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Research Article

A New Numerical Approach for Solving 1D Fractional Diffusion-Wave Equation

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Fractional derivative is nonlocal, which is more suitable to simulate physical phenomena and provides more accurate models of physical systems such as earthquake vibration and polymers. The present study suggested a new numerical approach for the fractional diffusion-wave equation (FDWE). The fractional-order derivative is in the Riemann-Liouville (R-L) sense. Discussed the theoretical analysis of stability, consistency, and convergence. The numerical examples demonstrate that the method is more workable and excellently holds the theoretical analysis, showing the scheme's feasibility.

1. Introduction

Nowadays, fractional calculus gains attention because of many applications in different sciences, such as physical and chemical phenomena [1–13]. Therefore, many numerical methods are used for the fractional differential equations [14–21]. The fractional diffusion-wave equations (FDWE) have many applications, for example, continuous-time random walks, unification of diffusion, and wave propagation phenomenon [22, 23]. To find the solution of FDWE, different researchers used different numerical methods; for example, Hu and Zhanga [24] implemented a higher-order difference approach on the system of the FDWE and proved its stability, solvability, and convergence. Their numerical solutions reported that the high-order difference method is more efficient than the Crank-Nicolson method. Yang et al. [25] first converted FDWE into the integrodifferential equation and then apply the fractional multisteps method to solve the FDWE. They discretized the transform equation's integration term and Lubich's fractional multistep approach for space derivative and central difference schemes, respectively. Also, they show that the proposed method is convergent stable and solvable for $\gamma \in (1,1.71832)$. Brunner et al. [26] considered the numerical simulation of the 2-D time FDWE. They introduced artificial boundary conditions to transform the problem into the initial boundary value problem (IBVP) and produced a stable and convergence scheme using finite difference approximation. Sweilam et al. [27] solved two-sided space FDWE using a finite difference method. They studied the consistency and stability of both explicit and implicit methods. Also, they concluded that finite difference methods are relatively easy to use and can apply to different fractional differential equations easily. Later, Sweilam et al. [28] solved FDWE with the help of a weighted average finite difference method, which is an extension of the weighted average for the ordinary DWE. The noninteger order derivative is discretized using RL derivative and found that the suggested approach is stable. Liu et al. [29] used the novel finite difference scheme to solve the IBVPs for the FDWE. They reported that the proposed technique is unconditionally stable and convergent, having the order of convergence $O(\tau^{3-\gamma}+h^2)$, where h and t represent the time step and mesh size, respectively. Dehghan and Abbaszadeh [30] applied A finite difference method to solve one and 2-D FDWE both space and tempered fractional forms. For fractional space derivative, they used Riemann-Liouville's fractional derivative. They show using error estimate that the suggested scheme is convergence and unconditionally stable.

Sun and Wu [31] researched the fully discrete scheme to solve the solution of the BVP of FDWE. They determined that the proposed numerical approach is solvable, stable, and L_∞ convergent with convergence order $O(\tau^{3-\gamma}+h^2)$ using the energy method. Furthermore, Du et al. [32] constructed a higher-order approximation to find the solution of FDWE with the order $O(\tau^{4-\gamma}+h^2)$, where the fractional derivative is approximated using Caputo fractional derivative. The numerical results show that the Caputo fractional derivative gives convergence $O(\tau^{4-\gamma})$, which can be many apply to other fractional differential equations. Zhang et al. [33] proposed two new schemes, namely, the Crank-Nicolson (C-N), alternating direction implicit (ADI), and compact ADI to get a solution of 2-D FDWE. The close ADI scheme used the ADI approach for the time, and the relative difference technique is used for the spatial discretizations. Furthermore, they reported that the compact ADI approach is solvable, unconditionally stable, and convergent. The experiments confirmed that the ADI approach is computationally efficient in terms of CPU timing compared to the C-N ADI scheme for different values of γ without affecting the results. Huang et al. [34] used two approximations for the initial-boundary value time FDWEs, which is based on its integrodifferential equations. The proposed schemes were convergent with first and second order in temporal and spatial directions, respectively. Wei [35] used the semidiscrete strategy with the help of the finite difference method and fully implicit discrete scheme using the local Galerkin method for space time-fractional order derivatives, respectively, for the FDWE. The proposed approach results in a fully discrete scheme, which is convergent and unconditionally stable. Ali and Abdullah [36] formulated a numerical scheme for the fractional-order wave equation. They provided numerical solutions for the feasibility of the proposed scheme.

The above-cited literature means that fractional calculus is still a novel topic and needs more efficient numerical techniques to investigate the more feasible fractional differential equations (FDEs). This study is aimed at examining the highly accurate and robust numerical approach for fractional-order diffusion-wave equation. So far, the attempt has not been made to discretize the R-L integral and implement in the Riemann-Liouville fractional order derivative to approximate FDWE.

The remaining paper is organized as follows: Section 2 discussed the related preliminaries. Section 3 explains the methodology of the proposed scheme. Section 4 discusses the stability analysis. In Section 5, consistency and convergence are discussed. Section 6 provides numerical results, and finally, the conclusion is presented in Section 7.

Here, we consider the FDWE as following [36]:

$$\frac{\partial^2 Y(x,t)}{\partial t^2} = D_t^{2-q}\left(\frac{\partial^2 Y(x,t)}{\partial x^2}\right) + F(x,t), \tag{1}$$

$$q\varepsilon(1,2], x\varepsilon[0,L], t\varepsilon[0,T], \tag{2}$$

where the conditions are

$$Y(x,0)=v_1(x), Y_t(x,0)=v_2(x), Y(0,t)=Y(L,t)=0, \tag{3}$$

where $F(x,t)$ is denoted the source term and D_t^{2-q} represented the R-L of order q lying between 1 and 2.

