Hindawi
Journal of Function Spaces
Volume 2020, Article ID 8871563, 10 pages
https://doi.org/10.1155/2020/8871563

Research Article

Kannan Prequasi Contraction Maps on Nakano Sequence Spaces

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Received 29 September 2020; Revised 28 November 2020; Accepted 19 December 2020; Published 31 December 2020

Academic Editor: Nawab Hussain

In this article, we explore the concept of the prequasi norm on Nakano special space of sequences (sss) such that its variable exponent in $(0, 1]$. We evaluate the sufficient setting on it with the definite prequasi norm to configuration prequasi Banach and closed (sss). The Fatou property of different prequasi norms on this (sss) has been investigated. Moreover, the existence of a fixed point of Kannan prequasi norm contraction maps on the prequasi Banach (sss) and the prequasi Banach operator ideal constructed by this (sss) and $s-$ numbers have been examined.

1. Introduction

Ideal maps and summability theorems [1–6] are extremely significant in mathematical models and have more achievements, such as ideal transformations, normal series, fixed point theory, geometry of Banach spaces, and approximation theory. By $\mathfrak{R}^{\mathcal{N}}$, we mark the spaces of all sequences of real numbers. We denote the space of all bounded linear maps from a Banach space Z into a Banach space M by $\mathscr{L}(Z, M)$, and if $Z = M$, we indicate $\mathscr{L}(Z)$, the d-th s number by $s_d(W)$ [7], the d-th approximation number by $\alpha_d(W)$, and $e_d = \{0, 0, \cdots, 1, 0, 0, \cdots\}$, where 1 shows at the d^{th} place, for every $d \in \mathcal{N} = \{0, 1, 2, \cdots\}$.

Notations 1. The sets S_A, $S_A(Z, M)$, S_A^{app} and $S_A^{\text{app}}(Z, M)$, (cf. [8]) denote

$$\begin{aligned} S_A &:= \{S_A(Z, M)\}, \quad \text{where } S_A(Z, M) \\ &:= \left\{W \in \mathscr{L}(Z, M): ((s_d(W))_{d=0}^{\infty} \in A\right\}. \quad \text{Also} \\ S_A^{\text{app}} &:= \left\{S_A^{\text{app}}(Z, M)\right\}, \quad \text{where } S_A^{\text{app}}(Z, M) \\ &:= \left\{W \in \mathscr{L}(Z, M): ((\alpha_d(W))_{d=0}^{\infty} \in A\right\}. \end{aligned} \tag{1}$$

Let $r = (r_a) \in (0, 1]^{\mathcal{N}}$, the Nakano sequence space defined and studied in [9–11] is denoted by:

$$\ell(r) = \left\{v = (v_a) \in \mathfrak{R}^{\mathcal{N}} : \phi(\mu v) < \infty, \quad \text{for any} \quad \mu > 0\right\}, \tag{2}$$

where $\phi(v) = \sum_{a \in \mathcal{N}} |v_a|^{r_a}$. And $(\ell(r), \|.\|)$ is a Banach space, however, $\|v\| = \inf \{\kappa > 0 : \phi(v/\kappa) \leq 1\}$. Faried and Bakery [8] assumed the hypothesis of prequasi operator ideal that is more established than the quasi operator ideal. Bakery and Abou Elmatty [9] demonstrated the strictly inclusion of the prequasi operator ideal $S_{\ell(r)}^{\text{app}}$, for inconsistent powers. It was a small prequasi operator ideal. As the literature of the Banach fixed point theorem [12], many mathematicians created on many actions. Haghi et al. [13, 14] showed that some generalizations in fixed point theory are not real generalizations and investigated some fixed point generalizations to partial metric spaces, which are obtained from the corresponding results in metric spaces. Kannan [15] presented a representation of a class of operators with the same fixed point actions as contractions nevertheless that fails to be continuous. They only try to illustrate Kannan maps [16] in modular vector spaces. The target of this paper is to appraise the concept of prequasi norm on $\ell(r)$. The Fatou property of different prequasi norms on this (sss) has been examined. We are delving the sufficient set-up on $\ell(r)$ equipped with the definite prequasi norm to pattern prequasi Banach and closed (sss). The existence of a fixed point of Kannan prequasi norm contraction mapping on the prequasi Banach (sss) has been given. Finally, the existence of a fixed point of Kannan prequasi norm contraction mapping on the prequasi Banach operator ideal $S_{(\ell(r))_\phi}$ has been made current.

2. Definitions and Preliminaries

Definition 2 (see [2]). The linear space of sequences $\mathfrak{A}$ is detailed as a special space of sequences (sss), if

$$\{e_a\}_{a\in\mathcal{N}} \subseteq \mathfrak{A}, \quad (3)$$

(1) $\mathfrak{A}$ is solid, i.e., let $v=(v_a)\in\mathfrak{R}^{\mathcal{N}}$, $t=(t_a)\in\mathfrak{A}$, and $|v_a|\le|t_a|$, for every $a\in\mathcal{N}$, then $v\in\mathfrak{A}$

(2) $(v_{[a/2]})_{a=0}^{\infty}\in\mathfrak{A}$, where $[a/2]$ marks the integral part of $a/2$, if $(v_a)_{a=0}^{\infty}\in\mathfrak{A}$

Definition 3 (see [8]). A subclass $\mathfrak{A}_\phi$ of $\mathfrak{A}$ is definite a premodular (sss), if there is $\phi\in[0,\infty)^{\mathfrak{A}}$ verifying the set-up:

(i) For $v\in\mathfrak{A}$, $v=\theta \Leftrightarrow \phi(v)=0$ with $\phi(v)\ge 0$, where θ is the zero vector of $\mathfrak{A}$

(ii) For every $v\in\mathfrak{A}$ and $\eta\in\mathfrak{R}$, we have $B\ge 1$ for which $\phi(\eta v)\le B|\eta|\phi(v)$,

(iii) $\phi(v+t)\le J(\phi(v)+\phi(t))$, for each $v, t\in\mathfrak{A}$, for some $J\ge 1$

(iv) For $a\in\mathcal{N}$ and $|v_a|\le|t_a|$, then $\phi((v_a))\le\phi((t_a))$

(v) The inequality, $\phi((v_a))\le\phi((v_{[a/2]}))\le J_0\phi((v_a))$ holds, for some $J_0\ge 1$

(vi) Assume F be the space of finite sequences, then $\bar{F}=\mathfrak{A}_\phi$

(vii) There is $\varsigma>0$ such that $\phi(\beta,0,0,0,\cdots)\ge\varsigma|\beta|\phi(1,0,0,0,\cdots)$, for every $\beta\in\mathfrak{R}$

Definition 4 (see [17]). Suppose $\mathfrak{A}$ be a (sss). The function $\phi\in[0,\infty)^{\mathfrak{A}}$ is called prequasi norm on $\mathfrak{A}$, if it provides the conditions (i), (ii), and (iii) of Definition 3.

Theorem 5 (see [17]). *Pick up $\mathfrak{A}$ be a premodular (sss), then it is prequasi normed (sss).*

Theorem 6 (see [17]). *$\mathfrak{A}$ is a prequasi normed (sss), if it is quasinormed (sss).*

Definition 7 (see [3]). Let $\mathcal{L}$ be the class of all bounded linear operators between any two arbitrary Banach spaces. A subclass $\mathcal{U}$ of $\mathcal{L}$ is named an operator ideal, if every vector $\mathcal{U}(\mathrm{Z},\mathrm{M})=\mathcal{U}\cap\mathcal{L}(\mathrm{Z},\mathrm{M})$ verifies the next setting:

(i) $I_\Gamma\in\mathcal{U}$ where Γ denotes Banach space of one dimension

(ii) The space $\mathcal{U}(Z,M)$ is linear over $\mathfrak{R}$

(iii) Assume $W\in\mathcal{L}(Z_0,Z)$, $X\in\mathcal{U}(Z,M)$, and $Y\in\mathcal{L}(M,M_0)$, then, $YXW\in\mathcal{U}(Z_0,M_0)$, where Z_0 and M_0 are normed spaces (see [18, 19])

The theory of prequasi operator ideal, which is more general than the quasi operator ideal.

