} + \frac{\lambda_2(\ln(t/a))(\ln(b/a))^{1-\gamma_1}}{\Gamma(2-\gamma_1)}\right]A_3 \\ & + \frac{\mu_2}{\Delta}\left[\frac{\mu_1\mu_3\lambda_1(\ln(b/a))\sum_{i=1}^M(\ln(\xi_i/a))^{1-\delta_2}}{\Gamma(2-\delta_2)(1-\mu_1\lambda_1)} + \frac{\mu_1\lambda_3(\ln(b/a))^{2-\gamma_2}}{\Gamma(2-\gamma_2)(1-\mu_1\lambda_1)} + \frac{\mu_3(\ln(t/a))\sum_{i=1}^M(\ln(\xi_i/a))^{1-\delta_2}}{\Gamma(2-\delta_2)}\right]A_2 \\ & - \frac{\lambda_2}{\Delta}\left[\frac{\mu_1\mu_3\lambda_1(\ln(b/a))\sum_{i=1}^M(\ln(\xi_i/a))^{1-\delta_2}}{\Gamma(2-\delta_2)(1-\mu_1\lambda_1)} + \frac{\mu_1\lambda_3(\ln(b/a))^{2-\gamma_2}}{\Gamma(2-\gamma_2)(1-\mu_1\lambda_1)} + \frac{\mu_3(\ln(t/a))\sum_{i=1}^M(\ln(\xi_i/a))^{1-\delta_2}}{\Gamma(2-\delta_2)}\right]B_2 + \frac{\mu_1}{1-\mu_1\lambda_1}(\lambda_1A_1 + B_1) \\ & + \int_a^t \frac{(\ln(t/s))^{\beta-1}}{\Gamma(\beta)}y(s)\frac{ds}{s}, \end{aligned} \tag{15}$$

where

$$\begin{aligned} B_1 &= \int_a^b \frac{(\ln(b/s))^{\alpha-1}}{\Gamma(\alpha)}x(s)\frac{ds}{s},\ A_1 = \int_a^b \frac{(\ln(b/s))^{\beta-1}}{\Gamma(\beta)}y(s)\frac{ds}{s}, \\ B_2 &= \int_a^b \frac{(\ln(b/s))^{\alpha-\gamma_1-1}}{\Gamma(\alpha-\gamma_1)}x(s)\frac{ds}{s},\ A_2 = \sum_{i=1}^N\int_a^{\eta_i} \frac{(\ln(\eta_i/s))^{\beta-\delta_1-1}}{\Gamma(\beta-\delta_1)}y(s)\frac{ds}{s}, \\ B_3 &= \sum_{i=1}^M\int_a^{\xi_i} \frac{(\ln(\xi_i/s))^{\alpha-\delta_2-1}}{\Gamma(\alpha-\delta_2)}x(s)\frac{ds}{s},\ A_3 = \int_a^b \frac{(\ln(b/s))^{\beta-\gamma_2-1}}{\Gamma(\beta-\gamma_2)}y(s)\frac{ds}{s}. \end{aligned} \tag{16}$$

Proof. We apply Lemma [6] that the general solution of the Caputo-Hadamard fractional differential equation in (13) can be written as:

$$u(t) = c_0 + c_1\left(\ln \frac{t}{a}\right) + \int_a^t \frac{(\ln(t/s))^{\alpha-1}}{\Gamma(\alpha)}x(s)\frac{ds}{s}, \tag{17}$$

$$v(t) = d_0 + d_1\left(\ln \frac{t}{a}\right) + \int_a^t \frac{(\ln(t/s))^{\beta-1}}{\Gamma(\beta)}y(s)\frac{ds}{s}, \tag{18}$$

where $c_i, d_i, i = 0, 1$, are arbitrary real constants. From (17) and (18) we have

$${}^c\mathscr{D}_{a^+}^{\gamma_1} u(t) = c_1\frac{1}{\Gamma(2-\gamma_1)}\left(\ln \frac{t}{a}\right)^{1-\gamma_1} + \int_a^t \frac{(\ln(t/s))^{\alpha-\gamma_1-1}}{\Gamma(\alpha-\gamma_1)}x(s)\frac{ds}{s}, \tag{19}$$

$${}^c\mathscr{D}_{a^+}^{\gamma_2} v(t) = d_1\frac{1}{\Gamma(2-\gamma_2)}\left(\ln \frac{t}{a}\right)^{1-\gamma_2} + \int_a^t \frac{(\ln(t/s))^{\beta-\gamma_2-1}}{\Gamma(\beta-\gamma_2)}y(s)\frac{ds}{s}, \tag{20}$$

$${}^c\mathscr{D}_{a^+}^{\delta_1} v(t) = d_1\frac{1}{\Gamma(2-\delta_1)}\left(\ln \frac{t}{a}\right)^{1-\delta_1} + \int_a^t \frac{(\ln(t/s))^{\beta-\delta_1-1}}{\Gamma(\beta-\delta_1)}y(s)\frac{ds}{s}, \tag{21}$$

$$ {}^{c}\mathscr{D}_{a^+}^{\delta_2}u(t) = c_1 \frac{1}{\Gamma(2-\delta_2)}\left(\ln\frac{t}{a}\right)^{1-\delta_2} + \int_a^t \frac{(\ln(t/s))^{\alpha-\delta_2-1}}{\Gamma(\alpha-\delta_2)}x(s)\frac{ds}{s}. \tag{22} $$

Using the boundary conditions $u(a) = \lambda_1 v(b)$ and $v(a) = \mu_1 u(b)$ from (17) and (18), we have

$$ \Rightarrow c_0 = \lambda_1\left[d_0 + d_1\left(\ln\frac{b}{a}\right) + A_1\right], \tag{23} $$

$$ \Rightarrow d_0 = \mu_1\left[c_0 + c_1\left(\ln\frac{b}{a}\right) + B_1\right]. \tag{24} $$

Using the boundary conditions $\lambda_2 {}^{\mathscr{C}}\mathscr{D}_{a^+}^{\gamma_1}u(b) = \mu_2\sum_{i=1}^{N} {}^{\mathscr{C}}\mathscr{D}_{a^+}^{\delta_1}v(\eta_i)$ and $\lambda_3 {}^{\mathscr{C}}\mathscr{D}_{a^+}^{\gamma_2}v(b) = \mu_3\sum_{i=1}^{M} {}^{\mathscr{C}}\mathscr{D}_{a^+}^{\delta_2}u(\xi_i)$ from (19) to (22), we have

$$ \begin{aligned} &\Rightarrow c_1 \frac{\lambda_2}{\Gamma(2-\gamma_1)}\left(\ln\frac{b}{a}\right)^{1-\gamma_1} - d_1\frac{\mu_2}{\Gamma(2-\delta_1)}\sum_{i=1}^{N}\left(\ln\frac{\eta_i}{a}\right)^{1-\delta_1} = A_2\mu_2 - \lambda_2 B_2, \\ &\Rightarrow c_1 \frac{-\mu_3}{\Gamma(2-\delta_2)}\sum_{i=1}^{M}\left(\ln\frac{\xi_i}{a}\right)^{1-\delta_2} + d_1\frac{\lambda_3}{\Gamma(2-\gamma_2)}\left(\ln\frac{b}{a}\right)^{1-\gamma_2} = B_3\mu_3 - \lambda_3 A_3. \end{aligned} \tag{25} $$