2. Preliminaries

The R-L fractional order derivative is defined as follows:

$$D_t^{2-q}Y(x,t)=\frac{1}{\Gamma(q)}\frac{d^2}{dt^2}\int_0^t\frac{Y(x,z)}{(t-z)^{1-q}}dz=\frac{d^2}{dt^2}I_0^qY(x,t). \tag{4}$$

Here, the R-L integral can be defined as:

$$I_0^qY(x,t)=\frac{1}{\Gamma(q)}\int_0^t\frac{Y(x,z)}{(t-z)^{1-q}}dz. \tag{5}$$

To approximate the Riemann-Liouville integral in equation (5), following the approach in [37]:

$$I_0^qY(x,t)=\frac{1}{\Gamma(q)}\int_0^{t_n}(t_n-z)^{q-1}Y(x,z)dz, \tag{6}$$

using the Jumarie property [38] as:

$$\begin{aligned}
&=\frac{1}{q\Gamma(q)}\int_0^{t_n}Y(x,z)(dz)^q,\\
&=\frac{1}{\Gamma(1+q)}\sum_{k=0}^{n-1}\int_{t_k}^{t_{k+1}}Y(x,z)(dz)^q,\\
&=\frac{1}{\Gamma(1+q)}\sum_{k=0}^{n-1}Y(x,t_{n-k})\int_{t_k}^{t_{k+1}}z^0(dz)^q,
\end{aligned} \tag{7}$$

applying the Jumarie property $\int_0^t z^m(dz)^n=(\Gamma(1+m)\Gamma(1+n)/\Gamma(1+n+m))t^{m+n}$,

$$\begin{aligned}
&=\frac{\tau^q}{\Gamma(1+q)}\sum_{k=0}^{n-1}Y(x,t_{n-k})((k+1)^q-(k)^q),\\
&=\frac{\tau^q}{\Gamma(1+q)}\sum_{k=0}^{n-1}C_k^{(q)}Y(x,y,t_{n-k}),
\end{aligned} \tag{8}$$

and $C_k^{(q)}=(k+1)^q-(k)^q$,$k=0,1,2,\cdots,n-1$.

Lemma 1. *The $q(0<q<1)$ order R-L fractional integral of $Y(x,t)$ in $[0,T]$ defined as:*

$$I_0^q Y(x_i,t_n)=\frac{\tau^q}{\Gamma(1+q)}\sum_{k=0}^{n-1}C_k^{(q)}Y(x_i,t_{n-k}). \tag{9}$$

Lemma 2. *The coefficient $C_k^{(q)}(k=0,1,\cdots)$ satisfies the below properties [37]:*

(i) $C_0^{(q)}=1, C_k^{(q)}>0, k=0,1,\cdots$.

(ii) $C_{k-1}^{(q)}>C_k^{(q)}, k=1,2,\cdots$.

(iii) *Here, $\exists$ the +ve constant $C_1>0$. as $\tau\le C_1C_k^{(q)}\tau^q$, $k=1,2,\cdots$.*

(iv) $\sum_{k=0}^{n}C_k^{(q)}\tau^q=(n+1)^q\le T^q$

3. The Proposed Scheme

The IDS for FDWE is developed in this section. The fractional order part is replaced by Lemma 1, and space derivative is approximated by central difference approximation. The step for space is $x_i=i\Delta x$, and the step for time is $t_k=k\tau$, where $1\le i\le M-1$, $\Delta x=L/M$ and $0\le k\le N$, $\tau=T/N$, respectively. Using equation (4) in (1) at grid point $Y(x_i,t_k)$, as follows

$$\frac{\partial^2 Y(x_i,t_k)}{\partial t^2}=\frac{d^2}{dt^2}I_0^q\frac{\delta_x^2Y(x_i,t_k)}{\Delta x^2}+F(x_i,t_k). \tag{10}$$

Eliminating the second order time derivative by applying backward difference approximation w.r.t time and then integrating from t_k to t_{k+1} and using trapezoidal rule for source term, we obtained

$$\begin{aligned}Y_i^{k+1}-2Y_i^k+Y_i^{k-1}&=\frac{1}{\Delta x^2}I_0^q\left(\delta_x^2Y_i^{k+1}-2\delta_x^2Y_i^k+\delta_x^2Y_i^{k-1}\right)\\&\quad+\frac{\tau^2}{2}\left(F_i^{k+1}+F_i^k\right),\end{aligned} \tag{11}$$

substituting Lemma 1 in place of RL fractional integral in equation (11), the simplified approximated difference scheme is obtained for fractional diffusion-wave equation, as follows

$$\begin{aligned}&Y_i^{k+1}-2Y_i^k+Y_i^{k-1}\\&\quad=S_1\left(Y_{i+1}^{k+1}-2Y_i^{k+1}+Y_{i-1}^{k+1}\right)-S_1\left(Y_{i+1}^k-2Y_i^k+Y_{i-1}^k\right)\\&\qquad+S_1\sum_{j=0}^{k-1}\left(C_{j+1}^{(q)}-C_j^{(q)}\right)\left(Y_{i+1}^{k-j}-2Y_i^{k-j}+Y_{i-1}^{k-j}\right)\\&\qquad-S_1\sum_{j=1}^{k-1}\left(C_j^{(q)}-C_{j-1}^{(q)}\right)\left(Y_{i+1}^{k-j}-2Y_i^{k-j}+Y_{i-1}^{k-j}\right)\\&\qquad+\frac{\tau^2}{2}\left(F_i^{k+1}+F_i^k\right),\end{aligned} \tag{12}$$

we know that $Y_t(x,0)=v_2(x)$, therefore

$$Y_i^{-1}=Y_i^1+2\tau v_2(x_i), \tag{13}$$

$$Y_i^0=v_1(x_i),\ Y_0^k=Y_M^k=0, \tag{14}$$

where $S_1=\tau^q/(\Gamma(1+q)\Delta x^2)$, $i=1\cdots M-1$, $k=0\cdots N-1$.