Definition 8 (see [8]). A function $\phi\in[0,\infty)^{\mathcal{U}}$ is named a prequasi norm on the ideal $\mathcal{U}$ if the following setting includes

(1) Assume $W\in\mathcal{U}(Z,M)$, $\phi(W)\ge 0$, and $\phi(W)=0\Leftrightarrow W=0$

(2) There is $D\ge 1$ so as to $\phi(\eta W)\le D|\eta|\phi(W)$, for every $W\in\mathcal{U}(Z,M)$ and $\eta\in\mathfrak{R}$

(3) There is $J\ge 1$ such that $\phi(W_1+W_2)\le J[\phi(W_1)+\phi(W_2)]$, for each $W_1, W_2\in\mathcal{U}(Z,M)$,

(4) There is $\sigma\ge 1$ for to if $W\in\mathcal{L}(Z_0,Z)$, $X\in\mathcal{U}(Z,M)$ and $Y\in\mathcal{L}(M,M_0)$, then $\phi(YXW)\le\sigma\|Y\|\phi(X)\|W\|$

Theorem 9 (see [20]). *Pick up $\mathfrak{A}_\phi$ be a premodular (sss), then $\phi(W)=\phi(s_a(W))_{a=0}^{\infty}$ be a prequasi norm on $S_{\mathfrak{A}_\phi}$.*

Theorem 10 (see [9]). *Suppose Z and M be Banach spaces, and $\mathfrak{A}_\phi$ be a premodular (sss), then $(S_{\mathfrak{A}_\phi},\phi)$ be a prequasi Banach operator ideal, such that $\phi(W)=\phi((s_a(W))_{a=0}^{\infty})$.*

Theorem 11 (see [8]). *ϕ is a prequasi norm on the ideal $\mathcal{U}$, if ϕ is a quasinorm on the ideal $\mathcal{U}$.*

The agreeable inequality [21] will be used in the consequence: Suppose $(r_a)\in(0,1]^{\mathcal{N}}$ and $v_a, t_a\in\mathfrak{R}$, for every $a\in\mathcal{N}$, then $|v_a+t_a|^{r_a}\le|v_a|^{r_a}+|t_a|^{r_a}$.

3. Main Results

3.1. Prequasi Normed (sss). We illustrate the adequate set-up on $\ell(r)$ equipped with a prequasi norm ϕ to generate prequasi Banach and closed (sss).

Definition 12. (a) $\{v_a\}_{a\in\mathcal{N}}\subseteq(\ell(r))_\phi$ is ϕ-convergent to $v\in(\ell(r))_\phi\Leftrightarrow\lim_{a\longrightarrow\infty}\phi(v_a-v)=0$. If the ϕ-limit exists, then it is unique

(b) $\{v_a\}_{a\in\mathcal{N}}\subseteq(\ell(r))_\phi$ is ϕ-Cauchy, if $\lim_{a,b\longrightarrow\infty}\phi(v_a-v_b)=0$

(c) $\Lambda\subseteq(\ell(r))_\phi$ is ϕ-closed, if for all ϕ-converging $\{v_a\}_{a\in\mathcal{N}}\subset\Lambda$ to v, then $v\in\Lambda$

Theorem 13. $(\ell(r))_\phi$, *where* $\phi(v)=\sum_{a\in\mathcal{N}}|v_a|^{r_a}$, *for all* $v\in\ell(r)$, *is a premodular (sss), if* $(r_a)_{a\in\mathcal{N}}\in(0,1]^{\mathcal{N}}$ *is an increasing.*

Proof. First, we have to prove $\ell(r)$ is a (sss):

(1) Suppose $v,t\in\ell(r)$. Since $(r_a)\in(0,1]^{\mathcal{N}}$, we have

$$\phi(v+t)=\sum_{a\in\mathcal{N}}|v_a+t_a|^{r_a}\le\sum_{a\in\mathcal{N}}|v_a|^{r_a}+\sum_{a\in\mathcal{N}}|t_a|^{r_a}=\phi(v)+\phi(t)<\infty, \tag{4}$$

so $v+t\in\ell(r)$.

(2) Assume $\eta\in\mathfrak{R}$ and $v\in\ell(r)$. As $(r_a)\in(0,1]^{\mathcal{N}}$, one has

$$\phi(\eta v)=\sum_{a\in\mathcal{N}}|\eta v_a|^{r_a}\le\sup_a|\eta|^{r_a}\sum_{a\in\mathcal{N}}|v_a|^{r_a}\le D|\eta|\phi(v)<\infty. \tag{5}$$

Hence, $\eta v\in\ell(r)$. So, by using Parts (1) and (2), we get $\ell(r)$ is linear. Also $e_a\in\ell(r)$, for all $a\in\mathcal{N}$, since $\phi(e_a)=\sum_{j=0}^{\infty}|e_a(j)|^{r_j}=1$.

(3) Let $|v_a|\le|t_a|$, for every $a\in\mathcal{N}$ and $t\in\ell(r)$. One can see

$$\phi(v)=\sum_{a\in\mathcal{N}}|v_a|^{r_a}\le\sum_{a\in\mathcal{N}}|t_a|^{r_a}=\phi(t)<\infty, \tag{6}$$

we have $v\in\ell(r)$. This implies the sequence space $\ell(r)$ is solid.

(4) Suppose $(v_a)\in\ell(r)$ and (r_a) be an increasing sequence, one has

$$\begin{aligned}\phi\left(\left(v_{[a/2]}\right)\right)&=\sum_{a\in\mathcal{N}}\left|v_{[a/2]}\right|^{r_a}=\sum_{a\in\mathcal{N}}|v_a|^{r_{2a}}+\sum_{a\in\mathcal{N}}|v_a|^{r_{2a+1}}\\&\le2\sum_{a\in\mathcal{N}}|v_a|^{r_a}=2\phi((v_a)),\end{aligned} \tag{7}$$

then $(v_{[a/2]})\in\ell(r)$. Secondly, we show that the functional ϕ on $\ell(r)$ is a premodular:

(i) Evidently, $\phi(v)\ge0$ and $\phi(v)=0\Leftrightarrow v=\theta$

(ii) We have $D=\max\{1,\sup_a|\eta|^{r_a-1}\}\ge1$ such that $\phi(\eta v)\le D|\eta|\phi(v)$, for every $v\in\ell(r)$ and $\eta\in\mathfrak{R}\setminus\{0\}$. For $\eta=0$, there is $D\ge1$ such that $\phi(\eta v)\le D|\eta|\phi(v)$, for every $v\in\ell(r)$

(iii) We have $J\ge1$ so that $\phi(v+t)\le J(\phi(v)+\phi(t))$, for every $v,t\in\ell(r)$

(iv) Clearly, since $\ell(r)$ is solid

(v) From (49), we have $J_0=2\ge1$

(vi) Clearly, $\bar{F}=\ell(r)$

(vii) There is $0<\varsigma\le|\beta|^{r_0-1}$, for $\beta\ne0$ or $\varsigma>0$, for $\beta=0$ such that $\phi(\beta,0,0,0,\cdots)\ge\varsigma|\beta|\phi(1,0,0,0,\cdots)$

Theorem 14. *Assume* $(r_a)\in(0,1]^{\mathcal{N}}$ *be an increasing, then* $(\ell(r))_\phi$ *be a prequasi Banach (sss), where* $\phi(v)=\sum_{a\in\mathcal{N}}|v_a|^{r_a}$, *for every* $v\in\ell(r)$.