Solving the resulting equations for c_1 and d_1, we find that

$$ \begin{aligned} c_1 &= \frac{1}{\Delta}\left[\frac{\lambda_3(A_2\mu_2-\lambda_2B_2)}{\Gamma(2-\gamma_2)}\left(\ln\frac{b}{a}\right)^{1-\gamma_2} + \frac{\mu_2(B_3\mu_3-\lambda_3A_3)}{\Gamma(2-\delta_1)}\sum_{i=1}^{N}\left(\ln\frac{\eta_i}{a}\right)^{1-\delta_1}\right], \\ d_1 &= \frac{1}{\Delta}\left[\frac{\lambda_2(B_3\mu_3-\lambda_3A_3)}{\Gamma(2-\gamma_1)}\left(\ln\frac{b}{a}\right)^{1-\gamma_1} + \frac{\mu_3(A_2\mu_2-\lambda_2B_2)}{\Gamma(2-\delta_2)}\sum_{i=1}^{M}\left(\ln\frac{\xi_i}{a}\right)^{1-\delta_2}\right], \end{aligned} \tag{26} $$

substituting c_1 and d_1 in (23) and (24), we have

$$ \begin{aligned} c_0 = \frac{\lambda_1}{1-\mu_1\lambda_1}\Bigg[&\Bigg[\frac{\mu_1\mu_2\mu_3}{\Delta\Gamma(2-\delta_1)}\sum_{i=1}^{N}\left(\ln\frac{\eta_i}{a}\right)^{1-\delta_1}\left(\ln\frac{b}{a}\right) + \frac{\lambda_2\mu_3}{\Delta\Gamma(2-\gamma_1)} \\ &\cdot\left(\ln\frac{b}{a}\right)^{2-\gamma_1}\Bigg]B_3 - \Bigg[\frac{\mu_1\mu_2\lambda_3}{\Delta\Gamma(2-\delta_1)}\sum_{i=1}^{N}\left(\ln\frac{\eta_i}{a}\right)^{1-\delta_1}\left(\ln\frac{b}{a}\right) \\ &+ \frac{\lambda_2\lambda_3}{\Delta\Gamma(2-\gamma_1)}\left(\ln\frac{b}{a}\right)^{2-\gamma_1}\Bigg]A_3 + \Bigg[\frac{\mu_1\mu_2\lambda_3}{\Delta\Gamma(2-\gamma_2)}\left(\ln\frac{b}{a}\right)^{2-\gamma_2} \\ &+ \frac{\mu_2\mu_3}{\Delta\Gamma(2-\delta_2)}\sum_{i=1}^{M}\left(\ln\frac{\xi_i}{a}\right)^{1-\delta_2}\left(\ln\frac{b}{a}\right)\Bigg]A_2 \\ &- \Bigg[\frac{\mu_1\lambda_2\lambda_3}{\Delta\Gamma(2-\gamma_2)}\left(\ln\frac{b}{a}\right)^{2-\gamma_2} + \frac{\lambda_2\mu_3}{\Delta\Gamma(2-\delta_2)}\sum_{i=1}^{M}\left(\ln\frac{\xi_i}{a}\right)^{1-\delta_2} \\ &\cdot\left(\ln\frac{b}{a}\right)\Bigg]B_2 + (\mu_1B_1 + A_1)\Bigg], \end{aligned} \tag{27} $$

and

$$ \begin{aligned} d_0 = \frac{\lambda_1}{1-\mu_1\lambda_1}\Bigg[&\Bigg[\frac{\lambda_1\lambda_2\mu_3}{\Delta\Gamma(2-\gamma_1)}\left(\ln\frac{b}{a}\right)^{2-\gamma_1} + \frac{\mu_2\mu_3}{\Delta\Gamma(2-\delta_1)}\sum_{i=1}^{N} \\ &\cdot\left(\ln\frac{\eta_i}{a}\right)^{1-\delta_1}\left(\ln\frac{b}{a}\right)\Bigg]B_3 - \Bigg[\frac{\lambda_1\lambda_2\lambda_3}{\Delta\Gamma(2-\gamma_1)}\left(\ln\frac{b}{a}\right)^{2-\gamma_1} \\ &+ \frac{\mu_2\lambda_3}{\Delta\Gamma(2-\delta_1)}\sum_{i=1}^{N}\left(\ln\frac{\eta_i}{a}\right)^{1-\delta_1}\left(\ln\frac{b}{a}\right)\Bigg]A_3 + \Bigg[\frac{\lambda_1\mu_2\mu_3}{\Delta\Gamma(2-\delta_2)} \\ &\cdot\sum_{i=1}^{M}\left(\ln\frac{\xi_i}{a}\right)^{1-\delta_2}\left(\ln\frac{b}{a}\right) + \frac{\mu_2\lambda_3}{\Delta\Gamma(2-\gamma_2)}\left(\ln\frac{b}{a}\right)^{2-\gamma_2}\Bigg]A_2 \\ &- \Bigg[\frac{\lambda_1\lambda_2\mu_3}{\Delta\Gamma(2-\delta_2)}\sum_{i=1}^{M}\left(\ln\frac{\xi_i}{a}\right)^{1-\delta_2}\left(\ln\frac{b}{a}\right) + \frac{\lambda_2\lambda_3}{\Delta\Gamma(2-\gamma_2)} \\ &\cdot\left(\ln\frac{b}{a}\right)^{2-\gamma_2}\Bigg]B_2 + (\lambda_1A_1 + B_1)\Bigg]. \end{aligned} \tag{28} $$

Inserting the values of $c_i, d_i, i = 0, 1$ in (17) and (18), which leads to the solution system (14), (15). The converse follows by direct computation. The proof is completed.

3. Existence and Uniqueness Results

This section is concerned with the main results of the paper. First of all, we fix our terminology. Let $\mathscr{C} = C([a,b], R)$, $a > 0$ be the Banach space of all continuous functions from $[a,b]$ to R. Space $X = \{u(t)\colon u(t) \in C^2([a,b], R)\}$ endowed with the norm $\|u\| = \sup\{|u(t)|, t \in [a,b]\}$ is a Banach space. In addition, let $Y = \{v(t)\colon v(t) \in C^2([a,b], R)\}$ with the norm $\|v\| = \sup\{|v(t)|, t \in [a,b]\}$. It is obvious that product space $(X \times Y, \|(u,v)\|)$ is a Banach space with the norm $\|(u,v)\| = \|u\| + \|v\|$.

In view of Lemma 7, we introduce an operator $\mathscr{T} : X \times Y \to X \times Y$ as follows:

$$ \mathscr{T}(u,v)(t) = (\mathscr{T}_1(u,v)(t), \mathscr{T}_2(u,v)(t)), \tag{29} $$

where

$$ \begin{aligned} \mathscr{T}_1(u,v)(t) = \frac{\mu_3}{\Delta}&\Bigg[\frac{\mu_1\mu_2\lambda_1(\ln(b/a))\sum_{i=1}^{N}(\ln(\eta_i/a))^{1-\delta_1}}{\Gamma(2-\delta_1)(1-\mu_1\lambda_1)} \\ &+ \frac{\lambda_1\lambda_2(\ln(b/a))^{2-\gamma_1}}{\Gamma(2-\gamma_1)(1-\mu_1\lambda_1)} + \frac{\mu_2(\ln(t/a))\sum_{i=1}^{N}(\ln(\eta_i/a))^{1-\delta_1}}{\Gamma(2-\delta_1)}\Bigg]B_{3f} \\ &- \frac{\lambda_3}{\Delta}\Bigg[\frac{\mu_1\mu_2\lambda_1(\ln(b/a))\sum_{i=1}^{N}(\ln(\eta_i/a))^{1-\delta_1}}{\Gamma(2-\delta_1)(1-\mu_1\lambda_1)} + \frac{\lambda_1\lambda_2(\ln(b/a))^{2-\gamma_1}}{\Gamma(2-\gamma_1)(1-\mu_1\lambda_1)} \\ &+ \frac{\mu_2(\ln(t/a))\sum_{i=1}^{N}(\ln(\eta_i/a))^{1-\delta_1}}{\Gamma(2-\delta_1)}\Bigg]A_{3g} + \frac{\mu_2}{\Delta}\Bigg[\frac{\mu_1\lambda_1\lambda_3(\ln(b/a))^{2-\gamma_2}}{\Gamma(2-\gamma_2)(1-\mu_1\lambda_1)} \\ &+ \frac{\lambda_1\mu_3(\ln(b/a))\sum_{i=1}^{M}(\ln(\xi_i/a))^{1-\delta_2}}{\Gamma(2-\delta_2)(1-\mu_1\lambda_1)} + \frac{\lambda_3(\ln(t/a))(\ln(b/a))^{1-\gamma_2}}{\Gamma(2-\gamma_2)}\Bigg]A_{2g} \\ &- \frac{\lambda_2}{\Delta}\Bigg[\frac{\mu_1\lambda_1\lambda_3(\ln(b/a))^{2-\gamma_2}}{\Gamma(2-\gamma_2)(1-\mu_1\lambda_1)} + \frac{\lambda_1\mu_3(\ln(b/a))\sum_{i=1}^{M}(\ln(\xi_i/a))^{1-\delta_2}}{\Gamma(2-\delta_2)(1-\mu_1\lambda_1)} \\ &+ \frac{\lambda_3(\ln(t/a))(\ln(b/a))^{1-\gamma_2}}{\Gamma(2-\gamma_2)}\Bigg]B_{2f} + \frac{\lambda_1}{1-\mu_1\lambda_1}(\mu_1B_{1f} + A_{1g}) \\ &+ \int_a^t \frac{(\ln(t/s))^{\alpha-1}}{\Gamma(\alpha)}f(s,u(s),v(s))\frac{ds}{s}, \end{aligned} \tag{30} $$