4. Stability

The Von Neumann technique is using to find the stability of the suggested scheme. Let y_i^k represent the exact solution for equation (12), we get

$$\begin{aligned}&y_i^{k+1}-2y_i^k+y_i^{k-1}\\&\quad=S_1\left(y_{i+1}^{k+1}-2y_i^{k+1}+y_{i-1}^{k+1}\right)-S_1\left(y_{i+1}^k-2y_i^k+y_{i-1}^k\right)\\&\qquad+S_1\sum_{j=0}^{k-1}\left(C_{j+1}^{(q)}-C_j^{(q)}\right)\left(y_{i+1}^{k-j}-2y_i^{k-j}+y_{i-1}^{k-j}\right)\\&\qquad-S_1\sum_{j=1}^{k-1}\left(C_j^{(q)}-C_{j-1}^{(q)}\right)\left(y_{i+1}^{k-j}-2y_i^{k-j}+y_{i-1}^{k-j}\right).\end{aligned} \tag{15}$$

The error is defined as $E_i^k=y_i^k-Y_i^k$. And E_i^k satisfies equation (15) as follows:

$$\begin{aligned}&E_i^{k+1}-2E_i^k+E_i^{k-1}\\&\quad=S_1\left(E_{i+1}^{k+1}-2E_i^{k+1}+E_{i-1}^{k+1}\right)-S_1\left(E_{i+1}^k-2E_i^k+E_{i-1}^k\right)\\&\qquad+S_1\sum_{j=0}^{k-1}\left(C_{j+1}^{(q)}-C_j^{(q)}\right)\left(E_{i+1}^{k-j}-2E_i^{k-j}+E_{i-1}^{k-j}\right)\\&\qquad-S_1\sum_{j=1}^{k-1}\left(C_j^{(q)}-C_{j-1}^{(q)}\right)\left(E_{i+1}^{k-j}-2E_i^{k-j}+E_{i-1}^{k-j}\right).\end{aligned} \tag{16}$$

Consider the growth factor in the form of a one Fourier mode as

$$E_i^k=\lambda^k e^{\sqrt{-1}i\sigma\Delta x}, \tag{17}$$

where σ and Δx are the mode number and step size, respectively. Equation (17) can be the solution of the above error equation (16), as

$$\begin{aligned}&\lambda^{k+1}e^{\sqrt{-1}i\sigma\Delta x}-2\lambda^k e^{\sqrt{-1}i\sigma\Delta x}+\lambda^{k-1}e^{\sqrt{-1}i\sigma\Delta x}\\&\quad=S_1\left(\lambda^{k+1}e^{\sqrt{-1}i+1\sigma\Delta x}-2\lambda^{k+1}e^{\sqrt{-1}i\sigma\Delta x}+\lambda^{k+1}e^{\sqrt{-1}i-1\sigma\Delta x}\right)\\&\qquad-S_1\left(\lambda^k e^{\sqrt{-1}i+1\sigma\Delta x}-2\lambda^k e^{\sqrt{-1}i\sigma\Delta x}+\lambda^k e^{\sqrt{-1}i-1\sigma\Delta x}\right)\\&\qquad+S_1\sum_{j=0}^{k-1}\left(C_{j+1}^{(q)}-C_j^{(q)}\right)\\&\qquad\cdot\left(\lambda^{k-j}e^{\sqrt{-1}i+1\sigma\Delta x}-2\lambda^{k-j}e^{\sqrt{-1}i\sigma\Delta x}+\lambda^{k-j}e^{\sqrt{-1}i-1\sigma\Delta x}\right)\\&\qquad-S_1\sum_{j=1}^{k-1}\left(C_j^{(q)}-C_{j-1}^{(q)}\right)\\&\qquad\cdot\left(\lambda^{k-j}e^{\sqrt{-1}i+1\sigma\Delta x}-2\lambda^{k-j}e^{\sqrt{-1}i\sigma\Delta x}+\lambda^{k-j}e^{\sqrt{-1}i-1\sigma\Delta x}\right).\end{aligned} \tag{18}$$

Dividing both sides by $e^{\sqrt{-1}i\sigma\Delta x}$, then replace $e^{\sqrt{-1}\sigma\Delta x}+e^{-\sqrt{-1}i\sigma\Delta x}=2-4S_1 sin^2(\sigma\Delta x/2)$, we obtained

$$\begin{aligned}\lambda^{k+1}={}&\frac{1}{1+4S_1\sin^2(\sigma\Delta x/2)}\\&\cdot\left[\left(2+4S_1\sin^2\left(\frac{\sigma\Delta x}{2}\right)\right)\lambda^k-\lambda^{k-1}-4S_1\sin^2\left(\frac{\sigma\Delta x}{2}\right)\right.\\&\left.\cdot\left(\sum_{j=0}^{k-1}\left(C_{j+1}^{(q)}-C_j^{(q)}\right)\lambda^{k-j}-\sum_{j=1}^{k-1}\left(C_j^{(q)}-C_{j-1}^{(q)}\right)\lambda^{k-j}\right)\right].\end{aligned}\tag{19}$$