Proof. Let the set-up be verified. From Theorem 13, the space $(\ell(r))_\phi$ is a premodular (sss). By Theorem 5, the space $(\ell(r))_\phi$ is a prequasi normed (sss). To prove that $(\ell(r))_\phi$ is a prequasi Banach (sss), assume $v^p=(v_a^p)_{a=0}^{\infty}$ be a Cauchy sequence in $(\ell(r))_\phi$. Hence, for every $\varepsilon\in(0,1)$, we have $p_0\in\mathcal{N}$ such that for all $p,q\ge p_0$, one has

$$\phi(v^p-v^q)=\sum_{a\in\mathcal{N}}|v_a^p-v_a^q|^{r_a}<\varepsilon. \tag{8}$$

Therefore, for $p,q\ge p_0$ and $a\in\mathcal{N}$, we get $|v_a^p-v_a^q|<\varepsilon$. So (v_a^q) is a Cauchy sequence in $\mathfrak{R}$, for constant $a\in\mathcal{N}$. Which implies $\lim_{q\longrightarrow\infty}v_a^q=v_a^0$, for fixed $a\in\mathcal{N}$. Hence, $\phi(v^p-v^0)<\varepsilon$, for every $p\ge p_0$. Then, to show that $v^0\in\ell(r)$, we have $\phi(v^0)=\phi(v^0-v^p+v^p)\le\phi(v^p-v^0)+\phi(v^p)<\infty$. So $v^0\in\ell(r)$. This explains that $(\ell(r))_\phi$ is a prequasi Banach (sss).

Theorem 15. *Pick up* $(r_a)\in(0,1]^{\mathcal{N}}$ *be an increasing, then* $(\ell(r))_\phi$ *be a prequasi closed (sss), where* $\phi(v)=\sum_{a\in\mathcal{N}}|v_a|^{r_a}$, *for every* $v\in\ell(r)$.

Proof. Assume the conditions be verified. From Theorem 13, the space $(\ell(r))_\phi$ be a premodular (sss). By Theorem 5, the space $(\ell(r))_\phi$ is a prequasi normed (sss). To show that $(\ell(r))_\phi$ is a prequasi closed (sss), suppose $v^p=(v_a^p)_{a=0}^{\infty}\in(\ell(r))_\phi$ and $\lim_{p\longrightarrow\infty}\phi(v^p-v^0)=0$, then for all $\varepsilon\in(0,1)$, we have $p_0\in\mathcal{N}$ so that for all $p\ge p_0$, we have $\varepsilon>\phi(v^p-v^0)=\sum_{a\in\mathcal{N}}|v_a^p-v_a^0|^{r_a}$. Hence, for $p\ge p_0$ and $a\in\mathcal{N}$, one has $|v_a^p-v_a^0|<\varepsilon$. Therefore, (v_a^p) is a convergent sequence in $\mathfrak{R}$, for fixed $a\in\mathcal{N}$. Hence, $\lim_{p\longrightarrow\infty}v_a^p=v_a^0$, for constant $a\in\mathcal{N}$. Finally, to prove that $v^0\in\ell(r)$, we obtain

$$\phi\left(v^0\right)=\phi\left(v^0-v^p+v^p\right)\le\phi\left(v^p-v^0\right)+\phi\left(v^p\right)<\infty, \tag{9}$$

hence, $v^0\in\ell(r)$. This gives that $(\ell(r))_\phi$ is a prequasi closed (sss).

Example 16. The functional $\phi(\mathrm{v})=\sum_{\mathrm{a}\in\mathcal{N}}|\mathrm{v_a}|^{\mathrm{a+1/a+2}}$ is a prequasi norm (not a quasinorm) on Nakano special space of sequences $\ell((\mathrm{a}+1/\mathrm{a}+2)_{\mathrm{a}=0}^{\infty})$.

Example 17. The functional $\phi(\mathrm{v})=[\sum_{\mathrm{a}\in\mathcal{N}}|\mathrm{v_a}|^{\mathrm{a+1/2a+4}}]^4$ is a prequasi norm (not a quasinorm) on Nakano special space of sequences $\ell((\mathrm{a}+1/2\mathrm{a}+4)_{\mathrm{a}=0}^{\infty})$.

Example 18. The functional $\phi(\mathrm{v})=\sum_{\mathrm{a}\in\mathcal{N}}|\mathrm{v_a}|^{\mathrm{r}}$ is a prequasi norm (not a norm) on r-absolutely summable sequences of real numbers ℓ_{r}, for all $0<\mathrm{r}\le1$.

Example 19. For $(r_a) \in (0,1]^{\mathcal{N}}$, the functional $\phi(v) = \inf\{\kappa > 0 : \sum_{a\in\mathcal{N}} |v_a/\kappa|^{r_a} \le 1\}$ is a prequasi norm (a quasinorm and a norm) on Nakano special space of sequences $\ell(r)$.

4. The Fatou Property

We investigate here the Fatou property of different prequasi norms ϕ on $\ell(r)$.

Definition 20. A prequasi norm ϕ on $\ell(r)$ provides the Fatou property, if for all sequence $\{t^a\} \subseteq (\ell(r))_\phi$ with $\lim_{a\longrightarrow\infty}\phi(t^a - t) = 0$ and any $v \in (\ell(r))_\phi$ then $\phi(v-t) \le \sup_j \inf_{a\ge j}\phi(v - t^a)$.

Theorem 21. *The function $\phi(v) = \sum_{a\in\mathcal{N}} |v_a|^{r_a}$ provides the Fatou property, if $(r_a) \in (0,1]^{\mathcal{N}}$ is an increasing, for all $v \in \ell(r)$.*

Proof. Let the set-up be satisfied and $\{t^b\} \subseteq (\ell(r))_\phi$ with $\lim_{b\longrightarrow\infty}\phi(t^b - t) = 0$. Since the space $(\ell(r))_\phi$ is a prequasi closed space, then $t \in (\ell(r))_\phi$. So for every $v \in (\ell(r))_\phi$, one has

$$\phi(v-t) = \sum_{a\in\mathcal{N}} |v_a - t_a|^{r_a} \le \sum_{a\in\mathcal{N}} \left|v_a - t_a^b\right|^{r_a} + \sum_{a\in\mathcal{N}} \left|t_a^b - t_a\right|^{r_a} \le \sup_j \inf_{b\ge j} \phi\left(v - t^b\right). \tag{10}$$

Theorem 22. *The function $\phi(v) = [\sum_{a\in\mathcal{N}} |v_a|^{r_a}]^{1/\inf_a r_a}$ does not fulfill the Fatou property, for all $v \in \ell(r)$, if $(r_a) \in (0,1]^{\mathcal{N}}$ with $\inf_a r_a > 0$.*

Proof. Suppose the set-up be confirmed and $\{t^b\} \subseteq (\ell(r))_\phi$ with $\lim_{b\longrightarrow\infty}\phi(t^b - t) = 0$. Since the space $(\ell(r))_\phi$ is a prequasi closed space, then $t \in (\ell(r))_\phi$. Then, for each $v \in (\ell(r))_\phi$, we get

$$\begin{aligned}\phi(v-t) &= \left[\sum_{a\in\mathcal{N}} |v_a - t_a|^{r_a}\right]^{1/\inf_a r_a} \\ &\le 2^{1/\inf_a r_a - 1}\left(\left[\sum_{a\in\mathcal{N}} \left|v_a - t_a^b\right|^{r_a}\right]^{1/\inf_a r_a} + \left[\sum_{a\in\mathcal{N}} \left|t_a^b - t_a\right|^{r_a}\right]^{1/\inf_a r_a}\right) \\ &\le 2^{1/\inf_a r_a - 1} \sup_j \inf_{b\ge j}\phi\left(v - t^b\right).\end{aligned} \tag{11}$$

So, ϕ does not indulge the Fatou property.