and

$$\begin{aligned}
\mathcal{T}_2(u,v)(t) = {} & \frac{\mu_3}{\Delta}\left[\frac{\mu_1\lambda_1\lambda_2(\ln(b/a))^{2-\gamma_1}}{\Gamma(2-\gamma_1)(1-\mu_1\lambda_1)} + \frac{\mu_1\mu_2(\ln(b/a))\sum_{i=1}^{N}(\ln(\eta_i/a))^{1-\delta_1}}{\Gamma(2-\delta_1)(1-\mu_1\lambda_1)}\right. \\
& \left. + \frac{\lambda_2(\ln(t/a))(\ln(b/a))^{1-\gamma_1}}{\Gamma(2-\gamma_1)}\right] B_{3f} - \frac{\lambda_3}{\Delta}\left[\frac{\mu_1\lambda_1\lambda_2(\ln(b/a))^{2-\gamma_1}}{\Gamma(2-\gamma_1)(1-\mu_1\lambda_1)}\right. \\
& \left. + \frac{\mu_1\mu_2(\ln(b/a))\sum_{i=1}^{N}(\ln(\eta_i/a))^{1-\delta_1}}{\Gamma(2-\delta_1)(1-\mu_1\lambda_1)} + \frac{\lambda_2(\ln(t/a))(\ln(b/a))^{1-\gamma_1}}{\Gamma(2-\gamma_1)}\right] A_{3g} \\
& + \frac{\mu_2}{\Delta}\left[\frac{\mu_1\mu_3\lambda_1(\ln(b/a))\sum_{i=1}^{M}(\ln(\xi_i/a))^{1-\delta_2}}{\Gamma(2-\delta_2)(1-\mu_1\lambda_1)} + \frac{\mu_1\lambda_3(\ln(b/a))^{2-\gamma_2}}{\Gamma(2-\gamma_2)(1-\mu_1\lambda_1)}\right. \\
& \left. + \frac{\mu_3(\ln(t/a))\sum_{i=1}^{M}(\ln(\xi_i/a))^{1-\delta_2}}{\Gamma(2-\delta_2)}\right] A_{2g} \\
& - \frac{\lambda_2}{\Delta}\left[\frac{\mu_1\mu_3\lambda_1(\ln(b/a))\sum_{i=1}^{M}(\ln(\xi_i/a))^{1-\delta_2}}{\Gamma(2-\delta_2)(1-\mu_1\lambda_1)} + \frac{\mu_1\lambda_3(\ln(b/a))^{2-\gamma_2}}{\Gamma(2-\gamma_2)(1-\mu_1\lambda_1)}\right. \\
& \left. + \frac{\mu_3(\ln(t/a))\sum_{i=1}^{M}(\ln(\xi_i/a))^{1-\delta_2}}{\Gamma(2-\delta_2)}\right] B_{2f} + \frac{\mu_1}{1-\mu_1\lambda_1}(\lambda_1 A_{1g} + B_{1f}) \\
& + \int_a^t \frac{(\ln(t/s))^{\beta-1}}{\Gamma(\beta)} g(s,u(s),v(s))\frac{ds}{s}.
\end{aligned} \tag{31}$$

Here,

$$\begin{aligned}
B_{1f} &= \int_a^b \frac{(\ln(b/s))^{\alpha-1}}{\Gamma(\alpha)} f(s,u(s),v(s))\frac{ds}{s},\ A_{1g} \\
&= \int_a^b \frac{(\ln(b/s))^{\beta-1}}{\Gamma(\beta)} g(s,u(s),v(s))\frac{ds}{s}, \\
B_{2f} &= \int_a^b \frac{(\ln(b/s))^{\alpha-\gamma_1-1}}{\Gamma(\alpha-\gamma_1)} f(s,u(s),v(s))\frac{ds}{s},\ A_{2g} \\
&= \sum_{i=1}^{N}\int_a^{\eta_i} \frac{(\ln(\eta_i/s))^{\beta-\delta_1-1}}{\Gamma(\beta-\delta_1)} g(s,u(s),v(s))\frac{ds}{s}, \\
B_{3f} &= \sum_{i=1}^{M}\int_a^{\xi_i} \frac{(\ln(\xi_i/s))^{\alpha-\delta_2-1}}{\Gamma(\alpha-\delta_2)} f(s,u(s),v(s))\frac{ds}{s},\ A_{3g} \\
&= \int_a^b \frac{(\ln(b/s))^{\beta-\gamma_2-1}}{\Gamma(\beta-\gamma_2)} g(s,u(s),v(s))\frac{ds}{s}.
\end{aligned} \tag{32}$$

For computational convenience, we set

$$\begin{aligned}
K_1 = {} & \frac{|\mu_3|}{|\Delta|}\left[\frac{|\mu_1||\mu_2||\lambda_1|(\ln(b/a))\sum_{i=1}^{N}(\ln(\eta_i/a))^{1-\delta_1}}{\Gamma(2-\delta_1)|1-\mu_1\lambda_1|}\right. \\
& \left. + \frac{|\lambda_1||\lambda_2|(\ln(b/a))^{2-\gamma_1}}{\Gamma(2-\gamma_1)|1-\mu_1\lambda_1|} + \frac{|\mu_2|(\ln(b/a))\sum_{i=1}^{N}(\ln(\eta_i/a))^{1-\delta_1}}{\Gamma(2-\delta_1)}\right] \\
& \times \frac{\sum_{i=1}^{M}(\ln(\xi_i/a))^{\alpha-\delta_2}}{\Gamma(\alpha-\delta_2+1)} + \frac{|\lambda_2|}{|\Delta|}\left[\frac{|\mu_1||\lambda_1||\lambda_3|(\ln(b/a))^{2-\gamma_2}}{\Gamma(2-\gamma_2)|1-\mu_1\lambda_1|}\right. \\
& \left. + \frac{|\lambda_1||\mu_3|(\ln(b/a))\sum_{i=1}^{M}(\ln(\xi_i/a))^{1-\delta_2}}{\Gamma(2-\delta_2)|1-\mu_1\lambda_1|} + \frac{|\lambda_3|(\ln(b/a))^{2-\gamma_2}}{\Gamma(2-\gamma_2)}\right] \\
& \times \frac{(\ln(b/a))^{\alpha-\gamma_1}}{\Gamma(\alpha-\gamma_1+1)} + \left[\frac{|\mu_1||\lambda_1|}{|1-\mu_1\lambda_1|}+1\right]\frac{(\ln(b/a))^{\alpha}}{\Gamma(\alpha+1)},
\end{aligned}$$