Proposition 3. *Suppose λ^{k+1}, $k=0,1,\cdots,N-1$ is the solution of equation (19), then we need to prove that*

$$|\lambda^{k+1}|\le|\lambda^0|.\tag{20}$$

Proof. Let take $k=0$ in equation (19), to prove the proposition by induction method, as

$$\lambda^1=\frac{\lambda^0\left(2+4S_1\sin^2(\sigma\Delta x/2)\right)-\lambda^{-1}}{1+4S_1\sin^2(\sigma\Delta x/2)}.\tag{21}$$

By utilizing equations (13) and (17), we get $\lambda^{-1}=\lambda^1$ and putting in equation (21). After simplification, the above equation becomes

$$\lambda^1=\frac{\lambda^0\left(2+4S_1\sin^2(\sigma\Delta x/2)\right)}{2+4S_1\sin^2(\sigma\Delta x/2)},\tag{22}$$

obtained the following relation

$$|\lambda^1|\le|\lambda^0|.\tag{23}$$

Let $|\lambda^1|\le|\lambda^0|$ holds for $k=0,1,\cdots,n-1$.

Using equations (19) and (23) and Lemma 2, we have

$$\begin{aligned}\left|\lambda^{k+1}\right|={}&\frac{1}{1+4S_1\sin^2(\sigma\Delta x/2)}\\&\cdot\left[\left(2+4S_1\sin^2\left(\frac{\sigma\Delta x}{2}\right)\right)\left|\lambda^k\right|-\left|\lambda^{k-1}\right|\right.\\&-4S_1\sin^2\left(\frac{\sigma\Delta x}{2}\right)\left(\sum_{j=0}^{k-1}\left(C_{j+1}^{(q)}-C_j^{(q)}\right)\left|\lambda^{k-j}\right|\right.\\&\left.\left.-\sum_{j=1}^{k-1}\left(C_j^{(q)}-C_{j-1}^{(q)}\right)\left|\lambda^{k-j}\right|\right)\right],\left|\lambda^{k+1}\right|\\\le{}&\frac{1}{1+4S_1\sin^2(\sigma\Delta x/2)}\\&\cdot\left[\left(2+4S_1\sin^2\left(\frac{\sigma\Delta x}{2}\right)\right)\left|\lambda^0\right|-\left|\lambda^0\right|\right.\\&-4S_1\sin^2\left(\frac{\sigma\Delta x}{2}\right)\left(\sum_{j=0}^{k-1}\left(C_{j+1}^{(q)}-C_j^{(q)}\right)\left|\lambda^0\right|\right.\\&\left.\left.-\sum_{j=1}^{k-1}\left(C_j^{(q)}-C_{j-1}^{(q)}\right)\left|\lambda^0\right|\right)\right],\\\le{}&\frac{1}{1+4S_1\sin^2(\sigma\Delta x/2)}\\&\cdot\left[2+4S_1\sin^2\left(\frac{\sigma\Delta x}{2}\right)-1-4S_1\sin^2\left(\frac{\sigma\Delta x}{2}\right)\right.\\&\left.\cdot\left(\sum_{j=0}^{k-1}\left(C_{j+1}^{(q)}-C_j^{(q)}\right)-\sum_{j=1}^{k-1}\left(C_j^{(q)}-C_{j-1}^{(q)}\right)\right)\right]\left|\lambda^0\right|,\\\le{}&\frac{1}{1+4S_1\sin^2(\sigma\Delta x/2)}\\&\cdot\left[1+4S_1\sin^2\left(\frac{\sigma\Delta x}{2}\right)-4S_1\sin^2\left(\frac{\sigma\Delta x}{2}\right)\right.\\&\left.\cdot\left(\left(C_1^{(q)}-C_{k-1}^{(q)}\right)-\left(C_1^{(q)}-C_{k-2}^{(q)}\right)\right)\right]\left|\lambda^0\right|,\\\le{}&\frac{1}{1+4S_1\sin^2(\sigma\Delta x/2)}\\&\cdot\left[1+4S_1\sin^2\left(\frac{\sigma\Delta x}{2}\right)-4S_1\sin^2\left(\frac{\sigma\Delta x}{2}\right)\right.\\&\left.\cdot\left(C_{k-2}^{(q)}-C_{k-1}^{(q)}\right)\right]\left|\lambda^0\right|.\end{aligned}\tag{24}$$

From Lemma 2, the value $0<C_{k-2}^{(q)}-C_{k-1}^{(q)})<1$, so it is clear that the value

$$\begin{aligned}&0<\frac{1+4S_1\sin^2(\sigma\Delta x/2)-4S_1\sin^2(\sigma\Delta x/2)\left(C_{k-2}^{(q)}-C_{k-1}^{(q)}\right)}{1+4S_1\sin^2(\sigma\Delta x/2)}<1,\\&\left|\lambda^{k+1}\right|\le\left|\lambda^0\right|.\end{aligned}\tag{25}$$

Here, $|\lambda^{k+1}|\le|\lambda^0|$, also $|E_i^{k+1}|\le|E_i^0|$, so it can written as $\|E_i^{k+1}\|_2\le\|E_i^0\|_2$. It shows that the suggested approach is unconditionally stable.

5. Consistency

To find the consistency analysis, suppose w is the closed form solution, and W is the approximated solution, and the function $Y(W)=0$ is the approximated scheme for the proposed equation at the mesh point (x_i,t_k). Then, $Y(W)=T_i^k$ denoted the local truncation error at (x_i,t_k).