5. Kannan Prequasi ϕ-Contraction Operator

Now, we explain the definition of Kannan ϕ-contraction mapping on the prequasi normed (sss). We study the sufficient setting on $(\ell(r))_\phi$ constructed with definite prequasi norm so that there is one and only one fixed point of Kannan prequasi norm contraction mapping.

Definition 23. An operator $W : \mathfrak{A}_\phi \longrightarrow \mathfrak{A}_\phi$ is called a Kannan ϕ-contraction, if there is $\xi \in [0, 1/2)$, so that $\phi(Wv - Wt) \le \xi(\phi(Wv - v) + \phi(Wt - t))$, for all $v, t \in \mathfrak{A}_\phi$.

An element $v \in \mathfrak{A}_\phi$ is named a fixed point of W, if $W(v) = v$.

Theorem 24. *Assume $(r_a) \in (0,1]^{\mathcal{N}}$ be an increasing, and $W : (\ell(r))_\phi \longrightarrow (\ell(r))_\phi$ be Kannan ϕ-contraction mapping, where $\phi(v) = \sum_{a\in\mathcal{N}} |v_a|^{r_a}$, for all $v \in \ell(r)$, then W has one fixed point.*

Proof. Let the setting be satisfied. For each $v \in \ell(r)$, then $W^p v \in \ell(r)$. As W is a Kannan ϕ-contraction operator, one has

$$\begin{aligned}\phi\left(W^{p+1}v - W^p v\right) &\le \xi\left(\phi\left(W^{p+1}v - W^p v\right) + \phi\left(W^p v - W^{p-1}v\right)\right) \\ &\Rightarrow \phi\left(W^{p+1}v - W^p v\right) \le \frac{\xi}{1-\xi}\phi\left(W^p v - W^{p-1}v\right) \\ &\le \left(\frac{\xi}{1-\xi}\right)^2 \phi\left(W^{p-1}v - W^{p-2}v\right) \\ &\le \le \left(\frac{\xi}{1-\xi}\right)^p \phi(Wv - v).\end{aligned} \tag{12}$$

So, for all $p, q \in \mathcal{N}$ with $q > p$, one can see

$$\begin{aligned}\phi\left(W^p v - W^q v\right) &\le \xi\left(\phi\left(W^p v - W^{p-1}v\right) + \phi\left(W^q v - W^{q-1}v\right)\right) \\ &\le \xi\left(\left(\frac{\xi}{1-\xi}\right)^{p-1} + \left(\frac{\xi}{1-\xi}\right)^{q-1}\right)\phi(Wv - v).\end{aligned} \tag{13}$$

Therefore, $\{W^p v\}$ is a Cauchy sequence in $(\ell(r))_\phi$. As the space $(\ell(r))_\phi$ is prequasi Banach space. Hence, there is $t \in (\ell(r))_\phi$ so that $\lim_{p\longrightarrow\infty} W^p v = t$. To prove that $Wt = t$. Since ϕ has the Fatou property, we have

$$\begin{aligned}\phi(Wt - t) &\le \sup_i \inf_{p\ge i} \phi\left(W^{p+1}v - W^p v\right) \\ &\le \sup_i \inf_{p\ge i} \left(\frac{\xi}{1-\xi}\right)^p \phi(Wv - v) = 0,\end{aligned} \tag{14}$$

hence, $Wt = t$. Then, t is a fixed point of W. To show that the fixed point is unique. Let we have two distinctive fixed points $b, t \in (\ell(r))_\phi$ of W. So, we have

$$\phi(b - t) \le \phi(Wb - Wt) \le \xi(\phi(Wb - b) + \phi(Wt - t)) = 0. \tag{15}$$

Therefore, $b = t$.

Corollary 25. *Let* $(r_a) \in (0,1]^{\mathcal{N}}$ *be an increasing, and* $W : (\ell(r))_\phi \longrightarrow (\ell(r))_\phi$ *be Kannan* ϕ*-contraction mapping, where* $\phi(v) = \sum_{a\in\mathcal{N}} |v_a|^{r_a}$, *for all* $v \in \ell(r)$, *then* W *has unique fixed point* b *with*

$$\phi(W^p v - b) \le \xi \left(\frac{\xi}{1-\xi}\right)^{p-1} \phi(Wv - v). \tag{16}$$

Proof. Pick up the conditions be satisfied. By Theorem 24, we have a unique fixed point b of W. Hence, one has

$$\begin{aligned}\phi(W^p v - b) &= \phi(W^p v - Wb) \le \xi\left(\phi\left(W^p v - W^{p-1} v\right) + \phi(Wb - b)\right)\\ &= \xi\left(\frac{\xi}{1-\xi}\right)^{p-1} \phi(Wv - v).\end{aligned} \tag{17}$$

Definition 26. Suppose $\mathfrak{A}_\phi$ be a prequasi normed (sss), $W : \mathfrak{A}_\phi \longrightarrow \mathfrak{A}_\phi$ and $b \in \mathfrak{A}_\phi$. The operator W is called ϕ-sequentially continuous at b, if and only if, when $\lim_{a\longrightarrow\infty}\phi(v_a - b) = 0$, then $\lim_{a\longrightarrow\infty}\phi(Wv_a - Wb) = 0$.

Theorem 27. *Pick up* $(r_a) \in (0,1]^{\mathcal{N}}$ *with* $\inf_a r_a > 0$, *and* $W : (\ell(r))_\phi \longrightarrow (\ell(r))_\phi$, *where* $\phi(v) = [\sum_{a\in\mathcal{N}} |v_a|^{r_a}]^{1/\inf_a r_a}$, *for all* $v \in \ell(r)$. *The point* $g \in (\ell(r))_\phi$ *is the only fixed point of* W, *if the following conditions are satisfied:*

(a) *W is Kannan* ϕ*-contraction mapping*

(b) *W is* ϕ*-sequentially continuous at* $g \in (\ell(r))_\phi$

(c) *There is* $v \in (\ell(r))_\phi$ *so that the sequence of iterates* $\{W^p v\}$ *has a subsequence* $\{W^{p_i} v\}$ *converging to* g

Proof. Let the set-up be verified. Suppose g be not a fixed point of W, then $Wg \ne g$. By the set-up (b) and (c), we have

$$\lim_{p_i\longrightarrow\infty} \phi(W^{p_i} v - g) = 0 \text{ and } \lim_{p_i\longrightarrow\infty} \phi\left(W^{p_i+1} v - Wg\right) = 0. \tag{18}$$

As the operator W is Kannan ϕ-contraction, one has

$$\begin{aligned}0 < \phi(Wg - g) &= \phi\left(\left(Wg - W^{p_i+1}v\right) + \left(W^{p_i}v - g\right) + \left(W^{p_i+1}v - W^{p_i}v\right)\right)\\ &\le 2^{2/\inf_a r_a - 2}\phi\left(W^{p_i+1}v - Wg\right) + 2^{2/\inf_a r_a - 2}\phi\left(W^{p_i}v - g\right)\\ &\quad + 2^{1/\inf_a r_a - 1}\xi\left(\frac{\xi}{1-\xi}\right)^{p_i - 1}\phi(Wv - v).\end{aligned} \tag{19}$$

As $p_i \longrightarrow \infty$, we have a contradiction. Therefore, g is a fixed point of W. To prove that the fixed point g is unique. Assume we have two different fixed points $g, b \in (\ell(r))_\phi$ of W. So, one can see

$$\phi(g - b) \le \phi(Wg - Wb) \le \xi(\phi(Wg - g) + \phi(Wb - b)) = 0. \tag{20}$$

Therefore, $g = b$.