$$\begin{aligned}
K_2 = {} & \frac{|\lambda_3|}{|\Delta|}\left[\frac{|\mu_1||\mu_2||\lambda_1|(\ln(b/a))\sum_{i=1}^{N}(\ln(\eta_i/a))^{1-\delta_1}}{\Gamma(2-\delta_1)|1-\mu_1\lambda_1|}\right. \\
& \left. + \frac{|\lambda_1||\lambda_2|(\ln(b/a))^{2-\gamma_1}}{\Gamma(2-\gamma_1)|1-\mu_1\lambda_1|} + \frac{|\mu_2|(\ln(b/a))\sum_{i=1}^{N}(\ln(\eta_i/a))^{1-\delta_1}}{\Gamma(2-\delta_1)}\right] \\
& \times \frac{(\ln(b/a))^{\beta-\gamma_2}}{\Gamma(\beta-\gamma_2+1)} + \frac{|\mu_2|}{|\Delta|}\left[\frac{|\mu_1||\lambda_1||\lambda_3|(\ln(b/a))^{2-\gamma_2}}{\Gamma(2-\gamma_2)|1-\mu_1\lambda_1|}\right. \\
& \left. + \frac{|\lambda_1||\mu_3|(\ln(b/a))\sum_{i=1}^{M}(\ln(\xi_i/a))^{1-\delta_2}}{\Gamma(2-\delta_2)|(1-\mu_1\lambda_1)|} + \frac{|\lambda_3|(\ln(b/a))^{2-\gamma_2}}{\Gamma(2-\gamma_2)}\right] \\
& \times \frac{\sum_{i=1}^{N}(\ln(\eta_i/a))^{\beta-\delta_1}}{\Gamma(\beta-\delta_1+1)} + \frac{|\mu_1|}{|1-\mu_1\lambda_1|}\frac{(\ln(b/a))^{\beta}}{\Gamma(\beta+1)},
\end{aligned}$$

$$\begin{aligned}
K_3 = {} & \frac{|\mu_3|}{|\Delta|}\left[\frac{|\mu_1||\lambda_1||\lambda_2|(\ln(b/a))^{2-\gamma_1}}{\Gamma(2-\gamma_1)|1-\mu_1\lambda_1|}\right. \\
& \left. + \frac{|\mu_1||\mu_2|(\ln(b/a))\sum_{i=1}^{N}(\ln(\eta_i/a))^{1-\delta_1}}{\Gamma(2-\delta_1)|1-\mu_1\lambda_1|} + \frac{|\lambda_2|(\ln(b/a))^{2-\gamma_1}}{\Gamma(2-\gamma_1)}\right] \\
& \cdot \frac{\sum_{i=1}^{M}(\ln(\xi_i/a))^{\alpha-\delta_2}}{\Gamma(\alpha-\delta_2+1)} + \frac{|\lambda_2|}{|\Delta|}\left[\frac{|\mu_1||\mu_3||\lambda_1|(\ln(b/a))\sum_{i=1}^{M}(\ln(\xi_i/a))^{1-\delta_2}}{\Gamma(2-\delta_2)|1-\mu_1\lambda_1|}\right. \\
& \left. + \frac{|\mu_1||\lambda_3|(\ln(b/a))^{2-\gamma_2}}{\Gamma(2-\gamma_2)|1-\mu_1\lambda_1|} + \frac{|\mu_3|(\ln(b/a))\sum_{i=1}^{M}(\ln(\xi_i/a))^{1-\delta_2}}{\Gamma(2-\delta_2)}\right] \\
& \times \frac{(\ln(b/a))^{\alpha-\gamma_1}}{\Gamma(\alpha-\gamma_1+1)} + \frac{|\mu_1|}{|1-\lambda_1\mu_1|}\frac{(\ln(b/a))^{\alpha}}{\Gamma(\alpha+1)},
\end{aligned}$$

$$\begin{aligned}
K_4 = {} & \frac{|\lambda_3|}{|\Delta|}\left[\frac{|\mu_1||\lambda_1||\lambda_2|(\ln(b/a))^{2-\gamma_1}}{\Gamma(2-\gamma_1)|1-\mu_1\lambda_1|}\right. \\
& \left. + \frac{|\mu_1||\mu_2|(\ln(b/a))\sum_{i=1}^{N}(\ln(\eta_i/a))^{1-\delta_1}}{\Gamma(2-\delta_1)|1-\mu_1\lambda_1|} + \frac{\lambda_2(\ln(b/a))^{2-\gamma_1}}{\Gamma(2-\gamma_1)}\right] \\
& \cdot \frac{(\ln(b/a))^{\beta-\gamma_2}}{\Gamma(\beta-\gamma_2+1)} + \frac{|\mu_2|}{|\Delta|}\left[\frac{|\mu_1||\mu_3||\lambda_1|(\ln(b/a))\sum_{i=1}^{M}(\ln(\xi_i/a))^{1-\delta_2}}{\Gamma(2-\delta_2)|1-\mu_1\lambda_1|}\right. \\
& \left. + \frac{|\mu_1||\lambda_3|(\ln(b/a))^{2-\gamma_2}}{\Gamma(2-\gamma_2)|1-\mu_1\lambda_1|} + \frac{\mu_3(\ln(b/a))\sum_{i=1}^{M}(\ln(\xi_i/a))^{1-\delta_2}}{\Gamma(2-\delta_2)}\right] \\
& \times \frac{\sum_{i=1}^{N}(\ln(\eta_i/a))^{\beta-\delta_1}}{\Gamma(\beta-\delta_1+1)} + \left[\frac{|\lambda_1||\mu_1|}{|1-\lambda_1\mu_1|}+1\right]\frac{(\ln(b/a))^{\beta}}{\Gamma(\beta+1)}.
\end{aligned} \tag{33}$$

Now, we are in a position to present our main results. The methods used to prove the existence and uniqueness solutions of boundary value problem (3), (4) via Banach's contraction principle.

Theorem 8. *Suppose that $f, g : [a,b]\times R\times R \to R$ are continuous functions. In addition, we assume that:*

(H1) there exist constants m_i and n_i, $i = 1, 2$, such that for all $t \in [a,b]$ and $u_i, v_i \in R$, $i = 1, 2$, we have

$$\begin{aligned}
|f(t,u_1,v_1) - f(t,u_2,v_2)| &\le m_1|u_1-u_2| + m_2|v_1-v_2|, \\
|g(t,u_1,v_1) - g(t,u_2,v_2)| &\le n_1|u_1-u_2| + n_2|v_1-v_2|.
\end{aligned} \tag{34}$$

Then, the system (3), (4) has a unique solution on $[a, b]$, *if*

$$(K_1 + K_3)(m_1 + m_2) + (K_2 + K_4)(n_1 + n_2) < 1. \tag{35}$$

Proof. Define $\sup_{t\in[a,b]} f(t,0,0) = \sigma_1 < \infty$ and $\sup_{t\in[a,b]} g(t,0,0) = \sigma_2 < \infty$ and $r > 0$ such that

$$r > \frac{(K_1 + K_3)\sigma_1 + (K_2 + K_4)\sigma_2}{1 - [(K_1 + K_3)(m_1 + m_2) + (K_2 + K_4)(n_1 + n_2)]}. \tag{36}$$

Now, we show that $\mathcal{T} B_r \subset B_r$, where $B_r = \{(u, v) \in X \times Y : \|(u, v)\| \le r\}$.

By assumption $(H1)$, for $(u, v) \in B_r, t \in [a, b]$, we have that

$$\begin{aligned} |f(t, u(t), v(t))| &\le |f(t, u(t), v(t)) - f(t, 0, 0)| + |f(t, 0, 0)| \\ &\le m_1|u(t)| + m_2|v(t)| + \sigma_1 \le m_1\|u\| + m_2\|v\| + \sigma_1 \end{aligned}$$