Theorem 4. *The local transcation error $T(x,t)$ for the suggested scheme is $T_i^k=O(\Delta t^2)+O(\Delta x)^2$.*

Proof. The obtained difference scheme in equation (12), can also be written as:

$$\begin{aligned} T_i^{k+1} &= Y_i^{k+1} - 2Y_i^k + Y_i^{k-1} \\ &\quad - S_1 \sum_{j=0}^{k} C_j^{(q)} \left(u_{i+1}^{k-j+1} - 2u_i^{k-j+1} + u_{i-1}^{k-j+1} \right) \\ &\quad - 2 \sum_{j=0}^{k-1} C_j^{(q)} \left(u_{i+1}^{k-j} - 2u_i^{k-j} + u_{i-1}^{k-j} \right) \\ &\quad + \sum_{j=0}^{k-2} C_j^{(q)} \left(u_{i+1}^{k-j-1} - 2u_i^{k-j-1} + u_{i-1}^{k-j-1} \right), \end{aligned} \tag{26}$$

using Taylor series, we obtained

$$\begin{aligned} T_i^{k+1} &= \left(u_i^k + (\Delta t) \frac{\partial u}{\partial t}\bigg|_i^k + \frac{(\Delta t)^2}{2!} \frac{\partial^2 u}{\partial t^2}\bigg|_i^k + \frac{(\Delta t)^3}{3!} \frac{\partial^3 u}{\partial t^3}\bigg|_i^k + \cdots \right) \\ &\quad - 2u_i^k - \left(u_i^k - (\Delta t) \frac{\partial u}{\partial t}\bigg|_i^k + \frac{(\Delta t)^2}{2!} \frac{\partial^2 u}{\partial t^2}\bigg|_i^k - \frac{(\Delta t)^3}{3!} \frac{\partial^3 u}{\partial t^3}\bigg|_i^k + \cdots \right) \\ &\quad - S_1 \sum_{j=0}^{k} C_j^{(q)} \left(u_i^{k-j+1} + (\Delta x) \frac{\partial u}{\partial x}\bigg|_i^{k-j+1} + \frac{(\Delta x)^2}{2!} \frac{\partial^2 u}{\partial x^2}\bigg|_i^{k-j+1} \right. \\ &\quad + \frac{(\Delta x)^3}{3!} \frac{\partial^3 u}{\partial x^3}\bigg|_i^{k-j+1} + \cdots - 2u_i^{k-j+1} + u_i^{k-j+1} - (\Delta x) \frac{\partial u}{\partial x}\bigg|_i^{k-j+1} \\ &\quad \left. + \frac{(\Delta x)^2}{2!} \frac{\partial^2 u}{\partial x^2}\bigg|_i^{k-j+1} - \frac{(\Delta x)^3}{3!} \frac{\partial^3 u}{\partial x^3}\bigg|_i^{k-j+1} + \cdots \right) \\ &\quad - 2S_1 \sum_{j=0}^{k-1} C_j^{(q)} \left(u_i^{k-j} + (\Delta x) \frac{\partial u}{\partial x}\bigg|_i^{k-j} + \frac{(\Delta x)^2}{2!} \frac{\partial^2 u}{\partial x^2}\bigg|_i^{k-j} \right. \\ &\quad + \frac{(\Delta x)^3}{3!} \frac{\partial^3 u}{\partial x^3}\bigg|_i^{k-j} + \cdots - 2u_i^{k-j} + u_i^{k-j} - (\Delta x) \frac{\partial u}{\partial x}\bigg|_i^{k-j} \\ &\quad \left. + \frac{(\Delta x)^2}{2!} \frac{\partial^2 u}{\partial x^2}\bigg|_i^{k-j} - \frac{(\Delta x)^3}{3!} \frac{\partial^3 u}{\partial x^3}\bigg|_i^{k-j} + \cdots \right) \\ &\quad - S_1 \sum_{j=0}^{k-2} C_j^{(q)} \left(u_i^{k-j-1} + (\Delta x) \frac{\partial u}{\partial x}\bigg|_i^{k-j-1} + \frac{(\Delta x)^2}{2!} \frac{\partial^2 u}{\partial x^2}\bigg|_i^{k-j-1} \right. \\ &\quad + \frac{(\Delta x)^3}{3!} \frac{\partial^3 u}{\partial x^3}\bigg|_i^{k-j-1} + \cdots - 2u_i^{k-j-1} + u_i^{k-j-1} - (\Delta x) \frac{\partial u}{\partial x}\bigg|_i^{k-j-1} \\ &\quad \left. + \frac{(\Delta x)^2}{2!} \frac{\partial^2 u}{\partial x^2}\bigg|_i^{k-j-1} - \frac{(\Delta x)^3}{3!} \frac{\partial^3 u}{\partial x^3}\bigg|_i^{k-j-1} + \cdots \right), \\ T_i^{k+1} &= \left((\Delta t)^2 \frac{\partial^2 u}{\partial t^2}\bigg|_i^k + \frac{(\Delta t)^4}{12} \frac{\partial^4 u}{\partial t^4}\bigg|_i^k + \cdots \right) \\ &\quad + S_1 \sum_{j=0}^{k} C_j^{(q)} \left((\Delta x)^2 \frac{\partial^2 u}{\partial x^2}\bigg|_i^{k-j+1} + \frac{(\Delta x)^4}{12} \frac{\partial^4 u}{\partial x^4}\bigg|_i^{k-j+1} + \cdots \right) \\ &\quad - 2S_1 \sum_{j=0}^{k-1} C_j^{(q)} \left((\Delta x)^2 \frac{\partial^2 u}{\partial x^2}\bigg|_i^{k-j} + \frac{(\Delta x)^4}{12} \frac{\partial^4 u}{\partial x^4}\bigg|_i^{k-j} + \cdots \right) \\ &\quad - S_1 \sum_{j=0}^{k-2} C_j^{(q)} \left((\Delta x)^2 \frac{\partial^2 u}{\partial x^2}\bigg|_i^{k-j-1} + \frac{(\Delta x)^4}{12} \frac{\partial^4 u}{\partial x^4}\bigg|_i^{k-j-1} + \cdots \right). \end{aligned} \tag{27}$$

Table 1: The numerical results of the IDS for Example 6 at $T = 1.0$.