Example 28. Let $W : (\ell((a + 1/2a + 4)_{a=0}^\infty))_\phi \longrightarrow (\ell((a + 1/2a + 4)_{a=0}^\infty))_\phi$, where $\phi(v) = \sum_{a\in\mathcal{N}} |v_a|^{a+1/2a+4}$, for all $v \in \ell((a + 1/2a + 4)_{a=0}^\infty)$ and

$$W(v) = \begin{cases} \dfrac{v}{18}, & \phi(v) \in [0,1),\\ \dfrac{v}{20}, & \phi(v) \in [1,\infty).\end{cases} \tag{21}$$

Since for all $v_1, v_2 \in (\ell((a + 1/2a + 4)_{a=0}^\infty))_\phi$ with $\phi(v_1), \phi(v_2) \in [0,1)$, we have

$$\begin{aligned}\phi(Wv_1 - Wv_2) &= \phi\left(\frac{v_1}{18} - \frac{v_2}{18}\right) \le \frac{1}{\sqrt[4]{17}}\left(\phi\left(\frac{17v_1}{18}\right) + \phi\left(\frac{17v_2}{18}\right)\right)\\ &= \frac{1}{\sqrt[4]{17}}(\phi(Wv_1 - v_1) + \phi(Wv_2 - v_2)).\end{aligned} \tag{22}$$

For all $v_1, v_2 \in (\ell((a + 1/2a + 4)_{a=0}^\infty))_\phi$ with $\phi(v_1), \phi(v_2) \in [1,\infty)$, we have

$$\begin{aligned}\phi(Wv_1 - Wv_2) &= \phi\left(\frac{v_1}{20} - \frac{v_2}{20}\right) \le \frac{1}{\sqrt[4]{19}}\left(\phi\left(\frac{19v_1}{20}\right) + \phi\left(\frac{19v_2}{20}\right)\right)\\ &= \frac{1}{\sqrt[4]{19}}(\phi(Wv_1 - v_1) + \phi(Wv_2 - v_2)).\end{aligned} \tag{23}$$

For all $v_1, v_2 \in (\ell((a + 1/2a + 4)_{a=0}^\infty))_\phi$ with $\phi(v_1) \in [0,1)$ and $\phi(v_2) \in [1,\infty)$, we have

$$\begin{aligned}\phi(Wv_1 - Wv_2) &= \phi\left(\frac{v_1}{18} - \frac{v_2}{20}\right) \le \frac{1}{\sqrt[4]{17}}\phi\left(\frac{17v_1}{18}\right) + \frac{1}{\sqrt[4]{19}}\phi\left(\frac{19v_2}{20}\right)\\ &\le \frac{1}{\sqrt[4]{17}}\left(\phi\left(\frac{17v_1}{18}\right) + \phi\left(\frac{19v_2}{20}\right)\right)\\ &= \frac{1}{\sqrt[4]{17}}(\phi(Wv_1 - v_1) + \phi(Wv_2 - v_2)).\end{aligned} \tag{24}$$

Therefore, the map W is Kannan ϕ-contraction mapping. Since ϕ satisfies the Fatou property. By Theorem 24, the map W has a unique fixed point $\theta \in (\ell((a + 1/2a + 4)_{a=0}^\infty))_\phi$.

Let $\{v^{(n)}\} \subseteq (\ell((a + 1/2a + 4)_{a=0}^\infty))_\phi$ be such that $\lim_{n\longrightarrow\infty} \phi(v^{(n)} - v^{(0)}) = 0$, where $v^{(0)} \in (\ell((a + 1/2a + 4)_{a=0}^\infty))_\phi$ with

$\phi(\nu^{(0)}) = 1$. Since the prequasi norm ϕ is continuous, we have

$$\lim_{n\longrightarrow\infty}\phi\left(W\nu^{(n)} - W\nu^{(0)}\right) = \lim_{n\longrightarrow\infty}\phi\left(\frac{\nu^{(n)}}{18} - \frac{\nu^{(0)}}{20}\right) = \phi\left(\frac{\nu^{(0)}}{180}\right) > 0. \quad (25)$$

Hence, W is not ϕ-sequentially continuous at $\nu^{(0)}$. So, the map W is not continuous at $\nu^{(0)}$.

If $\phi(\nu) = [\sum_{a\in\mathcal{N}} |\nu_a|^{a+1/2a+4}]^4$, for all $\nu \in \ell((a+1/2a+4)_{a=0}^{\infty})$. Since for all $\nu_1, \nu_2 \in (\ell((a+1/2a+4)_{a=0}^{\infty}))_\phi$ with $\phi(\nu_1)$, $\phi(\nu_2) \in [0,1)$, we have

$$\begin{aligned}\phi(W\nu_1 - W\nu_2) &= \phi\left(\frac{\nu_1}{18} - \frac{\nu_2}{18}\right) \le \frac{8}{17}\left(\phi\left(\frac{17\nu_1}{18}\right) + \phi\left(\frac{17\nu_2}{18}\right)\right) \\ &= \frac{8}{17}(\phi(W\nu_1 - \nu_1) + \phi(W\nu_2 - \nu_2)).\end{aligned} \quad (26)$$

For all $\nu_1, \nu_2 \in (\ell((a+1/2a+4)_{a=0}^{\infty}))_\phi$ with $\phi(\nu_1), \phi(\nu_2) \in [1,\infty)$, we have

$$\begin{aligned}\phi(W\nu_1 - W\nu_2) &= \phi\left(\frac{\nu_1}{20} - \frac{\nu_2}{20}\right) \le \frac{8}{19}\left(\phi\left(\frac{19\nu_1}{20}\right) + \phi\left(\frac{19\nu_2}{20}\right)\right) \\ &= \frac{8}{19}(\phi(W\nu_1 - \nu_1) + \phi(W\nu_2 - \nu_2)).\end{aligned} \quad (27)$$

For all $\nu_1, \nu_2 \in (\ell((a+1/2a+4)_{a=0}^{\infty}))_\phi$ with $\phi(\nu_1) \in [0,1)$ and $\phi(\nu_2) \in [1,\infty)$, we have

$$\begin{aligned}\phi(W\nu_1 - W\nu_2) &= \phi\left(\frac{\nu_1}{18} - \frac{\nu_2}{20}\right) \le \frac{8}{17}\phi\left(\frac{17\nu_1}{18}\right) + \frac{8}{19}\phi\left(\frac{19\nu_2}{20}\right) \\ &\le \frac{8}{17}\left(\phi\left(\frac{17\nu_1}{18}\right) + \phi\left(\frac{19\nu_2}{20}\right)\right) \\ &= \frac{8}{17}(\phi(W\nu_1 - \nu_1) + \phi(W\nu_2 - \nu_2)).\end{aligned} \quad (28)$$

Therefore, the map W is Kannan ϕ-contraction mapping and $W^p(\nu) = \begin{cases} \nu/18^p, & \phi(\nu) \in [0,1), \\ \nu/20^p, & \phi(\nu) \in [1,\infty). \end{cases}$

It is clear that W is ϕ-sequentially continuous at $\theta \in (\ell((a+1/2a+4)_{a=0}^{\infty}))_\phi$ and $\{W^p\nu\}$ has a subsequence $\{W^{p_i}\nu\}$ converging to θ. By Theorem 27, the point $\theta \in (\ell((a+1/2a+4)_{a=0}^{\infty}))_\phi$ is the only fixed point of W.