$$|g(t, u(t), v(t))| \le n_1\|u\| + n_2\|v\| + \sigma_2, \tag{37}$$

which leads to

$$\begin{aligned} |\mathcal{T}_1(u,v)(t)| \le{}& \frac{|\mu_3|}{|\Delta|}\left[\frac{|\mu_1||\mu_2||\lambda_1|(\ln(b/a))\sum_{i=1}^{N}(\ln(\eta_i/a))^{1-\delta_1}}{\Gamma(2-\delta_1)\,|(1-\mu_1\lambda_1)|}\right. \\ &\left. + \frac{|\lambda_1||\lambda_2|(\ln(b/a))^{2-\gamma_1}}{\Gamma(2-\gamma_1)\,|(1-\mu_1\lambda_1)|} + \frac{|\mu_2|(\ln(b/a))\sum_{i=1}^{N}(\ln(\eta_i/a))^{1-\delta_1}}{\Gamma(2-\delta_1)}\right] \\ &\times \frac{\sum_{i=1}^{M}(\ln(\xi_i/a))^{\alpha-\delta_2}}{\Gamma(\alpha-\delta_2+1)}(m_1\|u\| + m_2\|v\| + \sigma_1) \\ &+ \frac{|\lambda_3|}{|\Delta|}\left[\frac{|\mu_1||\mu_2||\lambda_1|(\ln(b/a))\sum_{i=1}^{N}(\ln(\eta_i/a))^{1-\delta_1}}{\Gamma(2-\delta_1)\,|(1-\mu_1\lambda_1)|}\right. \\ &\left. + \frac{|\lambda_1||\lambda_2|(\ln(b/a))^{2-\gamma_1}}{\Gamma(2-\gamma_1)\,|(1-\mu_1\lambda_1)|} + \frac{|\mu_2|\,(\ln(b/a))\sum_{i=1}^{N}(\ln(\eta_i/a))^{1-\delta_1}}{\Gamma(2-\delta_1)}\right] \\ &\times \frac{(\ln(b/a))^{\beta-\gamma_2}}{\Gamma(\beta-\gamma_2+1)}(n_1\|u\| + n_2\|v\| + \sigma_2) \\ &+ \frac{|\mu_2|}{|\Delta|}\left[\frac{|\mu_1||\lambda_1||\lambda_3|(\ln(b/a))^{2-\gamma_2}}{\Gamma(2-\gamma_2)\,|(1-\mu_1\lambda_1)|}\right. \\ &\left. + \frac{|\lambda_1||\mu_3|(\ln(b/a))\sum_{i=1}^{M}(\ln(\xi_i/a))^{1-\delta_2}}{\Gamma(2-\delta_2)\,|(1-\mu_1\lambda_1)|} + \frac{|\lambda_3|\,(\ln(b/a))^{2-\gamma_2}}{\Gamma(2-\gamma_2)}\right] \\ &\times \frac{\sum_{i=1}^{N}(\ln(\eta_i/a))^{\beta-\delta_1}}{\Gamma(\beta-\delta_1+1)}(n_1\|u\| + n_2\|v\| + \sigma_2) \\ &+ \frac{|\lambda_2|}{|\Delta|}\left[\frac{|\mu_1||\lambda_1||\lambda_3|(\ln(b/a))^{2-\gamma_2}}{\Gamma(2-\gamma_2)\,|(1-\mu_1\lambda_1)|} + \frac{|\lambda_1||\mu_3|(\ln(b/a))\sum_{i=1}^{M}(\ln(\xi_i/a))^{1-\delta_2}}{\Gamma(2-\delta_2)\,|(1-\mu_1\lambda_1)|}\right. \\ &\left. + \frac{|\lambda_3|\,(\ln(b/a))^{2-\gamma_2}}{\Gamma(2-\gamma_2)}\right] \times \frac{(\ln(b/a))^{\alpha-\gamma_1}}{\Gamma(\alpha-\gamma_1+1)}(m_1\|u\| + m_2\|v\| + \sigma_1) \\ &+ \frac{|\lambda_1|}{|1-\mu_1\lambda_1|}\left[|\mu_1|\,\frac{(\ln(b/a))^{\alpha}}{\Gamma(\alpha+1)}(m_1\|u\| + m_2\|v\| + \sigma_1)\right. \\ &\left. + \frac{(\ln(b/a))^{\beta}}{\Gamma(\beta+1)}(n_1\|u\| + n_2\|v\| + \sigma_2)\right] + \frac{(\ln(b/a))^{\alpha}}{\Gamma(\alpha+1)}(m_1\|u\| + m_2\|v\| + \sigma_1) \\ ={}& K_1(m_1\|u\| + m_2\|v\| + \sigma_1) + K_2(n_1\|u\| + n_2\|v\| + \sigma_2) \\ ={}& (K_1 m_1 + K_2 n_1)\|u\| + (K_1 m_2 + K_2 n_2)\|v\| \\ &+ K_1\sigma_1 + K_2\sigma_2 \le [K_1(m_1 + m_2) + K_2(n_1 + n_2)]r + K_1\sigma_1 + k_2\sigma_2. \end{aligned} \tag{38}$$

Hence,

$$\|\mathcal{T}_1(u, v)\| \le [K_1(m_1 + m_2) + K_2(n_1 + n_2)]r + K_1\sigma_1 + K_2\sigma_2. \tag{39}$$

In the same way, we can obtain that

$$\|\mathcal{T}_2(u, v)\| \le [K_3(m_1 + m_2) + K_4(n_1 + n_2)]r + K_3\sigma_1 + K_4\sigma_2. \tag{40}$$

Consequently, it follows that

$$\begin{aligned} \|\mathcal{T}(u, v)\| \le{}& [K_1(m_1 + m_2) + K_2(n_1 + n_2)]r + K_1\sigma_1 + K_2\sigma_2 \\ &+ [K_3(m_1 + m_2) + K_4(n_1 + n_2)]r + K_3\sigma_1 + K_4\sigma_2 \le r, \end{aligned} \tag{41}$$

which implies $\mathcal{T} B_r \subset B_r$. Next, we show that operator $\mathcal{T}$ is contraction mapping.

For any $(u_1, v_1), (u_2, v_2) \in X \times Y$ and for any $t \in [a, b]$, we obtain

$$\begin{aligned} |\mathcal{T}_1(u_1,v_1)(t) - \mathcal{T}_1(u_2,v_2)(t)| \le{}& \frac{|\mu_3|}{|\Delta|} \\ &\cdot \left[\frac{|\mu_1||\mu_2||\lambda_1|(\ln(b/a))\sum_{i=1}^{N}(\ln(\eta_i/a))^{1-\delta_1}}{\Gamma(2-\delta_1)\,|(1-\mu_1\lambda_1)|}\right. \\ &+ \frac{|\lambda_1||\lambda_2|(\ln(b/a))^{2-\gamma_1}}{\Gamma(2-\gamma_1)\,|(1-\mu_1\lambda_1)|} \\ &\left. + \frac{|\mu_2|\,(\ln(b/a))\sum_{i=1}^{N}(\ln(\eta_i/a))^{1-\delta_1}}{\Gamma(2-\delta_1)}\right] \times \frac{\sum_{i=1}^{M}(\ln(\xi_i/a))^{\alpha-\delta_2}}{\Gamma(\alpha-\delta_2+1)} \\ &\cdot (m_1\|u_1 - u_2\| + m_2\|v_1 - v_2\|) + \frac{|\lambda_3|}{|\Delta|} \\ &\cdot \left[\frac{|\mu_1||\mu_2||\lambda_1|(\ln(b/a))\sum_{i=1}^{N}(\ln(\eta_i/a))^{1-\delta_1}}{\Gamma(2-\delta_1)\,|(1-\mu_1\lambda_1)|} + \frac{|\lambda_1||\lambda_2|(\ln(b/a))^{2-\gamma_1}}{\Gamma(2-\gamma_1)\,|(1-\mu_1\lambda_1)|}\right. \\ &\left. + \frac{|\mu_2|\,(\ln(b/a))\sum_{i=1}^{N}(\ln(\eta_i/a))^{1-\delta_1}}{\Gamma(2-\delta_1)}\right] \times \frac{(\ln(b/a))^{\beta-\gamma_2}}{\Gamma(\beta-\gamma_2+1)} \\ &\cdot (n_1\|u_1 - u_2\| + n_2\|v_1 - v_2\|) + \frac{|\mu_2|}{|\Delta|}\left[\frac{|\mu_1||\lambda_1||\lambda_3|(\ln(b/a))^{2-\gamma_2}}{\Gamma(2-\gamma_2)\,|(1-\mu_1\lambda_1)|}\right. \\ &\left. + \frac{|\lambda_1||\mu_3|(\ln(b/a))\sum_{i=1}^{M}(\ln(\xi_i/a))^{1-\delta_2}}{\Gamma(2-\delta_2)\,|(1-\mu_1\lambda_1)|} + \frac{|\lambda_3|\,(\ln(b/a))^{2-\gamma_2}}{\Gamma(2-\gamma_2)}\right] \\ &\times \frac{\sum_{i=1}^{N}(\ln(\eta_i/a))^{\beta-\delta_1}}{\Gamma(\beta-\delta_1+1)}(n_1\|u_1 - u_2\| + n_2\|v_1 - v_2\|) + \frac{|e_2|}{|\Delta|} \\ &\cdot \left[\frac{|\mu_1||\lambda_1||\lambda_3|(\ln(b/a))^{2-\gamma_2}}{\Gamma(2-\gamma_2)\,|(1-\mu_1\lambda_1)|} + \frac{|\lambda_1||\mu_3|(\ln(b/a))\sum_{i=1}^{M}(\ln(\xi_i/a))^{1-\delta_2}}{\Gamma(2-\delta_2)\,|(1-\mu_1\lambda_1)|}\right. \\ &\left. + \frac{|\lambda_3|(\ln(b/a))^{2-\gamma_2}}{\Gamma(2-\gamma_2)}\right] \times \frac{(\ln(b/a))^{\alpha-\gamma_1}}{\Gamma(\alpha-\gamma_1+1)}(m_1\|u_1 - u_2\| + m_2\|v_1 - v_2\|) \\ &+ \frac{|\lambda_1|}{|1-\mu_1\lambda_1|}\left[|\mu_1|\,\frac{(\ln(b/a))^{\alpha}}{\Gamma(\alpha+1)}(m_1\|u_1 - u_2\| + m_2\|v_1 - v_2\|)\right. \\ &\left. + \frac{(\ln(b/a))^{\beta}}{\Gamma(\beta+1)}(n_1\|u_1 - u_2\| + n_2\|v_1 - v_2\|)\right] + \frac{(\ln(b/a))^{\alpha}}{\Gamma(\alpha+1)} \\ &\cdot (m_1\|u_1 - u_2\| + m_2\|v_1 - v_2\|) = K_1(m_1\|u_1 - u_2\| + m_2\|v_1 - v_2\|) \\ &+ K_2(n_1\|u_1 - u_2\| + n_2\|v_1 - v_2\|) = (K_1 m_1 + K_2 n_1)\|u_1 - u_2\| \\ &+ (K_1 m_2 + K_2 n_2)\|v_1 - v_2\|. \end{aligned} \tag{42}$$