$\tau = (\Delta x)^2$	$q = 0.25$	$q = 0.5$	$q = 0.75$
1/16	1.9802×10^{-2}	1.9053×10^{-2}	1.8120×10^{-2}
1/64	5.0811×10^{-3}	4.8647×10^{-3}	4.6537×10^{-3}
1/144	2.2611×10^{-3}	3.1168×10^{-3}	2.0756×10^{-3}
1/256	1.1894×10^{-3}	1.1418×10^{-3}	1.1693×10^{-3}

The above equation can also be written as

$$\begin{aligned} T_i^{k+1} &= (\Delta t)^2 \left(\frac{\partial^2 u}{\partial t^2}\bigg|_i^k + \frac{(\Delta t)^2}{12} \frac{\partial^4 u}{\partial t^4}\bigg|_i^k + \cdots \right) \\ &\quad + (\Delta x)^2 \left(S_1 \sum_{j=0}^{k} C_j^{(q)} \frac{\partial^2 u}{\partial x^2}\bigg|_i^{k-j+1} - 2S_1 \sum_{j=0}^{k-1} C_j^{(q)} \frac{\partial^2 u}{\partial x^2}\bigg|_i^{k-j} \right. \\ &\quad \left. + S_1 \sum_{j=0}^{k-2} C_j^{(q)} \frac{\partial^2 u}{\partial x^2}\bigg|_i^{k-j-1} \right). \end{aligned} \tag{28}$$

We obtained the following truncation error

$$T_i^{k+1} = O(\Delta t)^2 + O(\Delta x)^2. \tag{29}$$

The IDS of fractional diffusion-wave equation is consistent if space step Δx and time step Δt approaches zero; then, truncation error approaches zero.

Theorem 5. *The consistency and stability analysis are both the necessary and the sufficient for convergence. By Lax-equivalence theorem [33], the proposed scheme is convergent.*

6. Numerical Results

Here, the FDWE examples reported to find the accuracy and feasibility of the scheme. The numerical examples are coded in Maple 15, i.e., the maximum error, is defined as follows:

$$E_\infty = \max_{0 \le i \le M-1, 0 \le k \le N} \left| y(x_i, t_k) - Y_i^k \right|. \tag{30}$$

The E_2 error is:

$$E_2 = \left(\sum_{i=1}^{M-1} \left(y(x_i, t_k) - Y_{i,j}^k \right)^2 (\Delta x) \right)^{1/2}. \tag{31}$$

The convergence order is:

$$C-\text{order} = \frac{\log\,(e_1/e_2)}{\log\,(h_1/h_2)}. \tag{32}$$

Example 6. In equations (1)–(3), take the source term $F(x, t) = (\Gamma(3+q)/\Gamma(1+q)) e^x x^2 (1-x)^2 t^q - (\Gamma(3+q)/\Gamma(1+2q)) e^x t^{2q} (2 - 8x + x^2 + 6x^3 + x^4)$ and the closed solution $Y(x, t) = e^x x^2 (1-x)^2 t^{q+2}$.

Table 2: The numerical results of the proposed scheme for Example 6 at $T = 1.0$.

$\tau = (\Delta x)$	$q = 0.25$	$q = 0.5$	$q = 0.75$
1/10	4.9799×10^{-3}	4.8584×10^{-3}	4.6045×10^{-3}
1/30	9.3870×10^{-4}	8.3367×10^{-4}	8.0913×10^{-4}
1/60	3.7440×10^{-4}	3.1316×10^{-4}	3.1540×10^{-4}
1/90	2.0584×10^{-4}	1.8973×10^{-4}	1.0265×10^{-4}
1/120	1.7578×10^{-4}	1.3173×10^{-4}	9.7360×10^{-5}

Table 3: Order of convergence of the IDS for Example 6 at $q = 0.5$, $\Delta x = 1/100$, $T = 1.0$.

τ	E_∞	$C-$ order	E_2	$C-$ order
1/5	1.6672×10^{-2}	—	1.3142×10^{-2}	—
1/10	4.5646×10^{-3}	1.8791	3.1168×10^{-3}	2.0762
1/20	1.2792×10^{-3}	1.8543	7.7861×10^{-4}	2.0012
1/40	4.0253×10^{-4}	1.7823	2.0522×10^{-4}	1.9239
For $q = 0.75$				
1/5	1.5879×10^{-2}	—	1.2531×10^{-2}	—
1/10	4.3893×10^{-3}	1.8652	2.9858×10^{-3}	2.0695
1/20	1.2362×10^{-3}	1.8282	8.5184×10^{-4}	1.8157
1/40	3.4073×10^{-4}	1.8594	2.1507×10^{-4}	1.9926
For $q = 0.95$				
1/5	1.4832×10^{-2}	—	1.1722×10^{-2}	—
1/10	4.2321×10^{-3}	1.8652	2.8672×10^{-3}	2.0695
1/20	1.2239×10^{-3}	1.8282	8.4446×10^{-4}	1.8157
1/40	3.4853×10^{-4}	1.8183	2.3648×10^{-4}	1.8427

Table 4: The numerical results of the proposed scheme for Example 7 at $T = 1.0$.