Example 29. Let $W : (\ell((a+1/2a+4)_{a=0}^{\infty}))_\phi \longrightarrow (\ell((a+1/2a+4)_{a=0}^{\infty}))_\phi$, where $\phi(\nu) = [\sum_{a\in\mathcal{N}} |\nu_a|^{a+1/2a+4}]^4$ *, for all* $\nu \in \ell((a+1/2a+4)_{a=0}^{\infty})$ and

$$W(\nu) = \begin{cases} \frac{1}{18}(1+\nu_0, \nu_1, \nu_2, \cdots), & \nu_0 \in \left(-\infty, \frac{1}{17}\right), \\ \frac{1}{17}(1,0,0,0,\cdots), & \nu_0 = \frac{1}{17}, \\ \frac{1}{18}(1,0,0,0,\cdots), & \nu_0 \in \left(\frac{1}{17}, \infty\right). \end{cases} \quad (29)$$

Since for all $\nu, t \in (\ell((a+1/2a+4)_{a=0}^{\infty}))_\phi$ with $\nu_0, t_0 \in (-\infty, 1/17)$, we have

$$\begin{aligned}\phi(W\nu - Wt) &= \phi\left(\frac{1}{18}(\nu_0 - t_0, \nu_1 - t_1, \nu_2 - t_2, \cdots)\right) \\ &\le \frac{8}{17}\left(\phi\left(\frac{17\nu}{18}\right) + \phi\left(\frac{17t}{18}\right)\right) \\ &\le \frac{8}{17}(\phi(W\nu - \nu) + \phi(Wt - t)).\end{aligned} \quad (30)$$

For all $\nu, t \in (\ell((a+1/2a+4)_{a=0}^{\infty}))_\phi$ with $\nu_0, t_0 \in (1/17, \infty)$, then for any $\varepsilon > 0$, we have

$$\phi(W\nu - Wt) = 0 \le \varepsilon(\phi(W\nu - \nu) + \phi(Wt - t)). \quad (31)$$

For all $\nu, t \in (\ell((a+1/2a+4)_{a=0}^{\infty}))_\phi$ with $\nu_0 \in (-\infty, 1/17)$ and $t_0 \in (1/17, \infty)$, we have

$$\begin{aligned}\phi(W\nu - Wt) &= \phi\left(\frac{\nu}{18}\right) \le \frac{1}{17}\phi\left(\frac{17\nu}{18}\right) = \frac{1}{17}\phi(W\nu - \nu) \\ &\le \frac{1}{17}(\phi(W\nu - \nu) + \phi(Wt - t)).\end{aligned} \quad (32)$$

Therefore, the map W is Kannan ϕ-contraction mapping. It is clear that W is ϕ-sequentially continuous at $1/17e_0 \in (\ell((a+1/2a+4)_{a=0}^{\infty}))_\phi$ and there is $\nu \in (\ell((a+1/2a+4)_{a=0}^{\infty}))_\phi$ with $\nu_0 \in (-\infty, 1/17)$ such that the sequence of iterates $\{W^p\nu\} = \{\sum_{n=1}^{p} 1/18^n e_0 + 1/18^p \nu\}$ has a subsequence $\{W^{p_i}\nu\} = \{\sum_{n=1}^{p_i} 1/18^n e_0 + 1/18^{p_i}\nu\}$ converging to $1/17e_0$. Then, W has one fixed point $1/17e_0 \in (\ell((a+1/2a+4)_{a=0}^{\infty}))_\phi$. Note that W is not continuous at $1/17e_0 \in (\ell((a+1/2a+4)_{a=0}^{\infty}))_\phi$.

6. Kannan Contraction Maps on Prequasi Ideal

We account the being present of a fixed point of Kannan prequasi norm contraction operator on the prequasi Banach operator ideal investigated by $(\ell(r))_\phi$ and $s-$ numbers.

Theorem 30. *Let Z and M be Banach spaces, and $(r_a) \in (0,1]^{\mathcal{N}}$ be an increasing, then $(S_{(\ell(r))_\phi}, \Phi)$, where $\Phi(W) = \phi((s_a(W))_{a=0}^{\infty})$ be a prequasi Banach operator ideal.*

Proof. Pick up the conditions be verified. By Theorem 13, the space $(\ell(r))_\phi$ is a premodular (sss). Therefore, from Theorem

9, one has $\Phi(W)=\phi((s_a(W))_{a=0}^{\infty})$ is a prequasi norm on $S_{(\ell(r))_\phi}$. So, from Theorem 10, we obtain the space $(S_{(\ell(r))_\phi}, \Phi)$ is a prequasi Banach operator ideal.

Theorem 31. *Pick up Z and M be Banach spaces, and $(r_a)\in(0,1]^{\mathscr{N}}$ be an increasing, then $(S_{(\ell(r))_\phi},\Phi)$, where $\Phi(W)=\phi((s_a(W))_{a=0}^{\infty})$ be a prequasi closed operator ideal.*

Proof. By Theorem 13, the space $(\ell(r))_\phi$ is a premodular (sss). Therefore, from Theorem 9, we have $\Phi(W)=\phi((s_a(W))_{a=0}^{\infty})$ is a prequasi norm on $S_{(\ell(r))_\phi}$. Assume $W_q\in S_{(\ell(r))_\phi}(Z,M)$, for each $q\in\mathscr{N}$ and $\lim_{q\longrightarrow\infty}\Phi(W_q-W)=0$. Hence, we have $\varsigma>0$ and since $\mathscr{L}(Z,M)\supseteq S_{(\ell(r))_\phi}(Z,M)$, we have

$$\begin{aligned}\Phi(W_q-W)&=\phi\left(\left(s_a(W_q-W)\right)_{a=0}^{\infty}\right)\geq\phi(s_0(W_q-W),0,0,0,\cdots)\\&=\phi(\|W_q-W\|,0,0,0,\cdots)\geq\varsigma\|W_q-W\|.\end{aligned}\tag{33}$$

So $(W_q)_{q\in\mathscr{N}}$ is convergent in $\mathscr{L}(Z,M)$, i.e., $\lim_{q\longrightarrow\infty}\|W_q-W\|=0$ and as $(s_a(W_q))_{a=0}^{\infty}\in(\ell(r))_\phi$, for every $q\in\mathscr{N}$ and $(\ell(r))_\phi$ is a premodular (sss). Therefore, we get

$$\begin{aligned}\Phi(W)&=\phi((s_a(W))_{a=0}^{\infty})=\phi\left(\left(s_a\left(W-W_q+W_q\right)\right)_{a=0}^{\infty}\right)\\&\leq\phi\left(\left(s_{[a/2]}\left(W-W_q\right)\right)_{a=0}^{\infty}\right)+\phi\left(\left(s_{[a/2]}\left(W_q\right)_{a=0}^{\infty}\right)\right)\\&\leq\phi\left(\left(W_q-W\right)_{a=0}^{\infty}\right)+2\phi\left(\left(s_a\left(W_q\right)_{a=0}^{\infty}\right)\right)<\varepsilon,\end{aligned}\tag{34}$$

we have $(s_a(W))_{a=0}^{\infty}\in(\ell(r))_\phi$, so $W\in S_{(\ell(r))_\phi}(Z,M)$.