Therefore, we get the following inequality

$$\|\mathcal{T}_1(u_1,v_1)(t)-\mathcal{T}_1(u_2,v_2)(t)\| \le (K_1(m_1+m_2) + K_2(n_1+n_2))(\|u_1-u_2\|+\|v_1-v_2\|). \tag{43}$$

Similarly,

$$\|\mathcal{T}_2(u_1,v_1)(t)-\mathcal{T}_2(u_2,v_2)(t)\| \le (K_3(m_1+m_2) + K_4(n_1+n_2))(\|u_1-u_2\|+\|v_1-v_2\|). \tag{44}$$

From inequalities (43) and (44), it yields

$$\|\mathcal{T}(u_1,v_1)(t)-\mathcal{T}(u_2,v_2)\| \le [(K_1+K_3)(m_1+m_2) + (K_2+K_4)(n_1+n_2)](\|u_1-u_2\|+\|v_1-v_2\|). \tag{45}$$

Since $(K_1+K_3)(m_1+m_2)+(K_2+K_4)(n_1+n_2)<1$, therefore, $\mathcal{T}$ is a contraction operator. So, by applying Banach's fixed point theorem, the operator $\mathcal{T}$ has a unique fixed point in B_r. Hence, there exists a unique solution of problem (3), (4) on $[a,b]$.

Now, we prove our second existence result via the Leray-Schauder alternative.

Lemma 9 (Leray-Schauder alternative [39]). *Let $F: E \to E$ be a completely continuous operator (i.e., a map restricted to any bounded set in E is compact). Let*

$$\varepsilon(F)=\{x\in E : x=\lambda F(x) for some 0<\lambda<1\}. \tag{46}$$

Then, either the set $\varepsilon(F)$ is unbounded or F has at least one fixed point.

Theorem 10. *Assume that:*

(H2) $f,g:[a,b]\times R\times R\to R$ are continuous functions and there exist real constants $a_i, b_i \ge 0 (i=0,1,2)$ and $a_0>0$, $b_0>0$ such that $\forall x_i \in R\ (i=1,2,)$ we have

$$\begin{aligned}|f(t,x_1,x_2)| &\le a_0+a_1|x_1|+a_2|x_2|,\\ |g(t,x_1,x_2)| &\le b_0+b_1|x_1|+b_2|x_2|.\end{aligned} \tag{47}$$

If $(K_1+K_3)a_1+(K_2+K_4)b_1<1$ and $(K_1+K_3)a_2+(K_2+K_4)b_2<1$ then system (3), (4) has at least one solution on $[a,b]$.

Proof. By the continuity of functions f, g on $[a,b]\times R\times R$, the operator $\mathcal{T}$ is continuous. Now, we show that the operator $\mathcal{T}: X\times Y\to X\times Y$ is completely continuous. Let $\Omega\subset X\times Y$ be bounded. Then, there exist two positive constants, M_1 and M_2, such that

$$|f(t,u(t),v(t))|\le M_1,\ |g(t,u(t),v(t))|\le M_2 \forall (u,v)\in\Omega. \tag{48}$$

Then, for any $(u,v)\in\Omega$, we have

$$\begin{aligned}|\mathcal{T}_1(u,v)(t)| \le{}& \frac{|\mu_3|}{|\Delta|}\left[\frac{|\mu_1||\mu_2||\lambda_1|(\ln(b/a))\sum_{i=1}^N(\ln(\eta_i/a))^{1-\delta_1}}{\Gamma(2-\delta_1)|1-\mu_1\lambda_1|}\right.\\ &+\frac{|\lambda_1||\lambda_2|(\ln(b/a))^{2-\gamma_1}}{\Gamma(2-\gamma_1)|1-\mu_1\lambda_1|}\\ &\left.+\frac{|\mu_2|(\ln(b/a))\sum_{i=1}^N(\ln(\eta_i/a))^{1-\delta_1}}{\Gamma(2-\delta_1)}\right]\\ &\times\frac{\sum_{i=1}^M(\ln(\xi_i/a))^{\alpha-\delta_2}}{\Gamma(\alpha-\delta_2+1)}M_1+\frac{|\lambda_3|}{|\Delta|}\\ &\cdot\left[\frac{|\mu_1||\mu_2||\lambda_1|(\ln(b/a))\sum_{i=1}^N(\ln(\eta_i/a))^{1-\delta_1}}{\Gamma(2-\delta_1)|1-\mu_1\lambda_1|}\right.\\ &+\frac{|\lambda_1||\lambda_2|(\ln(b/a))^{2-\gamma_1}}{\Gamma(2-\gamma_1)|1-\mu_1\lambda_1|}\\ &\left.+\frac{|\mu_2|(\ln(b/a))\sum_{i=1}^N(\ln(\eta_i/a))^{1-\delta_1}}{\Gamma(2-\delta_1)}\right]\\ &\times\frac{(\ln(b/a))^{\beta-\gamma_2}}{\Gamma(\beta-\gamma_2+1)}M_2+\frac{|\mu_2|}{|\Delta|}\left[\frac{|\mu_1||\lambda_1|\lambda_3|(\ln(b/a))^{2-\gamma_2}}{\Gamma(2-\gamma_2)|1-\mu_1\lambda_1|}\right.\\ &+\frac{|\lambda_1||\mu_3|(\ln(b/a))\sum_{i=1}^M(\ln(\xi_i/a))^{1-\delta_2}}{\Gamma(2-\delta_2)|1-\mu_1\lambda_1|}\\ &\left.+\frac{|\lambda_3|(\ln(b/a))^{2-\gamma_2}}{\Gamma(2-\gamma_2)}\right]\times\frac{\sum_{i=1}^N(\ln(\eta_i/a))^{\beta-\delta_1}}{\Gamma(\beta-\delta_1+1)}M_2\\ &+\frac{|\lambda_2|}{|\Delta|}\left[\frac{|\mu_1||\lambda_1||\lambda_3|(\ln(b/a))^{2-\gamma_2}}{\Gamma(2-\gamma_2)|1-\mu_1\lambda_1|}\right.\\ &+\frac{|\lambda_1||\mu_3|(\ln(b/a))\sum_{i=1}^M(\ln(\xi_i/a))^{1-\delta_2}}{\Gamma(2-\delta_2)|1-\mu_1\lambda_1|}\\ &\left.+\frac{|\lambda_3|(\ln(b/a))^{2-\gamma_2}}{\Gamma(2-\gamma_2)}\right]\times\frac{(\ln(b/a))^{\alpha-\gamma_1}}{\Gamma(\alpha-\gamma_1+1)}M_1\\ &+\frac{|\lambda_1|}{|1-\mu_1\lambda_1|}\left[|\mu_1|\frac{(\ln(b/a))^{\alpha}}{\Gamma(\alpha+1)}M_1+\frac{(\ln(b/a))^{\beta}}{\Gamma(\beta+1)}M_2\right]\\ &+\frac{(\ln(b/a))^{\alpha}}{\Gamma(\alpha+1)}M_1,\end{aligned} \tag{49}$$

which yields,

$$\|\mathcal{T}_1(u,v)\|\le K_1M_1+K_2M_2. \tag{50}$$

In the same way, we can obtain that $\|\mathcal{T}_2(u,v)\|\le K_3M_1+K_4M_2$. Hence, from the above inequalities, we get that the operator $\mathcal{T}$ is uniformly bounded, since $\|\mathcal{T}(u,v)\|\le (K_1+K_3)M_1+(K_2+K_4)M_2$.