$\tau = (\Delta x)^2$	$q = 0.5$	$q = 0.75$
1/16	1.6221×10^{-2}	3.0081×10^{-2}
1/64	4.3498×10^{-3}	7.8153×10^{-3}
1/144	1.9566×10^{-3}	3.4826×10^{-3}
1/324	8.6632×10^{-4}	9.7248×10^{-4}

Example 7. In equations (1)–(3) taking the source term $F(x,t) = e^{(x)}((\Gamma(2+q)/\Gamma(q))t^{q-1} - (\Gamma(2+q)/\Gamma(2q))t^{2q-1})$ and the closed solution $u(x,t) = e^{(x)}t^{1+q}$.

7. Discussion

The numerical examples 5.1 and 5.2 are solved by the suggested numerical scheme in equation (12). Tables 1–4 represent the performance of the numerical results. The accuracy level increases as reducing the space and the time step; please refer to Tables 1 and 2 for different and same space and time steps, respectively. In Table 3, fixed $\Delta x = 1/100$ and for various values of τ and q, and find the convergence order completely demonstrates the analysis theoretically. Similarly, Table 4 also shows the accuracy for the second test example 5.2, which converges to the exact solution to reduce the error by reducing the step size. From the above discussion, it is clear that the proposed scheme proved our theoretical analysis.

8. Conclusion

A practical and fast numerical scheme has been developed for FDWE. The approximation is based on the discretization of Riemann-Liouville integral in Lemma 1. We have successfully proved the analysis theoretically of stability by mathematical induction, consistency, and convergence. The numerical result confirmed our theoretical analysis and demonstrated that the proposed scheme is fast convergent and more feasible. This approach can also apply to other types of FDWEs.

Data Availability

The data that support the findings of this study are available from the corresponding author upon reasonable request.

Conflicts of Interest

The authors declare no conflict of interest.

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References

[1] L. Yin, J. Gui, and Z. Zeng, "Improving energy efficiency of multimedia content dissemination by adaptive clustering and D2D multicast," *Mobile Information Systems*, vol. 2019, 16 pages, 2019.

[2] K. S. Miller and B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations*, John-wiley and sons, Inc, New York, 1993.

[3] M. A. Khan and N. H. M. Ali, "High-order compact scheme for the two-dimensional fractional Rayleigh-Stokes problem for a heated generalized second-grade fluid," *Advances in Difference Equations*, vol. 2020, no. 1, 21 pages, 2020.

[4] S. Jahanshahi and D. F. Torres, "A simple accurate method for solving fractional variational and optimal control problems," *Journal of Optimization Theory and Applications*, vol. 174, no. 1, pp. 156–175, 2017.

[5] B. Vinagre, I. Podlubny, A. Hernandez, and V. Feliu, "Some approximations of fractional order operators used in control theory and applications," *Fractional calculus and applied analysis*, vol. 3, no. 3, pp. 231–248, 2000.

[6] U. Ali, R. Kamal, and S. Mohyud-Din, "On nonlinear fractional differential equations," *International Journal of Modern Mathematical Sciences*, vol. 3, no. 3, 2012.

[7] Y. Liu and B. Xin, “Numerical solutions of a fractional predator-prey system,” *Advances in Difference Equations*, vol. 2011, 11 pages, 2011.

[8] A. Bhrawy, “A new spectral algorithm for a time-space fractional partial di_erential equations with subdiffusion and superdiffusion,” *Proceedings of the Romanian Academy - Series A: Mathematics, Physics, Technical Sciences, Information Science*, , pp. 39–47, 2016.

[9] S. Kumar, K. S. Nisar, R. Kumar, C. Cattani, and B. Samet, “A new Rabotnov fractional-exponential function-based fractional derivative for diffusion equation under external force,” *Mathematical Methods in the Applied Sciences*, vol. 43, no. 7, pp. 4460–4471, 2020.

[10] A. Shaikh, A. Tassaddiq, K. S. Nisar, and D. Baleanu, “Analysis of differential equations involving Caputo-Fabrizio fractional operator and its applications to reaction-diffusion equations,” *Advances in Difference Equations*, vol. 2019, no. 1, 14 pages, 2019.

[11] N. Valliammal and C. Ravichandran, “Results on fractional neutral integro-differential systems with state dependent delay in banach spaces,” *Nonlinear Stud*, vol. 25, no. 1, pp. 159–171, 2018.

[12] M. M. A. Khater and D. Baleanu, “On abundant new solutions of two fractional complex models,” *Advances in Difference Equations*, vol. 2020, no. 1, pp. 1–14, 2020.

[13] P. Veeresha, D. Prakasha, M. Qurashi, and D. Baleanu, “A reliable technique for fractional modified boussinesq and approximate long wave equations,” *Advances in Difference Equations*, vol. 2019, no. 1, Article ID 253, 2019.

[14] U. Ali, F. A. Abdullah, and S. T. Mohyud-Din, “Modified implicit fractional difference scheme for 2D modified anomalous fractional sub-diffusion equation,” *Advances in Difference Equations*, vol. 2017, no. 1, 14 pages, 2017.

[15] M. Cui, “Convergence analysis of high-order compact alternating direction implicit schemes for the two-dimensional time fractional diffusion equation,” *Numerical Algorithms*, vol. 62, no. 3, pp. 383–409, 2013.