Definition 32. A prequasi norm Φ on the ideal $S_{\mathfrak{A}_\phi}$, where $\Phi(W)=\phi((s_a(W))_{a=0}^{\infty})$, provides the Fatou property if for every sequence $\{W_a\}_{a\in\mathscr{N}}\subseteq S_{\mathfrak{A}_\phi}(Z,M)$ with $\lim_{a\longrightarrow\infty}\Phi(W_a-W)=0$ and all $V\in S_{\mathfrak{A}_\phi}(Z,M)$, then

$$\Phi(V-W)\leq\sup_a\inf_{i\geq a}\Phi(V-W_i).\tag{35}$$

Theorem 33. *The prequasi norm $\Phi(W)=\sum_{a\in\mathscr{N}}|s_a(W)|^{r_a}$, for all $W\in S_{(\ell(r))_\phi}(Z,M)$ does not satisfy the Fatou property, if $(r_a)\in(0,1]^{\mathscr{N}}$ is increasing.*

Proof. Let the setting be provided and $\{W_p\}_{p\in\mathscr{N}}\subseteq S_{(\ell(r))_\phi}(Z,M)$ with $\lim_{p\longrightarrow\infty}\Phi(W_p-W)=0$. Since the space $S_{(\ell(r))_\phi}$ is a prequasi closed ideal, so $W\in S_{(\ell(r))_\phi}(Z,M)$. Therefore, for every $V\in S_{(\ell(r))_\phi}(Z,M)$, we have

$$\begin{aligned}\Phi(V-W)&=\sum_{a\in\mathscr{N}}|s_a(V-W)|^{r_a}\leq\sum_{a\in\mathscr{N}}\left|s_{[a/2]}(V-W_i)\right|^{r_a}\\&\quad+\sum_{a\in\mathscr{N}}\left|s_{[a/2]}(W_i-W)\right|^{r_a}\\&\leq2\sup_p\inf_{i\geq p}\sum_{a\in\mathscr{N}}|s_a(V-W_i)|^{r_a}.\end{aligned}\tag{36}$$

Hence, Φ does not support the Fatou property.

Now, we introduce the definition of Kannan Φ-contraction operator on the prequasi operator ideal.

Definition 34. For the prequasi norm Φ on the ideal $S_{\mathfrak{A}_\phi}$, where $\Phi(W)=\phi((s_a(W))_{a=0}^{\infty})$. An operator $G:S_{\mathfrak{A}_\phi}(Z,M)\longrightarrow S_{\mathfrak{A}_\phi}(Z,M)$ is called a Kannan Φ -contraction, if we have $\xi\in[0,1/2)$ so that $\Phi(GW-GA)\leq\xi(\Phi(GW-W)+\Phi(GA-A))$, for all $W,A\in S_{\mathfrak{A}_\phi}(Z,M)$.

Definition 35. For the prequasi norm Φ on the ideal $S_{\mathfrak{A}_\phi}$, where $\Phi(W)=\phi((s_a(W))_{a=0}^{\infty})$, $G:S_{\mathfrak{A}_\phi}(Z,M)\longrightarrow S_{\mathfrak{A}_\phi}(Z,M)$ and $B\in S_{\mathfrak{A}_\phi}(Z,M)$. The operator G is called Φ -sequentially continuous at B , if and only if, when $\lim_{p\longrightarrow\infty}\Phi(W_p-B)=0$, then $\lim_{p\longrightarrow\infty}\Phi(GW_p-GB)=0$.

Theorem 36. *Set up $(r_a)\in(0,1]^{\mathscr{N}}$ be an increasing and $G:S_{(\ell(r))_\phi}(Z,M)\longrightarrow S_{(\ell(r))_\phi}(Z,M)$, where $\Phi(W)=\sum_{a\in\mathscr{N}}|s_a(W)|^{r_a}$, for every $W\in S_{(\ell(r))_\phi}(Z,M)$. The point $A\in S_{(\ell(r))_\phi}(Z,M)$ is the unique fixed point of G, if the following set up are satisfied:*

(a) G is Kannan Φ-contraction mapping

(b) G is Φ-sequentially continuous at a point $A\in S_{(\ell(r))_\phi}(Z,M)$

(c) There is $B\in S_{(\ell(r))_\phi}(Z,M)$ such that the sequence of iterates $\{G^pB\}$ has a subsequence $\{G^{p_i}B\}$ converging to A

Proof. Let the conditions be verified. If A is not a fixed point of G, then $GA\neq A$. From the setting (b) and (c), we have

$$\lim_{p_i\longrightarrow\infty}\Phi(G^{p_i}B-A)=0\text{ and }\lim_{p_i\longrightarrow\infty}\Phi\left(G^{p_i+1}B-GA\right)=0.\tag{37}$$

Since G is Kannan Φ-contraction mapping, one can see

$$\begin{aligned}0<\Phi(GA-A)&=\Phi\left(\left(GA-G^{p_i+1}B\right)+(G^{p_i}B-A)+\left(G^{p_i+1}B-G^{p_i}B\right)\right)\\&\leq2\Phi\left(G^{p_i+1}B-GA\right)+4\Phi(G^{p_i}B-A)\\&\quad+4\xi\left(\frac{\xi}{1-\xi}\right)^{p_i-1}\Phi(GB-B).\end{aligned}\tag{38}$$

As $p_i \longrightarrow \infty$, this implies a contradiction. Therefore, A is a fixed point of G. To show that the fixed point A is unique. Let we have two different fixed points $A, D \in S_{(\ell(r))_\phi}(Z, M)$ of G. Hence, one has

$$\Phi(A - D) \le \Phi(GA - GD) \le \xi(\Phi(GA - A) + \Phi(GD - D)) = 0. \tag{39}$$

Therefore, $A = D$.

Example 37. Let Z and M be Banach spaces, $G : S_{(\ell((a+1/a+2)_{a=0}^\infty))_\phi}(Z, M) \longrightarrow S_{(\ell((a+1/a+2)_{a=0}^\infty))_\phi}(Z, M)$, where $\Phi(W) = \sum_{a\in\mathcal{N}} (s_a(W))^{a+1/a+2}$, for every $W \in S_{(\ell((a+1/a+2)_{a=0}^\infty))_\phi}(Z, M)$ and

$$G(W) = \begin{cases} \dfrac{W}{26}, & \Phi(W) \in [0, 1), \\ \dfrac{W}{37}, & \Phi(W) \in [1,\infty). \end{cases} \tag{40}$$

Since for all $W_1, W_2 \in S_{(\ell((a+1/a+2)_{a=0}^\infty))_\phi}$ with $\Phi(W_1), \Phi(W_2) \in (0, 1]$, we have

$$\begin{aligned}\Phi(GW_1 - GW_2) &= \Phi\left(\frac{W_1}{26} - \frac{W_2}{26}\right) \le \frac{2}{5}\left(\Phi\left(\frac{25W_1}{26}\right) + \Phi\left(\frac{25W_2}{26}\right)\right) \\ &= \frac{2}{5}(\Phi(GW_1 - W_1) + \Phi(GW_2 - W_2)).\end{aligned} \tag{41}$$

For all $W_1, W_2 \in S_{(\ell((a+1/a+2)_{a=0}^\infty))_\phi}$ with $\Phi(W_1), \Phi(W_2) \in [1,\infty)$, we have

$$\begin{aligned}\Phi(GW_1 - GW_2) &= \Phi\left(\frac{W_1}{37} - \frac{W_2}{37}\right) \le \frac{1}{3}\left(\Phi\left(\frac{36W_1}{37}\right) + \Phi\left(\frac{36W_2}{37}\right)\right) \\ &= \frac{1}{3}(\Phi(GW_1 - W_1) + \Phi(GW_2 - W_2)).\end{aligned} \tag{42}$$