Next, we show that $\mathcal{T}$ is equicontinuous. For any $(u,v)\in\Omega$, and $\tau_1,\tau_2\in[a,b]$ with $\tau_1<\tau_2$. Then, we have

$$\begin{aligned}|\mathcal{T}_1(u,v)(\tau_2)-\mathcal{T}_1(u,v)(\tau_1)|\le{}&\frac{M_1|\mu_2||\mu_3|\sum_{i=1}^N(\ln(\eta_i/a))^{1-\delta_1}}{|\Delta|\,\Gamma(2-\delta_1)}\\ &\cdot\frac{\sum_{i=1}^M(\ln(\xi_i/a))^{\alpha-\delta_2}}{\Gamma(\alpha-\delta_2+1)}\left|\left(\ln\frac{\tau_2}{a}\right)-\left(\ln\frac{\tau_1}{a}\right)\right|\end{aligned}$$

$$
\begin{aligned}
&+ \frac{M_2|\mu_2||\lambda_3|\sum_{i=1}^{N}(\ln(\eta_i/a))^{1-\delta_1}}{|\Delta|\,\Gamma(2-\delta_1)} \frac{(\ln(b/a))^{\beta-\gamma_2}}{\Gamma(\beta-\gamma_2+1)} \left|\left(\ln\frac{\tau_2}{a}\right)\right. \\
&\left.- \left(\ln\frac{\tau_1}{a}\right)\right| + \frac{M_2||\mu_2|\lambda_3|(\ln(b/a))^{1-\gamma_2}}{|\Delta|\,\Gamma(2-\gamma_2)} \frac{\sum_{i=1}^{N}(\ln(\eta_i/a))^{\beta-\delta_1}}{\Gamma(\beta-\delta_1+1)} \\
&\cdot \left|\left(\ln\frac{\tau_2}{a}\right) - \left(\ln\frac{\tau_1}{a}\right)\right| + M_1 \frac{|\lambda_2||\lambda_3|(\ln(b/a))^{1-\gamma_2}}{|\Delta|\,\Gamma(2-\gamma_2)} \frac{(\ln(b/a))^{\alpha-\gamma_1}}{\Gamma(\alpha-\gamma_1+1)} \\
&\cdot \left|\left(\ln\frac{\tau_2}{a}\right) - \left(\ln\frac{\tau_1}{a}\right)\right| + M_1 \left|\frac{1}{\Gamma(\alpha)}\int_a^{\tau_2}\left(\ln\frac{\tau_2}{s}\right)^{\alpha-1}\right. \\
&\left.\cdot \frac{ds}{s} - \frac{1}{\Gamma(\alpha)}\int_a^{\tau_1}\left(\ln\frac{\tau_1}{s}\right)^{\alpha-1}\frac{ds}{s}\right| \le \frac{M_1|\mu_2||\mu_3|\sum_{i=1}^{N}(\ln(\eta_i/a))^{1-\delta_1}}{|\Delta|\,\Gamma(2-\delta_1)} \\
&\cdot \frac{\sum_{i=1}^{M}(\ln(\xi_i/a))^{\alpha-\delta_2}}{\Gamma(\alpha-\delta_2+1)} \left|\left(\ln\frac{\tau_2}{a}\right) - \left(\ln\frac{\tau_1}{a}\right)\right| \\
&+ \frac{M_2|\mu_2||\mu_3|\sum_{i=1}^{N}(\ln(\eta_i/a))^{1-\delta_1}}{|\Delta|\,\Gamma(2-\delta_1)} \frac{(\ln(b/a))^{\beta-\gamma_2}}{\Gamma(\beta-\gamma_2+1)} \left|\left(\ln\frac{\tau_2}{a}\right)\right. \\
&\left.- \left(\ln\frac{\tau_1}{a}\right)\right| + \frac{M_2||\mu_2|\lambda_3|(\ln(b/a))^{1-\gamma_2}}{|\Delta|\,\Gamma(2-\gamma_2)} \frac{\sum_{i=1}^{N}(\ln(\eta_i/a))^{\beta-\delta_1}}{\Gamma(\beta-\delta_1+1)} \\
&\cdot \left|\left(\ln\frac{\tau_2}{a}\right) - \left(\ln\frac{\tau_1}{a}\right)\right| + M_1 \frac{|\lambda_2||\lambda_3|(\ln(b/a))^{1-\gamma_2}}{|\Delta|\,\Gamma(2-\gamma_2)} \frac{(\ln(b/a))^{\alpha-\gamma_1}}{\Gamma(\alpha-\gamma_1+1)} \\
&\cdot \left|\left(\ln\frac{\tau_2}{a}\right) - \left(\ln\frac{\tau_1}{a}\right)\right| + M_1 \left|\frac{1}{\Gamma(\alpha)}\int_a^{\tau_1}\left[\left(\ln\frac{\tau_2}{s}\right)^{\alpha-1} - \left(\ln\frac{\tau_1}{s}\right)^{\alpha-1}\right]\frac{ds}{s}\right| \\
&+ M_1 \left|\frac{1}{\Gamma(\alpha)}\int_{\tau_1}^{\tau_2}\left(\ln\frac{\tau_2}{s}\right)^{\alpha-1}\frac{ds}{s}\right| = \frac{M_1|\mu_2||\mu_3|\sum_{i=1}^{N}(\ln(\eta_i/a))^{1-\delta_1}}{|\Delta|\,\Gamma(2-\delta_1)} \\
&\cdot \frac{\sum_{i=1}^{M}(\ln(\xi_i/a))^{\alpha-\delta_2}}{\Gamma(\alpha-\delta_2+1)} \left|\left(\ln\frac{\tau_2}{a}\right) - \left(\ln\frac{\tau_1}{a}\right)\right| \\
&+ \frac{M_2|\mu_2||\lambda_3|\sum_{i=1}^{N}(\ln(\eta_i/a))^{1-\delta_1}}{|\Delta|\,\Gamma(2-\delta_1)} \frac{(\ln(b/a))^{\beta-\gamma_2}}{\Gamma(\beta-\gamma_2+1)} \left|\left(\ln\frac{\tau_2}{a}\right)\right. \\
&\left.- \left(\ln\frac{\tau_1}{a}\right)\right| + \frac{M_2||\mu_2|\lambda_3|(\ln(b/a))^{1-\gamma_2}}{|\Delta|\,\Gamma(2-\gamma_2)} \frac{\sum_{i=1}^{N}(\ln(\eta_i/a))^{\beta-\delta_1}}{\Gamma(\beta-\delta_1+1)} \\
&\cdot \left|\left(\ln\frac{\tau_2}{a}\right) - \left(\ln\frac{\tau_1}{a}\right)\right| + M_1 \frac{|\lambda_2||\lambda_3|(\ln(b/a))^{1-\gamma_2}}{|\Delta|\,\Gamma(2-\gamma_2)} \frac{(\ln(b/a))^{\alpha-\gamma_1}}{\Gamma(\alpha-\gamma_1+1)} \\
&\cdot \left|\left(\ln\frac{\tau_2}{a}\right) - \left(\ln\frac{\tau_1}{a}\right)\right| + \frac{M_1}{\Gamma(\alpha+1)}\left[2\left(\ln\frac{\tau_2}{\tau_1}\right)^{\alpha} + \left|\left(\ln\frac{\tau_2}{a}\right)^{\alpha} - \left(\ln\frac{\tau_1}{a}\right)^{\alpha}\right|\right].
\end{aligned}
\tag{51}
$$

Therefore, we obtain

$$|\mathcal{T}_1(u,v)(\tau_2) - \mathcal{T}_1(u,v)(\tau_1)| \longrightarrow 0, as \tau_1 \longrightarrow \tau_2. \tag{52}$$

Analogously, we can get the following inequality:

$$|\mathcal{T}_2(u,v)(\tau_2) - \mathcal{T}_2(u,v)(\tau_1)| \longrightarrow 0, as \tau_1 \longrightarrow \tau_2. \tag{53}$$

Then we can easily show that the operator $\mathcal{T}(u,v)$ is equicontinuous. As a consequence of steps together with the Arzela'-Ascoli theorem, we get that the operator $\mathcal{T}(u,v$) is completely continuous.