[16] U. Ali and F. A. Abdullah, “Explicit Saul'yev finite difference approximation for two-dimensional fractional sub-diffusion equation,” *AIP Conference Proceedings*, , AIP Publishing LLC, 2018.

[17] M. Khan and N. Ali, “Fourth-order compact iterative scheme for the two-dimensional time fractional sub-diffusion equations,” *Mathematics and Statistics*, vol. 8, no. 2A, pp. 52–57, 2020.

[18] S. K. Panda, T. Abdeljawad, and C. Ravichandran, “Novel fixed point approach to Atangana-Baleanu fractional and Lp-Fredholm integral equations,” *Alexandria Engineering Journal*, vol. 59, pp. 1959–1970, 2020.

[19] K. Logeswari and C. Ravichandran, “A new exploration on existence of fractional neutral integro-differential equations in the concept of Atangana-Baleanu derivative,” *Physica A: Statistical Mechanics and Its Applications*, vol. 544, article 123454, 2020.

[20] R. Subashini, K. Jothimani, K. S. Nisar, and C. Ravichandran, “New results on nonlocal functional integrodifferential equations via Hilfer fractional derivative,” *Alexandria Engineering Journal*, vol. 59, pp. 2891–2899, 2020.

[21] C. Ravichandran, K. Logeswari, S. K. Panda, and K. S. Nisar, “On new approach of fractional derivative by Mittag-Leffler kernel to neutral integro-differential systems with impulsive conditions,” *Chaos, Solitons & Fractals*, vol. 139, article 110012, 2020.

[22] S. Momani, Z. Odibat, and V. S. Erturk, “Generalized differential transform method for solving a space-and time-fractional diffusion-wave equation,” *Physics Letters A*, vol. 370, no. 5-6, pp. 379–387, 2007.

[23] N. Valliammal, C. Ravichandran, and K. S. Nisar, “Solutions to fractional neutral delay differential nonlocal systems,” *Chaos, Solitons & Fractals*, vol. 138, article 109912, 2020.

[24] X. Hu and L. Zhang, “A compact finite difference scheme for the fourth-order fractional diffusion-wave system,” *Computer Physics Communications*, vol. 182, no. 8, pp. 1645–1650, 2011.

[25] J. Yang, J. Huang, D. Liang, and Y. Tang, “Numerical solution of fractional diffusion-wave equation based on fractional multistep method,” *Applied Mathematical Modelling*, vol. 38, no. 14, pp. 3652–3661, 2014.

[26] H. Brunner, H. Han, and D. Yin, “Artificial boundary conditions and finite difference approximations for a time-fractional diffusion-wave equation on a two-dimensional unbounded spatial domain,” *Journal of Computational Physics*, vol. 276, pp. 541–562, 2014.

[27] N. H. Sweilam, M. M. Khader, and A. Nagy, “Numerical solution of two-sided space-fractional wave equation using finite difference method,” *Journal of Computational and Applied Mathematics*, vol. 235, no. 8, pp. 2832–2841, 2011.

[28] N. Sweilam, M. Khader, and M. Adel, “On the stability analysis of weighted average finite difference methods for fractional wave equation,” *Fractional Differential Calculus*, vol. 2, no. 1, pp. 17–29, 2012.

[29] Z. Liu, A. Cheng, and X. Li, “A novel finite difference discrete scheme for the time fractional diffusion-wave equation,” *Applied Numerical Mathematics*, vol. 134, pp. 17–30, 2018.

[30] M. Dehghan and M. Abbaszadeh, “A finite difference/finite element technique with error estimate for space fractional tempered diffusion-wave equation,” *Computers & Mathematics with Applications*, vol. 75, no. 8, pp. 2903–2914, 2018.

[31] Z.-z. Sun and X. Wu, “A fully discrete difference scheme for a diffusion-wave system,” *Applied Numerical Mathematics*, vol. 56, no. 2, pp. 193–209, 2006.

[32] R. Du, Y. Yan, and Z. Liang, “A high-order scheme to approximate the Caputo fractional derivative and its application to solve the fractional diffusion wave equation,” *Journal of Computational Physics*, vol. 376, pp. 1312–1330, 2019.

[33] Y.-N. Zhang, Z.-Z. Sun, and X. Zhao, “Compact alternating direction implicit scheme for the two-dimensional fractional diffusion-wave equation,” *SIAM Journal on Numerical Analysis*, vol. 50, no. 3, pp. 1535–1555, 2012.

[34] J. Huang, Y. Tang, L. Vazquez, and J. Yang, “Two finite difference schemes for time fractional diffusion-wave equation,” *Numerical Algorithms*, vol. 64, no. 4, pp. 707–720, 2013.

[35] L. Wei, “Analysis of a new finite difference/local discontinuous Galerkin method for the fractional diffusion-wave equation,” *Applied Mathematics and Computation*, vol. 304, pp. 180–189, 2017.

[36] U. Ali and F. A. Abdullah, “Modified implicit difference method for one-dimensional fractional wave equation,” *AIP Conference Proceedings*, , AIP Publishing LLC, 2019.

[37] U. Ali, M. Sohail, M. Usman, F. A. Abdullah, I. Khan, and K. S. Nisar, “Fourth-order difference approximation for time-fractional modified sub-diffusion equation,” *Symmetry*, vol. 12, no. 5, p. 691, 2020.

[38] G. Jumarie, “Modified Riemann-Liouville derivative and fractional Taylor series of nondifferentiable functions further results,” *Computers & Mathematics with Applications*, vol. 51, no. 9-10, pp. 1367–1376, 2006.