For all $W_1, W_2 \in S_{(\ell((a+1/a+2)_{a=0}^\infty))_\phi}$ with $\Phi(W_1) \in [0, 1)$ and $\Phi(W_2) \in [1,\infty)$, we have

$$\begin{aligned}\Phi(GW_1 - GW_2) &= \Phi\left(\frac{W_1}{26} - \frac{W_2}{37}\right) \le \frac{2}{5}\Phi\left(\frac{25W_1}{26}\right) + \frac{1}{3}\Phi\left(\frac{36W_2}{37}\right) \\ &\le \frac{2}{5}\left(\Phi\left(\frac{25W_1}{26}\right) + \Phi\left(\frac{36W_2}{37}\right)\right) \\ &= \frac{2}{5}(\Phi(GW_1 - W_1) + \Phi(GW_2 - W_2)).\end{aligned} \tag{43}$$

Therefore, the map W is Kannan Φ-contraction mapping and $G^p(W) = \begin{cases} W/26^p, & \Phi(W) \in [0, 1), \\ W/37^p, & \Phi(W) \in [1,\infty). \end{cases}$

It is clear that G is Φ-sequentially continuous at the zero operator $\Theta \in S_{(\ell((a+1/a+2)_{a=0}^\infty))_\phi}$ and $\{G^p W\}$ has a subsequence $\{G^{p_i} W\}$ converging to Θ. By Theorem 36, the zero operator $\Theta \in S_{(\ell((a+1/a+2)_{a=0}^\infty))_\phi}$ is the only fixed point of G. Let $\{W^{(n)}\} \subseteq S_{(\ell((a+1/a+2)_{a=0}^\infty))_\phi}$ be such that $\lim_{n\longrightarrow\infty}\Phi(W^{(n)} - W^{(0)}) = 0$, where $W^{(0)} \in S_{(\ell((a+1/a+2)_{a=0}^\infty))_\phi}$ with $\Phi(W^{(0)}) = 1$. Since the prequasi norm Φ is continuous, we have

$$\begin{aligned}\lim_{n\longrightarrow\infty} \Phi\left(GW^{(n)} - GW^{(0)}\right) &= \lim_{n\longrightarrow\infty} \Phi\left(\frac{W^{(n)}}{26} - \frac{W^{(0)}}{37}\right) \\ &= \Phi\left(\frac{11W^{(0)}}{962}\right) > 0.\end{aligned} \tag{44}$$

Hence G is not Φ-sequentially continuous at $W^{(0)}$. So, the map G is not continuous at $W^{(0)}$.

7. Application to the Existence of Solutions of Summable Equations

Summable equations like (45) were studied by Salimi et al. [22], Agarwal et al. [23], and Hussain et al. [24]. In this section, we search for a solution to (45) in $(\ell(r))_\phi$, where $(r_a) \in (0, 1]^{\mathcal{N}}$ be an increasing and $\phi(v) = \sum_{a\in\mathcal{N}} |v_a|^{r_a}$, for all $v \in \ell(r)$. Consider the summable equations

$$v_a = p_a + \sum_{m=0}^{\infty} A(a, m)f(m, v_m), \tag{45}$$

and let $W : (\ell(r))_\phi \longrightarrow (\ell(r))_\phi$ defined by

$$W(v_a)_{a\in\mathcal{N}} = \left(p_a + \sum_{m=0}^{\infty} A(a, m)f(m, v_m)\right)_{a\in\mathcal{N}}. \tag{46}$$

Theorem 38. *The summable equations ((45)) has a solution in $(\ell(r))_\phi$, if $A : \mathcal{N}^2 \longrightarrow \mathfrak{R}, f : \mathcal{N} \times \mathfrak{R} \longrightarrow \mathfrak{R}, p : \mathcal{N} \longrightarrow \mathfrak{R}$, and for all $a \in \mathcal{N}$, there is $\xi \in [0, 1/2)$, so that*

$$\begin{aligned}&\left|\sum_{m\in\mathcal{N}} A(a, m)(f(m, v_m) - f(m, t_m))\right|^{r_a} \\ &\le \xi\left[\left|p_a - v_a + \sum_{m=0}^{\infty} A(a, m)f(m, v_m)\right|^{r_a} + \left|p_a - t_a + \sum_{m=0}^{\infty} A(a, m)f(m, t_m)\right|^{r_a}\right].\end{aligned} \tag{47}$$

Proof. Let the conditions be verified. Consider the mapping $W : (\ell(r))_\phi \longrightarrow (\ell(r))_\phi$ defined by (46). We have

$$\begin{aligned}\phi(Wv - Wt) &= \sum_{a\in\mathcal{N}} |Wv_a - Wt_a|^{r_a} \\ &= \sum_{a\in\mathcal{N}} \left| \sum_{m\in\mathcal{N}} A(a,m)[f(m,v_m) - f(m,t_m)] \right|^{r_a} \\ &\leq \xi \left(\sum_{a\in\mathcal{N}} \left[\left| p_a - v_a + \sum_{m=0}^{\infty} A(a,m) f(m,v_m) \right|^{r_a} \right.\right. \\ &\quad \left.\left. + \sum_{a\in\mathcal{N}} \left| p_a - t_a + \sum_{m=0}^{\infty} A(a,m) f(m,t_m) \right|^{r_a} \right] \right) \\ &= \xi(\phi(Wv - v) + \phi(Wt - t)).\end{aligned} \tag{48}$$

Then, from Theorem 24, we have a solution of equation (45) in $(\ell(r))_\phi$.

Example 39. Given the sequence space $(\ell((a+1/a+2)_{a=0}^{\infty}))_\phi$, where $\phi(v) = \sum_{a\in\mathcal{N}} |v_a|^{a+1/a+2}$, for all $v \in (\ell((a+1/a+2)_{a=0}^{\infty}))$. Consider the summable equations

$$v_a = e^{-(3a+6)} + \sum_{m=0}^{\infty} (-1)^{a+m} \left(\frac{v_a}{a^2 + m!+1} \right)^q, \tag{49}$$

where $q > 2$ and let $W : (\ell((a+1/a+2)_{a=0}^{\infty}))_\phi \longrightarrow (\ell((a+1/a+2)_{a=0}^{\infty}))_\phi$ defined by

$$W(v_a)_{a\in\mathcal{N}} = \left(e^{-(3a+6)} + \sum_{m=0}^{\infty} (-1)^{a+m} \left(\frac{v_a}{a^2 + m!+1} \right)^q \right)_{a\in\mathcal{N}}. \tag{50}$$

It is easy to see that

$$\begin{aligned}&\left| \sum_{m=0}^{\infty} (-1)^a \left(\frac{v_a}{a^2 + m!+1} \right)^q ((-1)^m - (-1)^m) \right|^{a+1/a+2} \\ &\leq \frac{1}{3} \left[\left| e^{-(3a+6)} - v_a + \sum_{m=0}^{\infty} (-1)^{a+m} \left(\frac{v_a}{a^2 + m!+1} \right)^q \right|^{a+1/a+2} \right. \\ &\quad \left. + \left| e^{-(3a+6)} - t_a + \sum_{m=0}^{\infty} (-1)^{a+m} \left(\frac{v_a}{a^2 + m!+1} \right)^q \right|^{a+1/a+2} \right].\end{aligned} \tag{51}$$

By Theorem 38, the summable equations (49) has a solution in $(\ell((a+1/a+2)_{a=0}^{\infty}))_\phi$.

Data Availability

Not applicable.

Ethical Approval

This article does not contain any studies with human participants or animals performed by any of the authors.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Acknowledgments

This work was funded by the University of Jeddah, Saudi Arabia, under grant No. (UJ-20-078-DR). The authors, therefore, acknowledge with thanks the University's technical and financial support. Also, the authors are extremely grateful to the reviewers for their valuable suggestions and leading a crucial role for a better presentation of this manuscript.

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