Finally, it will be verified that the set $\varepsilon = \{(u,v) \in X \times Y : (u,v) = \lambda\mathcal{T}(u,v), 0 \le \lambda \le 1\}$ is bounded. Let $(u,v) \in \varepsilon$, with $(u,v) = \lambda\mathcal{T}(u,v)$. For any $t \in [a,b]$, we have

$$u(t) = \lambda\mathcal{T}_1(u,v)(t), v(t) = \lambda\mathcal{T}_2(u,v)(t). \tag{54}$$

Then, we have

$$
\begin{aligned}
\|u(t)\| &\le K_1(a_0 + a_1\|u\| + a_2\|v\|) + K_2(b_0 + b_1\|u\| + b_2\|v\|) \\
&= K_1a_0 + K_2b_0 + (K_1a_1 + K_2b_1)\|u\| + (K_1a_2 + K_2b_2)\|v\|, \\
\|v(t)\| &\le K_3(a_0 + a_1\|u\| + a_2\|v\|) + K_4(b_0 + b_1\|u\| + b_2\|v\|) \\
&= K_3a_0 + K_4b_0 + (K_3a_1 + K_4b_1)\|u\| + (K_3a_2 + K_4b_2)\|v\|,
\end{aligned}
\tag{55}
$$

which implies that

$$
\begin{aligned}
\|u\| + \|v\| \le{}& (K_1 + K_3)a_0 + (K_2 + K_4)b_0 \\
&+ [(K_1 + K_3)a_1 + (K_2 + K_4)b_1]\|u\| \\
&+ [(K_1 + K_3)a_2 + (K_2 + K_4)b_2]\|v\|.
\end{aligned}
\tag{56}
$$

Consequently,

$$\|(u,v)\| \le \frac{(K_1 + K_3)a_0 + (K_2 + K_4)b_0}{K_0}, \tag{57}$$

where

$$
\begin{aligned}
K_0 = \min\{&1 - [(K_1 + K_3)a_1 + (K_2 + K_4)b_1], 1 \\
&- [(K_1 + K_3)a_2 + (K_2 + K_4)b_2]\},
\end{aligned}
\tag{58}
$$

which proves that ε is bounded. Therefore, by applying Lemma 9, the operator $\mathcal{T}$ has at least one fixed point in Ω. Therefore, we deduce that the boundary value problem (3), (4) has at least one solution on $[a,b]$.

4. Some Examples

In this section, we give an example to illustrate our main results.

Example 11. Consider the following system of Caputo-Hadamard boundary value problem:

$$
\begin{cases}
{}^{c}\mathcal{D}_{1^+}^{3/2}u(t) = f(t,u(t),v(t)), t \in [1,e], \\
{}^{c}\mathcal{D}_{1^+}^{3/2}v(t) = gf(t,u(t),v(t)), t \in [1,e], \\
u(1) = v(e), 1/2\,{}^{c}\mathcal{D}_{1^+}^{1/2}u(e) = 1/3\,{}^{c}\mathcal{D}_{1^+}^{1/3}v(3/2) + 1/3\,{}^{c}\mathcal{D}_{1^+}^{1/3}v(4/3), \\
v(1) = 2u(e), 1/4\,{}^{c}\mathcal{D}_{1^+}^{1/4}v(e) = 1/5\,{}^{c}\mathcal{D}_{1^+}^{1/5}u(5/3) + 1/5\,{}^{c}\mathcal{D}_{1^+}^{1/5}u(5/4).
\end{cases}
\tag{59}
$$

Here, $\alpha = \beta = 3/2$, $a = 1$, $b = e$, $\gamma_1 = 1/2$, $\gamma_2 = 1/4$, $\delta_1 = 1/3$, $\delta_2 = 1/5$, $N = M = 2$, $\eta_1 = 3/2$, $\eta_2 = 4/3$, $\xi_1 = 5/3$, $\xi_2 = 5/4$, $\lambda_1 = 1$, $\lambda_2 = 1/2$, $\lambda_3 = 1/4$, $\mu_1 = 2$, $\mu_2 = 1/3$, $\mu_3 = 1/5$. By simple calculation, we found that $\Delta = 0.078172$, $K_1 = 10.36402$, $K_2 = 8.38734$, $K_3 = 11.58173$, $K_4 = 11.18721$.

(i) Let two nonlinear functions $f, g : [1,e] \times R \times R \longrightarrow R$ be given by

$$f(t,x,y)=\frac{1}{15\sqrt{24+t^2}}\frac{|x|}{1+|x(t)|}+\frac{\sin y(t)}{(64+t^2)}+\frac{1}{2}, \quad (60)$$

$$g(t,x,y)=\frac{\sin(|x|)}{124+t^2}+\frac{\tan^{-1}(y)}{120t^2+2}+\frac{2}{3}. \quad (61)$$

Note that

$$\begin{aligned}|f(t,x_1,x_2)-f(t,y_1,y_2)| &\le \frac{1}{75}|x_1-x_2|+\frac{1}{65}|y_1-y_2|,\\ |g(t,x_1,x_2)-g(t,y_1,y_2)| &\le \frac{1}{125}|x_1-x_2|+\frac{1}{122}|y_1-y_2|,\end{aligned} \quad (62)$$

we obtain $(K_1+K_3)(1/75+1/65)+(K_2+K_4)(1/125+1/122)\approx 0.9552435311<1$. Thus, all the conditions of Theorem 8 are satisfied. Problem (59) with (60) and (61)has a unique solution on $[1,e]$.

(ii) Let two nonlinear functions $f,g:[1,e]\times R\times R\to R$ be given by

$$f(t,x,y)=\frac{e^{-2t}}{4}+\frac{x^2\cos^2 t}{39(1+|x|)}+\frac{|y|^4\sin^2 t}{45(1+y^3)}, \quad (63)$$

$$g(t,x,y)=\frac{2}{t^2+2}+\frac{\sin x}{12(t+4)}+\frac{\tan^{-1}y}{14(3+t^2)}. \quad (64)$$

Note that

$$\begin{aligned}|f(t,x,y)| &\le \frac{1}{4}+\frac{1}{39}|x|+\frac{1}{45}|y|,\\ |g(t,x,y)| &\le \frac{2}{3}+\frac{1}{60}|x|+\frac{1}{56}|y|.\end{aligned} \quad (65)$$

We get $a_1=1/39, a_2=1/45, b_1=1/60, b_2=1/56$. By simple calculation, we have $(K_1+K_3)a_1+(K_2+K_4)b_1\approx 0.8960930769<1$ and $(K_1+K_3)a_2+(K_2+K_4)b_2\approx 0.8434206667<1$. By Theorem 10, the coupled boundary value problem (59) with (63) and (64) has at least one positive solution on [1,e].

5. Conclusions

In this paper, we studied existence and uniqueness of solutions for the system of Caputo-Hadamard fractional differential equations with multipoint boundary conditions. The existence theory of solutions of a Caputo-Hadamard system using a variety of fixed point theorems. The Leray-Schauder alternative was applied to prove existence, while the uniqueness result was obtained via the Banach contradiction mapping principle. Finally, we have given two examples to demonstrate our result.

Data Availability

No data were used to support this study.

Conflicts of Interest

There is no competing interest among the authors regarding the publication of the article.

Authors' Contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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